ON THE THEORY OF EXPONENTIAL GROUPS[1]

BY

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1. **Introduction.** Let $\mathcal{L}$ be a real solvable Lie algebra. As it is known, there exists a sequence of ideals $\mathcal{L}_0 = \mathcal{L} \supset \mathcal{L}_1 \supset \cdots \supset \mathcal{L}_n = (0)$ of $\mathcal{L}$, such that the codimension of $\mathcal{L}_{i+1}$ in $\mathcal{L}_i$ is 1 or 2 ($i = 0, 1, 2, \ldots, m-1$), and such that in the latter case the representation induced by the adjoint representation of $\mathcal{L}$ on $\mathcal{L}_i/\mathcal{L}_{i+1}$ is irreducible. If for some $i = 0, 1, 2, \ldots, m-1$, the codimension of $\mathcal{L}_{i+1}$ in $\mathcal{L}_i$ is 1, and if l_0 is a nonzero element of $\mathcal{L}_i$ not contained in $\mathcal{L}_{i+1}$, we have for all l in $\mathcal{L}$

$$\operatorname{ad} l(l_0) = \gamma(l)l_0 \quad (\operatorname{mod} \mathcal{L}_{i+1})$$

where $\gamma(l)$ is a linear form vanishing on the first derived algebra of $\mathcal{L}$. If the codimension of $\mathcal{L}_{i+1}$ in $\mathcal{L}_i$ is 2, there exists a pair of elements l', l'', linearly independent mod $\mathcal{L}_{i+1}$, of $\mathcal{L}_i$ such that for all l in $\mathcal{L}$

$$\begin{aligned}\operatorname{ad} l(l') &= \gamma'(l)l' - \gamma''(l)l'' \\ \operatorname{ad} l(l'') &= \gamma''(l)l' + \gamma'(l)l''\end{aligned} \quad (\operatorname{mod} \mathcal{L}_{i+1}),$$

where γ' and γ'' are linear forms on $\mathcal{L}$ of the kind considered above. The collection of all linear forms $\{\gamma(l), \gamma'(l) \pm i\gamma''(l)\}$ where $\gamma, \gamma', \gamma''$ respectively have been obtained in the indicated fashion, is called the system of roots of $\mathcal{L}$. The set of values corresponding to a fixed l in $\mathcal{L}$ coincides with the collection of eigenvalues of the operator ad l on $\mathcal{L}$.

We say that a real solvable Lie algebra is exponential if its complex roots are of the form $(1 + i\alpha)\,\delta(l)$, where α is a nonzero real number and $\delta(l)$ is a real linear form on $\mathcal{L}$. We shall call a connected and simply connected Lie group exponential, if its Lie algebra is exponential. Let $\mathcal{L}$ be a solvable Lie algebra and G the corresponding connected and simply connected Lie group.

It is known (cf. [2] and [8]) that the exponential map establishes an analytic isomorphism between the underlying manifolds of $\mathcal{L}$ and G resp. if and only if $\mathcal{L}$ is exponential. One sees easily, that any subalgebra or factoralgebra of an exponential Lie algebra is again of the same kind.

An important special case is that of a nilpotent Lie algebra, which is characterized by the vanishing of all roots.

The study of the classification of all irreducible unitary representations of an exponential group was started by J. Dixmier's work on nilpotent groups, continued

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later by A. A. Kirillov (for a systematic account cf. [7]). The first result referring to not necessarily nilpotent exponential groups is due to O. Takenouchi, who showed that the factor representations of exponential groups are of type I (cf. [9]). In a recent work [1] P. Bernat gives a detailed study of the system of all irreducible unitary representations of an exponential group. His results, extending those found in the nilpotent case by Kirillov, can be summed up as follows. Let $\mathscr{L}$ be an exponential Lie algebra and $\mathscr{L}'$ the dual of the underlying space of $\mathscr{L}$; we denote by (l, l') ($l \in \mathscr{L}$, $l' \in \mathscr{L}'$) the canonical bilinear form on $\mathscr{L} \times \mathscr{L}'$. Let l_0' be a fixed element of $\mathscr{L}'$ and H a subalgebra of $\mathscr{L}$. We say, that H is subordinated to l_0', if the first derived algebra of H is orthogonal to l_0'; in this case we shall write $H < l_0'$. Let us denote by $\bar{\mathscr{L}}$ the connected and simply connected Lie group belonging to $\mathscr{L}$, and by $\bar{H}$ the connected subgroup corresponding to the subalgebra $H \subset \mathscr{L}$. If $H < l_0'$, the formula

$$\chi(\exp h) = \exp(i(h, l_0')) \qquad (h \in H)$$

defines a character of $\bar{H}$. Let us denote by ind (H, l_0') the unitary representation of $\bar{\mathscr{L}}$ induced by the character χ of $\bar{H}$. Takenouchi showed, using G. W. Mackey's results on systems of imprimitivity, that any irreducible unitary representation of $\bar{\mathscr{L}}$ can be written in the form ind (H, l_0'), by choosing H and l_0' appropriately. Bernat proves first that, conversely, for any given l_0' in $\mathscr{L}'$ there exists at least one choice of the subalgebra H, subordinated to l_0', for which ind (H, l_0') turns out to be irreducible. We denote by σ the adjoint representation of $\bar{\mathscr{L}}$ acting on $\mathscr{L}$, and by δ the representation, which is contragredient to σ, of $\bar{\mathscr{L}}$ on $\mathscr{L}'$; that is, for any $g \in \bar{\mathscr{L}}$ the operators $\delta(g)$ and $\sigma(g)$ are connected by the relation

$$(\sigma(g^{-1})l, l') = (l, \delta(g)l') \qquad (l \in \mathscr{L}, l' \in \mathscr{L}').$$

Suppose now that $H_0 < l_0'$ and $H_1 < l_1'$ such that ind (H_0, l_0') and ind (H_1, l_1') are irreducible. Then we have ind $(H_0, l_0') = $ ind (H_1, l_1'), the equality sign indicating the unitary equivalence of the corresponding representations, if and only if there exists an element g of $\bar{\mathscr{L}}$ such that $l_1' = \delta(g)l_0'$; in other words, if and only if l_0' and l_1' belong to the same orbit of $\mathscr{L}'$ with respect to δ. In order that, for a given l_0', the representation ind (H, l_0') be irreducible, it is necessary that H should be of a maximal dimension in the family of subalgebras subordinated to l_0'. Furthermore, for such an H, dim H depends only on the orbit determined by l_0'. Denoting this by $o(l_0')$, Kirillov proved that in the nilpotent case the dimension in question is given by dim $\mathscr{L} - \frac{1}{2}$ dim $o(l_0')$ [3, Lemma 5.2]. In the general case, however, the maximality of the dimension of H is far from being sufficient to imply the irreducibility of ind (H, l_0'); in fact, already the consideration of the affine group of the real line furnishes counterexamples. As Bernat shows, a sufficient condition assuring that the maximality of dim H implies the irreducibility of ind (H, l_0') is that all real roots of $\mathscr{L}$ be zero. An exponential Lie algebra satisfying this condition is called quasi-nilpotent; evidently a nilpotent algebra is quasi-nilpotent.

The purpose of the present paper is to give necessary and sufficient conditions on the subalgebra H, in order that ind (H, l_0') turn out to be irreducible. To describe these we denote by $H^\perp$ the orthogonal complement of H in $\mathscr{L}'$. Then we have first the following result (cf. Theorem 1): ind (H, l_0') is irreducible if and only if H is of a maximal dimension among the subalgebras subordinated to l_0', and the linear variety $l_0' + H^\perp$ is contained in $o(l_0')$ (= the orbit of l_0'). We show (cf. Lemma 5), that the first of these conditions, similarly as in the case of nilpotent groups, is equivalent to dim $H = $ dim $\mathscr{L} - \frac{1}{2}$ dim $o(l_0')$. As far as the necessity of the second condition is concerned, this follows immediately from the results of Bernat quoted above. In fact, to this end it suffices to observe that evidently

$$\text{ind } (H, l_0') = \text{ind } (H, l_0 + h^\perp)$$

for any choice of $h^\perp$ in $H^\perp$, and therefore, if these representations are irreducible, then l_0' and $l_0' + h^\perp$ must lie on the same orbit. Next we prove (cf. Proposition 3) that corresponding to Bernat's observation the first condition implies the second one if $\mathscr{L}$ is quasi-nilpotent. Another situation, where the same implication is true, is furnished by an l_0', for which the orbit $o(l_0')$ is closed (cf. Proposition 4). Since for a nilpotent $\mathscr{L}$ all orbits of $\mathscr{L}'$ according to δ are closed (even Zariski closed; cf. e.g., [7, Deuxième Partie, Chapitre I, §2]), the sufficiency to yield irreducibility, of dim $H = $ dim $\mathscr{L} - \frac{1}{2}$ dim $o(l_0')$ obtains here a new proof. We repeat, however, that in the general case the two conditions are independent of each other (cf. Remark 2 after Definition 2 in §3).

Theorem 2 shows that in the case of solvable algebras with real roots the necessary and sufficient condition of Theorem 1 can be alternatively described in close analogy with finite groups. For this we recall first the following theorem of Shoda (generalized by Mackey, cf. [5, Theorem 6]). Let G be a finite group, Γ a subgroup of G, and χ a homomorphism of Γ into the multiplicative group of complex numbers of absolute value one. For an element x of G, we denote by Γ_x the intersection of the subgroups Γ and $x^{-1}\Gamma x$, and by χ_x the character of Γ_x defined by $\chi_x(\gamma) = \chi(x\gamma x^{-1})$ $(\gamma \in \Gamma_x)$. Then the representation, induced by χ, of G turns out to be irreducible if and only if for all x not belonging to Γ, χ_x is different from the restriction of χ to Γ_x. Let G be the connected and simply connected Lie group which belongs to the solvable Lie algebra $\mathscr{L}$ with real roots. Suppose again that H is a subalgebra, subordinated to l_0', and denote by $\bar{H}$ the connected subgroup determined by H. Theorem 2 asserts that ind (H, l_0') is an irreducible unitary representation of G if and only if the condition to yield irreducibility of the above theorem is fulfilled upon replacing Γ by $\bar{H}$ and χ by the character

$$\chi(\exp h) = \exp [i(h, l_0')] \qquad (h \in H)$$

of $\bar{H}$ resp.

The reader is assumed to be familiar with the basic properties of induced representations, as exposed in [6]; we shall also use several reasonings of Takenouchi

[9], to be specified later. On the other hand, it is necessary for our purposes to reproduce the relevant parts of Bernat's paper [1] with some variations, and therefore, on this side, we shall use but a small number of elementary computations without proof. For the same reason, we give a complete proof of Theorem 1.

