THE CONSTRUCTION OF CERTAIN 0-DIMENSIONAL TRANSFORMATION GROUPS

BY
R. F. WILLIAMS

The purpose of this paper is to demonstrate the power of the techniques of construction introduced in [4] and [6] and hopefully, to encourage new research on the Hilbert-Smith Conjecture. The constructions make use of the functor described in [6] which is useful when a great deal of structure, or much computation is involved, e.g., infinite transformation groups or complicated local cohomology groups.

We give an example in Part I of a 1-dimensional space X (1 for simplicity; similar constructions work for any n) and a free action of the p-adic group A_p on X such that

A_1. dim $X/A_p = m = \dim X+1$:

B. $H_c^m(U) = Z_{p^\infty}$, for any connected open subset U of X/A_p.

Property B is hard to achieve; and this indicates the power of these techniques (e.g., see Lemma 1.3.2).

To see why one is interested in such properties, recall the famous and as yet unsolved

HILBERT-SMITH CONJECTURE. *If a compact group G acts freely*(1) *on a manifold, then G is a Lie group.*

To prove this conjecture, it would suffice, [5], [7] or [1], to show that no p-adic group can act freely on a manifold. Thus in the past, researchers have looked for whatever surprising consequences they could find from the assumption that a p-adic group A_p acts freely on an n-manifold X. In 1940, P. A. Smith [5] found,

A_0. dim $X/A_p \neq \dim X$.

Later, C. T. Yang [7] found

A_2. dim $X^n/A_p = m = \dim X+2$ (or ∞); and

B. $H^m(U) = Z_{p^\infty}$, U any connected open subset of X/A_p.

These two properties were also proved in [1] as easy (at least in the free case) consequences of the computation

$$(*)\ H^q(B_{A_p}) = \begin{cases} Z & \text{if } q = 0, \\ Z_{p^\infty} & \text{if } q = 2, \\ 0 & \text{otherwise,} \end{cases}$$

where B_{A_p} is the classifying space of A_p. The proof is as follows: using (*), a well-known spectral sequence has Z_{p^∞} as a corner term, and out pop A_2 and B. Thus,

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(1) Here and below we consider only the free case for simplicity. Similar remarks apply to *effective* transformation groups.

these two properties are the two most salient consequences of the assumption that A_p acts on a manifold.

In [4], Frank Raymond and the author gave an example in which A_2 holds. The space of course was not a manifold and B was far from true. The current example represents the author's best attempts to construct an example of a p-adic transformation group having these two properties.

Indeed, it is conjectured that A_2 and B cannot occur together for any space X, at least if dim $X/A_p < \infty$. A weak version of this conjecture is proved in Part II, to wit: *the current technique of construction cannot yield an example satisfying* A_2 *and* B.

Though very weak, this result is of interest for three reasons: (1) the existence of the example satisfying A_1 and B; (2) the "obstruction" is a rather delicate one; and (3) the success of this technique of construction is *very nearly* necessary for the existence of such an example. That is, this line of reasoning ("if there is an example, then it can be constructed in such-and-such a way") could well lead to a proof of the Hilbert-Smith Conjecture.

Finally we give yet another reason for a renewed attack on this old problem in the form of a compact, 2-dimensional "classifying space" B'_{A_p}, due to E. E. Floyd. The point is, ordinary cohomology does not distinguish between a "real" classifying space and B'_{A_p}, though all the results on this conjecture can be obtained via $H^*(B_{A_p})$ or its isomorph, $H^*(B'_{A_p})$. However, complex K-theory *does* distinguish between B_{A_p} and B'_{A_p}. Perhaps K-theory and some version of the spectral sequence alluded to above could lead to new results on this problem.

0. Notation and conventions. We will use S^n to denote a topological n-sphere. Z will be the integers, $Z_p = Z/pZ$, and the p-adic group A_p is the inverse limit of the sequence

$$Z_p \leftarrow Z_{p^2} \leftarrow \cdots$$

in which each map sends a generator into a generator. Z_{p^∞} is the dual of A_p and is thus the group of p^i-roots of unity, $i = 1, 2, \ldots$.

1 will denote any identity map as well as certain inclusions; if K is an n-circuit with boundary, ∂K will denote its boundary. Čech cohomology is used and H_c^* indicates compact supports. If a group π acts on a space X, X/π indicates its orbit space. If $\{f_i\}$, $i = 1, \ldots, n$, is a family of periodic maps defined on a subset A of X, then we may identify the orbits of A under the group π generated by $\{f_i\}$, $i = 1, \ldots, n$, to form a new space Y. This device is used throughout.

Finally we will make use of the functor defined in [6]. Let s^n be the standard n-simplex, thought of as a complex, and hence as closed. A simplex σ in a complex K is otherwise always taken as open. K' denotes the 1st barycentric subdivision of K. The notation differs from that of [6] in that the factors K, X are reversed, which now seems more natural. Thus for an n-complex K a space X and a map $q: X \to s^n$, $K\Delta_q X = \{(k, x) \in K \times X : \Phi k = qx\}$ where $\Phi: K \to s^n$ is the simplicial map determined on K' as follows. A typical vertex of K' is the barycenter $b(\sigma^i)$ of some i-dimensional simplex σ^i of K. Then $\Phi(b(\sigma^i))=v_i$, the ith vertex of s^n, relative to some fixed ordering of the vertices of s^n. (See [6] for other details.)

I. An Example Satisfying A_1 and B.

1. Description of the example.

1.1. First define the sequence $\{a_n\}$ of integers by

$$a_1 = 1; \quad a_{n+1} = a_n+n!.$$

Let π_n be cyclic of order p^{a_n}. Then A_p is the inverse limit of the sequence

$$\pi_1 \leftarrow \pi_2 \leftarrow \pi_3 \leftarrow \cdots$$

in which the map $\lambda_n\colon \pi_{n+1} \to \pi_n$ sends a generator, say g_{n+1}, of π_{n+1} into the generator g_n of π_n.

Next, we describe the building blocks:

$$X = S^1\times[0, 1],$$
$$B = S^1\times 1\subset X,$$
$$q_0 : X = S^1\times[0, 1]\to S^1\times[0, 1]/B=S^1 * b = \partial s^2 * b = s^2;$$
$$\phi_i\colon X\to X \text{ the rotation of } S^1 \text{ through } 2\pi/p^i \text{ radians},$$
$$r_i\colon X\to X/\phi_i = X.$$

We suppose X triangulated in accordance with [6, S01, p. 322].

1.2. Let K be a triangulated 2-manifold. We successively modify K getting $\{K_n\}$, then $\{L_n\}$. The example will be a limit of the L_n's. As the steps defining $\{K_n\}$ and $\{L_n\}$ are quite complicated, we first briefly describe these for $n=1$ and 2: Let K be a triangulated oriented 2-manifold, say a two-sphere,

$$K_1 = K\Delta_{q_0}X,$$
$$K_2 = K\Delta_{q_1}X\Delta_{q_0}X, \qquad (\text{here } q_1 = q_0r_1),$$
$$1\Delta r_1\Delta q_0\colon K\Delta_{q_1}X\Delta_{q_0}X\to K\Delta_{q_0}X$$

(see [6, §1]). Then $B_1=K\Delta B\subset K_1$, and $B_2=K\Delta B\Delta X\cup K\Delta X\Delta B\subset K_2$, and K_i is an orientable manifold with boundary B_i. Then form L_1 from $\pi_1\times K_1$ by identifying the points of $\pi_1\times B_1$ under $g_1\times 1\Delta\phi_1$; π_1 still acts on L_1, with generator induced by $g_1\times 1$. Finally $H^2(L_1/\pi_1)=Z_p$.

