

A RIGIDITY THEOREM FOR DISCRETE SUBGROUPS

BY

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0. Introduction. Let G denote an analytic group and let $\Gamma \subset G$ denote a discrete subgroup. A left invariant Haar measure on G then induces a measure on G/Γ , which is determined up to a positive constant. We let ν denote one such induced measure on G/Γ .

DEFINITION 0.1. Γ is called a *lattice*, in case $\nu(G/\Gamma) < \infty$.

Unless specified otherwise, Γ will denote a lattice in the analytic group G . Let W be a simply connected C^∞ manifold; let \mathcal{R} denote the set of all one-one homomorphisms

$$r: \Gamma \rightarrow G,$$

such that $r(\Gamma)$ is a lattice in G . We say a map $\chi: W \rightarrow \mathcal{R}$ is *smooth* if for all $\gamma \in \Gamma$, the map $w \rightarrow \chi(w)(\gamma)$, (where $w \in W$), is a C^∞ map of W into G .

DEFINITION 0.2. A *deformation* of Γ is the data (W, w_0, χ) , where W is a simply connected C^∞ manifold, w_0 is a point in W , and $\chi: W \rightarrow \mathcal{R}$ is a smooth map such that $\chi(w_0)$ is the inclusion map ι , of Γ into G .

DEFINITION 0.3. The deformation (W, w_0, χ) is called *trivial*, in case we can find a C^∞ map $c: W \rightarrow G$, such that for all $w \in W$ and for all $\gamma \in \Gamma$, we have

$$(0.4) \quad \chi(w)(\gamma) = c(w)\gamma c(w)^{-1}$$

We say Γ is *rigid* if all its deformations are trivial. We say that G is *admissible*, in case it is semisimple and has no compact or three dimensional factors. We make the convention that the Lie algebra of a Lie group is the tangent space of that group at the identity. An analytic subgroup and its corresponding Lie subalgebra will be denoted by the same capital Latin and capital German letter, respectively. Thus if G is an analytic group and $A \subset G$ an analytic subgroup, \mathfrak{G} will denote the Lie algebra of G , and $\mathfrak{A} \subset \mathfrak{G}$ will denote the subalgebra corresponding to A . Whenever G is admissible, we fix a Cartan decomposition $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$, and we let $M = K \backslash G$.

In this paper we will be interested in the following:

CONJECTURE (A. SELBERG). Let G be admissible and let $\Gamma \subset G$ be a lattice; then all deformations of Γ are trivial.

To attack this conjecture, we adopt the point of view in [13]. Thus let $\omega^1, \dots, \omega^n$ (where $n = \text{dimension } G$) be a basis of right-invariant one-forms on G . By the right-invariance of the ω^λ ($\lambda = 1, \dots, n$) we have everywhere independent, induced forms

on G/Γ , which we again denote by ω^λ . Thus we have a parallelism structure on G/Γ . Roughly speaking, we may think of a deformation (W, w_0, χ) of Γ as inducing a deformation of this parallelism structure. We show that if G is admissible, if Γ contains no nontrivial elements of finite order⁽¹⁾, and if this induced deformation satisfies a certain boundary condition, then (W, w_0, χ) is trivial (see §1, Main Theorem).

Our results may be regarded as an extension of those of A. Andreotti and E. Vesentini (see [2]) and our methods draw on theirs. In [2], one assumes that M is Hermitian, and considers the deformation of the complex structure on $M=M/\Gamma$ induced by a deformation (W, w_0, χ) of Γ (assume again that Γ contains no nontrivial elements of finite order). Roughly, their final result is that if this induced deformation satisfies a certain boundary condition (which they call rigidity at ∞), and if Γ satisfies certain conditions, then (W, w_0, χ) is trivial. At present it is not clear how our boundary condition relates to that in [2], though it is not as stringent as rigidity at ∞ in the Riemannian sense, as defined in [5]. In two respects, our results are stronger than those in [2]. First, we need not assume M is Hermitian and second, we need make no restrictions on the lattice Γ , other than assuming Γ contains no nontrivial elements of finite order. Moreover, in [2], the passage from cohomological results to deformation theoretic results is extremely complicated, involving delicate arguments from partial differential equations and several complex variables. However, in our framework this passage becomes relatively simple, and the fundamental idea is contained in Lemmas 5.6 and 5.23, below.

The text is organized into five sections. In §1 we show that when considering rigidity, it suffices to assume W is an open interval; we then state our Main Theorem. In §§2 and 3, we reformulate in terms of locally constant vector bundles, some results in [1], concerning holomorphic vector bundles; this reformulation is direct, but is included for the sake of completeness. In §4 we reformulate some results in [2] and in [9], to obtain some global estimates (in the noncompact case) which, together with the results in §§2 and 3, yield vanishing theorems for certain cohomology groups. Finally, we prove the Main Theorem in §5.

The author wishes to acknowledge stimulating discussions with S. Murakami and P. A. Griffiths. The author first proved Lemma 5.21 for the case when G is simple and of even dimension. The argument for the case when G is simple of odd dimension was shown to us by J. A. Wolf. Recently, G. D. Mostow found a somewhat simpler proof for Lemma 5.21, than the one presented here.

1. Statement of the Main Theorem. From now on we assume G is admissible. Otherwise, we continue to use the notation already introduced in §0. If M_1 and M_2 are C^∞ manifolds, and if $f: M_1 \rightarrow M_2$ is a C^∞ map, then f_* will denote the

⁽¹⁾ It seems likely that one can reformulate the results in §§3 and 4 for V -manifolds, and thereby do away with the assumption that Γ contains no elements of finite order; e.g. see [3].

tangent linear map to f . Now let V be a vector space and let S be a subset of $\text{End } V$. We then let $\mathcal{E}(S)$ denote the enveloping algebra of S .

DEFINITION 1.1. A subset $H \subset G$ is called ample, in case $\mathcal{E}(\text{Ad } H) = \mathcal{E}(\text{Ad } G)$.

Let $\zeta: G \rightarrow G/Z$ be the natural projection, where Z is the center of G .

LEMMA 1.2. Assume that g_i is a sequence in G , that $H \subset G$ is ample, and that for all $h \in H$, the sequence $g_i h g_i^{-1}$ approaches h ; then the sequence $\zeta(g_i)$ approaches the identity in G/Z .