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2. **Notations.** Besides those explained in the Introduction, we shall make constant use of the following notations.

a. Let V be a finite dimensional vector space over the real field and $B(v_1, v_2)$ $(v_1, v_2 \in V)$ a skew symmetric bilinear form on V. For a subspace W of V we denote by $W_B^\perp$ the orthogonal complement of W in V with respect to B. The subspace $R = V_B^\perp$ is called the radical of B. We shall often use the following elementary relation, valid for any choice of the subspace W of V:

$$\dim W_B^\perp = \dim V - \dim W + \dim (W \cap R).$$

W is called self orthogonal if $W \subseteq W_B^\perp$, and maximal self orthogonal if $W = W_B^\perp$. One verifies easily by aid of the above formula that a self orthogonal subspace W is maximal if and only if

$$\dim W = \tfrac{1}{2}(\dim V + \dim R).$$

b. All skew symmetric bilinear forms considered in this paper are going to be obtained by taking a Lie algebra $\mathscr{L}$, by fixing an element l_0' of the dual $\mathscr{L}'$ of the underlying space of $\mathscr{L}$, and by putting

$$B(l_1, l_2) = ([l_1, l_2], l_0') \qquad (l_1, l_2 \in \mathscr{L}).$$

The expressions self orthogonal and maximal self orthogonal with respect to l_0' have to be interpreted through the bilinear form determined by l_0'. Its radical will be denoted by $R(l_0')$, or simply by R, if its dependence on l_0' is indicated by the context. Let $\mathscr{L}_1$ be a subspace of $\mathscr{L}$; then $B(l_1, l_2; \mathscr{L}_1, l_0')$ will stand for the restriction of the above bilinear form to $\mathscr{L}_1$ (if $\mathscr{L}_1 = \mathscr{L}$, it will not be indicated).

c. Let H be a subspace of $\mathscr{L}$; then, as explained in a, $H_B^\perp$ will denote the orthogonal complement of H in $\mathscr{L}$ with respect to the bilinear form B. On the other hand, $H^\perp$ indicates the orthogonal complement of H in the dual of $\mathscr{L}$ with respect to the canonical bilinear form on $\mathscr{L} \times \mathscr{L}'$.

d. All Lie algebras considered in this paper are assumed to be exponential. If $\mathscr{L}$ is a Lie algebra, we denote by $\overline{\mathscr{L}}$ the corresponding connected and simply connected Lie group. Similarly, for any subalgebra H of $\mathscr{L}$, $\overline{H}$ will stand for the connected subgroup determined by H. Finally, if l is some element of $\mathscr{L}$, we shall write $\check{l}$ for the element $\exp l$ of $\overline{\mathscr{L}}$.

3. Let $\mathscr{L}$ be a Lie algebra and l_0' a nonzero element of the dual of $\mathscr{L}$; all these will be kept fixed throughout this section. We shall write mostly R for $R(l_0')$.

One sees at once that R is the collection of all elements of $\mathscr{L}$ satisfying $(\operatorname{ad} l)' l_0' = 0$. Hence R is a subalgebra of $\mathscr{L}$ and $g \in \bar{R}$ implies $\delta(g) l_0' = l_0'$. One can easily show, by making use of the assumption, according to which $\mathscr{L}$ is exponential, that conversely, if $\delta(g) l_0' = l_0'$ for some element g of $\bar{\mathscr{L}}$, then g belongs to $\bar{R}$, and thus the closed subgroup $\bar{R}$ of $\bar{\mathscr{L}}$ coincides with the stabilizer of l_0', but we are not going to use this fact directly. Sometimes R too will be referred to as the stabilizer of l_0'. Let us denote by O the orbit $o(l_0')$ of l_0' in $\mathscr{L}'$ according to the representation δ of $\mathscr{L}$. We have $\dim O = \dim \mathscr{L} - \dim R$; taking into account that the right hand side is just the rank of the skew symmetric bilinear form $B(l_1, l_2; l_0')$, we can conclude that $\dim O$ is always even.

Suppose now that H is a subalgebra of $\mathscr{L}$ satisfying $H < l_0'$; that is, that H is subordinated to l_0'. Let us observe, incidentally, that this is equivalent to saying that H is self orthogonal with respect to l_0' (cf. 2. b). If h_0 is some element of H, $\delta(\bar{h}_0) l_0' - l_0'$ is orthogonal to H. In fact, this is equivalent to $(h, \delta(\bar{h}_0) l_0') = (h, l_0')$ for all h in H. The left hand side is the same as

$$(\exp(-(\operatorname{ad} h_0))h, l_0') = (h, l_0') + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} ((\operatorname{ad} h_0)^j(h), l_0')$$

and to obtain the desired conclusion, it suffices to observe that $((\operatorname{ad} h_0)^j(h), l_0') = 0$ for $j > 0$ if $H < l_0'$.

DEFINITION 1. Let H be a subalgebra with $H < l_0'$. We shall say that H has the property F, and write $H = (F)$ if and only if

$$l_0' + H^\perp = \{\delta(\bar{h}) l_0'; h \in H\}.$$

REMARK 1. We have just proved that the right hand side is always contained in the left hand side. The converse, however, is false even if the dimensions of the two sides coincide. In fact, let $\mathscr{L}$ be the Lie algebra with the basis $\{e_0, e_1\}$ satisfying $[e_0, e_1] = e_1$; $\mathscr{L}$ is evidently exponential. If l_0' in $\mathscr{L}'$ is such that $(e_1, l_0') \neq 0$, then $R = 0$, and we have $H < l_0'$ for any 1-dimensional subalgebra H. On the other hand $H = (F)$ if and only if $H = \{e_1\}$.

Before proceeding we observe (cf. 2a) that $H < l_0'$ implies $H \subseteq H_{\bar{B}}^\perp$, hence (cf. 2a)

$$\dim H \leqq \tfrac{1}{2}(\dim \mathscr{L} + \dim (H \cap R)) \leqq \tfrac{1}{2}(\dim \mathscr{L} + \dim R) = \dim \mathscr{L} - \tfrac{1}{2}\dim O.$$

DEFINITION 2. Suppose that H is a subalgebra with $H < l_0'$. We say that H has the property J, and write $H = (J)$, if 1, $\dim H = \dim \mathscr{L} - \tfrac{1}{2} \dim O$, 2, $l_0' + H^\perp \subset O$ are satisfied at the same time.

REMARK 2. It is useful to observe that in general none of those two conditions implies the other. To see that 1 does not imply 2, let $\mathscr{L}$ and l_0' be as in Remark 1, and denote by $\{x_0, x_1\}$ the coordinates of an element x of $\mathscr{L}'$ with respect to the basis dual to $\{e_0, e_1\}$. Then

$$O = o(l_0') = \{x; \operatorname{sg}(x_0) = \operatorname{sg}((e_0, l_0'))\}$$

and thus

$$\dim \mathscr{L} - \tfrac{1}{2} \dim O = 1.$$

But for a 1-dimensional subalgebra H the linear variety $l_0' + H^\perp$ is contained in O again if and only if $H = \{e_1\}$. To show that 2 does not imply 1 consider the Lie algebra with the basis $\{e_1, e_2, e_3\}$, satisfying $[e_1, e_2] = e_3$, $[e_1, e_3] = [e_2, e_3] = 0$; it is nilpotent, hence, in particular, exponential. Choose l_0' such that $(e_3, l_0') \neq 0$; then with notations as above $O = \{x; x_3 = (e_3, l_0')\}$, and thus $\dim \mathscr{L} - \frac{1}{2} \dim O = 2$. On the other hand, defining H by $H = \{e_3\}$, we have $l_0' + H^\perp = O$, but $\dim H < 2$. This example is a special case of the following situation. Suppose that H is an ideal of $\mathscr{L}$ satisfying $R \subset H < l_0'$. One can show that in this case the intersection of O with $l_0' + H^\perp$ is open in the latter; hence, if O is closed, we have $l_0' + H^\perp \subset O$.

DEFINITION 3. Suppose that H is a subalgebra with $H < l_0'$. We say that H has the property G, and write $H = (G)$, if $\dim H = \frac{1}{2}(\dim \mathscr{L} + \dim R(l_0' + h^\perp))$ for all elements $h^\perp$ in $H^\perp$.

Notation. Suppose that $\mathscr{L}_0$ is a subalgebra of $\mathscr{L}$, such that $H \subseteq \mathscr{L}_0 \subseteq \mathscr{L}$. We shall write $H = (F, \mathscr{L}_0)$ to indicate that H has the property F as a subalgebra of $\mathscr{L}_0$ with respect to the element of $\mathscr{L}_0'$, which is the restriction of l_0' to $\mathscr{L}_0$; similarly for properties J and G.

Next we prove

PROPOSITION 1. *For any subalgebra H of $\mathscr{L}$, the properties described in Definitions 1–3 are equivalent.*

Proof. We shall carry it out in three steps.

A. $(F) \Rightarrow (J)$.

Since $l_0' + H^\perp \subset O$ is trivially fulfilled, all that we have to show is that

$$\dim H = \tfrac{1}{2}(\dim \mathscr{L} + \dim R),$$

or what amounts to the same (cf. 2a, 2b), that H is maximal self orthogonal with respect to l_0'. We denote by ρ the analytic map, defined for h in H by $\rho(h) = \delta(\bar{h})l_0'$, of the underlying space of H into $l_0' + H^\perp$. One sees easily that the rank of this map at the point h_0 of H is the same as the dimension of the subspace $T(h_0)$ of $\mathscr{L}'$ given by $T(h_0) = \{l'; l' = \delta(\bar{h}_0)(\mathrm{ad}\, h)'l_0'; h \in H\}$. Since $\dim T(h_0) = \dim T(0)$, this rank is constant. On the other hand, this must be the same as

$$\dim H^\perp = \dim \mathscr{L} - \dim H,$$

since, H having the property F, the map ρ is onto. Hence

$$\dim T(0) = \dim H - \dim (H \cap R) = \dim \mathscr{L} - \dim H,$$

and thus

$$\dim H = \dim \mathscr{L} - \dim H + \dim (H \cap R) = \dim H_B^\perp,$$

B indicating the skew symmetric bilinear form arising from l_0' (cf. 2b). Since $H < l_0'$ we also have $H \subseteq H_B^\perp$, and thus finally $H = H_B^\perp$ which is the desired result.

B. $(J) \Rightarrow (G)$.

If $H=(J)$, we have $l_0'+H^\perp\subset O$, and the stabilizer of any element of the form $l_0'+h^\perp$, with $h^\perp$ in $H^\perp$, has the same dimension. Thus,

$$\tfrac{1}{2}(\dim \mathscr{L}+\dim R) = \dim H = \tfrac{1}{2}(\dim \mathscr{L}+\dim R(l_0'+h^\perp))$$

for all $h^\perp \in H^\perp$; in other words, H has the property G.

C. $(G)\Rightarrow(F)$.

The following reasonings will be organized with a view of other applications in the next sections, and, for the same reason, we shall prove slightly more than actually needed.

