THE STRICT TOPOLOGY AND COMPACTNESS IN THE SPACE OF MEASURES. II

BY

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The strict topology β on $C(S)$, the bounded continuous complex valued functions on the locally compact Hausdorff space S, was first introduced by R. C. Buck [3], [4], [5]. It has also been studied by I. Glicksberg [10], J. Wells [20], and C. Todd [17]. This topology has been used in the study of various problems in spectral synthesis [11], spaces of bounded holomorphic functions [15], and multipliers of Banach algebras [18], [19]. This paper is a detailed account of results announced by the author in [6], [7] on the relationship of $C(S)_\beta$ with its dual $M(S)$, the bounded Radon measures on S. In particular, we are concerned with the question (posed by Buck) of whether or not $C(S)_\beta$ is a Mackey space and, consequently, with compactness criteria in $M(S)$.

The existence and description of the Mackey topology, the strongest topology yielding a given adjoint space, is known, and there are several properties (e.g., metrizable) which imply that a designated topology is the Mackey topology. However, the author knows of no example of a topological vector space with an intrinsically defined topology which is a Mackey space, except by virtue of some formally stronger property (e.g., metrizable, barrelled, bornological). This is not true for $C(S)_\beta$. In fact, we show that if S is paracompact then every β-weak* countably compact subset of $M(S)$ is β-equicontinuous; consequently, $C(S)_\beta$ is a Mackey space (Theorem 2.6). Also, if S is not compact then $C(S)_\beta$ is not barrelled, bornological, nor metrizable. It can also happen that $C(S)_\beta$ is not a Mackey space, as we show for the case when S is the space of ordinal numbers less than the first uncountable ordinal.

In §3 we examine the subspace problem for $C(S)_\beta$. That is, if $C(S)_\beta$ is a Mackey space, which subspaces of $C(S)$ are Mackey spaces when furnished with the relative strict topology? We are able to solve this problem when S is the space of positive integers. Also, we show that H^∞, the bounded holomorphic functions on the open unit disk D, is not a Mackey space when endowed with the β topology—even though $C(D)_\beta$ is. From these results we prove the existence of a closed subspace N of l^1 such that there is no bounded projection of l^1 onto N. Finally, (l^∞, β) is a semi-reflexive Mackey space with a closed subspace which is not a Mackey space.

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1. **Notation and preliminaries.** We will let $C_0(S)$ denote the subspace of $C(S)$ consisting of all those functions which vanish at infinity, and $C_c(S)$ those which vanish off some compact set. If ϕ is in $C(S)$ then $N(\phi)=\{s : \phi(s)\neq 0\}$, spt $(\phi)=$ the closure of $N(\phi)$, and $\|\phi\|_\infty=\sup\{|\phi(s)| : s\in S\}$. Finally, $V_\phi=\{f\in C(S) : \|\phi f\|_\infty\leqq 1\}$.

The strict topology β on $C(S)$ has as a neighborhood basis at the origin the collection of sets

$$\{V_\phi : \phi\in C_0(S), \phi\geqq 0\};$$

that is, it is defined by the seminorms $f\to\|\phi f\|_\infty$ on $C(S)$, where $\phi\in C_0(S)$. The compact open topology on $C(S)$ (denoted by c-op) has as a neighborhood basis at the origin the sets of the form V_ϕ where $\phi\in C_c(S)$ and $\phi\geqq 0$. It is known that $C(S)_\beta$ is complete, $C_c(S)$ is β-dense in $C(S)$, the norm bounded and β-bounded subsets of $C(S)$ are the same, the β and c-op topologies agree on norm bounded subsets of $C(S)$, and $C(S)^*_\beta=M(S)$ where the correspondence is by integration (see [5] for the proofs of these and other basic facts about the strict topology).

A topological vector space E is a Mackey space if and only if every weak* compact convex circled subset of E^* is equicontinuous. We will say that E is a *strong Mackey space* if and only if every weak* compact subset of E^* is equicontinuous.

If A is a subset of a topological space X then A is *sequentially compact* if and only if every sequence in A has a subsequence which converges to some point in X; A is *countably compact* if and only if every sequence in A has a cluster point in X; A is *conditionally compact* if and only if A^- (the closure of A) is compact. Our remaining terminology will be standard as is found in [8], [13], and [12].

2. **The Mackey topology on $C(S)_\beta$ and the main theorem.** In order to discover whether or not $C(S)_\beta$ is a Mackey space we must first characterize the β-equicontinuous subsets of $M(S)$. A set $H\subset M(S)$ is *β-equicontinuous* if and only if

$$H^0\equiv\left\{f\in C(S) : \left|\int f\,d\mu\right|\leqq 1 \text{ for all } \mu\in H\right\}$$

is a β-neighborhood of zero in $C(S)$. Since the sets V_ϕ form a neighborhood basis at zero, this is equivalent to saying that $V_\phi\subset H^0$ for some $\phi\in C_0(S)$; or, $H\subset V^0_\phi\subset M(S)$ where

$$V^0_\phi\equiv\left\{\mu\in M(S) : \left|\int f\,d\mu\right|\leqq 1 \text{ for all } f\in V_\phi\right\}.$$

Hence we must characterize the sets V^0_ϕ where ϕ is in $C_0(S)$. This is done in the following theorem of I. Glicksberg [10]. For the outline of a proof based on functional analysis rather than measure theory the reader may refer to [6, Theorem 1].

