L^p-CONJECTURE FOR LOCALLY COMPACT GROUPS. I

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Let G be a locally compact group with a left Haar measure 'μ'. Let 'p' and 'r' be two real numbers >1. By $L^{r,p}$ conjecture we mean the following assertion: whenever $f \in L^r(G)$ and $g \in L^p(G)$ we have that the convolution product $f * g$ of f and g is defined and belongs to $L^p(G)$ again if and only if G is compact. By L^p conjecture we mean the assertion above with $r=p$. Both the conjectures were widely believed to be true though there was no written statement about these conjectures until recently. But in 1960, Kunze and Stein [3] showed that the $L^{r,p}$ conjecture is false for the unimodular group of 2×2 real matrices. This naturally raises the question whether the L^p-conjecture is true in general. The first published result on the L^p-conjecture is by Zelazko [9] and Urbanik [7] in 1961. They proved the conjecture to be true for the abelian case. Then in [5] the author established the truth of the L^p-conjecture for discrete groups when $p \geqq 2$. The author announced in that paper that the conjecture is true for all groups when $p>2$ and presented this result to Amer. Math. Soc. in August of 1963 at the Boulder meeting. At the same time Zelazko [10] established the conjecture for all $p>2$ for all unimodular groups. He claims to have established the conjecture for $p>2$ for all groups in that paper but his crucial Lemma 1 of that paper contains a gap in the proof. In a private communication, Zelazko agreed to this gap. The problem is still open in general when $p>1$.

In this paper we prove the following:

The L^p-conjecture is true for all locally compact groups when $p>2$.

The L^p-conjecture is true for totally disconnected groups when $p=2$.

The methods used in this paper yield the truth of the conjecture for all nilpotent groups, and all unimodular C-groups of Iwasawa when $p>1$. But this result will appear elsewhere.

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Notations and conventions. All purely topological notions are taken from [2]. All topological spaces occurring in this paper are taken to be Hausdorff. All notions in topological groups and integration on locally compact groups are

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taken in general from [8]. By a normal algebra we mean a Banach space which is also a ring where multiplication is bicontinuous. A Banach algebra is a normed algebra where we have further the inequality $\|xy\| \leqq \|x\| \|y\|$ for all elements 'x' and 'y' of the algebra. The symbols μ, ν, θ are used for measures. When only one left Haar measure μ is used on a locally compact group G we write sometimes $\int_G f(x)\, dx$ or $\int f(x)\, dx$ instead of $\int_G f(x)\, d\mu(x)$. If $H \subset G$ is a subset of a group G, then $\chi_H(x)$ will denote the characteristic function of H. $L(G)$ will denote the class of complex valued continuous functions on G with compact support.

If G is a locally compact group with a left Haar measure 'μ' and $1 \leqq p < \infty$ then $L^p(G)$ will denote the equivalence classes of Borel measurable functions f on G with complex values such that $\int |f(x)|^p\, dx < \infty$. If $f \in L^p(G)$ then $\|f\|_p$ will denote $(\int_G |f|^p\, dx)^{1/p}$ when $1 \leqq p \leqq \infty$.

1. L^p-conjecture for the case $p > 2$, and some general results.

DEFINITION 1.1. Let G be a locally compact group with a left Haar measure 'μ'. Let f and g be two Borel measurable complex valued functions on G. Then the convolution $f * g$ of 'f' and 'g' is said to exist if the integral $\int_G |f(y)g(y^{-1}x)|\, dy$ exists for almost all $x \in G$. In this case $f * g(x)$ is defined to be $\int_G f(y)g(y^{-1}x)\, dy$. If $1 \leqq p < \infty$, we say that $L^p(G)$ is closed under convolution if whenever f and g belong to $L^p(G)$ we have that $f * g$ exists and again belongs to $L^p(G)$.

L^p-conjecture 1.2. This is the following statement: Let G be a locally compact group with a left Haar measure μ. Then $L^p(G)$ is closed under convolution for some p such that $1 < p < \infty$ if and only if G is compact.

REMARK. The "if" part of the L^p-conjecture is trivial to establish. So we consider the "only if" part in this paper.