LEMMA 1. *Let $\mathscr{L}_0$ be a subalgebra, of codimension one, of $\mathscr{L}$ containing H, and suppose $H=(F, \mathscr{L}_0)$. Then we have $H=(F)$ if and only if $R(l_0'+l_0^\perp)$ is contained in $\mathscr{L}_0$ for all $l_0^\perp$ in $\mathscr{L}_0^\perp$. The latter condition is automatically satisfied if $\mathscr{L}_0$ is an ideal and $R(l_0')\subseteqq\mathscr{L}_0$.*

Proof. I. To establish the necessity of the above condition first we observe that by virtue of A and B, $H=(F)$ implies $H=(G)$. In this fashion H is, in particular, maximal self orthogonal with respect to any element $l_0'+l_0^\perp$, where $l_0^\perp$ is arbitrary in $\mathscr{L}_0^\perp$, since $H\subseteqq\mathscr{L}_0$ implies $\mathscr{L}_0^\perp\subseteqq H^\perp$. On the other hand, it is evident that $H_B^\perp$ contains the radical of B. Thus, finally $R(l_0'+l_0^\perp)\subseteqq H\subseteqq\mathscr{L}_0$ for all $l_0^\perp\in\mathscr{L}_0$.

II. We shall prove the sufficiency in several steps. Observe that in what follows, to simplify the language, we shall often speak of l_0' instead of the restriction of l_0' to $\mathscr{L}_0$.

1. For $l_0^\perp=0$ we obtain $R=R(l_0')\subseteqq\mathscr{L}_0$. Denote by R_0 the radical of l_0' in $\mathscr{L}_0$; evidently $R\subseteqq R_0$. Since in any case $\dim \mathscr{L}-\dim R=\dim \mathscr{L}_0-\dim R_0+\eta$ ($\eta=0, 2$), or $\dim R-\dim R_0=1$ or -1, we have here $\dim R_0/R=1$.

2. Since $H=(F, \mathscr{L}_0)$ and since by A, $(F)\Rightarrow(J)$, H is maximal self orthogonal in $\mathscr{L}_0$ with respect to l_0' from which, as in I above, we conclude that $R_0\subset H$, $R_0\neq H$.

3. Let l' be a fixed nonzero element in $\mathscr{L}_0^\perp$. Then we have for all l_0 in $\mathscr{L}_0$: $-(\operatorname{ad} l_0)'l'=\alpha(l_0)l'$, where α is some linear form on $\mathscr{L}_0$. From this we conclude that $\delta(\exp l_0)l'=[\exp \alpha(l_0)]l'$ ($l_0\in\mathscr{L}_0$).

4. To obtain the proof of the sufficiency, it suffices to show the existence of an element r in $R_0\subseteqq H$ satisfying $\delta(\exp tr)l_0'=l_0'+tl'$ for all real t. In fact, we observe first that the dual of $\mathscr{L}_0$ can be canonically identified with the factor space of $\mathscr{L}'$ according to $\mathscr{L}_0^\perp$. We denote by π the projection from $\mathscr{L}'$ onto $\mathscr{L}_0'$. Writing δ_0 for the representation, corresponding to δ, of $\bar{\mathscr{L}}_0$ on $\mathscr{L}_0'$ one verifies easily that $\delta_0(\exp l_0)\pi=\pi\delta(\exp l_0)$ for all l_0 in $\mathscr{L}_0$. Since $H=(F, \mathscr{L}_0)$, taking into account that the orthogonal complement of H in $\mathscr{L}_0'$ is the image of $H^\perp$ under π, for any given $h^\perp$ in $H^\perp$ there exists an element h_0 in H such that $\delta_0(\bar{h}_0)\pi l_0'=\pi l_0'+\pi h^\perp$, which gives $\delta(\bar{h}_0)l_0'=l_0'+h^\perp+cl'$, where c is some real number. This gives (cf. 2) for any real t:

$$\delta(\bar{h}_0 \exp (tr))\, l_0' = \delta(\bar{h}_0)(l_0'+t\bar{l}') = l_0'+h^\perp+[t\exp (\alpha(h_0))+c]\bar{l}'.$$

Choosing $t = -c \exp(-\alpha(h_0))$ and writing $\bar{h} = \bar{h}_0 \exp(tr) \in \bar{H}$ this yields $\delta(\bar{h})l_0'$ $= l_0' + h^\perp$, and since $h^\perp$ was arbitrary in $H^\perp$, we have the desired conclusion.

5. R_0 being the stabilizer of the restriction of l_0' to $\mathscr{L}_0$, we have $-(\operatorname{ad} r)'l_0' = \gamma(r)l'$ for all r in R_0 where the linear form γ vanishes if and only if r belongs to R. On the other hand (cf. 3), $-(\operatorname{ad} r)'\bar{l}' = \alpha(r)l'$ ($r \in R_0$). From this, in view of 4, we conclude that it suffices to find an r in R_0 such that $\gamma(r) = 1$ and $\alpha(r) = 0$. In fact, in this case $-(\operatorname{ad} r)'l_0' = \bar{l}'$, $-(\operatorname{ad} r)'\bar{l}' = 0$, and thus $\delta(\exp tr)l' = l_0' + t\bar{l}'$ for all real t.

6. Suppose now that $\mathscr{L}_0$ is an ideal; then evidently $\alpha \equiv 0$ and so the problem just mentioned has certainly solutions. Hence in this case $R \subseteq \mathscr{L}_0$ suffices to imply $H = (F)$, if $H = (F, \mathscr{L}_0)$ is valid.

7. Suppose that in the general case no r with the required properties exists. Then there is a real number d such that $\gamma(r) = d\alpha(r)$ for all r in R_0 which gives $(\operatorname{ad} r)'(l_0' - d\bar{l}') = 0$ identically and hence $R(l_0' - d\bar{l}') \supseteq R_0$. By the reasoning of 1 above, in this case $\dim R(l_0' - d\bar{l}') = \dim R_0 + 1$, which excludes the equality sign. Since evidently $R(l_0' - d\bar{l}') \cap \mathscr{L}_0 \subseteq R_0$, $R(l_0' - d\bar{l}')$ then cannot be contained in $\mathscr{L}_0$, and this contradiction proves Lemma 1.   Q.E.D.

LEMMA 2. *Suppose that $\mathscr{L}_0$ is a subalgebra, containing H, of $\mathscr{L}$. Suppose also that $\dim \mathscr{L}_0 + 2 = \dim \mathscr{L}$, and that $\mathscr{L}/\mathscr{L}_0$ is irreducible under the restriction of the adjoint representation of $\mathscr{L}$ to $\mathscr{L}_0$. Denote by R_0 the radical of the restriction of l_0' to $\mathscr{L}_0$. Then $H = (F, \mathscr{L}_0)$ implies $H = (F)$ if and only if $R \subsetneqq R_0$.*

REMARK. In what follows we shall write $\mathscr{L}/\mathscr{L}_0 = \operatorname{irr}(\mathscr{L}_0)$ to indicate that $\mathscr{L}_0$ has the properties of the first two sentences of Lemma 2.

Proof. I. To establish the necessity we proceed as in Lemma 1. $H = (F)$ implies (cf. A) $\dim H = \frac{1}{2}(\dim \mathscr{L} + \dim R)$, $R \subseteq H$ and thus $R \subseteq R_0$. On the other hand $H = (F, \mathscr{L}_0)$ gives $\dim H = \frac{1}{2}(\dim \mathscr{L}_0 + \dim R_0)$ and so finally $\dim R + 2 = \dim R_0$ and $R \subsetneqq R_0$.

II. 1. To prove the sufficiency, we observe again that $H = (F, \mathscr{L}_0)$ gives $R_0 \subseteq H$. On the other hand, since

$$\dim \mathscr{L} - \dim R = \dim \mathscr{L}_0 - \dim R_0 + 2k \qquad (k = 0, 1, 2),$$

$R \subsetneqq R_0$ implies $\dim R + 2 = \dim R_0$.

2. We conclude from $\mathscr{L}/\mathscr{L}_0 = \operatorname{irr}(\mathscr{L}_0)$, that there exists a pair of linearly independent elements $\{l_1', l_2'\}$, orthogonal to $\mathscr{L}_0$, of $\mathscr{L}'$, such that for all l_0 in $\mathscr{L}_0$

$$-(\operatorname{ad} l_0)'l_1' = \delta(l_0)(l_1' - \alpha l_2'),$$

$$-(\operatorname{ad} l_0)'l_2' = \delta(l_0)(\alpha l_1' + l_2')$$

where the linear form δ on $\mathscr{L}_0$ is not identically 0 and $\alpha \neq 0$. This gives also

$$\delta(\exp l_0)l_1' = \exp \delta(l_0)(\cos[\alpha\delta(l_0)]l_1' + \sin[\alpha\delta(l_0)]l_2'),$$

$$\delta(\exp l_0)l_2' = \exp \delta(l_0)(-\sin[\alpha\delta(l_0)]l_1' + \cos[\alpha\delta(l_0)]l_2').$$

3. Next we observe that to prove $H=(F)$, it suffices to find a pair of elements r_1, r_2 in R_0 satisfying $\delta(\exp(t_1 r_1 + t_2 r_2))l_0' = l_0' + t_1 l_1' + t_2 l_2'$ for all real t_1 and t_2. In fact, reasoning as in 4 of part II of the proof of Lemma 1, using $H=(F, \mathscr{L}_0)$ we conclude that for any given $h^\perp$ in $H^\perp$ there exists an element $\bar{h}_0$ in $\bar{H}$, and real numbers c_1 and c_2 such that $\delta(\bar{h}_0)l_0' = l_0' + h^\perp + c_1 l_1' + c_2 l_2'$. This gives

$$\delta(\bar{h}_0 \exp(t_1 r_1 + t_2 r_2))l_0' = \delta(\bar{h}_0)(l_0' + t_1 l_1' + t_2 l_2') = l_0' + h^\perp + t_1' l_1' + t_2' l_2',$$

where

$$t_1' = c_1 + \exp[\delta(l_0)](\cos[\alpha\delta(h_0)]t_1 - \sin[\alpha\delta(h_0)]t_2),$$

$$t_2' = c_2 + \exp[\delta(l_0)](\sin[\alpha\delta(h_0)]t_1 + \cos[\alpha\delta(h_0)]t_2).$$

Choosing t_1 and t_2 such that $t_1' = t_2' = 0$, and writing $\bar{h} = \bar{h}_0 \exp(t_1 r_1 + t_2 r_2)$, we obtain in this fashion $\delta(\bar{h})l_0' = l_0' + h^\perp$. Since $\bar{h} \in \bar{H}$, in view of the arbitrariness of $h^\perp \in H^\perp$, this gives $H=(F)$.