THEOREM 2.1. *If ϕ is in $C_0(S)$ then*

$$V^0_\phi=\{\mu\in M(S) : \mu \textit{ vanishes off } N(\phi) \textit{ and } \|\mu/\phi\|\leqq 1\}.$$

(Here, μ/ϕ denotes the measure ν such that $\nu(A)=\int_A (1/\phi)\,d\mu$.)

This criterion for β-equicontinuity is, however, not sufficient for our purposes. An alternate characterization, which will be used in the proof of our main theorem, is provided by the following.

THEOREM 2.2. *A set $H \subset M(S)$ is β-equicontinuous if and only if H is uniformly bounded and for every $\varepsilon > 0$ there is a compact set $K \subset S$ such that $|\mu|(S \backslash K) \leqq \varepsilon$ for all μ in H.*

Proof. Suppose that H is β-equicontinuous; then there is a nonnegative function ϕ in $C_0(S)$ such that $H \subset V_\phi^0$. It easily follows from Theorem 2.1 that H is uniformly bounded by $\|\phi\|_\infty$. If $\varepsilon > 0$ and we let $K = \{s : \phi(s) \geqq \varepsilon\}$ then K is compact and, again applying Theorem 2.1, we have that

$$\begin{aligned} |\mu|(S \backslash K) &= \int_{S \backslash K} \phi(1/\phi)\, d|\mu| \\ &\leqq \|\mu/\phi\| \sup\{\phi(s) : s \notin K\} \\ &\leqq \varepsilon. \end{aligned}$$

For the converse suppose that $\|\mu\| \leqq 1$ for all μ in H and that the condition is satisfied. By induction we can obtain a sequence $\{K_n\}$ of compact subsets of S such that $K_n \subset \operatorname{int} K_{n+1}$ (the interior of K_{n+1}) and $|\mu|(S \backslash K_n) \leqq (\frac{1}{4})^n$ for all μ in H. For each integer $n \geqq 1$ let $\phi_n \in C_c(S)$ such that $\phi_n(K_n) = 1$, $0 \leqq \phi_n \leqq 1$, and $\phi_n(s) = 0$ for $s \notin K_{n+1}$. Put

$$\phi(s) = 2 \sum_{n=1}^{\infty} (\tfrac{1}{2})^n \phi_n(s), \qquad s \in S.$$

Then $\phi \in C_0(S)$ and $N(\phi) = \bigcup_{n=1}^{\infty} K_n$. We will now show that H is contained in V_ϕ^0 and is, therefore, β-equicontinuous. If $A \in \mathit{Borel}\,(S)$ (the Borel subsets of S) and $A \cap N(\phi) = \square$ then $A \cap K_n = \square$ for all $n \geqq 1$; thus $|\mu|(A) \leqq (\frac{1}{4})^n$ for all n and so each μ in H vanishes off $N(\phi)$. If $s \in K_n \backslash K_{n-1}$, $n \geqq 2$, then $\phi_1(s) = \cdots = \phi_{n-2}(s) = 0$ and $\phi_k(s) = 1$ for $k \geqq n$. Therefore

$$\begin{aligned} \phi(s) &= 2 \sum_{k=n-1}^{\infty} (\tfrac{1}{2})^k \phi_k(s) \\ &\geqq 2 \sum_{k=n}^{\infty} (\tfrac{1}{2})^k = 2(\tfrac{1}{2})^{n-1}. \end{aligned}$$

If μ is in H and $n \geqq 2$ then

$$\int_{K_n \backslash K_{n-1}} (1/\phi)\, d|\mu| \leqq (\tfrac{1}{2}) 2^{n-1} (\tfrac{1}{4})^{n-1} \leqq (\tfrac{1}{2})^n$$

and so

$$\begin{aligned} \int (1/\phi)\, d|\mu| &= \int_{K_1} (1/\phi)\, d|\mu| + \sum_{n=2}^{\infty} \int_{K_n \backslash K_{n-1}} (1/\phi)\, d|\mu| \\ &\leqq (\tfrac{1}{2})\|\mu\| + \sum_{n=2}^{\infty} (\tfrac{1}{2})^n \leqq 1. \end{aligned}$$

This completes the proof of the theorem.

The plan for proving Theorem 2.6 will be to first prove it when S is the space of positive integers and then reduce the general case to this one.

LEMMA 2.3. *The strong topology on $M(S)=C(S)^*_\beta$ is exactly the norm topology. Hence $C_0(S)$ and $C(S)_\beta$ have the same second adjoint space. Also, $C(S)_\beta$ is semi-reflexive if and only if S is discrete.*

Proof. The strong topology on $M(S)=C(S)^*_\beta$ is, by definition, the topology of uniform convergence on β-bounded subsets of $C(S)$. But the β-bounded and norm bounded subsets of $C(S)$ are the same [5, p. 98], and so the strong topology on $M(S)$ is the norm topology.

If S is discrete then it is immediate that $C(S)_\beta$ is semireflexive. For the converse let $s\in S$ and put $L(\mu)=\mu(\{s\})$ for all μ in $M(S)$. Then $L\in M(S)^*$ so that if $C(S)_\beta$ is semireflexive there is a function f in $C(S)$ such that $L(\mu)=\int f\,d\mu$ for all $\mu\in M(S)$. It now follows that f is the characteristic function of $\{s\}$, which, consequently, must be open. This completes the proof.

