# EXTREMAL LENGTH AND CONFORMAL CAPACITY(1)

BY

WILLIAM P. ZIEMER(2)

1. **Introduction.** The theory of extremal length has been successfully applied to conformal mappings, analytic functions of a complex variable, and in recent years it has found application in the study of quasiconformal mappings in space. Its application in quasiconformal mapping theory begins essentially with a theorem proved by Gehring [G3], that the conformal capacity of a space ring $R$ is directly related to the extremal length of the family of curves that join the boundary components of $R$. He has also shown that the conformal capacity is related to the extremal length of a family of surfaces that separate the boundary components of $R$. Gehring assumes that the separating surfaces are sufficiently smooth; Krivov [K] establishes a similar result under the assumption that the extremal metric is well-behaved. Under similar assumptions, other authors have dealt with the extremal length of separating surfaces, cf. [FU, Theorem 9], [SA], and [H].

The purpose of this paper is to eliminate the need for these assumptions. Thus, we consider the case of two disjoint compact sets $C_0$, $C_1$ contained in the closure of a bounded, open, connected set $G$. It is proved that the conformal capacity $C$ of $C_0$, $C_1$, relative to $G$ is related to the $n/n-1$-dimensional module $M$ of all closed sets that separate $C_0$ from $C_1$ in the closure of $G$ by

$$C^{-1/(n-1)} = M.$$

This is accomplished by using a technique of Gehring's [G1, Lemma 3] which eliminates all assumptions concerning the behavior of the extremal metric. Then, a surface-theoretic approximation theorem, first developed in [FF, 8.23], permits the consideration of arbitrary closed separating sets.

The author wishes to thank George Springer, who suggested the problem under consideration in this paper and who has shown a continued interest in the development of its solution.

2. **Preliminaries.**

2.1. $E^n$ will denote Euclidean $n$-space and $L_n$ is $n$-dimensional *Lebesgue measure*. *Hausdorff k-dimensional measure* in $E^n$ will be denoted by $H^k$ (for its definition and properties see [F2]) and in this paper, only $H^1$ and $H^{n-1}$ will be used. If $A \subset E^n$, then bdry $A$ will mean the boundary of $A$ and for $x \in E^n$, $\delta(x, A)$ will be the *distance from x to A*. More generally, $\delta(A, B)$ will denote the *distance between*

Presented to the Society, January 26, 1966; received by the editors May 24, 1966.

(1) Abstract appears in Notices Amer. Math. Soc. **13** (1966), 323 (66T–206).

(2) This work was supported in part by a grant from the National Science Foundation.

*the sets A and B.* $S(x, r)$ will stand for the open $n$-ball of radius $r$ and centered at $x$. If $A$ is an $L_n$-measurable subset of $E^n$, then $\mathscr{L}^p(A)$ will denote those functions $f$ for which $|f|^p$ is $L_n$-integrable over $A$ and $\|f\|_p$ will be its $\mathscr{L}^p$ norm.

2.2. A real valued function $u$ defined on an open set $U \subset E^n$ is said to be *absolutely continuous in the sense of Tonelli* (ACT) on U if it is ACT on every interval $I \subset U$, cf. [SS, p. 169]. Thus, the gradient of $u$, $\nabla u$, exists at $L_n$-almost every point in $U$. The following co-area formula, which was proved in [Z2], will be used frequently throughout this paper.

2.2.1. THEOREM. *If $A$ is an $L_n$-measurable subset of $U$ and $u$ is ACT on $U$, then*

$$\int_A |\nabla u(x)|\, dL_n(x) = \int_{-\infty}^{\infty} H^{n-1}[u^{-1}(s) \cap A]\, dL_1(s).$$

*Therefore,*

$$\int_U f(x)|\nabla u(x)|\, dL_n(x) = \int_{-\infty}^{\infty} \int_{u^{-1}(s)} f(x)\, dH^{n-1}(x)\, dL_1(s)$$

*whenever $f \in \mathscr{L}^1(U)$.*

2.3. *Conformal capacity.* The following will be standard notation throughout. $G$ is an open, bounded, connected set in $E^n$ and $C_0$, $C_1$, are disjoint compact sets in the closure of $G$. We will let $R = G - (C_0 \cup C_1)$ and $R^* = R \cup C_0 \cup C_1$. The *conformal capacity of $C_0$, $C_1$ relative to the closure of $G$* is defined as

$$C[G, C_0, C_1] = \inf \int_R |\nabla u|^n\, dL_n \tag{1}$$

where the infimum is taken over all functions $u$ which are continuous on $R^*$, ACT on $R$, and assume boundary values 1 on $C_1$ and 0 on $C_0$. Such functions are called *admissible* for $C[G, C_0, C_1]$. Sometimes we will write $C$ for $C[G, C_0, C_1]$.

If $C_0 \cup C_1 \subset G$ and if $C_0 \cup C_1$ consists of only a finite number of nondegenerate components, then the arguments of [G2, §§3–7] can be applied with only slight modifications to prove that the infimum in (1) is attained by a unique admissible function $u$ which is ACT in $G$. (By using the methods of Chapter III in [FU], one can prove the existence of an extremal for more general situations.) This extremal function $u$ satisfies the variational condition

$$\int_R |\nabla u|^{n-2} \nabla u \cdot \nabla w\, dL_n = 0 \tag{2}$$

for any function $w$ which is ACT on $G$, assumes boundary value 0 on $C_0 \cup C_1$, and for which $|\nabla w| \in \mathscr{L}^n(R)$.

2.4. *Integral currents.* The following notion of exterior normal, which was first introduced in [F1, p. 48], will be used later.