Form L_2 by identifying the points of $\pi_2\times B_2\subset\pi_2\times K_2$ as follows: the points of $\pi_2\times K\Delta B\Delta X$ are identified under the period p^2 map $g_2\times 1\Delta\phi_2\Delta 1$. We identify the points of $\pi_2\times K\Delta X\Delta B$ under the group of order p^2, generated by the two period p maps $g_2^p\times 1\Delta\phi_1\Delta 1$ and $g_2^p\times 1\Delta 1\Delta\phi_1$. π_2 still acts on L_2, generated by the map induced by $g_2\times 1$. We show below that $H^2(M_2/\pi_2)=Z_{p^2}$ and that the induced homomorphism

$$(\lambda_1\times 1\Delta R_1\Delta q_0)^*\colon H^2(M_1/\pi_1)\to H^2(M_2/\pi_2)$$

is the usual injection $Z_p\to Z_{p^2}$.

1.3. We return to the general description, and first define:

$$R_n\colon X \to X \quad \text{by } R_n = r_{n!},$$

$\{b_{i,j}\}$, a triangular array of integers, by

$$b_{11} = 0,$$
$$b_{n+1,i} = b_{n,i}+n!,$$
$$b_{n+1,n+1} = 0.$$

The maps $q_{i,j}\colon X \to s^2$ by

$$q_{ij} = q_0 r_{b_{ij}}.$$

(Note $q_{21}=q_1$ and $q_{11}=q_{22}=q_0$.) We can now define K_n:

$$K_n = K\Delta_{q_{n,1}}X\Delta_{q_{n,2}}X\Delta\cdots\Delta_{q_{n,n}}X.$$

From now on we will write $\Delta_{i,j}$ in place of $\Delta_{q_{ij}}$. Note that

$$1\Delta R_n\Delta\cdots\Delta R_n\Delta q_0\colon K_{n+1} \to K_n$$

is defined and yields

$$K_1 \leftarrow K_2 \leftarrow \cdots.$$

Next let B_n, $B_{n,i} \subset K_n$ be defined by

$$B_{n,i} = K\Delta_{n,1}X\Delta\cdots X\Delta_{n,i}B\Delta_{n,i+1}X\Delta\cdots\Delta_{n,n}X;$$
$$U_i B_{n,i}.$$

LEMMA 1.3.1. *K_n is an orientable 2-manifold with boundary B_n.*

Proof. Proceeding by induction, we need only prove:

LEMMA 1.3.2. *If K is a triangulated orientable 2-manifold with boundary ∂K, then $K\Delta_{i,j}X$ is a triangulated, orientable 2-manifold with boundary $\partial K\Delta_{i,j}X \cup K\Delta_{i,j}B$.*

Proof of 1.3.2. There is a map $q_0'\colon X \to s^2$ agreeing with q_0 on $S^1\times 0$, mapping X homeomorphically onto s^2-D, and mapping B onto ∂D, where D is a small open disk lying well in the interior of s^2. Then define $q_{ij}'\colon X\to s^2$ by $q_{ij}'=q_0' r_{b_{i,j}}$. Note that q_{ij}' is a local homeomorphism.

Then for any simplex σ of s^2, $q_i'^{-1}(\sigma)=q_i^{-1}(\sigma)$. It follows (see [6; M1, p. 320]) that both the maps

$$K\Delta_{q_i}X \xrightarrow[1\Delta 1]{} K\Delta_{q_i'}X, \qquad K\Delta_{q_i'}X \xrightarrow[1\Delta 1]{} K\Delta_{q_i}X$$

are defined and are inverses. Hence $K\Delta_{q_i'}X$ and $K\Delta_{q_i}X$ are homeomorphic. Now $K\Delta_1(s^2-D)\subset K\Delta_1 s^2$ and this last can be identified with K (actually K', the barycentric subdivision of K [6, p. 320]). Note that $K\Delta_1(s^2-D)$ is a manifold with boundary $\partial K\Delta_1 s^2 \cup K\Delta_1\,\partial D$. Finally the map $1\Delta q_{ij}'\colon K\Delta_{q_i'}X \to K\Delta_1(s^2-D)$ is a local homeomorphism, because q_{ij}' is a local homeomorphism. In detail, we need only check that $1\Delta q'_{ij}$ is locally 1-1, by compactness. But if U is open in X and $q'_{ij} \mid U$ is 1-1, then $K\Delta U$ is open in $K\Delta K$ and $1\Delta q'_{ij} \mid K\Delta U$ is 1-1.

Thus $K\Delta_{q'_i}X$ is a 2-manifold with boundary $\partial K\Delta_{q'_i}X \cup K\Delta_{q'_i}B$, as these are the sets which map onto $\partial K\Delta_1 s^2$ and $K\Delta_1\, \partial D$ under $1\Delta q'_{ij}$. The orientability of $K\Delta_{q'_i}X$ also follows from the fact that $1\Delta q'_{ij}$ is a local homeomorphism. This last is easily seen by assuming the contrary and looking at an orientation reversing curve in $K\Delta_{q'_i}X$.

1.4. Note that the map $K_{n+1} \to K_n$ does not send B_{n+1} into B_n. That is, $B_{n+1,i} \to B_{n,i}$, for $i \leqq n$, but $B_{n+1,n+1} \to$ (the barycenters of the two cells of K'_n). We call this last set $B_{n,n+1}$, and define $B_n^+ = B_n \cup B_{n,n+1}$. We now have a map of pairs $(K_{n+1}, B_{n+1}) \to (K_n, B_n^+)$. These are relative manifolds and our next task is to compute the degree of this map.

Lemma 1.4.1. *The map $(K_{n+1}, B_{n+1}) \to (K_n, B_n^+)$ is of degree $p^{n\cdot n!}$.*

Proof. Proceeding by induction, we need only prove the two lemmas:

Lemma 1.4.2. *If $(K_i, \partial K_i)$, $i=1, 2$, are triangulated, orientable relative 2-manifolds with $K_i - \partial K_i$ connected and $f\colon (K_2, \partial K_2) \to (K_1, \partial K_1)$ is a map of degree m, then*

(a) $(f\Delta R_n)\colon (K_2\Delta_{n+1,i}X, \partial K_2\Delta X \cup K_2\Delta B) \to (K_1\Delta_{n,i}X, \partial K_1\Delta X \cup K_1\Delta B)$ *is of degree $m\cdot p^{n!}$, $i=1,\dots,n$; and*

(b) $f\Delta q_0\colon (K_2\Delta_{n+1,n+1}X, \partial K_2\Delta X \cup K_2\Delta B) \to (K_1, \partial K_1^+)$ *is of degree m, where $\partial K_1^+ = \partial K_1$ together with all barycenters of the two simplexes of K'_1.*

Proof. Just as in the proof of Lemma 1.3.2 we note that the map $(f\Delta R_n)$ is a local homeomorphism with degree locally the degree of $R_n\colon X \to X$, which is $p^{n!}$. Thus the degree of $(f\Delta R_n)$ is everywhere $\pm p^{n!}$. If the degrees occurred with different signs, they would determine distinct components of the interior of K_2. But this is connected, as both $K_2 - \partial K_2$ and $X - B$ are connected. This proves (a). The proof of (b) is the same, except that q_0 has degree 1, instead of $p^{n!}$.

