SIMPLE-CONNECTIVITY AND THE BROWDER-NOVIKOV THEOREM(1)

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In this note we construct a family of odd-dimensional, closed, combinatorial manifolds, none of which has the homotopy type of a closed differentiable manifold. These manifolds all have an infinite cyclic fundamental group.

W. Browder [1] and S. P. Novikov [9] have proved that a *simply-connected* finite complex, K, satisfying Poincaré duality with respect to an odd-dimensional fundamental class $u \in H_{2n+1}(K)$, $(n \geqq 2)$ has the homotopy type of a closed $(2n+1)$-dimensional differential manifold provided there exists a vector bundle, ξ, over K whose Thom space $T(\xi)$ has a spherical top homology class (i.e., $\pi_q T(\xi) \to H_q T(\xi)$ is surjective for $q = 2n+1+\dim \xi$).

Denoting one of our combinatorial manifolds by M, we prove that SM, the suspension of M, has a spherical top homology class, so that the trivial real line bundle over M satisfies the Browder-Novikov hypothesis. The manifolds, M, show that the Browder-Novikov theorem cannot be extended to the nonsimply-connected case, at least not without additional hypotheses on K. (Recall that the Thom space of the trivial real line bundle over a space X has the homotopy type of $S^1 \vee SX$.)

The manifolds M^{2n+1} are constructed from certain knotted homotopy $(2n-1)$-spheres in S^{2n+1} (n is *odd*). Let A be (the space of) the tangent unit disk bundle over S^n, and W the differential manifold with boundary obtained by plumbing together two copies A_1, A_2 of A. (See [4], [5], or [8].) The images S_1, S_2 in W of the zero cross-sections in A_1 and A_2 respectively have a single (transversal) intersection point. Denote by Σ^{2n-1} the boundary of W. It was proved in [8], or more generally follows from Smale theory, that Σ^{2n-1} is combinatorially equivalent to S^{2n-1}.

We now imbed W into S^{2n+1}. It is well known, and easy to see, that W can be differentiably imbedded into S^{2n+1} so that if ν denotes a normal vector-field on W in S^{2n+1}, and if S_1', S_2' are the translates of S_1, S_2 by a small positive amount ε along ν, then $\lambda_i = L(S_i, S_i')$ for $i = 1, 2$, are *odd* integers, where $L(\ ,\ )$ denotes the linking coefficient in S^{2n+1}. (For further remarks on this see the lemma on normal bundles at the end of this paper.)

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Let $\phi\colon \Sigma^{2n-1}\times D^2 \to S^{2n+1}$ be a tubular neighborhood of bW, the boundary of W, and let $h\colon S^{2n-1}\to\Sigma^{2n-1}$ be a combinatorial equivalence. Then

$$\psi\colon S^{2n-1}\times D^2 \to S^{2n+1}$$

given by $\phi\circ(h\times \mathrm{id})$ is a combinatorial imbedding. Let $M^{2n+1}=\chi(S^{2n+1},\psi)$ be the combinatorial manifold obtained from S^{2n+1} by spherical modification:

$$M^{2n+1} = (S^{2n+1}-\psi(S^{2n-1}\times B^2))\cup D^{2n}\times S^1,$$

where $B^2=\operatorname{int} D^2$, and $(x,y)\in S^{2n-1}\times S^1\subset D^{2n}\times S^1$ is identified with $\psi(x,y)$. As usual we denote by $\psi'\colon D^{2n}\times S^1\to M$ the imbedding induced by the inclusion.

PROPOSITION. *The manifolds M^{2n+1} constructed above for odd n have the following properties for $n>1$:*

(1) $\pi_i M\cong\pi_i(S^1)$ *for* $i<n$;

(2) $\pi_n M$ *as a* $Z[J]$*-module, where* $J=\pi_1(S^1)$ *generated by* t, *is isomorphic to the* $Z[J]$*-module with presentation*

$$\{x_1, x_2;\ \lambda_1(t-1)x_1+(\lambda(t-1)+1)x_2,\ (\lambda(t-1)-t)x_1+\lambda_2(t-1)x_2\}.$$

(*Here* λ *is an integer depending on the imbedding* $W\to S^{2n+1}$, *and can be chosen arbitrarily.*)

(3) $H_*(M)\cong H_*(S^1\times S^{2n})$;

(4) $\pi_{2n+2}(\mathrm{SM})\to H_{2n+2}(\mathrm{SM})$ *is surjective, where* SM *denotes the suspension of* M.