Note that if S is the space of positive integers with the discrete topology then $C(S)=l^\infty$, $M(S)=l^1$, and $C_0(S)=c_0$. Also, from Theorem 2.2, a set $H\subset l^1$ is β-equicontinuous if and only if it is bounded and for every $\varepsilon>0$ there is an integer N such that $\sum_{n=N+1}^\infty |a_n|\leqq\varepsilon$ for every $\alpha=\{a_n\}_{n=1}^\infty$ in H.

Before proceeding let us make a distinction between two weak star topologies on $M(S)$. We will denote by *β-weak** the weak star topology which $M(S)$ has as the dual of $C(S)_\beta$. This will distinguish it from the *weak** topology which $M(S)$ has as the dual of the Banach space $C_0(S)$.

THEOREM 2.4. *If $H\subset l^1$ then the following are equivalent:*
(a) *H is weakly conditionally compact;*
(b) *H is β-weak* conditionally compact;*
(c) *H is norm conditionally compact;*
(d) *H is β-equicontinuous.*

Proof. It is clear that (a) and (b) are equivalent because (l^∞,β) is semireflexive; also it is trivial that (d) implies (b). To see that (a) implies (c) let $\{\xi_n\}$ be a sequence in H. Then there is a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ which converges weakly to some element ξ of l^1 [8, p. 430]. But it is a well-known result of J. Schür (e.g., see p. 296 of [8]) that weak and norm sequential convergence are the same in l^1. Thus, $\{\xi_{n_k}\}$ converges to ξ in norm and H is norm conditionally compact.

If (c) holds and we wish to establish (d) then let $\varepsilon>0$ and choose $\xi_1,\dots,\xi_n$ in H such that

$$H\subset\bigcup_{k=1}^n\{\xi:\|\xi-\xi_k\|<\varepsilon/2\}.$$

If for $k=1,\dots,n$ we let $\xi_k=\{x_i^{(k)}\}_{i=1}^\infty$ then we may find an integer N such that

$$\sum_{i=N+1}^\infty |x_i^{(k)}|<\varepsilon/2$$

for $1 \leqq k \leqq n$. From here it easily follows that

$$\sum_{i=N+1}^{\infty} |x_i| < \varepsilon$$

for all $\xi = \{x_i\}_{i=1}^{\infty}$ in H. This completes the proof of the theorem.

COROLLARY 2.5. *The space (l^{∞}, β) is a strong Mackey space.*

We are now in a position to prove the principal result of this paper.

THEOREM 2.6. *If S is paracompact and $H \subset M(S)$ is β-weak* countably compact then H is β-equicontinuous. Consequently, $C(S)_{\beta}$ is a strong Mackey space whenever S is paracompact.*

Proof. We will prove the theorem for the case when S is σ-compact (i.e., S is the countable union of compact sets), and, afterwards, indicate the proof for the general case. Therefore, let $S = \bigcup_{n=1}^{\infty} D_n$ where each D_n is compact and $D_n \subset \text{int } D_{n+1}$. Suppose that H is not β-equicontinuous. Since H is β-weak* countably compact, it follows that H is weak* bounded and hence uniformly bounded. Thus, Theorem 2.2 implies there is an $\varepsilon > 0$ such that for every compact set $K \subset S$, $|\mu|(S \backslash K) > \varepsilon$ for some μ in H. We can now prove the following

Claim 1. There exists a sequence of quadruples $\{(\mu_n, \phi_n, K_n, U_n)\}_{n=1}^{\infty}$ having the following properties:

(a) $\mu_n \in H$, $\phi_n \in C_c(S)$, K_n is compact, U_n is open in S with U_n^- compact and $U_n^- \cap K_n = \square$;

(b) $D_n \cup K_n \cup U_n^- \subset \text{int } K_{n+1}$;

(c) $|\mu_n(U_n)| > \varepsilon/4$;

(d) $\|\phi_n\|_{\infty} = 1$, spt $(\phi_n) \subset U_n$, and $|\mu_n|(U_n) < |\int \phi_n \, d\mu_n| + \varepsilon/8$.

If this is established, we can then prove the following facts.

Claim 2. $S = \bigcup_{n=1}^{\infty} \text{int } K_n$.

Claim 3. $F = \bigcup_{n=1}^{\infty} \text{spt } (\phi_n)$ is closed.

Claim 4. If $\xi = \{x_n\}_{n=1}^{\infty}$ is in l^{∞} then

$$f_{\xi}(s) = \sum_{n=1}^{\infty} x_n \phi_n(s),$$

for all s in S, defines an element f_{ξ} of $C(S)$ and $\|f_{\xi}\|_{\infty} = \|\xi\|_{\infty}$.

Claim 5. The map $T: (l^{\infty}, \beta) \to C(S)_{\beta}$ defined by $T(\xi) = f_{\xi}$ for all ξ in l^{∞} is a continuous linear map.