THEOREM 1.3. *Let G be a locally compact group with a left Haar measure 'μ'. Let $L^p(G)$ be closed under convolution for some $p > 1$ and $< \infty$. Then $L^p(G)$ is a normed algebra with convolution as multiplication. Moreover, in this case we can choose a suitable left Haar measure μ_1 such that $L^p(\mu_1)$ is a Banach algebra.*

Proof. Let $1/p + 1/q = 1$ and $\Delta(x)$ the modular function of G. Let $f, g \in L^p(G)$ and $h \in L^q(G)$. Let $(f, h) = \int_G f(x)h(x)\, dx$ and let T_f be the operator $g \to f * g$ in $L^p(G)$. Let $\tilde{f}(x) = f(x^{-1})\, \Delta(x^{-1})$. Then by a routine calculation it follows that $(T_f(g), h) = (f * g, h) = (g, \tilde{f} * h)$ for all $f, g \in L^p(G)$ and $h \in L^q(G)$. So by an easy application of the closed graph theorem we get that T_f is continuous in $L^p(G)$. Similarly we get that the right multiplication is continuous in $L^p(G)$. So by an application of the principle of uniform boundedness we get $L^p(G)$ is a normed algebra. So there is a constant K such that $\|f * g\|_p \leqq K\|f\|_p \|g\|_p$ for all $f, g \in L^p(G)$. Now choose a left Haar measure μ_1 on G by the relation $d\mu_1(x) = K^p\, d\mu(x)$. Then $L^p(\mu_1)$ will be a Banach algebra under convolution.

LEMMA 1.4. *Let G be a locally compact group with a left Haar measure μ. Let $H \subset G$ be an open subgroup of G. Let $L^p(G)$ be closed under convolution for some $p>1$. Then $L^p(H)$ is also closed under convolution. If G is the direct product $G_1 \times G_2$ of two closed subgroups G_1 and G_2 with left Haar measures μ_1 and μ_2 respectively and if $L^p(G)$ is closed under convolution then $L^p(G_1)$ and $L^p(G_2)$ are also closed under convolution.*

Proof. Obvious.

LEMMA 1.5. *Let G be a locally compact group with a left Haar measure μ. Let 'p' be a number such that $1<p<\infty$. Let $H \subset G$ be a compact normal subgroup of G. Let $L^p(G)$ be closed under convolution. Then $L^p(G/H)$ is also closed under convolution.*

Proof. Let ν be the normalized Haar measure of H and $\varphi: G \rightarrow G/H$ be the canonical map from G onto G/H. Let θ be a left Haar measure on G/H such that the relation $\int_G f(x)\, d\mu(x) = [\int_{G/H} (\int_H f(tx)\, d\nu(t))\, d\theta(\tilde{x})]$ holds for all $f \in L(G)$ where $\varphi(x)=\tilde{x}$. Then the following relations are easily deduced: If $T(f)=\int_H f(tx)\, d\nu(t)$ where $f \in L(G)$ then

1. $Tf \in L(G/H)$ whenever $f \in L(G)$.
2. T is linear from $L(G)$ onto $L(G/H)$.
3. $T(f * g) = T(f) * T(g)$ for all $f, g \in L(G)$.
4. $\|Tf\|_p = \|f\|_p$ for all $f \in L(G)$.

Now $L^p(G)$ is closed under convolution. So there is a constant K such that $\|f * g\|_p < K\|f\|_p\|g\|_p$ from Theorem 1.3. So we have that $\|\tilde{f} * \tilde{g}\|_p \leqq K\|\tilde{f}\|_p\|\tilde{g}\|_p$ for all $\tilde{f}$ and $\tilde{g} \in L(G/H)$ from 1, 2, 3, and 4 above. Since $p<\infty$, we have that $L(G/H)$ is dense in $L^p(G/H)$. Then we get by repeated use of Fatou's lemma, Fubini's theorem and monotone convergence theorem that if $\tilde{f}$ and $\tilde{g}$ belong to $L^p(G/H)$ then $\tilde{f} * \tilde{g}$ is defined and again belongs to $L^p(G/H)$.