(5) *For* $n=3$, 7 *the combinatorial equivalence* $h\colon\Sigma^{2n-1}\to S^{2n-1}$ *can be taken to be a diffeomorphism so that the manifold* M^{2n+1} *has a differential structure. For* $n=4k+1$, M^{2n+1} *does not have the homotopy type of any closed differential manifold.*

REMARK. Since isomorphic $Z[J]$-modules have identical elementary ideals, and the 0th elementary ideal of the module under (2), i.e., the ideal generated by the determinant of the relation matrix, is $(\lambda_1\lambda_2-\lambda^2+\lambda)(t-1)^2+t$ it follows that by varying the coefficients λ_1, λ_2, λ we can get infinitely many distinct homotopy types for the manifold M^{2n+1}.

The proofs of (1) and (2) rely on the method for calculating the homotopy groups of the complement of a knot, exposed in [4]. It is clear that

$$M-\psi'(D^{2n}\times S^1) = S^{2n+1}-\psi(S^{2n-1}\times D^2).$$

Hence $\pi_i M=\pi_i(S^{2n+1}-bW)$ for $i<2n-1$. Now, a homomorphism

$$I\colon \pi_1(S^{2n+1}-bW)\to Z$$

is given by assigning to $\alpha\in\pi_1(S^{2n+1}-bW)$ the intersection coefficient with W of a representative curve $f\colon S^1\to S^{2n+1}-bW$. Clearly, I is surjective. To prove that I is injective, let U be a tubular neighborhood of W and let $Y=S^{2n+1}-U$. Then first, $\pi_1 Y=\{1\}$ by the van Kampen theorem, since W and hence bY are simply connected. Secondly, $\pi_1(Y,bY)=\{1\}$ by the homotopy exact sequence of (Y,bY).

An element $\alpha \in \operatorname{Ker} I$ can be represented by a differentiably imbedded curve $[0, 1] \to S^{2n+1} - bW$ which intersects W transversally in a finite number of interior points. Since $I(\alpha)=0$, there will be a pair of consecutive intersection points with W having oppositive intersection coefficients. The arc joining these points represents an element of $\pi_1(Y, bY)$ and thus is homotopic to an arc in U which can then be pushed away from W, reducing the number of intersection points by 2. Eventually, we get a representative of the given element α whose image is contained in Y. Since Y is simply connected, $\alpha=1$.

To calculate $\pi_i M$ for $i \leqq n$, denote by W_+ and W_- the two copies of W in $bU=bY$. Let $X_1 = S^{2n+1} - bW$. Following Seifert, we construct the universal covering X of X_1 as the union of countably many copies of Y which we denote by Y^k, $k \in Z$, with W_+^k identified with W_-^{k+1} for every k. Then $\pi_i(S^{2n+1} - bW)=0$ for $2 \leqq i < n$ follows immediately from $H_i Y = 0$ for $i<n$. The module $\pi_n(S^{2n+1} - bW)$ is isomorphic to $H_n(X)$ for which we get a presentation by the Mayer-Vietoris theorem. If we denote by ξ_1, ξ_2 the generators of $H_n W$ represented by S_1, S_2 respectively, ν_+ and ν_- the obvious mappings of W onto W_+ and W_-, then a presentation for $H_n(X)$ is obtained from a presentation of $H_n Y$ (as an abelian group) by adjoining the relations $t\nu_+(\xi_i) = \nu_-(\xi_i)$. Now, it is easy to see that $H_n Y$ is free abelian on 2 generators x_1, x_2, and if S is a sphere in Y, the class of S is given by

$$(S) = L(S, S_1)x_1 + L(S, S_2)x_2.$$

The presentation for $H_n(X) \cong \pi_n M$ claimed in (2) follows readily, with $\lambda = L(S_1', S_2)$.

The proof of (3) is immediate.

To prove (4) we first observe that M has the homotopy type of a cell complex of the form

$$K = (S^1 \vee S_1^n \vee \cdots \vee S_\alpha^n) \cup e_1^{n+1} \cup \cdots \cup e_\alpha^{n+1} \cup e^{2n} \cup e^{2n+1}.$$