If we assume these claims for the time being (we will return to their proofs afterwards), we may complete the proof of the theorem. By Claim 5, T has a well-defined adjoint map $T^*: M(S) \to l^1$, which is continuous when both range and domain have their β-weak* topologies. Thus, $T^*(H)$ is a β-weak* (i.e., weak) countably compact subset of l^1. By a theorem of Eberlein [8, p. 430], $T^*(H)$ is weakly conditionally compact and hence, by Theorem 2.4, β-equicontinuous in l^1. Now if μ is in $M(S)$ and $\xi=\{x_n\}_{n=1}^{\infty}$ is in l^{∞} then

$$\langle \xi, T^*(\mu)\rangle = \int f_\xi \, d\mu = \sum_{n=1}^{\infty} x_n \int \phi_n \, d\mu,$$

so that

$$T^*(\mu) = \left\{\int \phi_n \, d\mu\right\}_{n=1}^{\infty}.$$

Interpreting our β-equicontinuity condition for $T^*(H)\subset l^1$, we can find an integer N such that

$$\sum_{n=N+1}^{\infty} \left|\int \phi_n \, d\mu\right| < \varepsilon/8$$

for all μ in H. In particular, if $n>N$ then $|\int \phi_n \, d\mu_n|<\varepsilon/8$; but

$$|\mu_n(U_n)| \leqq |\mu_n|(U_n) < \left|\int \phi_n \, d\mu_n\right| + \varepsilon/8$$

by Claim 1 (d). Combining these last two facts we obtain a contradiction to Claim 1 (c) whenever $n>N$. This completes the proof, modulo substantiating our five claims.

Proof of Claim 1. Let $K_1=D_1$; then there is a measure μ_1 in H such that $|\mu_1|(S\backslash K_1)>\varepsilon$. Hence, there is a compact set $C\subset S\backslash K_1$ such that $|\mu_1(C)|>\varepsilon/4$ (combine inner regularity with a well-known result [8, p. 97]). Choose an open set U_1 such that U_1^- is compact, $C\subset U_1\subset U_1^-\subset S\backslash K_1$, and $|\mu_1|(U_1\backslash C)<(\frac{1}{2})[|\mu_1(C)|-\varepsilon/4]$. Thus $|\mu_1(U_1)|>\varepsilon/4$. Restricting the measure μ_1 to U_1 and applying the Riesz representation theorem, we can find a function $\phi_1\in C_c(S)$ such that $\|\phi_1\|_\infty=1$, spt $(\phi_1)\subset U_1$ and $|\mu_1|(U_1)<|\int \phi_1 \, d\mu_1|+\varepsilon/8$.

Since $K_1 \cup U_1^-$ is compact we can find a compact set K_2 with $K_1 \cup U_1^- \subset \text{int } K_2$. The rest of the induction follows in a similar manner.

Proof of Claim 2. This is obvious from (b) of Claim 1, since $S=\bigcup_{n=1}^{\infty} D_n$.

Proof of Claim 3. If $s\in F^-$ then $s\in \text{int } K_n$ for some $n\geqq 1$. Hence, for every open neighborhood W of s contained in K_n we have that

$$W \cap \bigcup_{i=1}^{n-1} \text{spt}(\phi_i) = W\cap F \neq \square.$$

Thus, $s\in \bigcup_{i=1}^{n-1} \text{spt}(\phi_i)\subset F$ and F is closed.

Proof of Claim 4. Clearly f_ξ is well defined since at most one term in the sum is not zero. Also, for this same reason, $|f_\xi(s)|=\sum_{n=1}^{\infty} |x_n| \, |\phi_n(s)|$ for all $s\in S$; and, since each $|\phi_n|$ achieves its maximum of 1, $\|f_\xi\|_\infty=\|\xi\|_\infty$. To see that f_ξ is continuous let $s\in S$ and let $\{s_i\}$ be a net in S such that $s_i\to s$. If $s\notin F$ then F is closed implies there is an i_0 such that for $i\geqq i_0$ $s_i\notin F$. Hence, for $i\geqq i_0$ $f_\xi(s_i)=f_\xi(s)=0$ and $f_\xi(s_i)\to f_\xi(s)$.

If $s \in F$ then $s \in \operatorname{spt}(\phi_n) \subset U_n$ for a unique integer n. Therefore there is an i_0 such that for $i \geqq i_0$ $s_i \in U_n$. But then for $i \geqq i_0$ we have

$$f_\xi(s_i) = x_n\phi_n(s_i) \to x_n\phi_n(s) = f_\xi(s),$$

and so $f_\xi \in C(S)$.

Proof of Claim 5. Clearly T is linear. Let $\{\xi_i\}$ be a net in l^∞ such that $\xi_i \to 0\ (\beta)$; we must show that $T(\xi_i) \to 0\ (\beta)$ in $C(S)$. Toward this end, let $\phi \in C_0(S)$. If $\varepsilon > 0$ then there is an integer N such that for $s \notin K_N$, $|\phi(s)| \leqq \varepsilon$. Thus, for $n \geqq N$, $\operatorname{spt}(\phi_n) \cap K_N = \square$ implies $\|\phi\phi_n\|_\infty \leqq \varepsilon$. That is,

$$\xi = \{\|\phi\phi_n\|\}_{n=1}^\infty \in c_0.$$

Since $\xi_i \to 0\ (\beta)$, there is for every $\varepsilon > 0$ an i_ε such that $\|\xi\xi_i\|_\infty \leqq \varepsilon$ for $i \geqq i_\varepsilon$. If $\xi_i = \{x_n^{(i)}\}_{n=1}^\infty$ for every i then for $i \geqq i_\varepsilon$ we have

$$\begin{aligned}\|\phi T(\xi_i)\|_\infty &= \sup\left\{\sum_{n=1}^\infty |x_n^{(i)}|\,|\phi(s)\phi_n(s)| : s \in S\right\}\\ &= \sup\{|x_n^{(i)}|\,|\phi(s)\phi_n(s)| : s \in S, n \geqq 1\}\\ &= \sup\{|x_n^{(i)}|\,\|\phi\phi_n\|_\infty : n \geqq 1\} \leqq \varepsilon.\end{aligned}$$

Thus, $T(\xi_i) \to 0\ (\beta)$ and T is continuous.