1.5. Note that π_i acts on K_i, so that A_p acts on the inverse limit.

One easily sees (e.g., using 3.1.2, below) that the inverse limit of

$$K_1 \leftarrow K_2 \leftarrow K_3 \leftarrow \cdots,$$

is 1-dimensional, as is the limit of

$$\pi_1 \times K_1 \leftarrow \pi_2 \times K_2 \leftarrow \cdots.$$

We next introduce identification among the points of $\pi_n \times B_n \subset \pi_n \times K_n$ to form L_n. This is to be done in such a way that the inverse system

$$L_1 \leftarrow L_2 \leftarrow \cdots$$

is still defined and still has dimension 1. Clearly A_p acts on the second of these limits via π_i acting on $\pi_i \times K_i$; this action will induce an action via π_i on L_i and hence an action of A_p on the inverse limit. Finally the inverse limit

$$L_1/\pi_1 \leftarrow L_2/\pi_2 \leftarrow \cdots$$

is to be 2-dimensional and have the local groups as specified.

1.6. To specify the identifications among the points of B_n, we need the sequence $\{C^n\}$ of $n\times n$ matrices $\{C^n_{ij}\}$, defined by

$$C^1_{11} = 1,$$

(1.6.1)
$$C^{n+1}_{i,j} = C^n_{1,j}+\sum_k C^n_{i,k}, \qquad i,j = 1,\dots,n;$$
$$C^{n+1}_{n+1,j} = \sum_k C^n_{1,k}, \qquad j = 1,\dots,n+1;$$
$$C^{n+1}_{i,n+1} = 0, \qquad i = 1,\dots,n.$$

Lemma 1.6.2. $\sum_j C^n_{ij}=n!$, $i=1,2,\dots,n$.

Proof. This is trivially true for $n=1$. By induction, we have, for $i\leqq n$,

$$\sum_j C^{n+1}_{ij} = \sum_{j=1}^{n}\left(C^n_{ij}+\sum_{k=1}^{n} C^n_{1k}\right) = n!+n\cdot n! = (n+1)!.$$

For $i=n+1$,

$$\sum_j C^{n+1}_{n+1,j} = \sum_{j=1}^{n+1}\left(\sum_k C^n_{1k}\right) = (n+1)(n!) = (n+1)!.$$

Next, let $g_n(i)$ be the p^{a_n-i} power of g_n. Then

1.6.3. $g_n(i)$ has order p^i.

1.6.4. $\lambda_n g_{n+1}(i)=g_n(i-n!)$.

Proof. $g_{n+1}(i)=(g_{n+1})^{a_{n+1}-i}$ so that $\lambda_n g_{n+1}(i)=g_n^{a_{n+1}-i}=g_n^{a_n+n!-i}=g_n(i-n!)$.

1.6.5. Corollary.

$$\lambda_n g_{n+1}(C^{n+1}_{i,j}) = g_n(C^n_{i,j}), \qquad i,j\leqq n,$$
$$= 1 \qquad \textit{otherwise}.$$

This follows directly from 1.6.4 and the definition of $C^{n+1}_{i,j}$. Writing $\phi(k)$ for ϕ_k we have

1.6.6. $R_n\phi(i)=\phi(i-n!)R_n$, $i\leqq n!$

Proof. $\phi(i)$ is based on a rotation of $2\pi/p^i$ radians; R_n multiplies an angle by $p^{n!}$. Thus $R_n\phi(i)=\phi(i-n!)R_n$.

1.6.7. Corollary.

$$R_n\phi(C^{n+1}_{i,j}) = \phi(C^n_{i,j})R_n, \qquad i,j\leqq n,$$
$$= R_n \qquad \textit{otherwise}.$$

1.7. Then we let L_n be $\pi_n\times K_n$ with the points of $\pi_n\times B_n=\pi_n\times B_{n,1}\cup\cdots\cup\pi_n\times B_{n,n}$ identified as follows. The points of $\pi_n\times B_{n,i}$ are identified under the following maps:

$(1.7.1)_{n,i}$
$$g_{n+1}(C^n_{i,1})\times 1\Delta\phi(C^n_{i,1})\Delta 1\Delta\cdots\Delta 1,$$
$$g_n(C^n_{i2})\times 1\Delta 1\Delta\phi(C^n_{i2})\Delta 1\Delta\cdots\Delta 1,\dots, g_n(C^n_{in})\times 1\Delta 1\Delta\cdots\Delta 1\Delta\phi(C^n_{in}),$$

$i=1,2,\dots,n$.

Lemma 1.7.2. *The map* $\pi_{n+1}\times K_{n+1}\to\pi_n\times K_n$ *induces a map* $L_{n+1}\to L_n$.

Proof. Look first at the points of $B_{n+1,n+1}$ which are identified to form $L_{n+1,n+1}$. The map $\lambda_n \times 1\Delta R_{n+1}\Delta\cdots\Delta R_{n+1}\Delta q_0$ composed with the maps of $(1.7.1)_{n+1,n+1}$ yield

$$\lambda_n g_{n+1}(C^{n+1}_{n+1,1}) \times 1\Delta R_n\phi(C^{n+1}_{n+1,1})\Delta R_n\Delta\cdots\Delta R_n\Delta q_0, \ldots,$$
$$\lambda_n g_{n+1}(C^{n+1}_{n+1,n+1}) \times 1\Delta R_{n+1}\Delta\cdots\Delta R_{n+1}\Delta q_0\phi(C^{n+1}_{n+1,n+1}).$$

Using Corollaries 1.6.5 and 1.6.7, we have

$$1\Delta R_n\Delta\cdots\Delta R_n\Delta q_0, \ldots, 1\Delta R_n\Delta\cdots R_n\Delta q_0\phi(n!).$$

Now since $q_0(B)$ is a single point, these equations say that $(1.7.1)_{n+1,n+1}$-equivalent points are mapped into the same point.

Next look at $\pi_{n+1} \times B_{n+1,i}$, $i \leqq n$. Then the map $K_{n+1} \to K_n$ composed with the maps of (1.7.1) becomes

$$\lambda_n g_{n+1}(C^{n+1}_{i,1}) \times 1\Delta R_n\phi(C^{n+1}_{i,1})\Delta R_n\Delta\cdots\Delta R_n\Delta q_0, \ldots,$$
$$\lambda_n g_{n+1}(C^{n+1}_{i,n+1}) \times 1\Delta R_n\Delta\cdots\Delta R_n\Delta q_0\phi(C^{n+1}_{i,n+1}).$$

Now using Corollaries 1.6.5 and 1.6.7 and the fact that $C^{n+1}_{i,n+1}=0$, we have

$$g_n(C^n_{i,1})\Delta R_n\phi(C^n_{i,1})\Delta R_n\Delta\cdots\Delta R_n\Delta q_0, \ldots, 1\Delta R_n\Delta\cdots\Delta R_n\Delta q_0.$$

Thus $(1.7.1)_{n+1,i}$-equivalent points map into $(1.7.1)_{n,i}$-equivalent points.