An $L_n$-measurable set $E \subset E^n$ has the unit vector $n(x)$ as *exterior normal to E at x* if, letting

$$S(x, r) = \{y : |y-x| < r\},$$
$$S_+(x, r) = S(x, r) \cap \{y : (y-x)\cdot n(x) \geqq 0\}$$
$$S_-(x, r) = S(x, r) \cap \{y : (y-x)\cdot n(x) \leqq 0\}$$
$$\alpha = L_n[S(x, 1)]$$

we have

$$2 \lim_{r\to 0} L_n[S_-(x, r) \cap E]/\alpha r^n = 1, \qquad 2 \lim_{r\to 0} L_n[S_+(x, r) \cap E]/\alpha r^n = 0.$$

The set of points $x$ for which $n(x)$ exists is called the *reduced boundary of E* and will be denoted by $\beta(E)$. Obviously, $\beta(E) \subset \text{bdry } E$. The importance of the exterior normal is seen in the following general version of the Gauss-Green Theorem [F3], [DG].

2.4.1. THEOREM. *If E is a bounded $L_n$-measurable set with $H^{n-1}[\beta(E)] < \infty$, then*

$$\int_E \text{div} f(x)\, dL_n(x) = \int_{\beta(E)} f(x)\cdot n(x)\, dH^{n-1}(x)$$

*whenever $f: E^n \to E^n$ is continuously differentiable.*

This theorem enables us to regard a bounded, open set $U \subset E^n$ with

$$H^{n-1}(\text{bdry } U) < \infty$$

as an *integral current mod* 2 (or integral class), i.e., an integral current $T$ with coefficients in the group of integers mod 2, [Z1, §3.6] or [FL]. Thus, if $\phi$ is a differential $n$-form of class $C^\infty$, then

$$T(\phi) = \int_U \phi\, dL_n.$$

The *boundary of T*, $\partial T$, is defined as $\partial T(\omega) = T(d\omega)$ whenever $\omega$ is an $n-1$ form and $d\omega$ is its exterior derivative. Now $\beta(U)$ is a Hausdorff $(n-1)$-rectifiable set and therefore, 2.4.1 allows us to identify $\beta(U)$ with $\partial T$, [Z1, §3.6]. Thus, the *support* of $\partial T$ is cl $(\beta(U)) \subset \text{bdry } U$, the *mass* of $T$ is $M(T) = L_n(U)$, and $M(\partial T) = H^{n-1}[\beta(U)]$ [Z1, §3.6].

In view of this identification the following theorem is an immediate consequence of [Z1, 6.2] although the original version was given in [FF, 8.23]. An open set is called a *convex cell* if it can be expressed as the finite intersection of open half-spaces and an *n-dimensional polyhedral set* is the union of a finite number of convex cells.

2.4.2. THEOREM. *If $U \subset E^n$ is a bounded, open set with $H^{n-1}(\text{bdry } U) < \infty$, then there is a sequence of n-dimensional polyhedral sets $P_i$ such that*

(i) $P_i \subset \{x : \delta(x, U) < i^{-1}\}$,

(ii) $L_n(P_i) \to L_n(U)$ *as* $i \to \infty$,

(iii) $H^{n-1}(\text{bdry } P_i) \to H^{n-1}[\beta(U)]$ as $i \to \infty$.

*Moreover, by employing an argument similar to that of* [FL, (5.6) *and* (5.7)], *it can be arranged that*

(iv) bdry $P_i \subset \{x : \delta[x, \beta(U)] < i^{-1}\}$.

*Of course, one could apply* [FL, 5.6 *and p.* 170] *directly after verifying that the boundary of U in Fleming's sense is* $\beta(U)$. *Finally, defining the measure* $\mu_i$ *as the restriction of* $H^{n-1}$ *to* bdry $P_i$ *and* $\mu$ *as the restriction of* $H^{n-1}$ *to* $\beta(U)$, (iii) *above and* [Z1, 5.7] *or* [FL, 4.2] *imply that*

(v) $\mu_i \to \mu$ *weakly as* $i \to \infty$,

*that is,*

$$\int f\,d\mu_i \to \int f\,d\mu \quad \textit{for every continuous } f.$$

2.5. *The module of a system of measures.* Instead of dealing with extremal length, we prefer to work with the module as developed in [FU].

Let $\mathscr{M}$ be the class of all Borel measures on $E^n$. With an arbitrary system $E \subset \mathscr{M}$ of measures is associated a class of nonnegative Borel functions $f$ subject to the condition

$$\int_{E^n} f\,d\mu \geqq 1 \qquad \text{for every } \mu \in E. \tag{3}$$

We will write $f \wedge \mu$ if (3) is satisfied and $f \wedge E$ if (3) is satisfied for every $\mu \in E$. For $0 < p < \infty$, the *module of E*, $M_p(E)$, is defined as

$$M_p(E) = \inf\left\{\int_{E^n} f^p\,dL_n : f \wedge E\right\}. \tag{4}$$

A statement concerning measures $\mu \in \mathscr{M}$ is said to hold for $M_p$-a.e. $\mu$ if the statement fails to hold for only a set of measures $E_0$ where $M_p(E_0) = 0$. The proofs of the following statements can be found in [FU, Chapter 1].

(i) $M_p(E) \leqq M_p(E')$ if $E \subset E'$,

(ii) $M_p(E) \leqq \sum_{i=1}^{\infty} M_p(E_i)$ if $E = \bigcup_{i=1}^{\infty} E_i$.

(iii) If $\bar{\mu}$ is the completion of $\mu$ and $L_n(A) = 0$, then $\bar{\mu}(A) = 0$ for $M_p$-a.e. $\mu \in \mathscr{M}$.