This concludes the proof of the theorem.

Note that the reason for condition (b) on the sequence of quadruples was to ensure that $\bigcup_{n=1}^\infty K_n$ was both open and closed. From this it followed that F was closed and f_ξ was continuous for every ξ in l^∞. This same method yields a proof if S is a locally compact topological group, and (b) is replaced by the requirement that each K_n be a compact symmetric neighborhood of the identity and $K_n^2 \cup U_n^- \subset \operatorname{int} K_{n+1}$. Then $\bigcup_{n=1}^\infty K_n$ is an open subgroup of S and therefore also closed. Both of these proofs can be subsumed under the proof of the case where S is paracompact. By a standard theorem on paracompactness [1, p. 107], S can be expressed as the union of a collection $\{S_a : a \in A\}$ of pairwise disjoint open and closed σ-compact subsets of S. Let $S_a = \bigcup_{n=1}^\infty C(a, n)$, where each $C(a, n)$ is compact and $C(a, n) \subset \operatorname{int} C(a, n+1)$ for $n \geqq 1$ and $a \in \mathscr{A}$. By an induction argument similar to that used in the above proof, we obtain a strictly increasing sequence of integers $\{k_n\}$ and a sequence $\{a_n\}$ in $\mathscr{A}$, as well as the sequence $\{(\mu_n, \phi_n, K_n, U_n)\}$. This sequence of quadruples has all the properties it had in the proof of the theorem except that condition (b) is replaced by

$$\text{(b}'\text{)}\quad K_n = \bigcup_{i=1}^{k_n} C(k_n, a_i) \quad \text{and} \quad U_n^- \subset \operatorname{int} K_{n+1}.$$

We now proceed as above and

$$\begin{aligned}\bigcup_{n=1}^\infty K_n &= \bigcup_{n=1}^\infty \bigcup_{i=1}^{k_n} C(k_n, a_i)\\ &= \bigcup_{n=1}^\infty S_{a_n}\end{aligned}$$

since $k_{n+1} > k_n$. Thus $\bigcup_{n=1}^\infty K_n$ is both open and closed.

REMARKS. There is a class of spaces for which the preceding method of proof cannot succeed—namely, the pseudocompact, noncompact spaces. In such a space, no matter how the sets K_n are chosen F will never be closed [9].

There are spaces for which Theorem 2.6 does not hold, as the following example (of a pseudocompact space) illustrates. Let Ω_0 be the space of ordinal numbers less than the first uncountable ordinal with the order topology. We will show that $C(\Omega_0)_\beta$ is not a Mackey space.

Let H be the β-weak* closed convex circled hull of the set of all measures of the form

$$\tfrac{1}{2}[\delta_s-\delta_{s+1}]$$

where s is a nonlimit ordinal and $s+1$ is its immediate successor. We will need the following facts: Ω_0 is not σ-compact; every continuous function f on Ω_0 is eventually constant (i.e., there is an x in Ω_0 such that $f(y)=f(x)$ whenever $y \geqq x$), and hence, the Stone-Čech and one-point compactifications of Ω_0 are the same [12]. Also, if $s \in \Omega_0$ then the characteristic function of $[1, s]$ is continuous.

Let Ω be the first uncountable ordinal and Ω_1 the Stone-Čech compactification of Ω_0. Hence $\Omega_1=\Omega_0 \cup \{\Omega\}$ and $M(\Omega_1)=M(\Omega_0) \oplus \mathbf{C}\{\delta_\Omega\}$, where

$$\mathbf{C}\{\delta_\Omega\} = \{c\delta_\Omega : c \in \mathbf{C}\}.$$

We claim that if we consider H as a subset of $M(\Omega_1)$ then H is weak* closed. If this is so then we would have H weak* compact in $M(\Omega_1)$ since H is clearly contained in the unit ball [8, p. 424]. But the weak* topology of $M(\Omega_1)$ relativized to $M(\Omega_0)$ is the β-weak* topology and this would imply that H is β-weak* compact. Since, by Theorem 2.2, H is not β-equicontinuous, we would have that $C(\Omega_0)_\beta$ is not a Mackey space.