4. We have for all r in R_0: $-(\text{ad } r)'l_0' = \lambda_1(r)l_1' + \lambda_2(r)l_2'$, and we have $\lambda_1(r) = \lambda_2(r) = 0$ if and only if r belongs to R. Furthermore, since by 1 the codimension of R in R_0 is 2, λ_1 and λ_2 are linearly independent. By 3, to prove $H=(F)$ it suffices to find a pair of elements $\{r_1, r_2\}$ in R_0 satisfying $\lambda_j(r_i) = \delta_{ij}$ and $\delta(r_i) = 0$ $(i, j = 1, 2)$. Such a pair certainly exists if the triple $\{\lambda_1, \lambda_2, \delta\}$ in $\mathscr{L}'$ is linearly independent. In order to prove this we observe first that the system $\{l_1', l_2', l_0'\}$ is linearly independent in $\mathscr{L}'$. In fact, if this were not so, there would exist numbers b_1 and b_2 such that $l_0' = b_1 l_1' + b_2 l_2'$. This would imply (cf. 2)

$$0 = -(\text{ad } r)'(l_0' - b_1 l_1' - b_2 l_2')$$

$$= [\lambda_1(r) - \delta(r)[b_1 + \alpha b_2]]l_1' + [\lambda_2(r) - \delta(r)[-\alpha b_1 + b_2]]l_2'$$

for all r in R_0, or $\lambda_j(r) = d_j \delta(r)$ $(j = 1, 2)$, contradicting the linear independence of $\{\lambda_1, \lambda_2\}$. We denote by W the subspace of $\mathscr{L}'$ spanned by $\{l_1', l_2', l_0'\}$. W is invariant under $(\text{ad } r)'$ $(r \in R_0)$ and we have using the previous system as a basis

$$\psi(r) = -(\text{ad } r)'|_W = \begin{pmatrix} \delta(r) & \alpha\delta(r) & \lambda_1(r) \\ -\alpha\delta(r) & \delta(r) & \lambda_2(r) \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us write

$$(x_0, x_1, x_2) = \begin{pmatrix} x_0 & \alpha x_0 & x_1 \\ -\alpha x_0 & x_0 & x_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The collection $\hat{\mathscr{L}}$ of all matrices of this form is a Lie algebra with the basis $e_0 = (1, 0, 0)$, $e_1 = (0, 1, 0)$, $e_2 = (0, 0, 1)$ satisfying $[e_0, e_1] = e_1 - \alpha e_2$, $[e_0, e_2] = \alpha e_1 + e_2$, $[e_1, e_2] = 0$. If the system $\{\lambda_1, \lambda_2, \delta\}$ is linearly dependent, we have $\delta = a_1 \lambda_1 + a_2 \lambda_2$, and then the map ψ defined above establishes a homomorphism of R_0 onto a

2-dimensional subalgebra, different from $\{e_1, e_2\}$, of $\hat{\mathscr{L}}$. Therefore, a proof of the impossibility of this suffices to prove Lemma 2. Let $\mathscr{L}_1$ be a subalgebra of $\hat{\mathscr{L}}$ with the basis

$$f_1 = a_0 e_0 + a_1 e_1 + a_2 e_2, \qquad f_2 = b_0 e_0 + b_1 e_1 + b_2 e_2,$$

where $a_0 \neq 0$, say. Replacing, if necessary, $\{f_1, f_2\}$ by $\{(1/a_0)f_1, f_2 - (b_0/a_0)f_1\}$, we can assume $a_0 = 1$, $b_0 = 0$. But then we must have also $[f_1, f_2] = [e_0, f_2] = cf_2$ where c is a real number, which is impossible, since the roots of ad e_0 are $1 \pm i\alpha$, and $\alpha \neq 0$.

LEMMA 3. *Suppose that $H = (G)$ and that $\mathscr{L}_0$ is a subalgebra of $\mathscr{L}$ containing H. Then we have also $H = (G, \mathscr{L}_0)$.*

Proof. We observe again that the dual of $\mathscr{L}_0$ can be canonically identified with the quotient space of $\mathscr{L}'$ according to $\mathscr{L}_0^\perp$, and that the projection π of $\mathscr{L}'$ onto $\mathscr{L}_0'$, consisting in restriction to $\mathscr{L}_0$, maps $H^\perp$ onto the orthogonal complement of H in $\mathscr{L}_0'$. On the other hand (cf. 2a, b) $H = (G)$ means simply that H is maximal self orthogonal with respect to $l_0' + h^\perp$ for all $h^\perp$ in $H^\perp$; but then it is certainly maximal self orthogonal in $\mathscr{L}_0$ with respect to all $\pi(l_0' + h^\perp)$.

To finish the proof of the statement that $H = (G)$ implies $H = (F)$, we observe first that if $\mathscr{L}_1$ is a subalgebra of $\mathscr{L}$ such that

$$\dim \mathscr{L} > \dim \mathscr{L}_1 + 1 \quad \text{and} \quad \mathscr{L}/\mathscr{L}_1 \neq \text{irr}\,(\mathscr{L}_1)$$

(cf. Remark after Lemma 2), then there exists a subalgebra $\mathscr{L}_2$ containing $\mathscr{L}_1$ such that if $\dim \mathscr{L}_2 > \dim \mathscr{L}_1 + 1$, we have $\mathscr{L}_2/\mathscr{L}_1 = \text{irr}\,(\mathscr{L}_1)$ and $\dim \mathscr{L}_2 = \dim \mathscr{L}_1 + 2$; in fact, this follows from an easy application of the theorem of Lie. Therefore, by virtue of Lemmas 1–3, it suffices to prove that if $H = (G)$, and if $\mathscr{L}_0$ is a subalgebra of codimension 1 or 2 of $\mathscr{L}_1$ such that in the latter case $\mathscr{L}/\mathscr{L}_0 = \text{irr}\,(\mathscr{L}_0)$, then $\mathscr{L}_0$ satisfies the conditions of Lemmas 1–2. Indeed, observing that trivially $H = (F, H)$, we can finish then our proof by induction, using an increasing sequence of appropriately chosen subalgebras containing H. By arguments used repeatedly, $H = (G)$ implies that H is maximal self orthogonal with respect to $l_0' + l_0^\perp$ ($l_0^\perp \in \mathscr{L}_0^\perp$); hence $R(l_0' + l_0^\perp) \subseteq R_0 \subseteq H \subseteq \mathscr{L}_0 \subset \mathscr{L}$. Also,

$$\dim H = \tfrac{1}{2}(\dim \mathscr{L} + \dim R) = \tfrac{1}{2}(\dim \mathscr{L}_0 + \dim R_0)$$

and therefore $\dim R_0 - \dim R = \dim \mathscr{L} - \dim \mathscr{L}_0 = 1$ or 2; hence always positive.

For later use we list here the following two statements which are immediate consequences of Lemmas 1–2 resp. and of the equivalence of properties F and J:

LEMMA 1′. *Let $\mathscr{L}_0$ be a subalgebra, of codimension one, of $\mathscr{L}$, containing H, and suppose that $H = (J, \mathscr{L}_0)$. Then $H = (J)$ if and only if for all $l_0^\perp$ in $\mathscr{L}_0^\perp$ $R(l_0' + l_0^\perp)$ is contained in $\mathscr{L}_0$. If $\mathscr{L}_0$ is an ideal, the last condition follows from $R(l_0') \subseteq \mathscr{L}_0$.*

LEMMA 2′. *Suppose that the subalgebra $\mathscr{L}_0$ of codimension 2 in $\mathscr{L}$ contains H and that $\mathscr{L}/\mathscr{L}_0 = \text{irr}\,(\mathscr{L}_0)$. Then $H = (J, \mathscr{L}_0)$ implies $H = (J)$ if and only if $R \subset R_0$, $R \neq R_0$, where R and R_0 are the radicals of l_0' in $\mathscr{L}$ and $\mathscr{L}_0$ resp.*

4. Throughout this section, as in §3, we keep the exponential algebra $\mathscr{L}$, and the nonzero element l'_0 of $\mathscr{L}'$ fixed. We recall (cf. 1), that if the subalgebra H is subordinated to l'_0, ind (H, l'_0) denotes the unitary representation of $\bar{\mathscr{L}}$ induced by the character $\chi(\bar{h}) = \exp(i(h, l'_0))$ of $\bar{H}$ ($h \in H$).

One part of the proof of the following Proposition follows closely certain reasonings of Kirillov (cf. [3, particularly Lemma 5.1]) and Bernat (cf. [1, Chapitre III]).

PROPOSITION 2. *Suppose that the subalgebra H of $\mathscr{L}$ is subordinated to l'_0. Then the unitary representation* ind (H, l'_0) *of $\bar{\mathscr{L}}$ is irreducible if and only if $H = (J)$.*

Proof. We shall use induction according to the dimension of the groups considered. The Proposition being trivially true in the 1-dimensional case, in what follows we shall suppose dim $\mathscr{L} > 1$, and assume its validity for groups of dimension less than dim $\mathscr{L}$. We divide the proof into three parts.

A. We suppose first, that there exists a nonzero ideal I, orthogonal to l'_0 in $\mathscr{L}$; in what follows we denote by the same letter a minimal ideal having this property.

1. Let us prove that if either $H = (J)$, or ind $(H, l') = $ irreducible, we have $I \subseteq H$.

α. We know that if $H = (J)$, H as a subspace of $\mathscr{L}$ is maximal self orthogonal with respect to l'_0. Therefore, it suffices to verify that the subalgebra $H_1 = H + I$ is self orthogonal. But this is clear since $[H_1, H_1] = [H, H] + I$, and by virtue of $H < l'_0$, $[H, H]$ is orthogonal to l'_0.

β. Suppose now, that ind (H, l'_0) is irreducible. If $\mathscr{L}_0$ is a subalgebra, containing H, of $\mathscr{L}$, we write ind $(H, l'_0; \mathscr{L}_0)$ for the unitary representation induced in $\bar{\mathscr{L}}_0$ by the character of $\bar{H}$ corresponding to l'_0. We assume that I is not contained in H, and prove that then ind $(H, l'_0; H + I)$ is reducible. Since the representation induced in $\bar{\mathscr{L}}$ by the representation ind $(H, l'_0; H + I)$ of $\exp(H + I)$ is the same as ind (H, l'_0), and since the reducibility of the inducing representation implies that of the induced one in this fashion we shall obtain a contradiction. Observe that if $I \not\subseteq H$, then, by virtue of the minimality of I, we have $H \cap I = 0$. We distinguish two cases.

(a) If $[H, I] = 0$, $\exp(H + I)$ is isomorphic to the direct product of $\bar{H}$ and $\bar{I}$ and in this case ind $(H, l'_0; H + I)$ is clearly reducible.