LEMMA 1.6. *Suppose that the L^p-conjecture is true for a number p $(1<p<\infty)$ for all totally disconnected locally compact groups and all connected lie groups. Then the conjecture is true for that 'p' for all locally compact groups.*

Proof. By a theorem of Yamabe [4] every locally compact group G contains an open subgroup H and a compact normal subgroup $N \subset H$ such that H/N is a connected lie group (N is normal with respect to H). So if $L^p(G)$ is closed under convolution then $L^p(H)$ is closed under convolution by Lemma 1.4. So $L^p(H/N)$ is closed under convolution by Lemma 1.5. So H/N is compact by hypothesis of the lemma. So the connected component G_0 of the identity 'e' of G is compact. So $L^p(G/G_0)$ is closed under convolution by Lemma 1.5. So G/G_0 is compact by hypothesis of the lemma. So G is compact.

LEMMA 1.7. *Let G be a locally compact group with a left Haar measure μ. Let V be a compact symmetric neighborhood of the identity e of G such that the group generated by V is not compact. Then the following are true*:

(i) *If the set $\{u(V^{n+1})/\mu(V^n) \mid n=1, 2, \ldots\}$ is bounded then $L^p(G)$ is not closed under convolution for any $p>2$.*

(ii) *If the set $\{\mu(V^{2n})/\mu(V^n) \mid n=1, 2, 3, \ldots\}$ is bounded then $L^p(G)$ is not closed under convolution for any $p>1$.*

Proof. Let there be a constant $k>0$ such that $\mu(V^{n+1})/\mu(V^n) \leqq k$ for all $n=1, 2, 3, \ldots$. Let $\chi_{V^n}(x)$ be the characteristic function of V^n for $n=1, 2, 3, \ldots$. Then

$$\begin{aligned}\chi_{V^n} * \chi_{V^{n+1}}(x) &= \int_G \chi_{V^n}(y)\chi_{V^{n+1}}(y^{-1}x)\, d\mu(y)\\ &\geqq \mu(V^n)\chi_V(x) \quad \text{for all } x \in G \text{ and } n = 1, 2, \ldots.\end{aligned}$$

So $\|\chi_{V^n} * \chi_{V^{n+1}}\|_p \geqq \mu(V^n)(\mu(V))^{1/p}$. So

$$\begin{aligned}\frac{\|\chi_{V^n} * \chi_{V^{n+1}}\|_p}{\|\chi_{V^n}\|_p\|\chi_{V^{n+1}}\|_p} &\geqq \frac{\mu(V^n)(\mu(V))^{1/p}}{(\mu(V^n))^{1/p}(\mu(V^{n+1}))^{1/p}}\\ &= \left(\frac{\mu(V^n)}{\mu(V^{n+1})}\right)^{1/p}(\mu(V))^{1/p}(\mu(V^n))^{1-(2/p)}\\ &\geqq \left(\frac{\mu(V)}{k}\right)^{1/p}(\mu(V^n))^{1-(2/p)}.\end{aligned}$$

Thus $\mathrm{lt}_{n\to\infty} (\|\chi_{V^n} * \chi_{V^{n+1}}\|_p/\|\chi_{V^n}\|_p\|\chi_{V^{n+1}}\|_p)=\infty$ if $p>2$. Thus (i) follows from Theorem 1.3. The statement (ii) can be proved likewise.

LEMMA 1.8. *Let G be a totally disconnected locally compact group with a left Haar measure μ. Let $L^p(G)$ be closed under convolution for a real number p $(1<p<\infty)$. Then there is a maximal compact open subgroup H of G. (That is H is a compact, open subgroup of G and any open compact subgroup of G containing H is H itself.)*

Proof. Since G is totally disconnected, there are compact open subgroups in G (see p. 54 of [4]). Suppose there is an ascending sequence $H_1 \subset H_2 \subset \cdots \subset H_n \subset \cdots$ of compact open subgroups $H_1, H_2, \ldots, H_n, \ldots$ of G. Let $\chi_{H_n}(x)=1$ if $x \in H_n$ and 0 if $x \in G-H_n$. Put $\varphi_n(x)=(\chi_{H_n}(x)/\mu(H_n))$ for $n=1, 2, 3, \ldots$. Then $\varphi_n * \varphi_n=\varphi_n$ for all $n=1, 2, 3, \ldots$. So

$$\frac{\|\varphi_n * \varphi_n\|_p}{\|\varphi_n\|_p^2} = \frac{\|\varphi_n\|_p}{\|\varphi_n\|_p^2} = \frac{1}{\|\varphi_n\|_p} = \frac{\mu(H_n)}{(\mu(H_n))^{1/p}} = (\mu(H_n))^{1-(1/p)}.$$