Proof. From our assumption we have that for $h \in H$, the sequence $\text{Ad } g_i \text{ Ad } h \text{ Ad } g_i^{-1}$ approaches $\text{Ad } h$. Hence, since H is ample, $\text{Ad } g_i \text{ Ad } g \text{ Ad } g_i^{-1}$ approaches $\text{Ad } g$ for all $g \in G$. Identifying the Lie algebra of $\text{Ad } G$ with \mathfrak{G} , we have that $\text{Ad } g_i(X)$ approaches X for all $X \in \mathfrak{G}$. Hence $\text{Ad } g_i$ approaches the identity transformation on \mathfrak{G} , and the lemma follows. Q.E.D.

COROLLARY. Let H be an ample subset of G ; then the centralizer of H in G is Z .

Let $\theta: G \rightarrow \mathcal{R}$ be the map defined by

$$(1.3) \quad \theta(g)(\gamma) = g\gamma g^{-1}, \quad g \in G, \quad \gamma \in \Gamma.$$

From [4], Γ is ample. Hence, by the Corollary to Lemma 1.2, θ induces a one-one map, $\theta': G/Z \rightarrow \mathcal{R}$. If Γ_1 is any ample subgroup of G , let \mathcal{R}_1 denote the space of all homomorphisms of Γ_1 into G , and define $\theta_1: G \rightarrow \mathcal{R}_1$ just as we defined θ . Since Γ_1 is ample, we again have an induced one-one map, $\theta'_1: G/Z \rightarrow \mathcal{R}_1$. If Γ_1 is finitely generated, with generators $\gamma_1, \dots, \gamma_s$, we may identify \mathcal{R}_1 with a subset of $G^s = G \times \dots \times G$ (s times). Namely, we identify $\sigma \in \mathcal{R}_1$ with the s -tuple $(\sigma(\gamma_i))_i$. When convenient, we regard θ'_1 as a map from G/Z to G^s , and we observe that as such, θ'_1 is a C^∞ map.

LEMMA 1.4. Let Γ_1 be a finitely generated, ample subgroup; then θ'_1 is everywhere of maximal rank, and is a topological imbedding of G/Z into G^s .

Proof. In order to show θ'_1 is of maximal rank, it suffices to show θ_1 is of maximal rank. Let $g \cdot X$ be the left translate of $X \in \mathfrak{G}$ by $g \in G$, and assume $(\theta_1)_*(g \cdot X) = 0$. It then follows that $\text{Ad } \gamma(X) = X$, for all $\gamma \in \Gamma_1$. Since Γ_1 is ample, we must therefore have $X = 0$, and thus θ'_1 is everywhere of maximal rank. The fact that θ'_1 is a topological imbedding follows from Lemma 1.2. Q.E.D.

LEMMA 1.5. If (W, w_0, χ) is a deformation of Γ , and if $\chi(W) \subset \theta'(G/Z) = \theta(G)$, then (W, w_0, χ) is trivial.

Proof. It suffices to find a C^∞ map $c: W \rightarrow G$, such that $\theta \circ c = \chi$. We use the following observation of H. C. Wang in [12]: Γ contains a finitely generated, ample subgroup Γ_1 . In fact, just choose $\gamma_1, \dots, \gamma_s \in \Gamma$ so that the $\text{Ad } \gamma_i$ ($i = 1, \dots, s$)

generate $\mathcal{E}(\text{Ad } \Gamma)$, and then let Γ_1 be the subgroup generated by the γ_i . Clearly Γ_1 is ample. We then have a commutative diagram

$$\begin{array}{ccc} & G/Z & \\ \theta' \swarrow & & \searrow \theta'_1 \\ \mathcal{R} & \xrightarrow{\lambda} & \mathcal{R}_1 \end{array}$$

where λ is a restriction map.

By our definition of smoothness for χ , it follows that $\lambda \circ \chi$ is a C^∞ map of W into G^s . By assumption, $\chi(W) \subset \text{Image } \theta'$, so $\lambda \circ \chi(W) \subset \text{Image } \theta'_1$. Thus $\rho = (\theta'_1)^{-1} \circ \lambda \circ \chi$ is a C^∞ map from W to G/Z , where we are here utilizing Lemma 1.4. Since W is simply connected, we can find a C^∞ map $c: W \rightarrow G$, such that $\zeta \circ c = \rho$. Moreover, $\theta \circ c = (\theta' \circ \zeta) \circ c = \theta' \circ (\zeta \circ c) = \theta' \circ \rho = \theta' \circ ((\theta'_1)^{-1} \circ \lambda \circ \chi)$. On the image of χ , which is contained in the image of θ' , we have $(\theta'_1)^{-1} \circ \lambda = (\theta')^{-1}$. Therefore, $\theta \circ c = \chi$. Q.E.D.

For $w \in W$, let $\chi_w: \Gamma \rightarrow G$ be the monomorphism

$$\chi_w(\gamma) = \chi(w)(\gamma), \quad \gamma \in \Gamma.$$

We define an action of Γ on $G \times W$ by

$$(1.6) \quad (g, w)\gamma = (g\chi_w(\gamma), w), \quad g \in G, \quad w \in W, \quad \gamma \in \Gamma.$$

Let $\mathfrak{B} = (G \times W)/\Gamma$ denote the quotient space of $G \times W$ by this action of Γ , and let $\pi: G \times W \rightarrow \mathfrak{B}$ denote the natural projection. We make the regularity assumption

$$(1.7) \quad \pi \text{ is a covering map, and } \mathfrak{B} \text{ is Hausdorff.}$$

Thus \mathfrak{B} inherits the structure of a C^∞ manifold. We remark that (1.7) has been proved in the case when G/Γ is compact (see [13] and [14]). Let $\text{pr}_2: G \times W \rightarrow W$ denote the coordinate projection onto the second factor. Obviously pr_2 commutes with the action of Γ defined in (1.6), and hence induces a map $\varpi: \mathfrak{B} \rightarrow W$, such that the diagram

$$\begin{array}{ccc} \mathfrak{B} & \xrightarrow{\varpi} & W \\ \pi \swarrow & & \nearrow \text{pr}_2 \\ & G \times W & \end{array}$$

is commutative.

REMARK. In order to show that every lattice in G is rigid, it suffices to show that for any open interval I , and for any lattice $\Gamma \subset G$, a deformation $(I, 0, \chi)$ of Γ is locally trivial, in the sense that for some open interval I' with $0 \in I' \subset I$, the deformation $(I', 0, \chi)$ is trivial. This remark follows from Lemma 1.5. Hence, from now

on we will take W to be an open interval containing zero, and we will shrink I (about 0) whenever convenient.