(iv) If $f \in \mathscr{L}^p(E^n)$, then $f$ is $\bar{\mu}$ integrable for $M_p$-a.e. $\mu \in \mathscr{M}$.

(v) If $\|f_i - f\|_p \to 0$, then there is a subsequence $f_{i_j}$ such that

$$\int_{E^n} |f_{i_j} - f|\,d\bar{\mu} \to 0 \qquad \text{for } M_p\text{-a.e. } \mu \in \mathscr{M}.$$

(vi) If $E \subset \mathscr{M}$, then $M_p(E) = 0$ if and only if there is a nonnegative Borel function $f \in \mathscr{L}^p$ such that

$$\int_{E^n} f\,d\mu = \infty \qquad \text{for every } \mu \in E.$$

(vii) If $p > 1$ and $E \subset \mathscr{M}$, then there is a nonnegative Borel function $f$ such that

$$\int_{E^n} f^p\,dL_n = M_p(E)$$

and $\int f\,d\mu \geqq 1$ for $M_p$-a.e. $\mu \in E$.

For the applications in this paper, the measures $\mu$ are complete (in fact, they are the restrictions of Hausdorff measure to compact sets) and for such measures, we have the following

2.5.1 THEOREM. *If* $p \geqq 2$, $E_1 \subset E_2 \subset \cdots$ *are sets of complete measures, and* $E = \bigcup E_i$, *then*

$$M_p(E) = \lim_{i \to \infty} M_p(E_i).$$

**Proof.** In view of (i) above, observe that the limit exists and is dominated by $M_p(E)$. Therefore, only the case where the limit is finite needs to be considered.

For each $i$, let $f_i$ be the Borel function associated with $E_i$ as in (vii) above. Clarkson's inequality [C] states, for any $i$ and $j$,

$$\int_{E^n} \left| \frac{f_i + f_j}{2} \right|^p dL_n + \int_{E^n} \left| \frac{f_i - f_j}{2} \right|^p dL_n \leqq 2^{-1} \int_{E^n} |f_i|^p \, dL_n + 2^{-1} \int_{E^n} |f_j|^p \, dL_n. \tag{5}$$

If $i > j$, then $2^{-1} \cdot (f_i + f_j) \wedge \mu$ for $M_p$-a.e. $\mu \in E_j$. Therefore, because of (5),

$$\int_{E^n} \left| \frac{f_i - f_j}{2} \right|^p dL_n \leqq 2^{-1} M_p(E_i) - 2^{-1} M_p(E_j). \tag{6}$$

The right side of (6) tends to zero as $i, j \to \infty$ with $i > j$ and therefore, there is a nonnegative function $f$ such that $\|f_i - f\|_p \to 0$. Thus, from the above properties of module (especially (v)), we have that $f \wedge \mu$ for $M_p$-a.e. $\mu \in E$. This implies

$$M^p(E) \leqq \int_{E^n} f^p \, dL_n = \lim_{i \to \infty} M_p(E_i)$$

which is all that is required to prove.

3. **Module of separating sets and conformal capacity.** In this section we will establish the relationship between conformal capacity and arbitrary closed separating sets. $G$, $R^*$, $R$, $C_1$, and $C_0$ will be as defined in §2.3.

We will say that a *set* $\sigma \subset E^n$ *separates* $C_0$ *from* $C_1$ *in* $R^*$ if $\sigma \cap R$ is closed in $R$ and if there are disjoint sets $A$ and $B$ which are open in $R^* - \sigma$ such that $R^* - \sigma = A \cup B$, $C_0 \subset A$, and $C_1 \subset B$. Let $\Sigma$ denote the class of all sets that separate $C_0$ from $C_1$ in $R^*$. With every $\sigma \in \Sigma$, associate a complete measure $\mu$ in the following way: for every $H^{n-1}$-measurable set $A \subset E^n$, define

$$\mu(A) = H^{n-1}(A \cap \sigma \cap R).$$

From the properties of Hausdorff measure, it is clear that the Borel sets of $E^n$ are $\mu$-measurable and therefore the module of $\Sigma$ can be as defined as in 2.5. Thus, for $n' = n/n-1$,

$$M_{n'}(\Sigma) = \inf \left\{ \int_{E^n} f^{n'} \, dL_n : f \wedge \Sigma \right\}$$

where $f \wedge \Sigma$ means that $f$ is a nonnegative Borel function on $E^n$ such that

$$\int_{\sigma \cap R} f \, dH^{n-1} \geqq 1 \qquad \text{for every } \sigma \in \Sigma.$$

3.1. As in [V1, 2.3] it can be shown that if $\Sigma'$ denotes those $\sigma \in \Sigma$ for which $H^{n-1}(\sigma \cap R) = \infty$, then $M_{n'}(\Sigma') = 0$.

3.2. LEMMA. *Let $u$ be an admissible function for $C[G, C_0, C_1]$ (see 2.3) and let $S \subset [0, 1]$ be an $L_1$-measurable set. If $\Sigma(S) = \{u^{-1}(s) : s \in S\}$ and $M_{n'}[\Sigma(S)] = 0$, then $L_1(S) = 0$.*

**Proof.** Since $M_{n'}[\Sigma(S)] = 0$, (vi) of 2.5 provides a nonnegative Borel function $f \in \mathscr{L}^{n'}$ such that

$$\int_{u^{-1}(s) \cap R} f \, dH^{n-1} = \infty \qquad \text{for every } s \in S.$$

However, Hölder's inequality and the co-area formula (2.2.1) imply

$$\infty > \int_R f|\nabla u| \, dL_n \geqq \int_0^1 \int_{u^{-1}(s) \cap R} f(x) \, dH^{n-1}(x) \, dL_1(s)$$

and therefore $L_1(S) = 0$.