(b) Suppose now that $[H, I] \neq 0$. We have either

$$\dim I = 1, \quad I = \{e\} \quad \text{and} \quad \operatorname{ad} h(e) = \lambda(h)e,$$

or dim $I = 2$, $I = \{l_1, l_2\}$ and

$$\operatorname{ad} h(l_1) = \delta(h)(l_1 - \alpha l_2), \qquad \operatorname{ad} h(l_2) = \delta(h)(\alpha l_1 + l_2), \quad [l_1, l_2] = 0,$$

where λ and δ are nontrivial 1-dimensional representations of H, and $\alpha \neq 0$. We consider here in detail the second case only; the first can be settled in a similar fashion. We denote by H_δ the kernel of δ, and by H_0 the subspace of elements, orthogonal to l'_0, of H. Observe that $H_0 = H$ is impossible; in fact in this case, by virtue of the irreducibility of ind (H, l'_0) we would have $H = \mathscr{L}$ and hence $l'_0 = 0$

which we excluded at the beginning of this section. If $H_\delta = H_0$, $H + I/H_0$ is isomorphic to the algebra $\hat{\mathcal{L}} = \{f_0, f_1, f_2\}$ with $[f_0, f_1] = f_1 - \alpha f_2$, $[f_0, f_2] = \alpha f_1 + f_2$, and we obtain ind $(H, l_0'; H + I)$ by lifting up to exp $(H + I)$ a representation, induced by a character of exp $\{f_0\}$, of exp $\hat{\mathcal{L}}$, but this is always reducible. If $S = H_0 \cap H_\delta$ is properly contained in H_δ, we can write $H = X + Y + S$, where $X = \{x\}$, $Y = \{y\}$ with $\delta(x) = 0$ and $\delta(y) = 1$, $(y, l_0') = 0$ resp. $H < l_0'$ implies $[H, H] \subseteqq S$; hence S is an ideal in $H + I$. $H + I/S$ is isomorphic to the direct product of $\hat{\mathcal{L}}$ with a 1-dimensional algebra R, and ind $(H, l_0'; H + I)$ is obtained by lifting up to exp $(H + I)$ a representation of exp $(\hat{\mathcal{L}} \times R)$ induced by a character of $\bar{R}$, which is again reducible.

2. For what follows observe that the dual of $\hat{\mathcal{L}} = \mathcal{L}/I$ can be canonically identified to the subspace $I^\perp$ of $\mathcal{L}'$, and that by virtue of our assumptions we have $l_0' \in I^\perp$.

LEMMA 4. *Suppose that the subalgebra H of $\mathcal{L}$ is subordinated to the element l_0' of $\mathcal{L}'$, and that H contains the ideal I, orthogonal to l_0', in $\mathcal{L}$. We denote by $\tilde{H}$ the canonical image of H in $\tilde{\mathcal{L}} = \mathcal{L}/I$. Then we have $H = (J)$ if and only if $\tilde{H} = (J)$.*

Proof. We denote by $\tilde{\delta}$ the representation corresponding to δ of exp $\tilde{\mathcal{L}}$ on $\tilde{\mathcal{L}}'$. Writing $\tilde{l}$ for the image in $\tilde{\mathcal{L}}$ of $l \in \mathcal{L}$, we have with the identification explained above $\tilde{\delta}(\tilde{l})l_0' = \delta(l)l_0'$; hence the orbits of l_0' with respect to δ and $\tilde{\delta}$ coincide. Since also $l_0' + \tilde{H}^- = l_0' + H^\perp$, to finish the proof it suffices to observe that, by virtue of $I \subset H$, dim $\tilde{H}$ + dim I = dim H, and therefore dim $\tilde{H}$ = dim $\tilde{\mathcal{L}} - \frac{1}{2}$dim O is equivalent to dim H = dim $\mathcal{L} - \frac{1}{2}$dim O.

3. We denote by ρ the canonical homomorphism of $\tilde{\mathcal{L}}$ onto exp $\tilde{\mathcal{L}}$; we have ind (H, l_0') = ind $(\tilde{H}, l_0') \cdot \rho$. To finish the proof of Proposition 2 in case A, suppose first, that ind (H, l_0') is irreducible. Then so is ind $(\tilde{H}, l_0')$; hence by virtue of the hypothesis of the induction, we have $\tilde{H} = (J)$ which by Lemma 4 implies $H = (J)$. Conversely, $H = (J)$ gives $\tilde{H} = (J)$; hence by induction ind $(\tilde{H}, l_0')$ and along with it ind (H, l_0') is irreducible.

B. Here we suppose that the orthogonal complement of l_0' does not contain any nontrivial ideal but that there exists a minimal ideal I not contained in the center C of $\mathcal{L}$. If dim $I = 1$, and $I = \{e\}$, we have ad $l(e) = \lambda(l)e$ $(l \in \mathcal{L})$, where $\lambda \not\equiv 0$, since I is not contained in C. Also, (e, l_0') is nonzero since otherwise we would have case A. If dim $I = 2$, we can choose a basis $\{l_1, l_2\}$ in I, such that

$$\text{ad } l(l_1) = \delta(l)(l_1 - \alpha l_2), \text{ ad } l(l_2) = \delta(l)(\alpha l_1 + l_2) \qquad (l \in \mathcal{L}; \alpha \neq 0),$$

where as before $\delta \not\equiv 0$ and $(l_1, l_0')^2 + (l_2, l_0')^2 \neq 0$.

We denote by $\mathcal{L}_0$ the kernel of λ or of δ resp.

1. We prove first, that if either $H = (J)$ or ind (H, l_0') is irreducible, H is contained in $\mathcal{L}_0$. We shall do this by assuming that $H \not\subseteq \mathcal{L}_0$ and by showing that this leads to a contradiction. We give the proof in three steps.

a. Let us show that if $H < l_0'$ and $H \not\subseteq \mathcal{L}_0$, then $H \cap I = 0$. We take the case when dim $I = 2$; the other case can be settled similarly. Since I is a minimal ideal, if

$H \cap I \neq 0$, we have $I \subset H$. Since, by virtue of our assumption, H is not contained in $\mathscr{L}_0$, there exists an element h in H with $\delta(h) = 1$. We have then, since $H < l_0'$:

$$0 = ([h, l_1], l_0') = (l_1, l_0') - \alpha(l_2, l_0'),$$
$$0 = ([h, l_2], l_0') = \alpha(l_1, l_0') + (l_2, l_0')$$

which gives $(l_j, l_0') = 0$ $(j = 1, 2)$, contradicting the assumption that I is not orthogonal to l_0'.

b. Suppose now that the representation ind (H, l_0') is irreducible. We observe that $H \nsubseteq \mathscr{L}_0$ implies $[H, I] \neq 0$. Since by virtue of $H \cap I = 0$, a word by word repetition of the argument of A.1.β.b. shows that ind $(H, l_0'; H + I)$ is reducible, contradicting the irreducibility of ind (H, l_0'). Hence, in this case $H \subset \mathscr{L}_0$.

c. Next we consider the case when $H = (J)$. Then by virtue of Proposition 1 and Lemma 3, also $H = (J, H + I)$. We shall show that this is impossible, if $H \cap I = 0$. Here we shall restrict ourselves to the discussion of the case dim $I = 1$; the case of an I with dim $I = 2$ can be settled similarly (cf. for this also A.1.β.b.). We denote again by H_0 the subspace of elements, orthogonal to l_0' of H, and H_λ the intersection of H and of the kernel of λ; we have then $H_\lambda \subsetneq H$. If $H_\lambda \cap H_0 = H_\lambda$, the factor algebra of $H + I$ according to H_λ is isomorphic to $\hat{\mathscr{L}} = \{J_0, J_1\}$, with $[J_0, J_1] = J_1$, the image of H being $\{J_0\}$. Then by Lemma 4, $\{J_0\}$ must have the property J in $\hat{\mathscr{L}}$ with respect to an element J' of $\hat{\mathscr{L}}'$ with $(J_1, J_0') \neq 0$ (since $(e, l_0') \neq 0$), which can be shown to be impossible by a simple computation. If $S = H_\lambda \cap H_0$ is properly contained in H_λ, we can write $H = X + Y + S$, $X = \{x\}$, $Y = \{y\}$ with $\lambda(x) = 0$, $\lambda(y) = 1$ and $(J, l_0') = 0$. Since $H < l_0'$, we have $[H, H] \subset [S, S]$, from which we conclude that S is an ideal in $H + I$. The factor algebra of $H + I$ according to S is isomorphic to $\hat{\mathscr{L}}_1 = \{J_0, J_1, J_2\}$, where $[J_0, J_1] = J_1$ and $[J_0, J_2] = [J_1, J_2] = 0$; the image of H is $\{J_0, J_2\}$. Again an application of Lemma 4 shows that $\{J_0, J_2\}$ must have the property in $\hat{\mathscr{L}}_1$ with respect to $J' \in \hat{\mathscr{L}}'$, satisfying $(J_1, J') \neq 0$, $(J_0, J') = 0$, which can easily be disproved.

Summing up, if either ind (H, l_0') is irreducible, or H has the property J, H must be contained in the centralizer of I.

2. We suppose now that ind (H, l_0') is irreducible, and prove, that this implies $H = (J)$. H being contained in $\mathscr{L}_0$, we can form the representation ind $(H, l_0'; \mathscr{L}_0)$ which, too, is irreducible and therefore, by the hypothesis of our induction, $H = (J, \mathscr{L}_0)$. Hence to finish the proof, by virtue of Lemma 1' it suffices to show, that the radical R of l_0' is contained in the ideal $\mathscr{L}_0$. If this is not so then, in the case of an I with dim $I = 2$, there exists an element r in R satisfying $\delta(r) = 1$, which gives

$$0 = ([r, l_1], l_0') = (l_1, l_0') - \alpha(l_2, l_0'),$$
$$0 = ([r, l_2], l_0') = \alpha(l_1, l_0') + (l_2, l_0')$$

implying that I is orthogonal to l_0'. The case dim $I = 1$ can be settled similarly.