(In fact, it can be proved using Smale theory that M has a handle decomposition inducing the above cell structure. See [4].) Since $H_n M = 0$, we have, up to homotopy type

$$\mathrm{SM} = (S^2 \cup_f e^{2n+1}) \cup_g e^{2n+2}.$$

Now, since $H^1 M = Z$, there is a map $M \to S^1$ whose composition with the inclusion $S^1 \to M$ is homotopic to the identity $S^1 \to S^1$. Taking the suspension of these maps we see that S^2 is a retract of M. It follows that the attaching map f is trivial, and thus

$$\mathrm{SM} = (S^2 \vee S^{2n+1}) \cup_g e^{2n+2},$$

up to homotopy type. Since $H_{2n+2} M = Z$, it follows that g must have degree 0 (on S^{2n+1}), and since $\pi_{2n+1}(S^2 \vee S^{2n+1}) = \pi_{2n+1}(S^2) + \pi_{2n+1}(S^{2n+1})$, g is homotopic to a mapping h into S^2. Thus SM has the homotopy type of

$$S^{2n+1} \vee (S^2 \cup_{gh} e^{2n+2}).$$

Using again the retraction SM $\to S^2$, we see that h is trivial. So SM has the homotopy type of

$$S^2 \vee S^{2n+1} \vee S^{2n+2}.$$

(4) is now obvious.

The proof of (5) will rely essentially on recent results of E. Brown and F. Peterson [2]. Suppose M^{2n+1} is a closed differential manifold (n odd, $\neq 3, 7$) with the properties (1), (2), and (3). We construct a knot as follows. Let $\phi: S^1 \times D^{2n} \to M$ be a differentiable imbedding representing the generator $t \in \pi_1 M$. Performing a spherical modification we obtain a manifold $\Sigma^{2n+1} = \chi(M, \phi)$ which is easily seen to be a differential homotopy sphere. Replacing M^{2n+1} by the connected sum $(M \# (-\Sigma))$ if necessary, we may assume that Σ^{2n+1} is diffeomorphic to S^{2n+1}. (This operation is not in fact really necessary for what follows.) Since

$$S^{2n+1} = (M - \phi(S^1 \times B^{2n})) \cup D^2 \times S^{2n-1},$$

we have an imbedding $f: S^{2n-1} \to S^{2n+1}$. (It is however essential that S^{2n-1} here be the sphere with the usual differential structure.) An argument similar to the one used in the proof of (1) and (2) shows that $\pi_i(S^{2n+1} - f(S^{2n-1})) = \pi_i(S^1)$ for $i < n$, and $\pi_n(S^{2n+1} - f(S^{2n-1}))$ is the $Z[J]$-module whose presentation is given in (3). It is well known that $f(S^{2n-1})$ is the boundary of an orientable submanifold V of S^{2n+1}. Moreover J. Levine has proved that the manifold V can be taken to be $(n-1)$-connected as a consequence of $\pi_i(S^{2n+1} - f(S^{2n-1})) = \pi_i(S^1)$ for $i < n$. It follows from Poincaré duality (for V) that we may find a basis, $\{\xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s\}$, for $H_n(V)$ having the intersection numbers $\delta_{ij} = I(\xi_i, \eta_j)$, $O = I(\xi_i, \xi_j) = I(\eta_i, \eta_j)$. Using a normal vector field along V we may imbed $V \times [-\varepsilon, +\varepsilon]$ in S^{2n+1} in such a way that V corresponds to $V \times \{0\}$. If $\alpha \in H_n(V)$ we denote by $\alpha^{\pm}$ the elements of $H_n(S^{2n+1} - V)$ represented by the corresponding class in $V \times \{\pm \varepsilon\} \subset S^{2n+1} - V$. Since any $\alpha \in H_n(V)$ can be represented by an imbedded sphere and any two such can be put in general position it is clear that $L(\alpha^+, \beta) - L(\alpha^-, \beta) = L(\alpha^+ - \alpha^-, \beta)$ $=$ the intersection number of the chain $[-\varepsilon, \varepsilon] \times \alpha$ with $\beta = I(\alpha, \beta)$. It now follows from the methods used to establish (3) that there is a presentation of $\pi_n(S^{2n+1} - \partial V)$ with $2s$ generators and the relation matrix

$$R = \left\| \begin{matrix} (t-1)A & (t-1)B + E \\ (t-1)B' - tE & (t-1)C \end{matrix} \right\|$$

where E is the $s \times s$ identity matrix, B' is the transpose of B and $A = \|L(\xi_i^+, \xi_j)\|$, $B = \|L(\xi_i^+, \eta_j)\|$ and $C = \|C_{ij}\| = \|L(\eta_i^+, \eta_j)\|$. (We remark that the matrices A and C are symmetric (because n is odd).) Using the lemma below we recognize $a_{ii} \bmod 2$ as the obstruction to trivializing the normal bundle in V to an imbedded sphere representing ξ_i. A similar relation holds between c_{ii} and η_i and so

$$c(V) = \sum_{i=1}^{s} a_{ii} c_{ii} \bmod 2$$

is the Arf invariant of the quadratic form of V as defined in [6].