3.3. THEOREM. $M_{n'}(\Sigma) \geqq C[G, C_0, C_1]^{-1/n-1}$.

**Proof.** Choose $\varepsilon > 0$ and let $f$ be any function for which $f \wedge \Sigma$. Let $u$ be an admissible function for $C = C[G, C_0, C_1]$ such that

$$\int_R |\nabla u|^n \, dL_n < C + \varepsilon.$$

$R^*$ is connected since $G$ is and therefore it is clear that $u^{-1}(s) \in \Sigma$ for all $0 < s < 1$. Hence, by Hölder's inequality and 2.2.1, we have

$$\begin{aligned}\left(\int_{E^n} f^{n'} \, dL_n\right)^{1/n'} (C+\varepsilon)^{1/n} &\geqq \left(\int_{E^n} f^{n'} \, dL_n\right)^{1/n'} \left(\int_R |\nabla u|^n \, dL_n\right)^{1/n} \\ &\geqq \int_R f|\nabla u| \, dL_n \\ &\geqq \int_0^1 \int_{u^{-1}(s)} f \, dH^{n-1}(x) \, dL_1(s) \geqq 1.\end{aligned}$$

Since $\varepsilon$ is arbitrary,

$$\int_R f^{n'} \, dL_n \geqq C^{-1/n-1}$$

which is also true for the infimum over all $f \wedge \Sigma$, and thus the result follows.

3.4. The opposite inequality will be established by means of a sequence of approximations and *we will begin by first assuming that $C_0 \cup C_1$ consists only of a finite number of nondegenerate continua and that $C_0 \cup C_1 \subset G$. We will also assume initially that $H^{n-1}(\text{bdry } G) < \infty$.*

Let $V$ be an open connected set such that $G \supset \text{cl } V \supset V \supset C_0 \cup C_1$ and let $v$ be the extremal function for $C[V, C_0, C_1]$ (see 2.3). Since $v$ satisfies the variational condition (2), the proof of the following lemma is very similar to that of Lemma 3 in [G3] and will not be given here. It is possible to obtain a stronger result, but the following is sufficient for our purposes. Recall from 2.4 the definition of an $n$-dimensional polyhedral set.

3.5. LEMMA. *Let $\pi$ be the boundary of an n-dimensional polyhedral set $P$ ($P$ not necessarily contained in $V$) such that $C_0 \subset P$ and $C_1 \subset E^n - \text{cl } P$. Then $\pi$ separates $C_0$ from $C_1$ in $V$ and*

$$\int_{\pi(b)} |\nabla v|^{n-1} \, dL_n \geqq 2bC[V, C_0, C_1]$$

*whenever $0 < b < \delta(\pi, C_0 \cup C_1)$ and where $\pi(b) = \{x : \delta(x, \pi) < b\}$.*

3.6. REMARK. In Lemmas 3.7 and 3.8, the integral average, $f_r$, of $|\nabla v|^{n-1}$ will be used. Thus, defining $\nabla v = 0$ on the complement of $V$, for each $r > 0$ we have

$$f_r(x) = \alpha(r)^{-1} \int_{S(x,r)} |\nabla v(y)|^{n-1} \, dL_n(y)$$

where $\alpha(r) = L_n[S(x, r)]$. It is well known that $f_r$ is continuous and that

$$f_r \to |\nabla v|^{n-1} \; L_n\text{-a.e.} \qquad \text{as } r \to 0.$$

Also, by a result of K. T. Smith [S] and Lebesgue's dominated convergence theorem, $\|f_r\|_{n'} \to \| \, |\nabla v|^{n-1}\|_{n'}$ as $r \to 0$ and consequently, $\|f_r - |\nabla v|^{n-1}\|_{n'} \to 0$.

3.7. LEMMA. *With $\pi$ as in 3.5,*

$$\int_{\pi} f_r \, dH^{n-1} \geqq C[V, C_0, C_1]$$

*whenever $r < \delta(\pi, C_0 \cup C_1)$.*

**Proof.** Choose $b > 0$, $r > 0$ so that $b + r < \delta(\pi, C_0 \cup C_1)$. If $\pi_y$ denotes the translation of $\pi$ through the vector $y$, then Fubini's Theorem and 3.5 imply

$$\begin{aligned} \int_{\pi(b)} f_r(x) \, dL_n(x) &= \alpha(r)^{-1} \int_{S(0,r)} \int_{\pi(b)} |\nabla v(x+y)|^{n-1} \, dL_n(x) \, dL_n(y) \\ &= \alpha(r)^{-1} \int_{S(0,r)} \int_{\pi_y(b)} |\nabla v(x)|^{n-1} \, dL_n(x) \, dL_n(y) \qquad (7) \\ &\geqq 2bC[V, C_0, C_1] \end{aligned}$$

since $\pi_y$ satisfies the conditions of 3.5. In addition to this, if $d(x) = \delta(x, \pi)$, then $|\nabla d| = 1$ $L_n$-a.e. on $E^n - \pi$ [F4, (3) of 4.8] and therefore 2.2.1 gives

$$\int_{\pi(b)} f_r(x) \, dL_n(x) = \int_0^b \int_{d^{-1}(s)} f_r(x) \, dH^{n-1}(x) \, dL_1(s). \tag{8}$$

Let $F(s)$ denote the inner integral on the right. Since $f_r$ is continuous on $E^n$ and $\pi$ is the boundary of a polyhedral set, it is clear that

$$\lim_{s\to 0} F(s) = 2\int_{\pi} f_r\, dH^{n-1}.$$

Hence, from (7) and (8) we have

$$C[V, C_0, C_1] \leqq \lim_{b\to 0}(2b)^{-1}\int_0^b F(s)\, dL_1(s) = \int_{\pi} f_r\, dH^{n-1}.$$

3.8. Lemma. *If $\Sigma$ is the class of sets that separate $C_0$ from $C_1$ in $G$, then*

$$\int_{\sigma\cap R} |\nabla v|^{n-1}\, dH^{n-1} \geqq C[V, C_0, C_1]$$

for $M_{n'}$-a.e. $\sigma\in\Sigma$.