3. Suppose finally that $H = (J)$. Then also $H = (J, \mathscr{L}_0)$ and therefore by induction, ind $(H, l_0'; \mathscr{L}_0)$ is irreducible. We denote this representation by T and show that T

induces an irreducible representation in $\bar{\mathscr{L}}$. Since this is precisely ind (H, l_0') in this fashion we prove also that $H = (J)$ implies the irreducibility of ind (H, l_0'). To obtain the desired conclusion, by results of Mackey [4], it suffices to observe that

(a) the orbits of the representation induced by δ on $I' = \mathscr{L}'/I^{\perp}$ are countably separated and simply connected (cf. [9, p. 156]); hence

(b) the stabilizer of the image $\pi l_0'$ of l_0' in I' is connected, and therefore coincides with $\bar{\mathscr{L}}_0$,

(c) the ideal I being in the center of $\mathscr{L}_0$, the restriction of T to $\bar{I}$ is unit operator times the character χ of $\bar{I}$ corresponding to $\pi l_0'$; that is $\chi(\exp(te)) = \exp[it(e, l_0')]$, if dim $I = 1$, and $\chi(\exp(t_1 l_1 + t_2 l_2)) = \exp[i(t_1 l_1 + t_2 l_2, l_0')]$, if dim $I = 2$.

d. We continue to denote by C the center of $\mathscr{L}$. If we do not have case A, then dim $C = 0$, or 1. The former possibility implying B, it remains to consider the case when dim $C = 1$, $C = \{c\}$ and $(c, l_0') \neq 0$. Then there exists an ideal I, containing C, such that dim $I/C = 1$ or dim $I/C = 2$. In the first case we write $I = \{l_0, c\}$, ad $l(l_0) = \delta(l)l_0 + \gamma(l)c$ $(l \in \mathscr{L})$; observe that we can suppose $(l_0, l_0') = 0$. We denote by $\mathscr{L}_0$ the kernel of γ; one sees easily that this is the stabilizer in $\mathscr{L}$ of the image $\pi l_0'$ in I' of l_0' under the representation induced in I' by $d\delta(l) = -(\text{ad } l)'$ $(l \in \mathscr{L})$. If dim $I/C = 2$ we can assume

$$I = \{l_1, l_2, c\}, \quad \text{ad } l(l_1) = \delta(l)(l_1 + \alpha l_2) + \gamma_1(l)c,$$
$$\text{ad } l(l_2) = \delta(l)(-\alpha l_1 + l_2) + \gamma_2(l)c \quad (l \in \mathscr{L}),$$

where $\delta \not\equiv 0$ and $(l_j, l_0') = 0$ $(j = 1, 2)$. In this case, too, I is abelian (cf. [9, p. 155]). In fact, to this end it suffices to observe that for all l in $\mathscr{L}$

$$\text{ad } l([l_1, l_2]) = [\text{ad } l(l_1), l_2] + [l_1, \text{ad } l(l_2)] = 2\delta(l)[l_1, l_2],$$

hence, since $\delta \not\equiv 0$, we have A or B. We denote by $\mathscr{L}_0$ the intersection of the kernels of γ_1 and γ_2, and observe, that it can be interpreted as a stabilizer exactly as above. We divide the discussion in two major parts.

1. First we show that if H is a subalgebra not contained in $\mathscr{L}_0$ in either of the above cases, of $\mathscr{L}$, such that either ind (H, l_0') is irreducible, or $H = (J)$, then there exists a subalgebra H_1 contained in $\mathscr{L}_0$ such that ind $(H, l_0') = \text{ind } (H_1, l_0')$ and that $H_1 = (J)$ is equivalent to $H = (J)$. We write $\hat{H} = H + I$ and observe, that it suffices to find a subalgebra H_1 of $\hat{H} \cap \mathscr{L}_0$, such that

$$\text{ind } (H, l_0', \hat{H}) = \text{ind } (H_1, l_0', \hat{H}) \quad \text{and} \quad H = (J, \hat{H}), \ H_1 = (J, \hat{H}).$$

In fact, we have then certainly ind $(H, l_0') = \text{ind } (H_1, l_0')$. Suppose now, that $H = (J)$, say. To prove, that this implies $H_1 = (J)$, we can proceed, as at the end of the proof of Proposition 1. In fact, if the subalgebras $\mathscr{L}_1$ and $\mathscr{L}_2$ are such, that $\hat{H} \subseteq \mathscr{L}_1 \subset \mathscr{L}_2$ and that either dim $\mathscr{L}_1 + 1 = \dim \mathscr{L}_2$, or dim $\mathscr{L}_1 + 2 = \dim \mathscr{L}_2$ and $\mathscr{L}_2/\mathscr{L}_1 = \text{irr } (\mathscr{L}_1)$, then using $H = (J)$ we conclude, that $\mathscr{L}_1$ satisfies, with respect to $\mathscr{L}_2$, the conditions of Lemma 1' or Lemma 2' on $\mathscr{L}_0$, necessary that $H_1 = (J, \mathscr{L}_1)$ imply $H_1 = (J, \mathscr{L}_2)$. Starting with $H_1 = (J, \hat{H})$, we can then finish the proof of $H_1 = (J)$

by induction. Let us write $H_\gamma = H \cap \mathscr{L}_0$; we are going to prove that $H_1 = H_\gamma + I$ has the required properties. We begin by observing that, since I is abelian, we have $H_1 \subset \hat{H} \cap \mathscr{L}_0$. To prove, that $H_1 < l'_0$ it suffices to remark, that

$$[H_1, H_1] = [H_\gamma, H_\gamma] + [H_\gamma, I];$$

$H < l'_0$ implies $[H_\gamma, H_\gamma] \subset [H, H] \perp l'_0$, and $[H_\gamma, I]$ is contained in $\{l_0\}$ or in $\{l_1, l_2\}$ resp.; hence is again orthogonal to l'_0. If $H = (J)$, then $C \subset R(l'_0) \subset H$, hence $C \subset H$. Similarly, if ind $(H, l'_0) =$ irreducible, then $C \subset H$ (cf. A.1.β.a.). We denote by H_δ the intersection of the kernel of δ and of H. Then $Z = H_\delta \cap H_\gamma$ is the centralizer of I in H, and hence it is an ideal in H. We denote by S the subspace of elements, orthogonal to l'_0, of Z; we have $Z = C + S$. Next we show, that S is an ideal in H; since $[S, I] = 0$, this implies, that it is an ideal in $\hat{H} = H + I$, too. But

$$[S, H] = [S + C, H] = [Z, H] \subseteqq Z = S + C,$$

and thus, by virtue of $H < l'_0$, $[S, H] \subset S$, which proves our statement. Observe, that evidently $S \subset H_1$. Let ϕ denote the canonical homomorphism of $\hat{H}$ on its factor algebra $\mathscr{L}$ according to S. We write $\tilde{H}$ and $\tilde{H}_1$ for the images, under ϕ of H and H_1 resp., and denote by $\tilde{l}'_0$ the elements, corresponding to l'_0, of $\tilde{\mathscr{L}}'$; evidently $\tilde{H}, \tilde{H}_1 < \tilde{l}'_0$. In order to have ind $(H, l'_0; \hat{H}) =$ ind $(H_1, l'_0; \hat{H})$; by known properties of induced representations it suffices to show ind $(\tilde{H}, \tilde{l}'_0) =$ ind $(\tilde{H}_1, \tilde{l}'_0)$. Similarly, using Lemma 4 we conclude, that $\tilde{H} = (J)$ implies $H = (J)$, and $\tilde{H}_1 = (J)$ implies $H_1 = (J)$. Next we turn to the discussion of the structure of $\mathscr{L}$; here we distinguish two cases.

α. dim $I/C = 1$. With the notations introduced before, $I = \{l_0, e\}$ and

$$\text{ad } l(l_0) = \delta(l)l_0 + \gamma(l)c \qquad (l \in \mathscr{L}), \ (l_0, l'_0) = 0.$$

Since we exclude A and B, we must have $\gamma \neq 0$, and thus, because of $H \not\subset \mathscr{L}_0$, $H_\gamma \subsetneqq H$. If $H_\gamma = H_\delta$, then we have on $\hat{H}$: $\gamma = d\delta$, where d is a nonzero constant, and therefore ad $\hat{h}(l_0 + dc) = \delta(\hat{h})(l_0 + dc)$ for all $\hat{h}$ in $\hat{H}$, and $(l'_0 + dc_1 l'_0) \neq 0$. Hence, by B above if either ind $(H, l'_0; \hat{H})$ is irreducible, or $H = (J, \hat{H})$, H must be contained in $H_\delta + I = H_\gamma + I \subset \mathscr{L}_0$, contradicting $H \not\subset \mathscr{L}_0$. Observe, that $H < l'_0$ excludes $l_0 \in H$; indeed, otherwise we would have for all h in H: $0 = ([h_1 l_0], l'_0) = \gamma(h)(c, l'_0)$, and therefore, because of $(c, l'_0) \neq 0$, $H \subset \mathscr{L}_0$. Summing up, we have to consider two subcases.

a. $H_\delta = H$. We can write $H = X + C + S$, where $X = \{x\}$ with $(x, l'_0) = 0$ and $\gamma(x) = 1$. Thus we have $\mathscr{L} = \{e_1, e_2, e_3\}$, with the only nonvanishing bracket $[e_1, e_2] = e_3$, and $(e_j, l'_0) = 0$ $(j = 1, 2)$, $(e_3, l'_0) \neq 0$, finally $\tilde{H} = \{e_1, e_3\}$, $\tilde{H}_1 = \{e_2, e_3\}$.

b. $H_\delta \cap H_\gamma \subset H_\gamma$, $H_\delta \neq H_\gamma$. We have then $H = X + Y + C + S$, $X = \{x\}$, $Y = \{y\}$, $0 = (x, l'_0) = (y, l'_0)$, $\gamma(x) = 1$, $\delta(x) = 0$, $\gamma(y) = 0$, $\delta(y) = 1$. Observe, that $[x, y] \equiv x$ (mod S); in fact, $H < l'_0$ gives $[x, y] \equiv ax$ (mod S) and $a = \gamma([x, y]) = 1$. Therefore $\mathscr{L} = \{e_0, e_1, e_2, e_3\}$ with the nonvanishing brackets $[e_0, e_1] = -e_1$, $[e_0, e_2] = e_2$, $[e_1, e_2] = e_3$. Furthermore $H = \{e_0, e_2, e_3\}$, $\tilde{H}_1 = \{e_0, e_2, e_3\}$ and $(e_j, l'_0) = 0$ $(j = 0, 1, 2)$ $(e_3, l'_0) \neq 0$.

β. Suppose now dim $I/C=2$. Using the notations introduced before, we denote by V the subspace, spanned by γ_1, γ_2, and δ, of $\mathscr{L}'$. We are going to prove that, if dim $V<3$, there exists a choice of $\{l_1, l_2\}$, such that $\gamma_i \equiv 0$ $(i=1, 2)$. In fact, since $\delta \not\equiv 0$, we have dim $V>0$. If dim $V=1$ we have $\gamma_j = d_j \delta$ $(j=1, 2)$, hence it suffices to replace $\{l_1, l_2\}$ by $\{l_1 + x_1 c, l_2 + x_2 c\}$, where $\{x_1, x_2\}$ satisfy $d_1 = x_1 - \alpha x_2$, and $d_2 = -\alpha x_1 + x_2$. If dim $V=2$, the map $\psi(l) = -((\text{ad } l)|_I)'$ $(l \in \mathscr{L})$ defines a homomorphism of $\mathscr{L}$ into the 3 dimensional algebra $\hat{\mathscr{L}}$ considered in the proof of Lemma 2, such that dim $\psi(\mathscr{L})=2$ and, because of $\delta \not\equiv 0$, $\psi(\mathscr{L}) \neq \{e_1, e_2\}$ which is impossible. Replacing $\mathscr{L}$ by $\hat{H}$, and using $H \not\subset \mathscr{L}_0$ we conclude that H is properly contained in $H_\delta \supseteq H_\gamma$, and dim $H/H_\gamma \neq 1$, $H_\delta \neq H$ is impossible. If dim $H/H_\gamma = 2$, no nonzero element $a_1 l_1 + a_2 l_2$ can belong to H. In fact, otherwise we would have

$$0 = ([h, a_1 l_1 + a_2 l_2], l_0') = (a_1 \gamma_1(h) + a_2 \gamma_2(h))(c, l_0')$$

since $H < l_0'$ and $(l_j, l_0')=0$ $(j=1, 2)$.