Lemma 1.8. *The action of π_n on $\pi_n \times K_n$ induces a free action of π_n on L_n.*

Proof. On $\pi_n \times K_n$, π_n is generated by $g_n \times 1$ which clearly commutes with all the identification maps of (1.7.1), so that this induces an action of π_n on M_n. To see that this action is free, note first the obvious criterion:

Lemma 1.8.1. *If π and ρ act freely on a space X and commute, then π acts freely on X/ρ if and only if the direct product $\pi \times \rho$ acts freely on X.*

To see that Lemma 1.8.1 applies to Lemma 1.8, note that the ingredients of the maps of $(1.7.1)_{n,i}$, that is, the $g_n(C^n_{i,j})$'s and $\phi(C^n_{i,1})$'s all act freely on $B_{n,i}$. Next, a typical element of the group generated by the various maps of $(1.7.1)_{n,i}$ together with $g_n\Delta 1\Delta\cdots\Delta 1$, is

$$g_n^{k_0} g_n(C^n_{i1})^{k_1}\cdots g_n(C^n_{in})^{k_n} \times 1\Delta\phi(C^n_{i1})^{k_1}\Delta\cdots\Delta\phi(C^n_{in})^{k_n}$$

and for this to be the identity, $\phi(C^n_{ij})^{k_j}=1$, for all j. But then $g_n(C^n_{ij})^{k_j}=1$ for all j, so that finally $g_n^{k_0}=1$.

1.9. The action of π_{n+1} and π_n on M_{n+1} and M_n is equivariant.

Proof. This is just the fact that $\lambda_n g_{n+1}=g_n$.

We have therefore proved

1.10. A_p acts freely on the inverse limit, L, of

$$L_1 \leftarrow L_2 \leftarrow L_3 \leftarrow \cdots.$$

2. **The 2-dimensional cohomology of** L/A_p. We have shown above that K_n is an oriented 2-manifold with boundary B_n. To study L_n/π_n, note that there are two maps $\pi_n \times K_n \to L_n/\pi_n$, forming a commutative diagram:

$$\begin{array}{ccc} \pi_n \times K_n \to (\pi_n \times K_n)/\pi_n & = & K_n \\ \downarrow & & \downarrow \\ L_n & \longrightarrow & L_n/\pi_n \end{array}$$

Note that the map $K_n \to L_n/\pi_n$ is just the quotient map of the equivalences induced on $B_{n,i}$ by those of (1.7.1) on $\pi_n \times B_{ni}$, to wit:

$(2.1.1)_{n,i}$ $\quad 1\Delta\phi(C^n_{i,1})\Delta 1\Delta\cdots\Delta 1,\ 1\Delta 1\Delta\phi(C^n_{i2})\Delta 1\Delta\cdots\Delta 1, \ldots, 1\Delta 1\cdots\Delta 1\Delta\phi(C^n_{i,n}).$

As these generate a free group action of order $\sum_j C^n_{ij} = n!$, we see that L_n/π_n is a relative, oriented manifold with boundary B'_n = the image of B_n/π_n, and with boundary map

(2.1.2) $$L_n/\pi_n \to B'_n$$

of order $n!$, i.e.,

Lemma 2.1.3. *$H^1(B'_n) \to H^2(L_n/\pi_n, B'_n)$ has image $n!\,Z$.*

Lemma 2.1.4. *The map $H^2(M_n/\pi_n) \to H^2(M_{n+1}/\pi_{n+1})$ is the injection*

$$Z_{n!} \to Z_{(n+1)!}.$$

Proof. There is the commutative diagram

(2.1.5) $$\begin{array}{ccccccc} H^1(B'_n) & \longrightarrow & H^2(L_n/\pi_n, B'_n) & \longrightarrow & H^2(L_n/\pi_n) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^1(B'_{n+1}) & \to & H^2(L_{n+1}/\pi_{n+1}, B'_{n+1}) & \to & H^2(L_{n+1}/\pi_{n+1}) & \to & 0 \end{array}$$

in which the middle map may be replaced by $H^2(K_n, B_n) \to H^2(K_{n+1}, B_{n+1})$ as $(K_n, B_n) \to (L_n/\pi_n, B'_n)$ is a relative homeomorphism. Now by (1.4) and Lemma 2.1.3 the latter part of Lemma 2.1.5 becomes

$$\begin{array}{ccccc} Z & \longrightarrow & Z_{p^{n!}} & \longrightarrow & 0 \\ \downarrow{\scriptstyle p^{nn!}} & & \downarrow & & \\ Z & \longrightarrow & Z_{p^{(n+1)!}} & \longrightarrow & 0. \end{array}$$

It follows that the last vertical map is an injection, as required. Thus we have proved

Theorem 2.2. *$H^2(L/A_p) = Z_{p^\infty}$, the p-adic rationals* mod 1.

Actually there is a much stronger result: (H^2_c is Čech cohomology with compact supports):

Theorem 2.3. *$H^2_c(U/A_p) = Z_{p^\infty}$ for any connected invariant open set $U \subset L$.*

Proof. The inverse image of U under the map $A_p \times K \to L$ has the form $A_p \times V$, where V is open in K, by the invariance of U. Now by the definition of the topology for an inverse limit, $U = \{x \in L \mid x_n \in U_n\}$, where n is an integer and U_n is open in L_n. Define U_{n+i} by U_{n+i}=inverse image of U_n under the map $L_{n+i} \to L_n$. We claim that U_{n+i} is connected. For a separation would lead to a separation of U, because the maps $L_{n+1} \to L_n$ are *onto*, and the L_n's are compact. Similarly one defines V_{n+i} for all $i \geqq 0$ so that $A_p \times V_{n+i} \to U_{n+i}$ under the map $A_p \times K_{n+i} \to L_{n+i}$, and finds that all V_{n+i} are connected.

It follows that V_j is an orientable submanifold of K_j, and that

$$(V_{j+i}, V_{j+1} \cap B_{j+1}) \to (V_j, V_j \cap B_j^+)$$

is of degree $p^{(j \cdot j!)}$ for $j \geqq n$. This uses the fact that U_{n+i+1} is the *complete* inverse image of U_{n+i}. Now for sufficiently large i, $V_{n+i} \cap B_{n+i} \neq \varnothing$, as B_i is ε_i-dense in K_i where $\varepsilon_i \to 0$ as $i \to \infty$. But these facts play the role of 1.4 so that one gets analogies of (2.1.2), Lemma 2.1.3, and Lemma 2.1.4 for $U_j/\pi_j, j \geqq n$. Then Therem 2.3 follows just as Theorem 2.2.

3. **That L is 1-dimensional.**

3.0. $\dim L \geqq 1$.

Proof. For each n, the map $L_{n+1} - B'_{n+1} \to L_n - B_n'^{+}$ is a covering map. Thus the inverse image of a 1-simplex σ_1 under the map $L \to L_1$ is homeomorphic to $\sigma_1 \times A_p$. This is 1-dimensional and thus $\dim L \geqq 1$.

3.1. Thus it suffices to prove $\dim L \leqq 1$. We will do this via a general criterion for an inverse limit to be of dimension $\leqq n$. We also include an analogous criterion for the other inequality.