By Theorem 1.3 the set $\{\mu(H_n) \mid n=1, 2, 3, \ldots\}$ is bounded. Hence there is an n_0 such that $H_{n_0}=H_{n_0+1}=\cdots$. So every ascending sequence of compact, open subgroups of G is finite. Hence the lemma.

LEMMA 1.9. *Let G be a totally disconnected locally compact group with a left Haar measure μ. Let p be a real number $(2<p<\infty)$. Let $L^p(G)$ be closed under convolution. Then G contains an open compact normal subgroup.*

Proof. By Lemma 1.8 there is a maximal, compact, open subgroup H. Now take any element 'a' $\in G-H$ and consider the group generated by $H \cup a^{-1}Ha$. This group should be compact. If not put $V=H \cup a^{-1}Ha$. Then V is a compact symmetric open neighborhood of the identity $e \in G$. Since V^2 is compact there is a finite number of elements $a_1, a_2, \ldots, a_k$ of G such that $V^2 \subset (a_1V) \cup (a_2V) \cup \cdots \cup a_kV$. So $V^{n+1} \subset \bigcup_{i=1}^{k} a_iV^n$ for $n=1, 2, 3, \ldots$. So $\mu(V^{n+1}) \leqq k\mu(V^n)$ for all $n=1, 2, 3, \ldots$. So $(\mu(V^{n+1})/\mu(V^n)) \leqq k$ for all $n=1, 2, 3, \ldots$. Then $L^p(G)$ cannot be closed under convolution by Lemma 1.7 and Lemma 1.4 which contradicts our hypothesis on G. Since H is a maximal open compact subgroup of G we get that $H \cup a^{-1}Ha \subset H$. So $a^{-1}Ha \subset H$ for all 'a' $\in G$. So H is a compact, open, normal subgroup of G.

THEOREM 1.10. *Let G be a locally compact group with a left Haar measure μ. Let $L^p(G)$ be closed under convolution for a real number p $(2<p<\infty)$. Then G must be compact.*

Proof. Let us assume first that G is connected. Let V be a compact symmetric neighborhood of the identity e. Then adopting the proof of Lemma 1.9 we get that the set $\{(\mu(V^{n+1})/\mu(V^n)) \mid n=1, 2, 3, \ldots\}$ is bounded. Then by the connectedness of G and by Lemma 1.7 we get that G is compact. Now let us assume that G is totally disconnected. Then by Lemma 1.9 there exists a compact, open, normal subgroup H of G. Then G/H is a discrete group and $L^p(G/H)$ is closed under convolution by Lemma 1.5. So G/H is finite by Theorem 3 of [5]. Then G must be compact. So if G is either connected or totally disconnected the theorem is true. Now the result follows from Lemma 1.6.

2. The case $p=2$ of the L^p-conjecture.

DEFINITION 2.1. An involution $*$ in an algebra A over complex numbers is a one-to-one map from A onto A such that the following hold:

(i) $(x^*)^*=x$ for all $x \in A$.

(ii) $(\lambda x+\mu y)^*=\bar{\lambda}x^*+\bar{\mu}y^*$ for all complex numbers λ and μ and $x, y \in A$.

(iii) $(xy)^*=y^*x^*$ for all $x, y \in A$.

An A^*-algebra is a Banach algebra B with an involution $*$ and an auxiliary norm $|\,\|\cdot\|\,|$ such that $|\,\|xy\|\,| \leqq |\,\|x\|\,|\;|\,\|y\|\,|$ and $|\,\|xx^*\|\,|=|\,\|x\|\,|^2$ for all $x, y \in B$.