But this gives $a_1 = a_2 = 0$, by virtue of $(c, l_0') \neq 0$, and the independence of $\{\gamma_1, \gamma_2\}$ on H. Summing up, we have to consider the following subcases.

a. dim $H/H_\gamma = 2$ and $H_\delta = H$. Observe, that in this case evidently $[H, H] \subset Z$, hence also $[H, H] \subset S$. We can write $H = X_1 + X_2 + C + S$ with $X_j = \{x_j\}$, $\gamma_i(x_j) = \delta_{ij}$ and $x_i \perp l_0'$ $(i, j = 1, 2)$. Therefore $\hat{\mathscr{L}} = \{e_1, e_2, e_3, e_4, e_5\}$, where $[e_1, e_2] = e_5$, $[e_3, e_4] = e_5$, $\tilde{H} = \{e_1, e_2, e_5\}$, $\tilde{H}_1 = \{e_3, e_4, e_5\}$ and $(e_j, l_0') = 0$ $(j=1, 2, 3, 4)$.

b. dim $H/H_\gamma = 2$ and $H_\gamma \cap H_\delta$ is properly contained in H_γ. We have

$$H = X_1 + X_2 + Y + C + S,$$

where the x_i, beside conditions as in (a) satisfy $\delta(x_j)=0$ $(j=1, 2)$. Furthermore $Y = \{y\}$, $\delta(y) = 1$, $\gamma_j(y) = 0$ $(j=1, 2)$, and $(y, l_0') = 0$. To obtain the brackets among $\{x_1, x_2, y\}$ we note, that the map ψ considered above, when restricted to H, establishes an isomorphism between $H/(S+C)$ and $\hat{\mathscr{L}}$, and that, with the notations of the proof of Lemma 2, $\psi(h) = (-\delta(h), -\gamma_1(h), -\gamma_2(h))$ $(h \in H)$. We have therefore $[y, x_1] = -x_1 + \alpha x_2$, $[y_1 x_2] = -\alpha x_1 - x_2$ mod $(S+C)$ and hence, because of $H < l_0'$, also mod (S). We conclude in this fashion, that $\hat{\mathscr{L}} = \{e_1, e_2, e_3, e_4, e_5\}$ with the nonvanishing brackets $[e_0, e_1] = -e_1 + \alpha e_2$, $[e_0, e_2] = -\alpha e_1 - e_2$, $[e_0, e_3] = -e_3 + \alpha e_4$, $[e_0, e_4] = \alpha e_3 + e_4$, $[e_1, e_3] = e_5$, $[e_2, e_4] = e_5$. Furthermore $\tilde{H} = \{e_0, e_1, e_2, e_5\}$, $\tilde{H}_1 = \{e_0, e_3, e_4, e_5\}$ and $(e_j, l_0') \neq 0$ only for $j = 5$.

c. dim $H/H_\gamma = 1$, $H_\delta = H$. We have in this case $a_1 \gamma_1 + a_2 \gamma_2 \equiv 0$ on H, where $a_1^2 + a_2^2 \neq 0$. Replacing $\{l_1, l_2\}$ by $\{a_1 l_1 + a_2 l_2, -a_2 l_1 + a_1 l_2\}$ we can suppose $\gamma_1 \equiv 0$ on H, and we have then $[h, l_1] = 0$ $(h \in H)$. As shown above, H contains its centralizer in $\mathscr{L}$ if either $H = (J)$, or ind (H, l_0') is irreducible, and therefore $l_1 \in H$. On the other hand, $\gamma_2 \not\equiv 0$ implies $l_2 \bar{\in} H$. Proceeding, as in $\alpha \cdot a$, we conclude, that $\hat{\mathscr{L}} = \{e_1, e_2, e_3\}$, $[e_1, e_2] = e_3$, $\tilde{H} = \{e_1, e_3\}$, $\tilde{H}_1 = \{e_2, e_3\}$ and $(e_j, l_0') = 0$ $(j=1, 2)$.

The statement ind $(\tilde{H}, l_0') = $ ind $(\tilde{H}_1, l_0')$ is proved for cases α. a, β. a, and β. c in [1, Lemma 5, Chapitre II], and for cases α. b and β. b in [1, Lemma 7, Chapitre II].

To prove, that all subalgebras listed above possess property J, by virtue of Proposition 1 it suffices to show, that they have property G. We shall confine ourselves to verify $\tilde{H} = (G)$ in case β. b; the rest can be settled in a similar fashion. We write $(e_5, l_0') = x \neq 0$. Using a basis, dual to $\{e_j\}$, in $\tilde{\mathscr{L}}'$, we get $l_0' + \tilde{H}^\perp = \{(0, 0, 0, y, y_2, x)\}$, where y_1 and y_2 are arbitrary real numbers. If $\tilde{H}$ is not maximal self orthogonal with respect to an l' in $l_0' + \tilde{H}^\perp$, then there exists a nonzero element e of the form $a_3 e_3 + a_4 e_4$, such that $0 = ([h, e], l')$ for all h in $\tilde{H}$. Hence, in particular

$$0 = ([e_1, e], l') = a_3 x, \qquad 0 = ([e_2, e], l') = a_4 x,$$

contradicting $e \neq 0$.

2. To finish the proof of Proposition 2, we suppose first, that ind (H, l_0') is irreducible, and prove, that this implies $H = (J)$. According to 1 above, it suffices to consider the case, when $H \subset \mathscr{L}_0$. But then ind $(H, l_0'; \mathscr{L}_0)$, too is irreducible and hence, by induction, $H = (J, \mathscr{L}_0)$. Therefore to obtain the desired conclusion, it is enough to check, that $\mathscr{L}_0$ satisfies the conditions of Lemma 1' and Lemma 2' resp. We recall, that with the notations used above dim $\mathscr{L}/\mathscr{L}_0 = \dim I/C = 1$ or 2.

α. If dim $I/C = 1$, suppose, that r is a nonzero element of $R(l_0' + l_0^\perp) - \mathscr{L}_0$, where $l_0^\perp$ is arbitrary in $\mathscr{L}_0^\perp$. Then, in particular, since $l_0 \in \mathscr{L}_0$, $0 = ([r, l_0], l_0^\perp) = \gamma(r)(c_1 l_0')$, and so $\gamma(r) = 0$ and $r \in \mathscr{L}_0$.

β. Assume now, that dim $I/C = 2$. Since $(l_j, l_0') = 0$ $(j = 1, 2)$, we have

$$([l, l_j], l_0') = \gamma_j(l)(c, l_0') \qquad (j = 1, 2)$$

for all l in $\mathscr{L}$. Since $(c, l_0') \neq 0$, we conclude from this first, that $R \subseteq R_0 \subset \mathscr{L}_0$, and secondly, by virtue of the independence of γ_1 and γ_2, that dim $R + 2 = \dim R_0$.

3. Assume finally, that $H = (J)$. To prove that this implies that ind (H, l_0') is irreducible by 1 it suffices again to consider the case when H is contained in $\mathscr{L}_0$. Since we have also $H = (J, \mathscr{L}_0)$ we can conclude by induction, that ind $(H, l_0'; \mathscr{L}_0)$ is irreducible, and then finish the proof exactly as at the end of β by observing (cf. the beginning of c) that the stabilizer of $\pi l_0'$ in $\mathscr{L}$ is $\mathscr{L}_0$ (for the computation of the orbits cf. again [9, p. 156]).

The following Lemma follows from Proposition 2 above and from Théorème, Chapitre III in [1]; our purpose here is to give a direct proof, independent of representation theory.

LEMMA 5. *The subalgebra H is of a maximal dimension in the family of all sub-algebras subordinated to l_0' if and only if* dim $H = \dim \mathscr{L} - \frac{1}{2} \dim O$, *where O is the orbit of l_0'.*

Proof. If $H < l_0'$, H is self orthogonal with respect to l_0', and therefore dim $H \leq \frac{1}{2}(\dim \mathscr{L} + \dim O) = \dim \mathscr{L} - \frac{1}{2} \dim O$. In this fashion it suffices to show, that the value on the right hand side is assumed by an H, $H < l_0'$. In what follows we prove the existence of an H with $H = (J)$, which implies the desired conclusion. We proceed by induction and assume the validity of the statement to be proved for

algebras of dimension less than dim $\mathscr{L}$. Using the notations of the proof of Proposition 2, if we have case A, then $\tilde{\mathscr{L}} = \mathscr{L}/I$, by induction, contains a subalgebra $\tilde{H}$ of the desired sort. By Lemma 4, its complete inverse image H in $\mathscr{L}$ satisfies $H = (J)$. For each of the subcases of cases B and C we have constructed a subalgebra $\mathscr{L}_0$, of codimension 1 or 2, of $\mathscr{L}$, satisfying the conditions of Lemma 1' or Lemma 2' resp. Therefore to finish the proof of our Lemma it suffices to choose, using the hypothesis of our induction, a subalgebra having the property J, of $\mathscr{L}_0$.

REMARK. Observe, that the above proof yields also the following result: Any orbit O of the representation, contragredient to the adjoint representation, contains a linear variety of dimension $\frac{1}{2}$ dim O. In fact, it suffices to note, that if $H = (J)$ then, by definition, $l_0' + H^\perp \subset O$, and that

$$\dim H^\perp = \dim \mathscr{L} - \dim H = \tfrac{1}{2}(\dim \mathscr{L} - \dim R) = \tfrac{1}{2} \dim O.$$

Combining Proposition 2 with Lemma 5, we obtain the following:

THEOREM 1. *Suppose, that G is an exponential group with the Lie algebra $\mathscr{L}$, and let l_0' be a nonzero element of the dual $\mathscr{L}'$ of $\mathscr{L}$. Denote by O the orbit, containing l_0', of the representation, contragredient to the adjoint representation of G. Suppose, that H is a subalgebra of $\mathscr{L}$, such that its first derived algebra is orthogonal to l_0' with respect to the canonical bilinear form on $\mathscr{L} \times \mathscr{L}'$. Then the character*

$$\chi(\exp h) = \exp [i(h, l_0')] \qquad (h \in H)$$

induces an irreducible representation of G if and only if the dimension of H is maximal in the family of subalgebras satisfying the condition above, and if the linear variety $l_0' + $(orthogonal complement of H in $\mathscr{L}'$) is contained in $\mathscr{L}'$.

5. The purpose of this section is to give two sufficient conditions, under which a subalgebra H, subordinated to l_0' and satisfying dim $H = \dim \mathscr{L} - \frac{1}{2} \dim O$ (cf. Lemma 5) has the property J too.

We recall (cf. Introduction), that an exponential algebra $\mathscr{L}$ is called quasi-nilpotent, if it has no nonzero real roots. In particular, every nilpotent algebra is quasi-nilpotent.