To prove the above claim, suppose that μ is in the weak* closure of H in $M(\Omega_1)$. Then there is a unique measure ν in $M(\Omega_0)$ and a scalar c such that $\mu=\nu+c\delta_\Omega$. We must show that $c=0$. Clearly $\lambda(\Omega_1)=0$ for each λ in H, and so $\mu(\Omega_1)=0$. Thus, $c=-\nu(\Omega_1)=-\nu(\Omega_0)$. Since Ω_0 is not σ-compact and ν vanishes off a σ-compact set, there is a limit ordinal $x<\Omega$ such that ν vanishes off $[1, x]$. If s is any nonlimit ordinal then either $s<x$ or $x<s$. If $s<x$ then $s+1<x$; so if f is the characteristic function of $[1, x]$, $f(s)=f(s+1)=1$. If $x<s$ then $f(s)=f(s+1)=0$. In either case $\int f\,d\lambda=0$ for all λ in H and this implies

$$0 = \int f\,d\mu = \int f\,d\nu+cf(\Omega) = \int f\,d\nu = \nu([1, x]) = \nu(\Omega_0).$$

Thus $c=0$ and $\mu=\nu \in M(\Omega_0)$. But the weak* topology of $M(\Omega_1)$ relativized to $M(\Omega_0)$ is the β-weak* topology and H is β-weak* closed. Therefore $\mu \in H$ and $C(\Omega_0)_\beta$ is not a Mackey space.

3. **Subspaces of $C(S)_\beta$, the space (l^∞, β), and (H^∞, β).** A natural question to ask is whether or not a subspace of $C(S)_\beta$ is a Mackey space provided $C(S)_\beta$ is. Along these lines it is known that the completion of a Mackey space is a Mackey space but the converse is false. In fact, $C(S)_\beta$ is the completion of $C_0(S)_\beta$, but if S is not

compact then $C_0(S)_\beta$ is not a Mackey space since the norm topology yields $M(S)$ as the adjoint of $C_0(S)$ and is properly stronger than β.

The difficulties in attacking the general problem may be visualized as follows. Let E be a β-closed subspace of $C(S)$ and let $i: E_\beta \to C(S)_\beta$ be the injection mapping with $i^*: M(S) \to E_\beta^*$ its adjoint map. In order to show that a subset $H \subset E_\beta^*$ is β-equicontinuous it is necessary and sufficient to find a β-equicontinuous set $H_1 \subset M(S)$ such that $i^* H_1 = H$. Therefore if $C(S)_\beta$ is a Mackey space and $H \subset E_\beta^*$ is a β-weak* compact convex circled set then to show that H is β-equicontinuous we must find a β-weak* compact convex circled subset $H_1 \subset M(S)$ such that $i^* H_1 = H$. Since E_β^* with its β-weak* topology is topologically isomorphic to some quotient space of $M(S)$ and i^* acts like the canonical map, it would seem that what is needed is a version of a theorem of Bartle and Graves (e.g., see p. 375 of [14]), where both domain and range have their β-weak* topologies. No such result is presently available, although in the case of l^∞ this theorem can be used to great advantage (see Theorem 3.2 below).

Recall that we used a theorem of J. Schür to prove that (l^∞, β) is a strong Mackey space. It is not difficult to prove Schür's result if we assume that (l^∞, β) is a strong Mackey space. A statement similar to Schür's theorem will be what is needed to characterize the β-closed subspaces of l^∞ which are strong Mackey spaces.

LEMMA 3.1. *Let E be a β-closed subspace of l^∞. Then E_β is semireflexive and E_β^* with its strong topology is a Banach space. Consequently, the β-weak* topology on E_β^* is exactly its weak topology which it has as a Banach space.*

Proof. Since (l^∞, β) is semireflexive (Lemma 2.3) E_β is semireflexive. Also, E_β^* is topologically isomorphic to l^1/N with its quotient norm, where $N = E^\perp \subset l^1$ [13, p. 190]. Hence E_β^* is a Banach space.

THEOREM 3.2. *If E is a closed subspace of (l^∞, β) then a subset of E_β^* is β-equicontinuous if and only if it is norm conditionally compact.*

Proof. Let $H \subset E_\beta^*$ be β-equicontinuous and let $i: E_\beta \to (l^\infty, \beta)$ be the injection map. Then there is a $\phi \in c_0$ such that $i^*(V_\phi^0) \supset H$, where $i^*: l^1 \to E_\beta^*$ is the adjoint map of i. But V_ϕ^0 is norm compact in l^1 by Theorem 2.4 and, since i^* is norm continuous, we have that H has norm compact closure in E_β^*.

Assume that H is norm compact. Then E_β^* is a Banach space and i^* is a bounded linear map of l^1 onto E_β^*. By the Bartle-Graves selection theorem [14, p. 375], there exists a continuous function $f: E_\beta^* \to l^1$ such that $f(I) \in i^{*-1}(I)$ for all I in E_β^*. Therefore $f(H)$ is norm compact in l^1 and $i^*(f(H)) = H$ imply that H is β-equicontinuous.

THEOREM 3.3. *If E is a β-closed subspace of l^∞ then the following are equivalent:*

(a) *E_β is a Mackey space;*

(b) *E_β is a strong Mackey space;*

(c) *every β-weak* compact set in E_β^* is norm compact;*

(d) *every β-weak* convergent sequence in E_β^* is norm convergent.*

Proof. (a) $\Leftrightarrow$ (b). Since the β-weak* and weak topologies on E_β^* are the same (Lemma 3.1), the β-weak* closed convex circled hull of a β-weak* compact subset of E_β^* is β-weak* compact [8, p. 434].