3.1.1. Definition. Suppose K, L are simplicial complexes and $f: K \to L$ is a map (not necessarily simplicial). We say f is *simplicially of dimension* $\leqq n$ provided there is a map $g: K \to$ (n-skeleton of L) such that for all $k \in K$, $g(k) \in \bar{\sigma}$, where σ is the unique open simplex of L containing $f(k)$.

3.1.2. Suppose X is the inverse limit of a sequence

$$K_1 \xleftarrow{f_1} K_2 \xleftarrow{f_2} \cdots$$

of finite r-dimensional complexes, that f_i is simplicial relative to a subdivision of K_i and that $f_i: K_{i+1} \to K_i'$ is simplicially of dimension $\leqq n$. Then $\dim X \leqq n$.

Proof. Let $g_i: K_{i+1} \to K_i'$ be as guaranteed in the definition of simplicially of dimension $\leqq n$ and let $h_i: X \to K_i$ be the coordinate map $X \to K_{i+1}$ followed by g_i. Then h_i maps X into an n-complex, for all i. Thus it will complete the proof [2, p. 71] to show that h_i is an ε_i-map, with $\varepsilon_i \to 0$ as $i \to \infty$.

As this is independent of the metric, we first choose a metric ρ_i for K_i so that K_i has diameter $\leqq 1/2^i$, and then set $\rho(x, y) = \sum_{i=1}^{\infty} \rho_i(x_i, y_i)$, for $x, y \in X$. This yields a metric for X. Now for each m, i the map $f_i f_{i+1} \cdots f_m: K_{m+1} \to K_i$ is simplicial relative to the $(m+1-i)$th-barycentric subdivision of K_i. Thus if $x, y \in X$, and x_{m+i}, y_{m+i} are in an r-simplex $\bar{\sigma}$, then $\rho_i(x_i, y_i) \leqq (r/(r+1))^{n+1-i}/2^i$, $i = 1, 2, \ldots, m$.

Then $\rho(x, y) \leqq N_m = \sum_{j=m+1}^{\infty} 1/2^i + \sum_{j=1}^{m} (r/(r+1))^{m+1-i}/2^i$. One finds that $N_m \to 0$ as $m \to \infty$. But this shows that h_i is a $(1/2^i + N_i)$-map and thus completes the proof of 3.1.2.

3.2.1. One says that $f: K \to L$ is *simplicially of dimension* $\geqq n$, provided that $f^*: H^n(L, L') \to H^n(K, f^{-1}(L'))$ is monic for all subcomplexes $L' \subset L$.

3.2.2. Remark. If

$$K_1 \underset{f_1}{\longleftarrow} K_2 \underset{f_2}{\longleftarrow} \cdots$$

is as in 3.1.2 except that f_i is simplicially of dimension $\geqq n$, and dim $K_1 \geqq n$, then dim $X \geqq n$.

Proof. One chooses a subcomplex L, of K, so that $H^n(K_1, L_1) \neq 0$. Then all the homomorphisms of the system

$$H^n(K_1, L_1) \underset{f_1^*}{\longrightarrow} H^n(K_2, L_2) \underset{f_2^*}{\longrightarrow} \cdots$$

are monic, where $L_{i+1} = f_i^{-1}(L_i)$, for all i. Hence this limit is nontrivial. But this limit is just $H^n(X, L_\infty)$, where L_∞ is the inverse limit of $L_1 \leftarrow L_2 \leftarrow \cdots$, so that dim $X \geqq n$.

3.3. We now return to the task of establishing the dimension of our example, and will use the notation of §1.

3.3.1. Lemma. *The map* $L_{n+1} \to L'_n$ *is simplicially of dimension* $\leqq 1$.

Proof. Recall that the map $\pi_{n+1} \times K_{n+1} \to \pi_n \times K_n$ is given by

$$\lambda_n \times 1 \Delta R_n \Delta \cdots \Delta R_n \Delta q_0 = f_n.$$

We first define h_n to agree with f_n on f_n^{-1} (1-skeleton of $\pi_n \times K'_n$). Next, if σ is a 2-simplex of $\pi_n \times K'_n$, then $f_n^{-1}(\bar{\sigma})$ consists of $p^{(n+1)!}$ copies of X. The general fact follows from [6, p. 321] and the specific counting from the definitions of λ_n, R_n, (1.1), (1.3).

Let X_0 be one of these copies of X and let $\eta: X \to X_0$ and $\mu: s^2 \to \bar{\sigma}$ be homeomorphisms such that $f_n \mid X_0 = \mu q_0 \eta^{-1}$. Next let $\rho_1, \ldots, \rho_{n+1}$ be the first n maps of $(1.7.1)_{n+1,n+1}$ and

$$\nu_{n+1} = g_{n+1}(C^{n+1}_{n+1,n+1}) \times 1 \Delta 1 \Delta \cdots \Delta 1.$$

Then for each $0 \leqq j_1, \ldots, j_{n+1} \leqq p^{n!} - 1$,

$$\rho_1^{j_1} \cdots \rho_n^{j_n} \nu_{n+1}^{j_{n+1}} X_0 = X(j_1, \ldots, j_{n+1})$$

is a copy of X which maps onto $\bar{\sigma}$ under f_n. As ρ_i has order $p^{n!}$, by Lemma 1.6.2 and (1.6.3) there are $p^{(n+1)!}$ of these (all different; see Lemma 1.8.1) so that these constitute the total set $f_n^{-1}(\bar{\sigma})$.

Now define the map $t(j): X \to S^1 = \partial s^2$ by

$$t(j)(\theta, t) = \theta - 2\pi j t / p^{n!}, \qquad (\theta, t) \in S^1 \times [0, 1] = X.$$

Now the map f_n sends

$$X(j_1, \ldots, j_{n+1}) \to \bar{\sigma}$$

according to the formula $\mu q_0 \eta^{-1} \rho_1^{j_1} \cdots \nu_{n+1}^{-j_{n+1}}$. We define h_n on $X(j_1, \ldots, j_{n+1})$ to be $\mu t(j_{n+1})\eta^{-1}\rho_1^{-j_1}\nu_{n+1}^{-j_{n+1}}$. As for each j, $t(j) = q_0$ on $q_0^{-1}(\partial s^2)$, which is $S^1 \times 0 \subset S^1 \times [0, 1]$, this does agree with h_n on f_n^{-1} (1-skeleton of $\pi_n \times K_n'$). This then defines h_n on all of $\pi_{n+1} \times K_{n+1}$. Having only $\pi_n \times K\Delta X\Delta \cdots \Delta X\Delta(\partial s^2) =$ (1-skeleton of $\pi_n \times K_n'$) as image.

Note also that $f_n(x)$, $h_n(x) \in \bar{\sigma}$, where σ is the unique simplex containing $f_n(x)$, i.e., h_n is a "simplicial" approximation to h_n. Our only hedge is that h_n may not be simplicial—which does not harm its value as an approximation.

Finally we wish to show that h_n induces a map $L_{n+1} \to L_n$. Thus consider the identifications made on $\pi_{n+1} \times B_{n+1,i}$ in $(1.7.1)_{n+1,i}$. For $i \leqq n$, $\pi_{n+1} \times B_{n+1,i} \subset f_n^{-1}$ (1-skeleton of $\pi_n \times K_n'$) so h_n agrees with f_n here. For $i = n+1$, there is something to prove; let σ be a typical 2-simplex in $\pi_n \times K_n'$, and let the notation X_0, η, μ, et cetera, be as above.