PROPOSITION 3. *Suppose, that $\mathscr{L}$ is quasi-nilpotent, and that the subalgebra H, subordinated to $l_0' \in \mathscr{L}'$, satisfies* dim $H = \dim \mathscr{L} - \frac{1}{2} \dim O$. *Then we have also* $H = (J)$.

Proof. The statement being clear if dim $\mathscr{L} = 1$, in what follows we shall assume dim $\mathscr{L} > 1$, and suppose the validity of the proposition for algebras of smaller dimension. Also, we can assume dim $H < \dim \mathscr{L}$ since otherwise H clearly has property J. We denote by $\mathscr{L}_0$ a subalgebra, containing H, of $\mathscr{L}$, such that either dim $\mathscr{L}_0 + 1 = \dim \mathscr{L}$ or dim $\mathscr{L}_0 + 2 = \dim \mathscr{L}$ and $\mathscr{L}/\mathscr{L}_0 = \text{irr}(\mathscr{L}_0)$ (cf. Remark after Lemma 2). If dim $H = \dim \mathscr{L} - \frac{1}{2}$ dim O then H is maximal self orthogonal with respect to $B(l_1, l_2; l_0')$, hence also with respect to $B(l_1, l_2; \mathscr{L}_0, l_0')$ (cf. 2b). From this

we conclude, that $R \subseteq R_0 \subseteq \mathscr{L}_0$, and that, by induction, $H = (J, \mathscr{L}_0)$. Therefore it suffices to verify, that in each case $\mathscr{L}_0$ satisfies the conditions of Lemma 1' and Lemma 2' resp. If the codimension of $\mathscr{L}_0$ in $\mathscr{L}$ is 1, then it is enough to observe, that it is necessarily an ideal of $\mathscr{L}$. In fact, if l_1 is a nonzero element not belonging to $\mathscr{L}_0$, and if l_0 is arbitrary in $\mathscr{L}_0$, we have ad $l_0(l_1) \equiv \alpha l_1 (\text{mod } \mathscr{L}_0)$. Since α is a real eigenvalue of the operator ad l_0, and since $\mathscr{L}$ is quasi-nilpotent, we have $\alpha = 0$, proving $[\mathscr{L}, \mathscr{L}_0] \subseteq \mathscr{L}_0$. If dim $\mathscr{L}_0 + 2 = $ dim $\mathscr{L}$ then, as observed above, $R \subseteq R_0 \subseteq H$ and dim $H = \frac{1}{2}(\text{dim } \mathscr{L}_0 + \text{dim } R_0) = \frac{1}{2}(\text{dim } \mathscr{L} + \text{dim } R)$ and thus dim $R + 2 = $ dim R_0.

The following statement is an immediate consequence of Proposition 2 and Proposition 3, along with Lemma 5.

COROLLARY (BERNAT). *Suppose, that $\mathscr{L}$ is quasi-nilpotent, and that the dimension of the subalgebra H is maximal among the subalgebras subordinated to l_0'. Then* ind (H, l_0') *is irreducible.*

PROPOSITION 4. *Suppose, that the dimension of the subalgebra H of the exponential algebra $\mathscr{L}$ is maximal among the subalgebras subordinated to the nonzero element l_0' of $\mathscr{L}'$. Suppose also, that the orbit O of l_0' is closed. Then* ind (H, l_0') *is irreducible.*

Proof. By virtue of Proposition 2, it suffices to show, that the previous assumptions imply, that $H = (J)$. Since by Lemma 5 we have dim $H = $ dim $\mathscr{L} - \frac{1}{2}$ dim O, it is enough to prove, that $l_0' + H^\perp \subset O$. By the assumption on O, the intersection of O and of $l_0' + H^\perp$ is closed, hence it suffices to show, that at the same time it is open in $l_0' + H^\perp$. Let $h_0^\perp$ be an element of $H^\perp$, such that $l_0' + h_0^\perp$ belongs to O, and consider the map ϕ from H into $O \cap (l_0' + H^\perp)$ defined by $\phi(\bar{h}) = \delta(\bar{h})(l_0' + h_0^\perp)$ $(h \in H)$. Evidently it is enough to prove, that the rank r of ϕ at $h = 0$ is not smaller, than dim $H^\perp = $ dim $\mathscr{L} - $ dim R. But using the reasoning of A of the proof of Proposition 1 we conclude, that

$$r = \text{dim } H - \text{dim } (H \cap R(l_0' + h_0^\perp)) \geq \text{dim } H - \text{dim } R(l_0' + h_0^\perp)$$
$$= \text{dim } H - \text{dim } R(l_0') = \text{dim } H^\perp,$$

since $l_0' + h_0^\perp \in O$ implies dim $R = $ dim $R(l_0' + h_0^\perp)$.

6. In this section we suppose, that $\mathscr{L}$ has real roots. The motivation for the following considerations is described at the end of the Introduction.

Let H be a subalgebra, subordinated to $l_0' \in \mathscr{L}'$, of $\mathscr{L}$, and $\bar{H}$ the connected subgroup corresponding to H. We define again a character χ on $\bar{H}$ by $\chi(\exp h) = \exp [i(h, l_0')]$. Let x be a fixed element of $\bar{\mathscr{L}}$. We have $\chi(x^{-1} \exp hx) = \chi(\exp h)$ for all $\exp h \in x\bar{H}x^{-1} \cap \bar{H}$ if and only if $(\sigma(x^{-1})h, l_0') \equiv (h, l_0')$ for all $h \in \sigma(x)(H) \cap H$; but this is equivalent to $\delta(x)l_0' - l_0' \in (\sigma(x)(H) \cap H)^\perp = \delta(x)(H^\perp) + H^\perp$. This leads to the following

DEFINITION 4. We say, that the subalgebra H, subordinated to l_0', has the property C, and write $H = (C)$ if $\delta(x)(l_0' + h_1^\perp) = l_0' + h_2^\perp$ $(h_1^\perp, h_2^\perp \in H^\perp)$ implies $x \in \bar{H}$.

PROPOSITION 5. *We have $H = (C)$ if and only if $H = (J)$.*

Proof. A. $(J) \Rightarrow (C)$. By virtue of Proposition 1 it suffices to show, that $H = (F)$ implies $H = (C)$. If $h_j^\perp$ $(j = 1, 2)$ are elements in $H^\perp$, we can find elements h_k in H, such that $\delta(\bar{h}_k) l_0' = l_0' + h_k^\perp$ $(k = 1, 2)$. Hence $\delta(x)(l_0' + h_1^\perp) = l_0' + h_2^\perp$ implies

$$\delta(\bar{h}_2^{-1} x \bar{h}_1) l_0' = l_0',$$

and therefore $\bar{h}_2^{-1} x \bar{h}_1$ belongs to the connected subgroup corresponding to the radical R of l_0'. But if $H = (F)$ then, in particular, H is maximal self orthogonal with respect to l_0', and thus $R \subset H$. Hence finally $\bar{h}_2^{-1} x \bar{h}_1 \in \bar{R} \subset \bar{H}$ and $x \in \bar{H}$.

B. $(C) \Rightarrow (J)$. Again by Proposition 1 it suffices to prove, that $H = (C)$ implies $H = (G)$. We are going to do this by showing, that if $H = (C)$, then H is maximal self orthogonal with respect to l_0'. In fact, it will suffice then to observe, that $H < l_0'$ implies $H < l_0' + h^\perp$ for any $h^\perp$ in $H^\perp$, and thus the condition $H = (C)$ remains the same upon replacing l_0' by $l_0' + h_0^\perp$ $(h_0^\perp \in H^\perp)$.

Since the roots of $\mathscr{L}$ are real, there exists a subalgebra $\mathscr{L}_0$, of codimension 1, of $\mathscr{L}$, containing $H(^2)$. Next we prove, that H has the property C with respect to $\mathscr{L}_0$, or that $H = (C, \mathscr{L}_0)$. We denote by π the projection of $\mathscr{L}'$ onto $\mathscr{L}_0' = \mathscr{L}' | \mathscr{L}_0^\perp$, and observe, that the orthogonal complement of H in $\mathscr{L}_0'$ is the image of $H^\perp$ under π. Furthermore, if δ_0 is the representation, corresponding to δ, of $\bar{\mathscr{L}}_0$ on $\mathscr{L}_0'$, we have $\delta_0(\bar{l}_0)\pi = \pi\delta(\bar{l}_0)$ for all l_0 in $\mathscr{L}_0$. Therefore, if $H \neq (C, \mathscr{L}_0)$, there exist elements $h_j^\perp$ $(j = 1, 2)$ in $H^\perp$, and $l_0 \in \mathscr{L}_0$ not in H such that

$$\delta_0(\bar{l}_0)\pi(l_0' + h_1^\perp) = \pi(l_0' + h_2^\perp).$$

By the previous remarks, this implies

$$\delta(\bar{l}_0)(l_0' + h_1^\perp) = l_0' + h_2^\perp + l_0^\perp,$$

where $l_0^\perp$ is orthogonal to $\mathscr{L}_0$. Since $h_2^\perp + l_0^\perp \in H^\perp$, and $l_0 \bar{\in} H$, we have then a contradiction to $H = (C)$. Proceeding by induction, we conclude, that H is maximal self orthogonal in $\mathscr{L}_0$, or dim $H = \frac{1}{2}(\dim \mathscr{L}_0 + \dim R_0)$. Since $H = (C)$ trivially implies, that $R \subset H$, R is of codimension 1 in R_0, and dim $H = \frac{1}{2}(\dim \mathscr{L} + \dim R)$, proving, that H is maximal self orthogonal in $\mathscr{L}$.

Combining Proposition 5 with Proposition 2 we obtain the following.

THEOREM 2. *Let G be an exponential group with real roots, Γ a closed connected subgroup of G, and χ a continuous homomorphism of Γ into the multiplicative group of complex numbers of absolute value one. For an element x of G, we denote by Γ_x the intersection of the subgroups Γ and $x^{-1}\Gamma x$, and by χ_x the character of Γ_x defined by $\chi_x(\gamma) = \chi(x\gamma x^{-1})$ $(\gamma \in \Gamma_x)$. Then the representation, induced by χ, of G is irreducible if and only if, for all x not belonging to Γ, the character χ_x is different from the restriction of χ to Γ_x.*

$(^2)$ This is the only point, where the hypothesis of the real roots is used. Although likely to be valid, the author could not prove Theorem 2 without this restriction.

Proof. We denote by $\mathscr{L}$ the Lie algebra of G. Then Γ is the image, through the exponential map, of some subalgebra H of $\mathscr{L}$. According to what precedes, to obtain the desired result it suffices to observe, that χ can be written as $\exp\,[i(h,\,l'_0)]$ ($h \in H$) with an appropriately chosen l'_0 in $\mathscr{L}'$.

REMARK. As observed earlier (cf. the end of the Introduction), Theorem 2 extends to certain exponential groups a situation known for finite groups (cf. [5, Theorem 6]).

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