Let t denote the parameter on I , and $\partial/\partial t$ the corresponding canonical vector field on I . The ω^λ induce forms on $G \times I$, and since these induced forms are invariant under the action (1.6) of Γ , the ω^λ also induce forms on \mathfrak{B} . In each case these induced forms will again be denoted by ω^λ . From now on we make the convention that Greek indices λ, μ, ν run from 1 to n . We then choose a basis of right-invariant vector fields X_λ on G , such that with respect to the dual pairing with forms, we have $\langle X_\lambda, \omega^\mu \rangle = \delta_\lambda^\mu$. As with the ω^λ , the X_λ induce vector fields on $G \times I$ and on \mathfrak{B} , which we again denote by X_λ .

Using a partition of unity, we may define a C^∞ vector field Y on \mathfrak{B} , such that

$$(1.8) \quad \varpi_*(Y) = \partial/\partial t.$$

Letting $[,]$ denote the Poisson bracket, we have $\varpi_*([X_\lambda, Y]) = [\varpi_*X_\lambda, \partial/\partial t] = 0$, since X_λ is vertical. Hence we may define C^∞ functions f_λ^μ on \mathfrak{B} , by

$$(1.9) \quad [X_\lambda, Y] = f_\lambda^\mu X_\mu,$$

where we use the Einstein summation convention for repeated indices. Let $f_\lambda^{t\mu}$ denote the restriction of f_λ^μ to $\varpi^{-1}(t)$ and let $\Gamma_t = \chi_t(\Gamma)$. Then we may identify $\varpi^{-1}(t)$ with G/Γ_t . We fix a Haar measure on G , and let v_t denote the induced measure on G/Γ_t .

MAIN THEOREM. *Let Γ be a lattice in the admissible analytic group G ; assume Γ contains no nontrivial elements of finite order and let $(I, 0, \chi)$ be a deformation of Γ . Assume there exists a C^∞ vector field Y on \mathfrak{B} , such that (1.8) holds and such that for all λ, μ*

$$(1.10) \quad \int_{G/\Gamma_t} |f_\lambda^{t\mu}|^2 dv_t < \infty;$$

then $(I, 0, \chi)$ is trivial.

COROLLARY. *If $(I, 0, \chi)$ is rigid at ∞ (see [5, Definition 5]), then it is trivial.*

Using Lemma 1.5 and the subsequent remark, we see that the above Corollary strengthens the theorem in [5, p. 408] in that we need no longer assume Γ is finitely generated. To get some feeling for the boundary condition (1.10), consider the case when Y generates a one-parameter group of diffeomorphisms

$$\theta_t: G/\Gamma \rightarrow G/\Gamma_t.$$

If the θ_t are induced by right translations, the f_λ^μ are 0, and conversely (see [14]). In general, the f_λ^μ may be regarded as a measure of how far the θ_t differ from being induced by right translations.

2. General cohomological results. Let M be a connected, C^∞ , orientable, Riemannian manifold of dimension ℓ . We fix an orientation of M , and let ds^2 denote the Riemannian metric on M . Relative to local coordinates x^1, \dots, x^ℓ ,

we have a representation of this metric

$$ds^2 = \sum_{i,j} g_{ij} dx^i dx^j.$$

Let $(M, \mathbf{M}, \eta, \Gamma)$ denote a connected covering of M , where $\eta: \mathbf{M} \rightarrow M$ is the covering map, and Γ is the group of deck transformations of \mathbf{M} (acting to the right). Let $\rho: \Gamma \rightarrow \text{GL}(V_\rho)$ be a representation of Γ in the finite-dimensional, real vector space V_ρ . $(M, \mathbf{M}, \eta, \Gamma)$ may be regarded as a principal fiber bundle over M , with fiber Γ . With respect to ρ , we have the associated bundle $(M, E_\rho, \eta_\rho, V_\rho)$ with fiber V_ρ and bundle projection $\eta_\rho: E_\rho \rightarrow M$. For short, we will denote this bundle by E_ρ . Let $m = \text{dimension } V_\rho$ and fix a basis of V_ρ ; we then have identifications of V_ρ with \mathbf{R}^m and of $\text{GL}(V_\rho)$ with $\text{GL}(m, \mathbf{R})$ (where \mathbf{R} denotes the real numbers). Let ρ' denote the contragredient representation of ρ ; thus for $\gamma \in \Gamma$, $\rho'(\gamma) = {}^t\rho(\gamma)^{-1}$, the inverse transpose of $\rho(\gamma)$.

Let J be an index set, and let $\mathfrak{U} = \{U_a\}_{a \in J}$ be a covering of M by open coordinate neighborhoods U_a , such that for each U_a we have a diffeomorphism

$$\Phi_a: U_a \times V_\rho \rightarrow \eta_\rho^{-1}(U_a),$$

where $\eta_\rho \circ \Phi_a = \text{pr}_1$, the coordinate projection of $U_a \times V_\rho$ onto the first factor. If U_a meets U_b , we define

$$e_{ab}: U_a \cap U_b \rightarrow \text{GL}(V_\rho),$$

by

$$\Phi_a^{-1} \circ \Phi_b(x, v_b) = (x, e_{ab}(x)v_b), \quad x \in U_a \cap U_b, \quad v_b \in V_\rho.$$

We have

$$\begin{aligned} e_{aa} &= \text{identity}, \\ (2.1) \quad e_{ab}e_{ba} &= \text{identity} \quad (\text{on } U_a \cap U_b), \\ e_{ab}e_{bc} &= e_{ac} \quad (\text{on } U_a \cap U_b \cap U_c). \end{aligned}$$

Assume that for each $a \in J$, $h_a: U_a \rightarrow \text{GL}(m, \mathbf{R})$ is a C^∞ function, with values in the set of positive-definite, symmetric matrices. Assume, moreover, that on $U_a \cap U_b$

$$(2.2) \quad h_a = {}^te_{ba}h_b e_{ba}.$$

We can use the h_a to define a positive-definite inner product, h , on the fibers of E_ρ . Thus for $\mathbf{p} = \Phi_a(x, v_a)$, $\mathbf{p}' = \Phi_a(x, v'_a)$, with $x \in U_a$, and $v_a, v'_a \in \mathbf{R}^m$, we set $h(\mathbf{p}, \mathbf{p}') = {}^tv_a h_a(x) v'_a$. (2.2) then implies that h is well defined.