Now, det R is the 0th elementary ideal of $\pi_n(M)=\pi_n(S_{2n+1}-f(S^{2n-1}))$ and hence det R and $(\lambda_1\lambda_2-\lambda^2+\lambda)(t-1)^2+t$ must generate the same ideal in $Z[J]$. An argument of Robertello's [10], sketched below for the convenience of the reader, shows that $\det R=t^{s-1}(c(V)(t^2+1)+t)$ modulo the ideal generated by 2 and $(t-1)^4$. Since $(\lambda_1\lambda_2-\lambda^2+\lambda)(t-1)^2+t=(t^2+1)+t \bmod 2$, it follows that $C(V)=1 \bmod 2$ but since ∂V is diffeomorphic to S^{2n-1} this contradicts the Brown-Peterson result [2] when $n=4k+1$, $k\geqq 1$.

Robertello's argument in brief is this. Let $R=(x_{ij})$—thus x_{ij} is divisible by $t-1$ except if $1\leqq i\leqq s$ and $j=i+s$ or $1\leqq j\leqq s$ and $i=j+s$. Let $S_{\alpha,\beta}$ be the set of permutations $(i_1,\ldots,i_{2s})$ for which $i_k\neq s+k$ for exactly α values of $k\in[1,s]$ and $i_{s+k}\neq k$ for exactly β values of $k\in[1,s]$. Thus

$$\det R = \sum_{0\leqq\alpha,\beta\leqq k}\ \sum_{S_{\alpha,\beta}} x_{1,i_1}x_{2,i_2}\cdots x_{2s,i_{2s}}.$$

Now the individual terms in $s_{\alpha\beta}$ for $\alpha,\beta\in[2,s]$ are divisible by $(t-1)^4$ so that we need only consider the first few $S_{\alpha,\beta}$'s. $S_{0,0}$ contains only the permutation $(s+1,\ldots,2s,1,2,\ldots,s-1,s)$ which gives rise to the term

$$\prod_{i=1}^{s}\ [(t-1)b_{ii}t][(t-1)b_{ii}+1] = t^s \bmod 2.$$

The sets $S_{0,1}$ and $S_{1,0}$ are empty. The sum $\sum_{S_{1,1}}$ is

$$\sum_{i,k}\left\{\prod_{j=1;j\neq i}^{s}((t-1)b_{jj}-t)\right\}\left\{\prod_{j=1;j\neq k}^{s}((t-1)b_{jj}+1)\right\}(t-1)^2a_{ik}c_{ki}$$

which mod $\{2,(t-1)^4\}$ is $t^{s-1}(1+t^2)c(V)$. Further similar calculation shows that $\sum_{S_{2,1}}+\sum_{S_{1,2}}=0$. (Essential use is made of the fact that A and C are symmetric.)

Before stating the lemma on normal bundles, recall that an n-dimensional bundle, ξ^n, over an n-sphere determines an element $[\xi^n]\in\pi_n(B_{SOn})$—where B_{SOn} is a classifying space for the group SOn. We denote by T_n the tangent bundle of S^n.

Lemma on normal bundles. *Let $S^n\subset S^{2n+1}$ be a differentiable imbedding, ν a never vanishing normal field, $\bar{S}^n\subset S^{2n+1}$ a disjointly imbedded sphere obtained by "pushing" S^n along ν and finally η, the complementary normal bundle—i.e., $\eta(x)$ = the vectors normal to S^n at x but perpendicular to $\nu(x)$. Then*

$$[\eta] = L(S^n,\bar{S}^n)[T_n]\in\pi_n(B_{SOn}).$$

Proof. There is no loss in generality (see [3]) in assuming that the imbedding is the usual one—to wit $S^n\subset S^n\times R\subset R^{n+1}=R^{n+1}\times 0\subset R^{n+1}\times R^n=S^{2n+1}-\infty$. Thus we may refer the normal vector field, ν, to the *standard* framing of this normal bundle—thus ν and η are described completely by a function $f_\nu\colon S^n\to R^{n+1}-0$. Since the entities involved in our assertion are unchanged if we vary ν (through never-zero normal fields) we may assume that f_ν is a differentiable map to S^n having the south pole as a regular value. Then it is clear that $\bar{S}^n$ intersects $D^{n+1}\times 0$

transversally once for each inverse image of the south pole and in fact $L(S^n, \bar{S}^n)$ = the (algebraic) number of such inverse images = the degree of f_v. On the other hand, we clearly now have $\eta = \{(x, v) \in S^n \times R^{n+1} \mid f_v(x) \perp v\}$ and this is obviously the "pull-back" under f_v of the tangent bundle of S^n. Thus $[\eta] = \deg f_v[T_n]$.

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