(b) $\Rightarrow$ (c). If $H \subset E_\beta^*$ is β-weak* compact then H is β-equicontinuous and hence norm compact by Theorem 3.2.

(d) $\Rightarrow$ (b). If H is β-weak* compact then by Lemma 3.1 and [8, p. 430] it is β-weak* sequentially compact. It now follows from (d) that H is norm compact and, hence, β-equicontinuous.

To complete the proof of the theorem we need only show that (c) $\Rightarrow$ (d), and this is a triviality.

Now let us turn our attention to H^∞, the space of bounded holomorphic functions on the open unit disk D. Theorems 3.4 and 3.5 below were obtained by Rubel and Shields [15]. We present them here because they form a direct path to our result that (H^∞, β) is not a Mackey space.

We follow a method of Brown, Shields, and Zeller [2] and find a sequence $\{a_n\}$ in D having no limit points in D, and such that

$$\|f\|_\infty = \sup\{|f(a_n)| : n \geqq 1\}$$

for all f in H^∞. Hence, if E is the subspace of l^∞ consisting of all sequences $\{f(a_n)\}_{n=1}^\infty$ where f is in H^∞ then $T: H^\infty \to E$ defined by

$$T(f) = \{f(a_n)\}_{n=1}^\infty$$

for f in H^∞, is an isometry. Moreover, since $\{a_n\}$ has no limit points in D it is a discrete sequence. Thus, if $\{f_i\}$ is a net in H^∞ such that $f_i \to 0$ (β) and $\xi = \{x_n\}_{n=1}^\infty$ is a sequence in c_0, then there is a function ϕ in $C_0(D)$ such that $\phi(a_n) = x_n$ for all $n \geqq 1$ [5, p. 96]. If $\varepsilon > 0$ then there is an i_0 such that for $i \geqq i_0$ $\|f_i\phi\|_\infty \leqq \varepsilon$. But this implies that for $i \geqq i_0$

$$\begin{aligned}\|T(f_i)\xi\|_\infty &= \sup\{|f_i(a_n)x_n| : n \geqq 1\} \\ &= \sup\{|f_i(a_n)\phi(a_n)| : n \geqq 1\} \\ &\leqq \|f_i\phi\|_\infty \leqq \varepsilon,\end{aligned}$$

and so $T(f_i) \to 0$ (β) in E. Therefore

$$T: (H^\infty, \beta) \to E_\beta$$

is continuous. Not only will we use this fact later but the continuity of T can be used to give an alternate proof of the following results of Rubel and Shields (see p. 80 of [7]).

THEOREM 3.4. *A subset of (H^∞, β) is compact if and only if it is closed and bounded.*

THEOREM 3.5. *If I is a linear functional on H^∞ which is β-continuous on the unit ball of H^∞ then I is β-continuous on H^∞. Hence, $(H^\infty, \beta)^*$ with its strong topology is a Banach space and H^∞ is its adjoint.*

COROLLARY 3.6. *A linear functional I on H^∞ is β-continuous if and only if there is a Lebesgue integrable function g on $[-\pi, \pi]$ such that*

$$I(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})g(\theta)\, d\theta$$

or all f in H^∞.

Proof. Let g be a Lebesgue integrable function on $[-\pi, \pi]$ and let I be defined as above. If B is the closed unit ball in H^∞ then the β and c-op topologies agree on B and so (B, β) is metrizable. If $\{f_n\}$ is a sequence in B such that $f_n \to f$ in (B, β) then identify each f_n and f with its boundary values on the unit circle. Hence $\{f_n\}$ is a sequence in the unit ball of $L^\infty = L^\infty(-\pi, \pi)$, the adjoint of the separable Banach space $L^1 = L^1(-\pi, \pi)$. Hence, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and an element h in the unit ball of L^∞ such that $f_{n_k} \to h\ \sigma(L^\infty, L^1)$ (i.e., the topology induced on L^∞ by L^1). This implies that

$$\hat{h}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)e^{-in\theta}\, d\theta = \lim_k \hat{f}_{n_k}(n)$$

for $n=0, \pm 1, \pm 2, \ldots$ (here, $\hat{h}$ is the Fourier-Stieltjes transform of h [16]). Hence $\hat{h}(n)=0$ for $n<0$ and for $n \geqq 0$ $\hat{f}_{n_k}(n)=(1/n!)f_{n_k}^{(n)}(0)$. But $f_{n_k} \to f\ (\beta)$ in H^∞ implies

$$\begin{aligned}\hat{f}(n) &= (1/n!)f^{(n)}(0) = \lim_k\, (1/n!)f_{n_k}^{(n)}(0) \\ &= \lim_k \hat{f}_{n_k}(n).\end{aligned}$$

Therefore $\hat{h}(n)=\hat{f}(n)$ for all n and so $h(\theta)=f(e^{i\theta})$ for almost all θ. What we have shown is that every $\sigma(L^\infty, L^1)$ convergent subsequence of $\{f_n\}$ converges to f. Since the unit ball of L^∞ is $\sigma(L^\infty, L^1)$ compact, this implies that $f_n \to f\ \sigma(L^\infty, L^1)$. Therefore $I(f_n) \to I(f)$ and I is continuous on (B, β). By the above theorem we have that I is in $(H^\infty, \beta)^*$.