Let $\mathscr{A}^q(E_\rho)$ denote the set of C^∞ , E_ρ -valued q -forms on M . Thus if $\varphi \in \mathscr{A}^q(E_\rho)$, then on U_a , φ is given by a column vector

$$\varphi_a = \begin{pmatrix} \varphi_a^1 \\ \vdots \\ \varphi_a^m \end{pmatrix},$$

whose components are C^∞ scalar valued q -forms on U_a . On $U_a \cap U_b$ we then have

$$(2.3) \quad \varphi_a = e_{ab} \varphi_b.$$

We define

$$d: \mathcal{A}^q(E_\rho) \rightarrow \mathcal{A}^{q+1}(E_\rho)$$

by applying the ordinary exterior differentiation operator (which we also denote by d) componentwise. Since E_ρ is a locally constant vector bundle, equation (2.3) implies d is well defined.

Next, we define

$$\hat{*}: \mathcal{A}^q(E_\rho) \rightarrow \mathcal{A}^{\ell-q}(E_{\rho'}).$$

First, let $*$ denote the Hodge star operator defined on scalar forms by the Riemannian metric ds^2 . Then, for $\varphi \in \mathcal{A}^q(E_\rho)$, set

$$\hat{*}\varphi_a = h_a(*\varphi_a),$$

where $*$ is here defined componentwise. A direct computation, using (2.1), (2.2), and (2.3), shows that $\hat{*}$ is well defined. Now define $\delta: \mathcal{A}^q(E_\rho) \rightarrow \mathcal{A}^{q-1}(E_\rho)$, by $\delta = (-1)^q \hat{*}^{-1} d \hat{*}$. Finally, we define $\Delta: \mathcal{A}^q(E_\rho) \rightarrow \mathcal{A}^q(E_\rho)$, by $\Delta = d\delta + \delta d$.

Let dX denote the volume element on M , with respect to ds^2 (recall that we have a fixed orientation on M). If $\phi, \psi \in \mathcal{A}^q(E_\rho)$, we can define a scalar ℓ -form ${}^t\varphi \wedge \hat{*}\psi$ on M , as follows: on U_a , we let ${}^t\varphi \wedge \hat{*}\psi$ equal ${}^t\varphi_a \wedge \hat{*}\psi_a$, where the latter is the matrix product of the row vector ${}^t\varphi_a$ and the column vector $\hat{*}\psi_a$. A routine computation then shows that ${}^t\varphi \wedge \hat{*}\psi$ is well defined. We then define the C^∞ function $\mathcal{Q}(\varphi, \psi)$ on M , by

$$\mathcal{Q}(\varphi, \psi) dX = {}^t\varphi \wedge \hat{*}\psi.$$

Whenever $\mathcal{Q}(\varphi, \psi)$ is integrable (with respect to dX), we define the inner product, (φ, ψ) , of φ and ψ , by

$$(2.4) \quad (\varphi, \psi) = \int_M \mathcal{Q}(\varphi, \psi) dX.$$

The following is then a direct consequence of Stokes' theorem.

PROPOSITION 2.5. *Let φ and ψ be forms in $\mathcal{A}^q(E_\rho)$, such that support $\varphi \cap$ support ψ is compact; then $(d\varphi, \psi) = (\varphi, \delta\psi)$ and $(\Delta\varphi, \psi) = (\varphi, \Delta\psi) = (d\varphi, d\psi) + (\delta\varphi, \delta\psi)$.*

We conclude this section with a discussion of W -ellipticity and some of its consequences. Toward this end we first give some definitions and collect some elementary facts. Thus let $\mathcal{A}_c^q(E_\rho)$ denote the space of those forms in $\mathcal{A}^q(E_\rho)$ with compact support. We define a positive-definite, symmetric bilinear form, $a(\ , \)$, on $\mathcal{A}_c^q(E_\rho)$ by

$$a(\varphi, \psi) = (\varphi, \psi) + (d\varphi, d\psi) + (\delta\varphi, \delta\psi), \quad \varphi, \psi \in \mathcal{A}_c^q(E_\rho).$$

Let $\mathcal{L}^q = \mathcal{L}^q(E_\rho)$ and $\mathcal{W}^q = \mathcal{W}^q(E_\rho)$ denote the completions of $\mathcal{A}_c^q(E_\rho)$ with respect to the norms $\|\varphi\| = (\varphi, \varphi)^{1/2}$ and $N(\varphi) = a(\varphi, \varphi)^{1/2}$, respectively. We define $i: \mathcal{W}^q \rightarrow \mathcal{L}^q$ as follows: If $\varphi \in \mathcal{W}^q$, let φ_ν be a Cauchy sequence (with respect to $N(\cdot)$) in $\mathcal{A}_c^q(E_\rho)$, representing φ . Then clearly φ_ν is also Cauchy with respect to $\|\cdot\|$. Hence φ_ν has a limit $i(\varphi)$ in \mathcal{L}^q , and it is obvious that $i(\varphi)$ is independent of the Cauchy sequence φ_ν .

PROPOSITION 2.6. *i is one-one.*

Proof. Assume that $\varphi \in \mathcal{W}^q$ and that $i(\varphi) = 0$. Then, by definition of i , we have a sequence φ_ν in $\mathcal{A}_c^q(E_\rho)$, representing φ , and such that

$$\lim_{\nu, \mu \rightarrow \infty} N(\varphi_\nu - \varphi_\mu) = \lim_{\nu \rightarrow \infty} \|\varphi_\nu\| = 0.$$

But then, for all $\psi \in \mathcal{A}_c^q(E_\rho)$, we have $\lim_{\nu \rightarrow \infty} (\varphi_\nu, \psi) = 0$; so in particular,

$$\begin{aligned} \lim_{\nu \rightarrow \infty} (d\varphi_\nu, \psi') &= \lim_{\nu \rightarrow \infty} (\varphi_\nu, \delta\psi') = 0, \\ (2.7) \quad \lim_{\nu \rightarrow \infty} (\delta\varphi_\nu, \psi'') &= \lim_{\nu \rightarrow \infty} (\varphi_\nu, d\psi'') = 0, \\ \psi' &\in \mathcal{A}_c^{q+1}(E_\rho), \quad \psi'' \in \mathcal{A}_c^{q-1}(E_\rho). \end{aligned}$$

Since φ_ν is Cauchy with respect to the norm $N(\cdot)$, both $d\varphi_\nu$ and $\delta\varphi_\nu$ are Cauchy with respect to the norm $\|\cdot\|$. Let $d\varphi$ and $\delta\varphi$ denote the limits in \mathcal{L}^{q+1} and \mathcal{L}^{q-1} , respectively, of the sequences $d\varphi_\nu$ and $\delta\varphi_\nu$, respectively. Then (2.7) implies $d\varphi$ and $\delta\varphi$ are zero. Hence, $\lim_{\nu \rightarrow \infty} \|d\varphi_\nu\| = \lim_{\nu \rightarrow \infty} \|\delta\varphi_\nu\| = \lim_{\nu \rightarrow \infty} \|\varphi_\nu\| = 0$, and therefore $\varphi = 0$. Q.E.D.