**Proof.** Select $\sigma\in\Sigma$ and let $U$ be that part of a partition of $G-\sigma$ that contains $C_0$. Since $H^{n-1}(\text{bdry } G)<\infty$ by assumption, by appealing to 3.1 we can take $H^{n-1}(\text{bdry } U)<\infty$. Recall that $G\supset \text{cl } V$ and that $\nabla v=0$ on the complement of $V$. Hence, we can choose $r_0$ so small that the support of $f_{r_0}$ is contained in $G$ (and therefore for all $r\leqq r_0$) and $r_0<\delta$ (bdry $U$, $C_0\cup C_1$).

By applying 2.4.2 to the set $U$, we obtain a sequence of $n$-dimensional polyhedral sets $P_i$. Let $\pi_i=\text{bdry } P_i$. From properties (i), (ii), and (iv) of 2.4.2 it is clear that eventually $C_0\subset P_i$ and $C_1\subset E^n-\text{cl } P_i$. Thus Lemma 3.7 applies to $\pi_i$ for all large $i$. Now, $\beta(U)\subset\text{bdry } U\subset(\text{bdry } G)\cup\sigma$ and since the support of $f_r$ is contained in $G$ for all $r\leqq r_0$, it is clear that

$$\int_{\sigma} f_r\, dH^{n-1} \geqq \int_{\beta(U)} f_r\, dH^{n-1} \qquad \text{for all } r\leqq r_0. \tag{9}$$

Since $f_r$ is continuous, (v) of 2.4.2 and 3.7 imply

$$\int_{\beta(U)} f_r\, dH^{n-1} = \lim_{i\to\infty}\int_{\pi_i} f_r\, dH^{n-1} \geqq C[V, C_0, C_1] \qquad \text{for } r\leqq r_0.$$

Thus, from (9), we have

$$\int_{\sigma} f_r\, dH^{n-1} \geqq C[V, C_0, C_1] \qquad \text{for } r\leqq r_0. \tag{10}$$

In light of the fact that $\|f_r-|\nabla v|^{n-1}\|_{n'}\to 0$ as $r\to 0$ (3.6), the result follows from (iv) and (v) of 2.5 and (10).

3.9. Lemma. *Let $u$ be the extremal function for $C[G, C_0, C_1]$ (see 3.4 and 2.3). Then for $M_{n'}$-a.e. $\sigma\in\Sigma$,*

$$\int_{\sigma\cap R} |\nabla u|^{n-1}\, dH^{n-1} \geqq C[G, C_0, C_1] = C.$$

**Proof.** Let $V_i$ be a sequence of open connected sets such that

$$G \supset \text{cl } V_{i+1} \supset V_{i+1} \supset \text{cl } V_i \supset V_i \supset C_0 \cup C_1$$

and $G = \bigcup V_i$. Let $v_i$ be the extremal function for $C[V_i, C_0, C_1] = C_i$.

We will first show that $C_i \to C$ as $i \to \infty$. Recall that $C < \infty$. If $i > j$, then the restriction of $v_i$ to $V_j$ is an admissible function for $V_j$ and thus, so is $2^{-1} \cdot (v_i + v_j)$. As in the proof of 2.5.1, an application of Clarkson's inequality [C] gives

$$\int_R \left| \frac{\nabla v_i - \nabla v_j}{2} \right|^n dL_n \leqq \tfrac{1}{2} C_i - \tfrac{1}{2} C_j \qquad \text{for } i > j. \tag{11}$$

Since $C_i$ is a monotonically increasing sequence bounded above by $C$, (11) implies the existence of a vector-valued function $f$ such that

$$\int_R |\nabla v_i - f|^n \, dL_n \to 0 \qquad \text{as } i \to \infty. \tag{12}$$

In fact, since $C_0 \cup C_1$ consists only of a finite number of nondegenerate continua, an argument similar to that of [G2, p. 362] shows that there is an admissible function $u'$ such that $\nabla u' = f$ $L_n$-a.e. on $R$. Thus, (12) shows that

$$\lim_{i \to \infty} C_i = C. \tag{13}$$

This also implies that $u'$ is the extremal function for $C$, i.e., $u' = u$.

Since $\| \, |\nabla v_i| - |\nabla u| \, \|_n \to 0$ as $i \to \infty$, there is a subsequence of $|\nabla v_i|$ (which will still be denoted by $|\nabla v_i|$) such that $|\nabla v_i| \to |\nabla u|$ $L_n$-a.e. and therefore that $|\nabla v_i|^{n-1} \to |\nabla u|^{n-1}$ $L_n$-a.e. on $R$. This fact, along with

$$\| \, |\nabla v_i|^{n-1} \|_{n'} \to \| \, |\nabla u|^{n-1} \|_{n'}$$

leads to

$$\| \, |\nabla v_i|^{n-1} - |\nabla u|^{n-1} \|^{n'} \to 0 \qquad \text{as } i \to \infty.$$

Thus, with (iv) and (v) of 2.5, we have (for another subsequence)

$$\lim_{i \to \infty} \int_{\sigma \cap R} |\nabla v_i|^{n-1} \, dH^{n-1} = \int_{\sigma \cap R} |\nabla u|^{n-1} \, dH^{n-1} \qquad \text{for } M_{n'}\text{-a.e. } \sigma \in \Sigma. \tag{14}$$

Lemma 3.8 states that for each $i$,

$$\int_{\sigma \cap R} |\nabla v_i|^{n-1} \, dH^{n-1} \geqq C_i \qquad \text{for } M_{n'}\text{-a.e. } \sigma \in \Sigma,$$

and therefore, the result follows from (13), (14), and (ii) of 2.5.