For the converse let $u\colon H^\infty \to L^\infty$ be the imbedding which takes each function in H^∞ into the class in L^∞ represented by its boundary values. The first part of this proof established that u is continuous if H^∞ has its weak topology from $(H^\infty, \beta)^*$ and L^∞ its $\sigma(L^\infty, L^1)$ topology. If $E=u(H^\infty)$ and I is in $(H^\infty, \beta)^*$ then, by an argument similar to that used in the alternate proof of the preceding theorem of Rubel and Shields (see Theorem 5 of [7]), we have that $I \circ u^{-1}$ is a $\sigma(L^\infty, L^1)$-continuous linear functional on E. As such there is a Lebesgue integrable function g on $[-\pi, \pi]$ such that

$$I \circ u^{-1}(h) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)g(\theta)\, d\theta$$

for all h in E. Therefore if f is in H^∞ then

$$I(f) = I \circ u^{-1}(u(f)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})g(\theta)\, d\theta.$$

THEOREM 3.7. *A subset of $(H^\infty, \beta)^*$ is β-equicontinuous if and only if it is norm conditionally compact.*

Proof. Recall that by Theorem 3.5 $(H^\infty, \beta)^*$ is a Banach space where the norm of an element I is given by

$$\|I\| = \sup\{|I(f)| : f \in H^\infty, \|f\|_\infty \leqq 1\}.$$

If $A \subset (H^\infty, \beta)^*$ is norm compact and $T: (H^\infty, \beta) \to E_\beta$ is the map described prior to Theorem 3.4, then $T^*: E_\beta^* \to (H^\infty, \beta)^*$ is an isometry onto $(H^\infty, \beta)^*$. Therefore $T^{*-1}(A)$ is norm compact in E_β^* and hence, by Theorem 3.2, $T^{*-1}(A)$ is β-equicontinuous. This implies there is a $\phi \in c_0$ such that $T^{*-1}(A) \subset V_\phi^0$. But T is continuous and so $T^{-1}(V_\phi)$ is a β-neighborhood of zero in H^∞. It is routine to show that $A \subset [T^{-1}(V_\phi)]^0$ and hence A is β-equicontinuous.

Now suppose that $A \subset (H^\infty, \beta)^*$ is β-equicontinuous and $\{I_n\}$ is a sequence in A. Then A is β-weak* conditionally compact; since the β-weak* topology on $(H^\infty, \beta)^*$ is the same as its weak topology which it has as a Banach space (Theorem 3.5), the Eberlein-Smulyan theorem [8, p. 430] implies that A is β-weak* sequentially compact. Therefore some subsequence of $\{I_n\}$ converges β-weak* to an element I of $(H^\infty, \beta)^*$. Let us suppose, for notational reasons, that $I_n \to I$ β-weak*. Since the unit ball of H^∞ is β-compact and circled, there is for each integer n a function f_n in ball (H^∞) with

$$\|I - I_n\| = (I - I_n)(f_n).$$

Also, since ball (H^∞) with the β topology is a compact metric space, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and an element f of ball (H^∞) such that $f_{n_k} \to f$ (β). Since $\{I - I_n\}$ is β-equicontinuous, for any $\varepsilon > 0$ there is an integer N such that for $n_k \geqq N$

$$|(I - I_n)(f - f_{n_k})| < \varepsilon/2 \text{ for all } n \geqq 1; \text{ and also, } |I_{n_k}(f) - I(f)| < \varepsilon/2.$$

It now follows that for $n_k \geqq N$

$$\|I - I_{n_k}\| < \varepsilon,$$

and the theorem is proved.

Corollary 3.8. *If $\{I_n\}$, I are in $(H^\infty, \beta)^*$ then $I_n \to I$ in norm if and only if $I_n \to I$ β-weak* and $\{I_n\}$ is β-equicontinuous.*

Theorem 3.9. *(H^∞, β) is not a Mackey space.*

Proof. Since $(H^\infty, \beta)^*$ with its strong topology is a Banach space with H^∞ as adjoint, it is sufficient to show that (H^∞, β) is not a strong Mackey space [9, p. 434]. For each integer $n \geqq 1$ define

$$I_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta}\, d\theta$$

for all f in H^∞. Then $I_n \in (H^\infty, \beta)^*$ by Corollary 3.6. Also $I_n(f) = \hat{f}(n)$, the Fourier-Stieltjes transform of the function given by the boundary values of f. Therefore $I_n \to 0$ β-weak* (see [16]). Clearly $\|I_n\| \leqq 1$ and since $I_n(f) = 1$ when $f(z) = z^n$, we have $\|I_n\| = 1$ for all n. Therefore $\{I_n\}$ cannot converge to zero in norm and, by the preceding corollary, $\{I_n\}$ is not β-equicontinuous. This concludes the proof.

REMARK. Theorem 3.9 answers a question posed by Rubel and Shields.

If N is a closed subspace of l^1 such that there is a bounded projection onto N then E_β is a Mackey space, where $E=N^\perp\subset l^\infty$. If E is the image of H^∞ under the map T described prior to Theorem 3.4, then it is easy to see that E_β is not a Mackey space. Hence, if $N=E^\perp\subset l^1$ then there is no bounded projection of l^1 onto N. Also, this space E is an example of a closed subspace of a semireflexive Mackey space (l^∞, β) which is not a Mackey space.

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