REMARK. Using i , \mathcal{W}^q may be identified with the subspace of all $\varphi \in \mathcal{L}^q$, such that there exists a sequence φ_ν in $\mathcal{A}_c^q(E_\rho)$, so that the sequences φ_ν , $d\varphi_\nu$, and $\delta\varphi_\nu$ are Cauchy with respect to $\|\cdot\|$, and so that

$$\lim_{\nu \rightarrow \infty} \|\varphi - \varphi_\nu\| = 0.$$

Let $d\varphi$ and $\delta\varphi$ denote the limits (with respect to $\|\cdot\|$) of the sequences $d\varphi_\nu$ and $\delta\varphi_\nu$, respectively. Then Proposition 2.6 implies that $d\varphi$ and $\delta\varphi$ are independent of the sequence φ_ν . Thus d and δ have extensions to \mathcal{W}^q , in the sense of Friedrichs.

We define the Dirichlet inner product, $d(\cdot, \cdot)$ on \mathcal{W}^q , by

$$d(\varphi, \psi) = (d\varphi, d\psi) + (\delta\varphi, \delta\psi), \quad \varphi, \psi \in \mathcal{W}^q.$$

We note that $d(\varphi, \varphi)^{1/2}$ defines a seminorm on \mathcal{W}^q . Recall that we have fixed a Riemannian metric ds^2 on M .

DEFINITION 2.8. Let h be a positive-definite inner product on the fibers of E_ρ ; we then say that E_ρ is W^q -elliptic relative to h , if there exists a constant $c > 0$, so that for all $\varphi \in \mathcal{A}_c^q(E_\rho)$,

$$(2.9) \quad cd(\varphi, \varphi) \geq (\varphi, \varphi).$$

REMARK. Viewing \mathcal{W}^q as the subspace of all φ in \mathcal{L}^q , admitting $d\varphi$ and $\delta\varphi$ in the sense of Friedrichs, we note that (2.9) still has meaning on \mathcal{W}^q ; then if E_ρ is W^q -elliptic relative to h , (2.9) still holds on \mathcal{W}^q . Thus, if E_ρ is W^q -elliptic, $d(\varphi, \varphi)^{1/2}$ is a complete norm on \mathcal{W}^q , equivalent to $N(\varphi)$.

From now on, we will assume h is fixed, and will speak of W^q -ellipticity without referring to h explicitly.

THEOREM 2.10. *If E_ρ is W^q -elliptic, then for every $\alpha \in \mathcal{L}^q$, the equation*

$$(2.11) \quad \Delta x = \alpha$$

has a unique weak solution $x \in \mathcal{W}^q$; that is, for any $u \in \mathcal{W}^q$ we have

$$(2.12) \quad (dx, du) + (\delta x, \delta u) = (\alpha, u).$$

Moreover, if $\alpha \in \mathcal{A}^q(E_\rho) \cap \mathcal{L}^q$, then $x \in \mathcal{A}^q(E_\rho) \cap \mathcal{W}^q$, and (2.11) holds in the classical sense.

Proof. Let F be the linear functional on \mathcal{W}^q , defined by $F(u) = (\alpha, u)$, $u \in \mathcal{W}^q$. Then from the Schwarz inequality and W^q -ellipticity, we have

$$|F(u)| \leq c^{1/2} \|\alpha\| d(u, u)^{1/2}, \quad u \in \mathcal{W}^q.$$

Thus F is continuous on \mathcal{W}^q , with respect to the complete norm $d(u, u)^{1/2}$ (see the remark following Definition 2.8). Hence, by the Riesz representation theorem, there exists a unique $x \in \mathcal{W}^q$, such that $F(u) = d(u, x)$, $u \in \mathcal{W}^q$. But this proves the first assertion, while the last assertion follows from the fact that Δ is elliptic, and from the regularity theorem for such operators (see [7, pp. 268–269]). Q.E.D.

3. General cohomological results (continued). Fix a point $o \in M$ and for $p, q \in M$, let $d(p, q)$ denote the geodesic distance between p and q . For $c > 0$, let

$$B(q, c) = \{p \in M \mid d(p, q) < c\}, \quad B(c) = B(o, c).$$

Given q , we can assume $B(q, c)$ is relatively compact, *provided* c is sufficiently small. Moreover, if ds^2 is complete, then $B(q, c)$ is always relatively compact (see [6, p. 56, Theorem 10.3]).

A real-valued function f on R^ℓ is said to be locally Lipschitz, if for $p \in R^\ell$ we can find an open neighborhood U of p , and a constant $b > 0$, so that for p_1, p_2 in U , $|f(p_1) - f(p_2)| \leq b \sum_{i=1}^{\ell} |p_1^i - p_2^i|$, where $p_v = (p_v^1, \dots, p_v^\ell)$ for $v = 1, 2$. A real-valued function f on M , is called locally Lipschitz, if for $p \in M$, we can find an open neighborhood U of p , and a diffeomorphism $g: R^\ell \rightarrow U$, so that $f \circ g$ is locally Lipschitz on R^ℓ . It is easily seen that if f is locally Lipschitz on M , and if $g: R^\ell \rightarrow M$ is any C^∞ map, then $f \circ g$ is locally Lipschitz on R^ℓ .

LEMMA 3.1. $\rho(p) = d(p, o)$ is a locally Lipschitz⁽²⁾ function. At points where the

(2) In this section the symbol ρ is used in two capacities: as a distance function and as a representation. We trust that this double usage will not cause any confusion.

appropriate derivatives exist, we have in local coordinates

$$(3.2) \quad \sum_{i,j=1}^{\ell} g^{ij}(\partial\rho/\partial x^i)(\partial\rho/\partial x^j) \leq \ell.$$

Proof. We prove the second assertion first. Thus, let q be a point in M , and U a normal coordinate neighborhood of q , with normal coordinates (x^1, \dots, x^ℓ) on U . In particular, q has all coordinates zero. If $p \in U$ has coordinates (x^1, \dots, x^ℓ) , then

$$(3.3) \quad d(p, q) = ((x^1)^2 + \dots + (x^\ell)^2)^{1/2}.$$

If the first derivatives of ρ exist at q , we have at q

$$\begin{aligned} |\partial\rho/\partial x^i| &= \left| \lim_{y^i \rightarrow 0} (\rho(q + (0, \dots, y^i, \dots, 0)) - \rho(q))/y^i \right| \\ &\leq \lim_{y^i \rightarrow 0} |d(q + (0, \dots, y^i, \dots, 0), q)|/|y^i| = 1, \end{aligned}$$

where the last equality follows from (3.3). Since $g^{ij} = \delta_{ij}$ at q , (3.2) now follows.