Now $\pi_{n+1} \times B_{n+1,n+1} \cap X(j_1, \ldots, j_{n+1}) = \rho_1^{j_1} \cdots \rho_{n+1}^{j_{n+1}}\eta(B) = B(j_1, \ldots, j_{n+1})$. Thus each equivalence class of the relation $(1.7.1)_{n+1,n+1}$ hits each $B(j_1, \ldots, j_{n+1})$ in one and only one point. Let $\theta \times 1$ be a point of $B = S^1 \times 1 \subset X$. Then

$$\eta(\theta \times 1) \sim \rho_1^{j_1} \cdots \rho_{n+1}^{j_{n+1}}(\eta(\theta, 1)) = \rho_1^{j_1} \cdots \rho_n^{j_n}\nu_{n+1}^{j_{n+1}}(\eta(\theta + 2\pi j_{n+1}/p^{n!}, 1))$$

and these points both go into $\mu(\theta)$. Thus h_n induces a map $\bar{h}_n: L_{n+1} \to$ (1-skeleton L_n) and this completes the proof of Lemma 3.3.1.

II. Why This Construction Cannot Yield A_2 and B

One can follow the pattern of Part I, §1 a long way toward the construction of an example satisfying A_2 and B. But this process breaks down for a rather subtle reason, and it is this reason that we want to elucidate in this section.

Our method is to present the analogous construction, and show that one of the steps of §1 cannot be carried out. We will aim at a 2-dimensional example for simplicity even though 3 is the smallest dimension for which a counterexample to the Hilbert-Smith Conjecture could possibly exist [3, p. 249].

4.1. Let $\partial s^4 = S^3 = S_0^1 * S_1^1$ and let N be a small tubular neighborhood of S_0^1. The exact relation between these various structures will be specified below. Note that $S^3 - N = S^1 \times D^2$, where D^2 is a 2-disk. We use this in defining

$$X = S^1 \times D^2 \times [0, 1];$$
$$B = S^1 \times \partial D^2 \times [0, 1] \cup S^1 \times D^2 \times 1 \subset X.$$

$\phi_i: X \to X$ is to be a rotation, relative to S^1, through an angle of $2\pi/p^i$ radians. Define $q_0: X \to s^4$ by

$$q_0: (S^3 - N) \times [0, 1] \to S^3 \times [0, 1] \to S^3 * b = \partial s^4 * b = s^4.$$

Here the first map collapses the tubular neighborhood N back down along its fibers to S^2. The next collapses $S^3 \times 1$ to a single point b, the barycenter of s^4. Thus q_0 collapses each orbit of $\phi_i \mid B$ to a single point for all i.

S_0^1 is chosen to pass through the barycenters b_i, $i=0,\ldots,4$ of the 3-simplexes of s^4, to be orthogonal to the 2-simplexes it hits and to miss the 1-skeleton. Define $r_i, q_i\colon X\to s^4$ by

$$r_i\colon X\to X/\phi_i = X, \qquad q_i = q_0r_i.$$

4.2. One can now repeat the construction just as in Part I, §1; we will need to consider only L_1 and L_2. π_1 and π_2 are as in Part I, §1.

Let K be a 4-sphere with a fixed triangulation and let

$$\begin{aligned} K_1 &= K\Delta_{q_0}X, & B_1 &= K\Delta B\subset K_1,\\ K_2 &= K\Delta_{q_1}X\Delta_{q_0}X, & &\\ B_{21} &= K\Delta_{q_1}B\Delta_{q_0}X, & B_{22} &= K\Delta_{q_1}X\Delta_{q_0}B,\\ B_2 &= B_{21}\cup B_{22}\subset K_2. & & \end{aligned}$$

One can show just as in Part I, that K_i is an orientable 4-circuit with boundary B_i, $i=1,2$, and that the map $1\Delta r\Delta q_0\colon (K_2,B_2)\to(K_1,B_1^+)$ is of degree p, where $B_1^+=B_1$ together with a certain 2-dimensional subset, containing the dual 1-skeleton of K_1'.

Form L_1 from $\pi_1\times K_1$ by identifying the points of $\pi_1\times B_1$ under the period p map $g_1\times 1\Delta\phi_1$. Then $g_1\times 1\Delta 1$ generates the action of π_1 on $\pi_1\times K$ and induces an action on L_1.

Next consider the identification to be made on $\pi_2\times B_2$ to yield L_2 from $\pi_2\times K_2$. On $\pi_2\times B_{21}$ this must be generated by a map $g_2\times 1\Delta\phi_2\Delta r$, where $r\colon X\to X$ is some map of period p^2. This is true because the map $\lambda_1\times 1\Delta r_1\Delta q_0\colon \pi_2\times K_2\to\pi_1\times K_1$ must send these identifications into those generated by $g_1\times 1\times\phi_1$. Now as $q_0\colon X\to X$ is a homeomorphism away from B, it follows that $r\mid X-B$ is the identity. But $X-B$ is dense in X and therefore $\pi_2\times K\Delta B\Delta X-B$ is dense in $\pi_2\times K\Delta B\Delta X$. Thus $g_2\times 1\Delta\phi_2\Delta r=g_2\times 1\Delta\phi_2\Delta 1$ throughout $\pi_2\times B_{21}$.

So far there has been little choice of maps with which to make identifications. But with $\pi_2\times B_{22}$ this changes considerably. Thus we will list all the possible (uniform) choices and show that none works.

4.3. In order that a group action be used for this identification, its orbits must be collapsed under the map to $\pi_1\times K\Delta X$. There are three such group actions on $\pi_2\times B_{22}=\pi_2\times K\Delta X\Delta B$, each of order p; they have generators $\gamma_1=g_2(1)\times 1\Delta 1\Delta 1$, $\gamma_2=1\times 1\Delta\phi_1\Delta 1$, and $\gamma_3=1\times 1\Delta 1\Delta\phi_1$. Now modulo π_2, p^2 identifications must be made, in order that property B be satisfied.

Thus the identification group R has as generators

$$\gamma_1^{i_1}\gamma_2^{i_2}\gamma_3^{i_3},\quad \gamma_1^{j_1}\gamma_2^{j_2}\gamma_3^{j_3},$$

where i_n and j_n are integers mod p. We may therefore choose another pair of generators of R, having the form

$$\gamma_2\gamma_3^i,\quad \gamma_1\gamma_3^j,\qquad i=0,1,\ldots,p-1,\qquad j=1,2,\ldots,p-1.$$

The case $j=0$ need not be considered as then the group R, modulo π_2 would have only order p, because $\gamma_1\in\pi_2$.

We will distinguish two cases: (1) $i \not\equiv 0 \bmod p$ and (2) $i=0$ but $j \not\equiv 0 \bmod p$. First we must describe some additional structure of the building blocks.

4.4. We next use the fact that the map $L_2 \to L_1'$ must be simplicially of dimension $\leqq 2$, (3.1.1). We will actually work with the map $f\colon \pi_2 \times K_2 \to L_1'$, and require that the map f and maps g and g' below respect the identifications made in $\pi_2 \times K_2$ to form L_2.