Let B be a Banach algebra over the complex numbers. An element x is said to be in the radical of B if there is an ideal $I \subset B$ such that $x \in I$ and $\mathrm{lt}_{n\to\infty} (\|y^n\|)^{1/n} \to 0$ as $n \to \infty$ for all $y \in I$. The algebra B is said to be semisimple if 0 is the only element in the radical of B.

LEMMA 2.2. *Let G be a unimodular locally compact group with a left Haar measure μ. Let $L^p(G)$ be closed under convolution for some p $(1<p<\infty)$. Then $L^p(G)$ is a semisimple Banach algebra assuming that μ was properly chosen to make $L^p(G)$ a Banach algebra.*

Proof. Let $1/p+1/q=1$. Let $f^*(x)=\overline{f(x^{-1})}$ for all $f \in L^p(G)$. Then, from the fact that G is unimodular, one can check that $f \to f^*$ is an involution in $L^p(G)$. Moreover, by using standard theorems on integration one can show that $(f * g, h) = (g, f^* * h) = (f, h * g^*)$ for all $f, g \in L^p(G)$ and $h \in L^q(G)$ where $(f, h) = \int_G f(x)\overline{h(x)}\, dx$.

From this it follows easily that if $f \in L^p(G)$ and $g \in L^q(G)$ then $f * g \in L^q(G)$ and $\|f * g\|_q \leqq \|f\|_p \|g\|_q$. From this and the fact that $L^p(G)$ is a Banach algebra and the Riesz convexity theorem it follows that $f * g \in L^2(G)$, and $\|f * g\|_2 \leqq \|f\|_p \|g\|_2$ for all $f \in L^p(G)$ and $g \in L^2(G)$. Now put $| \|f\| | = \sup \{\|f * g\|_2 \mid g \in L^2(G) \text{ and } \|g\|_2 = 1\}$. Then it easily follows that $| \|\cdot\| |$ is a norm in $L^p(G)$ and $| \|f * f^*\| | = | \|f\| |^2$ and $| \|f * g\| | \leqq | \|f\| | \, | \|g\| |$ for all $f, g \in L^p(G)$. So $L^p(G)$ is an A^*-algebra. So it is semisimple by a theorem of Rickart (Theorem 4.1.15 of [6]).

THEOREM 2.3. *Let G be a totally disconnected locally compact group with a left Haar measure μ. Let $L^2(G)$ be closed under convolution. Then G is compact.*

Proof. Assume for the moment that G is unimodular. We may as well assume that 'μ' was properly chosen so as to make $L^2(G)$ a Banach algebra. Let $f^*(x) = \overline{f(x^{-1})}$ for all $f \in L^2(G)$. Then, as was shown in the proof of Lemma 2.2, $*$ is an involution in $L^2(G)$ and $(f * g, h) = (g, f^* * h)$ $(f, h * g^*)$ for all $f, g, h \in L^2(G)$ where (f, g) is the inner product in $L^2(G)$. By Lemma 2.2 we have that $L^2(G)$ is semisimple and hence it is a semisimple H^*-algebra of Ambrose (see [1]). Now let K be a compact, open subgroup of G and let $\varphi(x) = \chi_K(x)/\mu(K)$ where $\chi_K(x)$ is the characteristic function of K.

Then $\varphi = \varphi^*$ and $\varphi * \varphi = \varphi$ and $\varphi \in L^2(G)$. So $\varphi * L^2(G) * \varphi$ is a semisimple H^*-algebra with an identity element and hence is finite dimensional (see [1]). But $\varphi * L^2(G) * \varphi$ consists exactly of those functions in $L^2(G)$ which are constant on double cosets modulo K. So the number of such cosets has to be finite and hence G must be compact.

In the general case let $\Delta(x)$ be the modular function of G and let

$$H = \{x \mid \Delta(x) = 1; x \in G\}.$$

Then H contains all compact, open subgroups of G. So H is an open subgroup of G. So $L^2(H)$ is closed under convolution by Lemma 1.4. Clearly H is unimodular. So H is compact by what was shown above. So $L^2(G/H)$ is closed under convolution, by Lemma 1.5. But G/H is a discrete subgroup of the reals. So it is finite by Theorem 2 of [5]. So G is compact again.

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