Now assume c is small enough so that $B(q, c)$ is contained in a coordinate neighborhood U , with local coordinates x^1, \dots, x^ℓ . Let $T(M)$ denote the tangent bundle to M , and let $T(M)|B(q, c)$ denote the restriction of $T(M)$ to $B(q, c)$. Corresponding to the local coordinates x^1, \dots, x^ℓ on $B(q, c)$, we have induced coordinates $(x^1, \dots, x^\ell; z^1, \dots, z^\ell)$ on $T(M)|B(q, c)$, and we set $p = (x^1, \dots, x^\ell)$, and $Z = (z^1, \dots, z^\ell)$. If $c > 0$ is small enough, we can find a neighborhood Ω of $B(q, c)$ in $T(M)|B(q, c)$, so that the map

$$\Phi: \Omega \rightarrow B(q, c) \times B(q, c)$$

defined by $\Phi(p, Z) = (\exp_p(Z), p)$, is a diffeomorphism. Here \exp_p denotes the exponential map with respect to ds^2 , at p (see [8, p. 166]). Let $|Z|$ denote the length of the tangent vector Z , with respect to ds^2 . Passing to a smaller c , if necessary, we may assume that the matrix (g_{ij}) has bounded eigenvalues on $B(q, c)$. Thus, there exists $e > 0$, so that for $(p, Z) \in T(M)|B(q, c)$,

$$(3.4) \quad |Z| \leq e \left(\sum_{i=1}^{\ell} (z^i)^2 \right)^{1/2}.$$

If $p, p' \in B(q, c)$ and if $p' = \exp_p(Z)$, then $d(p', p) = |Z|$. By (3.4), $|Z|$ is a locally Lipschitz function on $T(M)|B(q, c)$, and hence using Φ , we see that $d(p', p)$ is a locally Lipschitz function on $B(q, c) \times B(q, c)$. Since

$$|\rho(p) - \rho(p')| \leq d(p', p),$$

$\rho(p)$ is a locally Lipschitz function on $B(q, c)$. Since $q \in M$ was chosen arbitrarily, ρ is a locally Lipschitz function on M . Q.E.D.

Let V be an ℓ -dimensional real vector space, let F be an m -dimensional real vector space, and let $\bigwedge V$ denote the exterior algebra of V . We assume that F and V are equipped with positive-definite inner products. We then have induced

inner products on $\bigwedge V$ and on $F \otimes \bigwedge V$. In each case we denote the induced inner product by \langle , \rangle , and the corresponding norm by $| \cdot |$. For $u \in \bigwedge V$, let $\varepsilon(u): \bigwedge V \rightarrow \bigwedge V$ denote the corresponding left multiplication operator. We also let $\varepsilon(u)$ denote the operator on $F \otimes \bigwedge V$, obtained by tensoring the identity operator on F with $\varepsilon(u)$ on $\bigwedge V$.

LEMMA 3.5. *If $u \in V$, $v \in F \otimes \bigwedge V$, then $|\varepsilon(u)v| \leq |u||v|$.*

Proof. Let $i(u): F \otimes \bigwedge V \rightarrow F \otimes \bigwedge V$ denote the adjoint of $\varepsilon(u)$ with respect to \langle , \rangle . One then has

$$(3.6) \quad \varepsilon(u)i(u) + i(u)\varepsilon(u) = \langle u, u \rangle \text{Id},$$

where Id denotes the identity on $F \otimes \bigwedge V$. Thus

$$\begin{aligned} |\varepsilon(u)v|^2 &= \langle \varepsilon(u)v, \varepsilon(u)v \rangle \\ &\leq \langle \varepsilon(u)v, \varepsilon(u)v \rangle + \langle i(u)v, i(u)v \rangle \\ &= \langle (i(u)\varepsilon(u) + \varepsilon(u)i(u))v, v \rangle = \langle u, u \rangle \langle v, v \rangle, \end{aligned}$$

where the last equality follows from (3.6). The lemma now follows by taking square roots. Q.E.D.

COROLLARY. *Let u be a scalar 1-form on M , and v a q -form with values in E_ρ . We define $u \wedge v$ pointwise by $u \wedge v = \varepsilon(u)v$. At every point p of M , we let $|u|$ denote the length of u at p with respect to ds^2 . We then have at every point in M*

$$\mathcal{Q}(u \wedge v, u \wedge v) \leq |u|^2 \mathcal{Q}(v, v).$$

The following lemma follows from the fact that Stokes' theorem holds for locally Lipschitz differential forms. In particular, we remark that a locally Lipschitz form α is differentiable almost everywhere, so $d\alpha$ is defined almost everywhere.

LEMMA 3.7. *Let α be a locally Lipschitz q -form on M , with values in E_ρ ; then for every $\varphi \in \mathcal{A}^q(E_\rho)$ such that support $\varphi \cap$ support α is compact, we have*

$$(3.8) \quad (d\varphi, d\alpha) + (\delta\varphi, \delta\alpha) = (\Delta\varphi, \alpha).$$

Let $\mu(t)$ be a C^∞ function on \mathbf{R} such that $0 \leq \mu(t) \leq 1$ and

$$(3.9) \quad \begin{aligned} \mu(t) &= 1, \quad t \leq 1, \\ &= 0, \quad t \geq 2. \end{aligned}$$

Let $m = \sup |d\mu/dt|$, and let

$$\omega(x) = \mu((\rho(x) + R - 2r)/(R - r)), \quad x \in M, \quad 0 < r < R.$$

Then (3.9) implies

$$(3.10) \quad \begin{aligned} 0 &\leq \omega(x) \leq 1, \\ \omega(x) &= 1, \quad x \in \overline{B(r)}, \\ &= 0, \quad x \in M - B(R). \end{aligned}$$

Also, a direct computation shows

$$(3.11) \quad |d\omega/d\rho| \leq m/(R-r).$$

For $\varphi \in \mathcal{A}^q(E_\rho)$, (3.10) implies that at every point in M , we have

$$(3.12) \quad \mathcal{Q}(\omega\varphi, \omega\varphi) \leq \mathcal{Q}(\varphi, \varphi).$$