Thus we assume we have a map $g\colon \pi_2 \times K_2 \to$ (2-skeleton of L_1') satisfying (3.1.1) and alter g slightly to get a map g' which is well behaved on B_{22}. Set $g'=g$ on f^{-1} (1-skeleton of L_1') as this set does not intersect B_{22}. Next, let σ be a 4-simplex of L_1'. Then $f^{-1}(\bar{\sigma})$ consists of p copies of X and on each of these f is a copy of the map $q_0\colon X \to s^4$. We will look at one of these copies of X and use this structure with no reference to the copying homomorphism.

Now $S_0^1 \cap$ (2-skeleton of s^4) consists of isolated points and if $x \in S_0^1 \cap \sigma^2$ is such a point then the fiber $S^1 \times x'$ of the tubular neighborhood N of S_0^1 at this point satisfies $S^1 \times x' \subset \sigma^2$. Then $g(S^1 \times x') \subset \bar{\sigma}^2$ and we may vary g by a homotopy so that $g'(S^1 \times x') = x \in \sigma^2$, maintaining the condition that $g'(y) \in \bar{\sigma}$ if $f(y) \in \sigma$. Thus we obtain g' which agrees with f on q_0^{-1} (2-skeleton s^4). We can go a bit further and have $g'=f$ on a neighborhood of q_0^{-1} (2-skeleton of s^4), or what is the same, $g'=f$ except near the dual 1-skeleton of $\pi_2 \times K_2$. This enlarges the image of g' to $L_1' - N'$, where N' is a neighborhood of the dual 1-skeleton of L_1'. This enlargement is harmless, as we may always retract $L_1' - N'$ back to the 2-skeleton of L_1'.

Now return to our copy of X. The balance of the $q_0(S^1 \times x') = x$, where

$$x' \in \partial D^2 \times [0,1] \cup D^2 \times 1,$$

are near the dual 1-skeleton of s^4. Furthermore they cannot be collapsed to x, as this would cover the dual 1-skeleton and it is essential that $g(L_2)$ misses the dual 1-skeleton. It would therefore appear that g' can be so chosen that for such an x', $S^1 \times x'$ maps to $S^1 \times x$ with degree 1, where $S^1 \times x$ is the corresponding fiber of the tubular neighborhood at $x \in s^4$. Another rather tedious argument shows that this is indeed the case.

4.5. This supplements (4.1). There is a unique point $y_i \in \partial D^2$ such that

$$q_0\colon X = S^1 \times D^2 \times [0,1] \to s^4$$

sends $S^1 \times y_i \times 0$ to b_i, $i=0, \ldots, 4$. Then

$$q_0^{-1}(b) = S^1 \times D^2 \times 1, \quad q_0^{-1}(c) = S^1 \times y_i \times t,$$

where $c = tb + (1-t)b_i$, $0 \leqq t < 1$. Note that b together with all such c's constitutes the dual 1-skeleton of s^4.

In D^2 we choose a point $y \neq y_i$, $i=0, \ldots, 4$, and five arcs α_i, $i=0, \ldots, 4$, where α_i has end points y and y_i, and $\alpha_i \cap \alpha_j = \varnothing$, $i \neq j$. Let $W = \bigcup_{i=0}^4 (S^1 \times y_i \times [0,1] \cup S^1 \times \alpha_i \times 1)$. Then $q_0(W) =$ (dual 1-skeleton of s^4) $= sk_1$ and W is homeomorphic to $S^1 \times sk_1$. Also the projection $q_0'\colon S^1 \times sk_1 \to sk_1$ differs only a little from q_0. In particular, $q_0(w)$ and $q_0'(w)$ lie in the same open simplex of s^4 for each $w \in W$. Finally, note that ϕ_i leaves W invariant and that ϕ_i on $W = S^1 \times sk_1$ is just a rotation on the first factor.

4.6. *Case* 1, $i \not\equiv 0 \bmod p$. We choose an arc β in the space $K\Delta q_1 X\Delta_1 sk_1$ with end points of the form x_0 and $1\Delta\phi_1\Delta 1(x_0)$. $(1\Delta\phi_1\Delta 1)$ *does* act on $K\Delta X\Delta sk_1$. Let $\beta^+ = \beta$ together with its iterates under $1\Delta\phi_1\Delta 1$. Then β^+ is a loop; we will assume that β^+ is a topological 1-sphere, which it is if β is carefully chosen. Now

$$1\Delta 1\Delta q_0 \colon K\Delta_{q_1} X\Delta_1 w \to K\Delta_{q_1} X\Delta_1 sk_1$$

and that part which maps onto β^+ is $S^1 \times \beta^+$ with $1\Delta 1\Delta q_0 \mid S^1 \times \beta^+$ being essentially the projection.

Now $S^1 \times \beta^+ \subset K\Delta X\Delta W \subset K\Delta X\Delta B \subset B_{22}$, and $\gamma_2\gamma_3^i$ leaves $S^1 \times \beta^+$ invariant. It is the specific way in which $\gamma_2\gamma_3^i$ acts on $S^1 \times \beta^+$ that makes it impossible to carry out this construction. To wit, γ_3^i effects a rotation of S^1 through an angle of $2\pi i/p$ radians and γ_2 a rotation of β^+ through an "angle" of $2\pi/p$.

Now g' maps the torus $S^1 \times \beta^+$ to another $S^1 \times \tilde{\beta}$ in L_1' and this map is of type $(1, p)$, i.e., of degree 1 on S^1 (proved in 4.4) and of degree p on β^+. Next choose coordinates (θ, ϕ) for $S^1 \times \beta^+$, θ, ϕ in the reals mod 1, and let S' be the set of all $(\theta, i\theta)$ in $S^1 \times \beta^+$. Then S' is a 1-sphere wrapping once around the S^1 factor and i times around the β^+ factor. Then $\gamma_2\gamma_3^i(\theta, i\theta) = (\theta + 1/p, i\theta + i/p)$ so that S' is invariant under $\gamma_2\gamma_3^i$. Now as the points of S' are identified (in forming L^2) we see that g' takes S' to $S^1 \times \tilde{\beta}$ by a map of type (pk, p). But by the above $g' \mid S'$ is of type (i, p). This contradicts the fact that $i \not\equiv 0 \bmod p$.

4.7. It remains only to consider case 2, i.e., $i \equiv 0 \bmod p$ but $j \not\equiv 0 \bmod p$. The argument in this case is much the same, except that β^+ must be defined differently. We choose an arc β in $K\Delta_{q_1} X\Delta_1 sk_1$ with endpoints of the form x_0 and $(1\Delta\phi_1\Delta 1)x_0$. Now let β^+ be β together with its iterates in the space $\pi_2 \times K\Delta_{q_1} X\Delta sk_1$ under the map $g_2(1) \times 1\Delta\phi_1\Delta 1$. (See 4.3.) Then $\gamma_1\gamma_2\gamma_3^j$ leaves $S^1 \times \beta^+$ invariant. Proceeding as in (4.6) we find a contradiction to the fact that $j \not\equiv 0 \bmod p$. This completes the proof.