3.10. Theorem. *If $G$ is a bounded, open, connected set, if $C_0 \cup C_1$ consists of only a finite number of nondegenerate continua, and if $C_0 \cup C_1 \subset G$, then*

$$M_{n'}(\Sigma) = C[G, C_0, C_1]^{-1/n-1}.$$

**Proof.** If it were the case that $H^{n-1}(\text{bdry } G)<\infty$, then the result would follow immediately from 3.3 and 3.9. For, if we let $f=|\nabla u|^{n-1}\cdot C^{-1}$, where $u$ is the extremal for $C$, then by 3.9 there is $\Sigma_0\subset\Sigma$ such that $f\wedge\Sigma_0$ and $M_{n'}(\Sigma_0)=M_{n'}(\Sigma)$. Hence, by 3.3,

$$C^{-1/n-1} \leqq M_{n'}(\Sigma) \leqq \int_R f^{n'}\, dL_n = C^{-1/n-1}.$$

In order to eliminate the assumption $H^{n-1}(\text{bdry } G)<\infty$, select a sequence of open, connected sets $V_1\subset V_2\subset\cdots$ such that $C_0\cup C_1\subset V_1$, $H^{n-1}(\text{bdry } V_i)<\infty$, and $G=\bigcup V_i$. As in the proof of 3.9, let $v_i$ be the extremal for $C[V_i, C_0, C_1]=C_i$ and again we have

$$(15)\qquad C_i\to C,\ \|\,|\nabla v_i|^{n-1}-|\nabla u|^{n-1}\|_{n'}\to 0 \qquad \text{as } i\to\infty.$$

Thus, for a subsequence

$$(16)\qquad \lim_{i\to\infty}\int_{\sigma\cap R}|\nabla v_i|^{n-1}\, dH^{n-1} = \int_{\sigma\cap R}|\nabla u|^{n-1}\, dH^{n-1} \qquad \text{for } M_{n'}\text{-a.e. } \sigma\in\Sigma.$$

For each $i$, every $\sigma\in\Sigma$ separates $C_0$ from $C_1$ in $V_i$, and thus applying 3.9 with $V_i$ replacing $G$, we have

$$\int_{\sigma\cap R}|\nabla v_i|^{n-1}\, dH^{n-1} \geqq \int_{\sigma\cap V_i}|\nabla v_i|^{n-1}\, dH^{n-1} \geqq C_i \qquad \text{for } M_{n'}\text{-a.e. } \sigma\in\Sigma.$$

(Observe that 3.9, with $V_i$ replacing $G$, applies only to $\Sigma_i$, which are those sets that separate $C_0$ from $C_1$ in $V_i$. However, a class in $\Sigma$ which is $M_{n'}$-zero relative to $\Sigma_i$ is $M_{n'}$-zero relative to $\Sigma$.) Hence, in view of (15) and (16),

$$(17)\qquad \int_{\sigma\cap R}|\nabla u|^{n-1}\, dH^{n-1} \geqq C \qquad \text{for } M_{n'}\text{-a.e. } \sigma\in\Sigma$$

which, as we have seen from above, is sufficient to establish the theorem.

3.11. Corollary. *With the hypotheses of* 3.10, *and if $u$ is the extremal for* $C[G, C_0, C_1]$, *then*

(i) $0\leqq u(x)\leqq 1$ *for all* $x\in G$,

(ii) $\int_{\sigma\cap R}|\nabla u|^{n-1}\, dH^{n-1}\geqq C[G, C_0, C_1]$ *for* $M_{n'}$*-a.e.* $\sigma\in\Sigma$,

(iii) $\int_{u^{-1}(s)}|\nabla u|^{n-1}\, dH^{n-1}=C[G, C_0, C_1]$, *for* $L_1$*-a.e.* $s\in[0, 1]$.

**Proof.** By truncating $u$ at levels 1 and 0 if necessary, a new admissible function would be formed whose gradient would be bounded above by the gradient of $u$. However, the extremal is unique and thus (i) follows. (ii) is just a restatement of (17).

In order to prove (iii), let

$$F(s) = \int_{u^{-1}(s)}|\nabla u|^{n-1}\, dH^{n-1} \qquad \text{for } L_1\text{-a.e. } s,$$

and observe that since $G$ is connected, $u^{-1}(s) \in \Sigma$ whenever $0<s<1$. Thus, (ii) above and 3.2 imply that $F(s) \geqq C$ for $L_1$-a.e. $s \in [0, 1]$. However, (i) and an application of 2.2.1 give

$$C = \int_R |\nabla u|^n \, dL_n = \int_0^1 F(s) \, dL_1(s)$$

which implies that $F(s)=C$ for $L_1$-a.e. $s \in [0, 1]$.