From (3.2) and (3.11), we have $|d\omega|^2 \leq \ell m^2/(R-r)^2$, wherever $d\omega$ is defined. From this and from the Corollary to Lemma 3.5, we have that wherever $d\omega$ is defined

$$(3.13) \quad \mathcal{Q}(d\omega \wedge \varphi, d\omega \wedge \varphi) \leq \mathcal{Q}(\varphi, \varphi) \ell m^2/(R-r)^2.$$

Set $\alpha = \omega^2 \varphi$; so we have almost everywhere

$$(3.14) \quad \begin{aligned} d\alpha &= \omega^2 d\varphi + 2\omega d\omega \wedge \varphi, \\ \delta\alpha &= \omega^2 \delta\varphi + (-1)^q \hat{*}^{-1}(2\omega d\omega \wedge \hat{*}\varphi). \end{aligned}$$

For $c > 0$, let $(\varphi, \varphi)_{B(c)} = \int_{B(c)} \mathcal{Q}(\varphi, \varphi) dX$, and set $\|\varphi\|_{B(c)} = ((\varphi, \varphi)_{B(c)})^{1/2}$. We then have

PROPOSITION 3.15. *There exists a constant $a > 0$, so that if $0 < r < R$ and if $B(R)$ is relatively compact, then for $\sigma > 0$ and $\varphi \in \mathcal{A}^q(E_\rho)$,*

$$(3.16) \quad (\|\omega d\varphi\|_{B(r)})^2 + (\|\omega \delta\varphi\|_{B(r)})^2 \leq \sigma(\|\Delta\varphi\|_{B(R)})^2 + ((1/\sigma) + a/(R-r)^2)(\|\varphi\|_{B(R)})^2.$$

Proof. If we consider $(d\varphi, d\alpha)$ and $(\delta\varphi, \delta\alpha)$ and apply (3.14) and Lemma 3.7, we get

$$(3.17) \quad \begin{aligned} (\|\omega d\varphi\|_{B(R)})^2 + (\|\omega \delta\varphi\|_{B(R)})^2 &\leq |(\Delta\varphi, \omega^2\varphi)_{B(R)}| + |(d\varphi, 2\omega d\omega \wedge \varphi)_{B(R)}| \\ &\quad + |(\delta\varphi, \hat{*}^{-1}2\omega d\omega \wedge \hat{*}\varphi)_{B(R)}|. \end{aligned}$$

Noting that for $\sigma > 0$, $(\Delta\varphi, \omega^2\varphi)_{B(R)} = (\sigma \Delta\varphi, (1/\sigma)\omega^2\varphi)_{B(R)}$, and applying the Schwarz inequality to each right-hand term in (3.17), we obtain

$$(3.18) \quad \begin{aligned} (\|\omega d\varphi\|_{B(R)})^2 + (\|\omega \delta\varphi\|_{B(R)})^2 &\leq \sigma(\|\Delta\varphi\|_{B(R)})^2 + (1/\sigma)(\|\omega^2\varphi\|_{B(R)})^2 \\ &\quad + 4(\|d\omega \wedge \varphi\|_{B(R)})^2 + 4(\|d\omega \wedge \hat{*}\varphi\|_{B(R)})^2. \end{aligned}$$

Applying (3.10) and (3.13) to the last three terms on the right of (3.18), we obtain

$$(\|\omega d\varphi\|_{B(R)})^2 + (\|\omega \delta\varphi\|_{B(R)})^2 \leq \sigma(\|\Delta\varphi\|_{B(R)})^2 + ((1/\sigma) + 8\ell m^2/(R-r)^2)(\|\varphi\|_{B(R)})^2.$$

But, since $\omega = 1$ on $B(r)$ and since $0 \leq \omega \leq 1$, we obtain (3.16), with $a = 8\ell m^2$. Q.E.D.

PROPOSITION 3.19. *If ds^2 is complete, then for all $\sigma > 0$ and $\varphi \in \mathcal{A}^q(E_\rho)$, we have*

$$\|d\varphi\|^2 + \|\delta\varphi\|^2 \leq \sigma\|\Delta\varphi\|^2 + (1/\sigma)\|\varphi\|^2.$$

Proof. Since ds^2 is complete, $B(R)$ is relatively compact for all $R > 0$. Thus, in (3.16), we can take $R = 2r$ and consider the limit of both sides as r approaches infinity. The proposition now follows. Q.E.D.

PROPOSITION 3.20. *Assume ds^2 is complete. If $\varphi \in \mathcal{L}^q \cap \mathcal{A}^q(E_\rho)$ and if $\Delta\varphi=0$, then $d\varphi=\delta\varphi=0$.*

Proof. Just apply Proposition 3.19 and let σ approach infinity. Q.E.D.

PROPOSITION 3.21. *If $x \in \mathcal{W}^q \cap \mathcal{A}^q(E_\rho)$, then $dx \in \mathcal{L}^{q+1}$, and $\delta x \in \mathcal{L}^{q-1}$.*

Proof. There is a sequence x_ν in $\mathcal{A}_c^q(E_\rho)$, such that $\lim_{\nu \rightarrow \infty} \|x_\nu - x\| = 0$, and such that the sequences dx_ν and δx_ν are Cauchy with respect to $\|\cdot\|$. But then for $\psi \in \mathcal{A}_c^{q+1}(E_\rho)$

$$\begin{aligned} \lim_{\nu \rightarrow \infty} (dx_\nu, \psi) &= \lim_{\nu \rightarrow \infty} (x_\nu, \delta\psi) \\ &= (dx, \psi), \end{aligned}$$

where we are here utilizing Proposition 2.5. On the other hand, let x' be the limit in \mathcal{L}^{q+1} of the Cauchy sequence dx_ν . Thus, for $\psi \in \mathcal{A}_c^{q+1}(E_\rho)$

$$\begin{aligned} (x', \psi) &= \lim_{\nu \rightarrow \infty} (dx_\nu, \psi) \\ &= \lim_{\nu \rightarrow \infty} (x_\nu, \delta\psi) = (dx, \psi). \end{aligned}$$

Hence $x' = dx$, so $dx \in \mathcal{L}^{q+1}$. Similarly $\delta x \in \mathcal{L}^{q-1}$. Q.E.D.