III. Classification Spaces for A_p

5. **A compact, 2-dimensional classifying space.** The following construction is due to E. E. Floyd. Let p be a fixed prime and let $n_i = p^{2^i}$, $i = 0, 1, \ldots$. We will construct regular, 2-dimensional CW-complexes K_i, free actions of Z_{n_i} on K_i and maps $f_i \colon K_{i+1} \to K_i$, equivariant with respect to the usual epimorphism $Z_{n_{i+1}} \to Z_{n_i}$. Let $K_i = Z_{n_i} * S_i$, where $*$ denotes the join and S_i is a 1-sphere. Z_{n_i} acts on K_i by translation on Z_{n_i} and by rotation through an angle of $2\pi/n_i$ on the second factor. Thus K_i is S_i together with n_i oriented 2-cells, $\{xe_i \colon x \in Z_{n_i}\}$ with the boundary formula $\partial(xe_i) = S_i$, for all $x \in Z_{n_i}$ and $i = 0, 1, 2, \ldots$.

The map $f_i \colon K_{i+1} \to K_i$ sends the 1-skeleton S_{i+1}, n_i times around S_i, and is thus equivariant on the 1-skeleton. To define f_i on the 2-cells consider $e_{i+1} = 1e_{i+1}$, let

T generate $Z_{n_{i+1}}$ and t generate Z_{n_i}. We think of e_{i+1} as $[0, 2n_i-1]\times S^1/0\times S^1$. First collapse $(2k)\times S^1$ to points, $k=1,2,\ldots,n_i-1$. This yields a "string" of n_i-1 2-spheres $S_k^2=[2k-1, 2k]\times S^1$, with ends collapsed, and one disk

$$[2n_i-2, 2n_i-1]\times S^1/(2n_i-2)\times S^1.$$

Then f_i sends this disk to $n_i t^{n_i-1} e_i$, extending f_i already defined on its boundary, $(2n_i-1)\times S^1$. We are now fairly free to map the spheres S_k^2 where we will, and send S_k^2 to the oriented sphere $t^{k-1}e_i - t^k e_i$ with degree k. Thus, on the chain level,

$$e_{i+1}\to \sum_k k(t^{k-1}-t^k)e_i + n_i t^{n_i-1}e_i.$$

Thus $f_{i\#}(e_i)=(1+t+\cdots+t^{n_i-1})e_i=\sigma e_i$ where $\sigma=1+t+\cdots+t^{n_i-1}$, as usual. Define f_i on the remaining 2-cells $\{xe_{i+1} : x\neq 1\}$ by equivariance:

$$f_i(xe_{i+1}) = xf_i(e_{i+1}).$$

Then on the chain level $f_{i\#}(xe_{i+1})=x\sigma e_i=\sigma e_i$. Now K_{i+1} has the homotopy type of a wedge of $n_{i+1}-1$ 2-spheres, which we take to be

$$\Sigma_j^2 = t^j(1-t)e_{i+1}, \qquad j = 0, 1, \ldots, n_{i+1}-2.$$

Then f_i sends Σ_j^2 to $\sigma t^j(1-t)e_i$ and is thus of degree zero on each of the n_i-1 2-spheres which make up K_i, again up to homotopy type. Thus the map $f_i^*\colon h(K_i)\to h(K_{i+1})$ is trivial in any generalized cohomology theory. Let E' be the inverse limit of $K_0\leftarrow K_1\leftarrow\cdots$. Then A_p acts freely on E' and we have proved the

5.1. Proposition (Floyd). *There is a 2-dimensional compact space E' and a free action of A_p on E' where E' is a cyclic in any cohomology theory, defined on the category of finite CW-complexes and extended in the Čech manner to compact pairs.*

5.2. Corollary (see also [1]).

$$\begin{aligned} H^q(B_{A_p}) &= Z, & q &= 0,\\ &= Z_{p^\infty}, & q &= 2,\\ &= 0 & &\textit{otherwise}. \end{aligned}$$

Proof. By the usual obstruction theory arguments, $H^*(B_{A_p})=H^*(B'_{A_p})$ where $B'_{A_p}=E'/A_p$. But E'/A_p is the inverse limit

$$P_p\leftarrow P_{p^2}\leftarrow\cdots$$

in which each P_k is a 1-sphere with a 2-cell attached by a map of degree k. The map $P_{n_{i+1}}\to P_{n_i}$ sends the 2-cell $[e_{i+1}]$ to $n_i[e_i]$. Hence, in the integral cohomology $H^2(E'/A_p)$ is the direct limit of

$$Z_p\to Z_{n_1}\to\cdots,$$

in which the maps are the usual injections. This completes the proof of 5.2.

6. **The complex K-theory of B_{A_p} and B'_{A_p}.**

6.1. $\tilde{K}^0(B'_{A_p})=Z_{p^\infty}$, with trivial products.

Proof. We let $[X, Y]$ denote the set of all homotopy classes of base point preserving maps of X into Y. Then, using the spaces P_m, defined in (5)

$$[P_m, BU] \approx [P_m, BU(1)] = [P_m, CP^\infty] \approx [P_m, CP^1] = [P_m, S^2],$$

inasmuch as P_m is 2-dimensional. Finally, by the Hopf classification theorem $[P_m, S^2] \approx Z_m$, and the map $P_{n_{i+1}} \to P_{n_i}$ induces the injection $Z_{n_i} \to Z_{n_{i+1}}$. This completes the proof of 6.1.

The "usual" classifying space, B_{A_p} (e.g., see [1]) can be achieved as a double limit in 2 ways:

(1) I L_iD $\mathrm{L}_j(S^{2i+1}/Z_{p^j})$,

(2) D L_jI $\mathrm{L}_i(S^{2i+1}/Z_{p^j})$.

Here I L means inverse limit and D L, direct limit. The action of Z_{p^j} on S^{2i+1} is that obtained by considering S^{2i+1} as the unit sphere in complex $(i+1)$-space. These two ways give two ways of "calculating" $K(B_{A_p})$. The first yields $\tilde{K}^0(B_{A_p}) \approx Z_{p^\infty}$, just as above. This can be seen in various ways, e.g., an easy spectral sequence argument, using the data found in [1].

The second yields

$$\tilde{K}^0(B_{A_p}) \approx \tilde{R}(Z_{p^\infty}) \otimes A_p,$$

in which $\tilde{R}$ is the reduced representation ring. This is a deeper computation, and is due to Don Anderson. Note this ring is torsion-free; more striking, it has arbitrarily long nontrivial products, $x_1 \otimes \cdots \otimes x_n$. This in particular means B_{A_p} is infinite dimensional.

Perhaps this last fact can be used to show that if A_p acts on a manifold M, then M/A_p is infinite dimensional. So far, all the author's attempts have failed. Indeed, there is the easy

6.2. Remark. If A_p acts freely on an n-manifold M, then the dimension of M/A_p is $n+2$, relative to any cohomology theory which satisfies the Vietoris mapping theorem.

Proof. There is the diagram

$$\begin{array}{ccccc} M & \longleftarrow & M \times E' & \longrightarrow & E' \\ \downarrow & & \downarrow & & \downarrow \\ M/A_p & \longleftarrow & M \times_{A_p} E' & \longrightarrow & B'_{A_p} \end{array}$$

in which $M \times_{A_p} E'$ is $(n+2)$-dimensional over integral cohomology, and at most $(n+2)$-dimensional because of the map $M \times_{A_p} E' \to B'_{A_p}$, see [2, p. 91]; hence $\dim M \times_{A_p} E' = n+2$. But the map $M \times_{A_p} E' \to M/A_p$ is a Vietoris map.

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Northwestern University,
Evanston, Illinois