3.12. REMARK. The following observation has some interest in view of Theorem 2 of [G2].

*In addition to the hypotheses of* 3.11, *assume that* $H^{n-1}(C_0)=0$. *Then there is a point* $x_0 \in C_0$ *such that*

$$\limsup_{x \to x_0} |\nabla u(x)| = \infty.$$

For if this were not the case, then, since $C_0$ is compact, there would be a constant $K>0$ and an open set $U$ such that $G-C_1 \supset \text{cl } U \supset U \supset C_0$ and $|\nabla u|^{n-1} < K \; L_n$-a.e. on $U$. Choose $\varepsilon > 0$. Since $H^{n-1}(C_0)=0$, $C_0$ can be covered by a countable number of open $n$-balls $B_i$ such that

$$\bigcup B_i \subset U \quad \text{and} \quad \Sigma \, H^{n-1}(\text{bdry } B_i) < \varepsilon K^{-1}. \tag{18}$$

Since $C_0$ is compact, a finite number of the $B_i$ will cover $C_0$, say $B_1, B_2, \ldots, B_k$. According to (vi) of 2.5, there is a nonnegative Borel function $f \in \mathscr{L}^{n'}(R)$ such that (ii) of 3.11 fails to hold for only those $\sigma \in \Sigma$ for which

$$\int_{\sigma \cap R} f \, dH^{n-1} = \infty.$$

By employing 2.2.1, we can replace each $n$-ball $B_i$, $i=1, 2, \ldots, k$, by a slightly larger one $B_i'$ such that

$$\int_{S_i'} f \, dH^{n-1} < \infty \qquad \text{where } S_i' = \text{bdry } B_i',$$

$|\nabla u|^{n-1} < K \; H^{n-1}$-a.e. on $S_i'$, and (18) still holds. Now let $\sigma = \text{bdry } (\bigcup B_i')$. Then $\sigma \in \Sigma$ and (ii) of 3.11 implies that $C[G, C_0, C_1] < \varepsilon$, which means that it is zero since $\varepsilon$ is arbitrary. This would mean that $\nabla u = 0 \; L_n$-a.e. on $G$. That is, since $G$ is connected and $u$ is ACT on $G$, $u$ would be constant, a contradiction.

In the following theorem, we will consider the general case of two disjoint compact sets $C_0$ and $C_1$ which are contained in the closure of an open, bounded, connected set $G$.

3.13. THEOREM. $M_{n'}(\Sigma) = C[G, C_0, C_1]^{-1/n-1}$.

**Proof.** In view of 3.3, we may assume that $C = C[G, C_0, C_1] \neq 0$. For each positive integer $i$ let

$K_0(i) = \text{cl} \{x : \delta(x, C_0) < (2i)^{-1}\}$ and

$H_0(i) = \{x : \delta(x, C_0) < i^{-1}\}$.

Define $K_1(i)$ and $H_1(i)$ similarly and let $G_i = G \cup H_0(i) \cup H_1(i)$. Since $G$ is connected, it is clear that $G_i$ is open and connected and notice that $K_0(i) \cup K_1(i)$ consists only of a finite number of nondegenerate continua. We will consider only those $i$ for which $K_0(i)$ and $K_1(i)$ are disjoint. Thus, let $v_i$ be the extremal function for $C_i = C[G_i, K_0(i), K_1(i)]$ and observe that if $i > j$, then the restriction of $v_j$ to $G_i$ is an admissible function for $C_i$. Finally, let $\Sigma_i$ be those sets $\sigma$ that separate $K_0(i)$ from $K_1(i)$ in $G_i$ and subject to the condition that $\sigma \cap [H_0(i) \cup H_1(i)] = 0$. The purpose for this requirement is that now an $M_{n'}$-null class in $\Sigma_i$ is also $M_{n'}$-null in $\Sigma$. It is clear that $\Sigma_1 \subset \Sigma_2 \subset \cdots, \Sigma_i \subset \Sigma$ for all $i$, and

$$\bigcup_{i=1}^{\infty} \Sigma_i = \Sigma. \tag{19}$$

Since $C_i$ is a monotonically decreasing sequence bounded below by $C$, we can employ again part of the argument of 3.9 to find a vector-valued function $f$ such that

$$\lim_{i\to\infty} \int_{E^n} |\nabla v_i - f|^n \, dL_n = 0$$

and therefore, for a subsequence,

$$\lim_{i\to\infty} \| \, |\nabla v_i|^{n-1} - |f|^{n-1} \|_{n'} = 0. \tag{20}$$

Hence, if $L = \lim_{i\to\infty} C_i$, $g_i = |\nabla v_i|^{n-1} \cdot C_i^{-1}$, and $g = |f|^{n-1} \cdot L^{-1}$ then (v) of 2.5 provides another subsequence such that

$$\lim_{i\to\infty} \int_{\sigma\cap R} g_i \, dH^{n-1} = \int_{\sigma\cap R} g \, dH^{n-1} \qquad \text{for } M_{n'}\text{-a.e. } \sigma \in \Sigma. \tag{21}$$

Now by employing 3.11 with $G$, $C_0$, $C_1$ replaced by $G_i$, $K_0(i)$, $K_1(i)$ respectively, we have for each $i$,

$$\int_{\sigma\cap R} g_i \, dH^{n-1} \geqq 1 \qquad \text{for } M_{n'}\text{-a.e. } \sigma \in \Sigma_i. \tag{22}$$

Therefore, (ii) of 2.5, (19), and (21) imply

$$\int_{\sigma\cap R} g \, dH^{n-1} \geqq 1 \qquad \text{for } M_{n'}\text{-a.e. } \sigma \in \Sigma. \tag{23}$$

Since $v_i$ is the extremal for $C_i$, (ii) of 3.11 and 3.3 show that, for each $i$,

$$\int_{E^n} (g_i)^{n'} \, dL_n = C_i^{-1/n-1}.$$

Thus, with (20), (22), and (23), we have

$$C^{-1/n-1} \geqq \lim_{i\to\infty} C_i^{-1/n-1} = \lim_{i\to\infty} \int_{E^n} (g_i)^{n'} \, dL_n = \int_{E^n} g^{n'} \, dL_n \geqq M(\Sigma).$$

Theorem 3.3 gives the opposite inequality, and thus the proof is complete.

We will conclude with a result concerning null sets for conformal capacity.

3.14. THEOREM. *Suppose $C_0$ and $C_1$ are disjoint compact sets in the closure of $G$. If $E \subset G-(C_0 \cup C_1)$ is a compact set with $H^{n-1}(E)=0$, then*

$$C[G, C_0, C_1] = C[G-E, C_0, C_1].$$

**Proof.** The topological dimension of $E$ is no more than $n-2$ since $H^{n-1}(E)=0$ and therefore, $G-E$ is connected. Thus, the right hand side of the equality has meaning. Clearly,

$$C[G, C_0, C_1] \geqq C[G-E, C_0, C_1]. \tag{24}$$

The opposite inequality will be established by considering separating sets. Let $\Sigma^*$ be those sets that separate $C_0$ from $C_1$ in

$$[(G-E)-(C_0 \cup C_1)] \cup [C_0 \cup C_1]$$

and let $\Sigma$ be those that separate $C_0$ from $C_1$ in $R^*$. In light of (24) and 3.13 it is sufficient to prove

$$M_{n'}(\Sigma) \geqq M_{n'}(\Sigma^*). \tag{25}$$

To this end, let $f$ be a function for which $f \wedge \Sigma$. In order to establish (25), we need only show $f \wedge \Sigma^*$. Choose $\sigma^* \in \Sigma^*$ and notice that $\sigma^* \cup E \in \Sigma$. Thus,

$$\int_{\sigma^* \cup E} f\, dH^{n-1} \geqq 1$$

and since $H^{n-1}(E)=0$,

$$\int_{\sigma^*} f\, dH^{n-1} \geqq 1.$$

This shows that $f \wedge \Sigma^*$ and consequently, proves the theorem.

If $G$ is compactified $E^n$, Bagby has shown that $C[G, C_0, C_1]=M_n(\Gamma)$ where $\Gamma$ is the family of all arcs that meet both $C_0$ and $C_1$ (Ph.D. thesis, Harvard Univ., Cambridge, Mass., 1966). By using 3.13, [FU], and [W] one can show that this result is valid when $G$ is an open, bounded, connected set and $C_0 \cup C_1 \subset G$. (Moreover, if $C_0 \cup C_1 \subset \operatorname{cl} G$, the result is also valid if certain conditions are imposed on the tangential behavior of $(\operatorname{bdry} G) \cap (C_0 \cup C_1)$.) Thus, if $\Gamma^*$ is the family of arcs that join $C_0$ to $C_1$ in $G-E$, then 3.14 implies

$$M_n(\Gamma^*) = M_n(\Gamma).$$

This result was obtained by Väisälä [V2] in the case where $C_0$ and $C_1$ are nondegenerate continua.

## BIBLIOGRAPHY

[C] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.

[DG] E. DeGiorgi, *Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni*, Ricerche Mat. **4** (1955), 95–113.

[F1] F. Federer, *The Gauss-Green theorem*, Trans. Amer. Math. Soc. **58** (1945), 44–76.

[F2] ———, *The* ($\varphi$, *K*) *rectifiable subsets of n space*, Trans. Amer. Math. Soc. **62** (1947), 114–192.

[F3] ———, *A note on the Gauss-Green theorem*, Proc. Amer. Math. Soc. **9** (1958), 447–451.

[F4] ———, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491.

[FF] H. Federer and W. H. Fleming, *Normal and integral currents*, Ann. of Math. **72** (1960), 458–520.

[FL] W. H. Fleming, *Flat chains over a finite coefficient group*, Trans. Amer. Math. Soc. **121** (1966), 160–186.

[FU] B. Fuglede, *Extremal length and functional completion*, Acta Math. **98** (1957), 171–219.

[G1] F. W. Gehring, *Symmetrization of rings in space*, Trans. Amer. Math. Soc. **101** (1961), 499–519.

[G2] ———, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.

[G3] ———, *Extremal length definitions for the conformal capacity in space*, Michigan Math. J. **9** (1962), 137–150.

[H] J. Hersch, *Longeurs extremales dans l'espace, resistance electrique et capacité*, C. R. Acad. Sci. Paris **238** (1954), 1639–1641.

[K] V. V. Krivov, *Some properties of moduli in space*, Dokl. Akad. Nauk. SSSR **154** (1964), 510–513 = Soviet Math. Dokl. **5** (1964), 113–117.

[SA] B. V. Šabat, *The modulus method in space*, Dokl. Akad. Nauk. SSSR **130** (1960), 1210–1213 = Soviet Math. Dokl. **1** (1960), 165–168.

[SS] S. Saks, *Theory of the integral*, Monografje Matematyczne, Warsaw, 1937.

[S] K. T. Smith, *A generalization of an inequality of Hardy and Littlewood*, Canad. J. Math. **8** (1956), 157–170.

[V1] J. Väisälä, *On quasiconformal mapping in space*, Ann. Acad. Sci. Fenn. Ser. AI **296** (1961), 1–36.

[V2] ———, *On the null-sets for extremal distances*, Ann. Acad. Sci. Fenn. Ser. AI **322** (1962), 1–12.

[W] H. Wallin, *A connection between* $\alpha$*-capacity and* $L^p$*-classes of differentiable functions*, Ark. Mat. **5** (1963–1965), 331–341.

[Z1] W. P. Ziemer, *Integral currents* mod 2, Trans. Amer. Math. Soc. **105** (1962), 496–524.

[Z2] ———, *Some lower bounds for Lebesgue area*, Pacific J. Math. **19** (1966), 381–390.

INDIANA UNIVERSITY,
BLOOMINGTON, INDIANA