THEOREM 3.22. *Assume ds^2 is complete and E_ρ is W^q -elliptic. If $\varphi \in \mathcal{L}^q \cap \mathcal{A}^q(E_\rho)$ and if $d\varphi=0$, then there exists $\psi \in \mathcal{L}^{q-1} \cap \mathcal{A}^{q-1}(E_\rho)$, such that $d\psi=\varphi$.*

Proof. By W^q -ellipticity and Theorem 2.10 there exists $x \in \mathcal{W}^q \cap \mathcal{A}^q(E_\rho)$, such that $\Delta x = \varphi$. By assumption, $0 = d\varphi = \Delta dx$, while from Proposition 3.21, we have $dx \in \mathcal{L}^{q+1}$. Hence, Proposition 3.20 implies that $\delta dx = 0$. Thus $\varphi = \Delta x = d\delta x = d\psi$, where $\psi = \delta x$ is in $\mathcal{L}^{q-1} \cap \mathcal{A}^{q-1}(E_\rho)$. Q.E.D.

4. Cohomology on locally symmetric spaces. Our purpose in this section is to prove a vanishing theorem for “square integrable” cohomology (Theorem 4.17). We employ Theorem 3.22 and the methods in [9] (we refer to this paper for details omitted in this section). In fact, Theorem 4.17 is a reformulation of Theorem 7.1 in [9, Part I].

From now on, G will denote an admissible analytic group and Γ will denote a lattice in G . Recall that we have fixed a Cartan decomposition $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$. We will let $r = \dim \mathfrak{P}$ and we will let Latin indices i, j, k run from 1 to r , Greek indices α, β, γ run from $r+1$ to n , and Greek indices λ, μ, ν run from 1 to n . Let \mathfrak{G}^* denote the dual space of \mathfrak{G} . The right-invariant one-forms ω^λ on G , determine corresponding element in \mathfrak{G}^* , which we again denote by ω^λ . Similarly, the right-invariant vector fields X_λ determine a basis of \mathfrak{G} , which is dual to the ω^λ (considered as elements of \mathfrak{G}^*), and which we again denote by X_λ . We assume the X_λ are chosen so that the X_i are a basis for \mathfrak{P} , and the X_α are a basis for \mathfrak{K} . Moreover, if $B(\cdot, \cdot)$

denotes the Killing form on \mathfrak{G} , we assume the X_λ are normalized so that

$$(4.1) \quad \begin{aligned} B(X_\alpha, X_\beta) &= -\delta_{\alpha\beta}, \\ B(X_i, X_j) &= \delta_{ij}, \\ B(X_i, X_\alpha) &= 0. \end{aligned}$$

We now define the positive-definite inner product, $(\ , \)_{\text{ad}}$, on \mathfrak{G} , by

$$(X_\lambda, X_\mu)_{\text{ad}} = \delta_{\lambda\mu}.$$

In this section, we assume that Γ contains no nontrivial elements of finite order; then the natural projection $\eta: M \rightarrow M/\Gamma$ is a covering map, and $M = M/\Gamma$ is a C^∞ manifold. (4.1) and the invariance of the Killing form, imply that $\sum_i (\omega^i)^2$ and $(\ , \)_{\text{ad}}$ are K -invariant. Hence, in particular, $\sum_i (\omega^i)^2$ induces a G -invariant, and hence complete, Riemannian metric on M . Hence $\sum_i (\omega^i)^2$ induces a complete Riemannian metric on M , and we take this metric to be the ds^2 of §§2 and 3.

Let $\rho: G \rightarrow GL(V_\rho)$ be a representation of G in the m -dimensional, real vector space V_ρ ; then ρ induces a representation of Γ , which we again denote by ρ . We then have a locally constant vector bundle $(M, E_\rho, \eta_\rho, V_\rho)$ associated to the principal bundle (M, M, η, Γ) by means of ρ . We let \mathcal{E}_ρ denote the sheaf of germs of locally constant sections in E_ρ . Let $\mathcal{A}(E_\rho)$ be the direct sum of the $\mathcal{A}^q(E_\rho)$; then let $H^q(E_\rho)$ denote the q th cohomology group of the complex $(\mathcal{A}(E_\rho), d)$. By the de Rham Theorem $H^q(E_\rho)$ is isomorphic to $H^q(M, \mathcal{E}_\rho)$, the q th cohomology group of M , with coefficients in \mathcal{E}_ρ .

To use the machinery of harmonic analysis, introduced in §§2 and 3, we need only introduce a metric h on the fibers of E_ρ . Toward this end, we first introduce the notion of an admissible, positive-definite inner product, $(\ , \)_\rho$, on V_ρ . Thus, we say $(\ , \)_\rho$ is admissible, in case the operators $\rho(X_i)$ and $\rho(X_\alpha)$ are symmetric and skew-symmetric, respectively, with respect to $(\ , \)_\rho$. For each ρ , we fix an admissible, positive-definite inner product $(\ , \)_\rho$ (see [9, Part I, Lemma 3.1], for the proof that such inner products exist). One notes that $(\ , \)_{\text{ad}}$, which we defined previously, is admissible, and for $\rho = \text{ad}$ we take this inner product to be our $(\ , \)_\rho$. It follows from Proposition 3.1, in [9, Part I], that $(\ , \)_\rho$ induces a positive-definite inner product on the fibers of E_ρ , and we take h to be this induced inner product.

Let $A^q(\Gamma, G, \rho)$ denote the space of all V_ρ -valued q -forms φ on G , such that

$$(4.2) \quad \varphi \circ R_\gamma = \rho(\gamma)^{-1} \varphi, \quad \gamma \in \Gamma,$$

where R_γ denotes the right translation on G , by γ . Let $A(\Gamma, G, \rho)$ denote the direct sum of the $A^q(\Gamma, G, \rho)$; then componentwise exterior differentiation induces a coboundary operator, \hat{d} , on $A(\Gamma, G, \rho)$. Let $\mathcal{A}(\Gamma, M, \rho)$ denote the set of all $\varphi \in A(\Gamma, G, \rho)$, such that

$$(4.3) \quad \varphi \circ L_k = \varphi, \quad k \in K, \quad i(X_\alpha)\varphi = 0,$$

where L_k denotes the left translation on G , by k , and where $i(X_\alpha)$ denotes interior

multiplication by the right-invariant vector field X_α ; then \hat{d} leaves $\mathcal{A}(\Gamma, \mathbf{M}, \rho)$ invariant. We let d denote the restriction of \hat{d} to $\mathcal{A}(\Gamma, \mathbf{M}, \rho)$, and we let $\mathcal{A}^q(\Gamma, \mathbf{M}, \rho) = \mathcal{A}(\Gamma, \mathbf{M}, \rho) \cap A^q(\Gamma, G, \rho)$. From [9, pp. 368–369], we have that the complexes $(\mathcal{A}(\Gamma, \mathbf{M}, \rho), d)$ and $(\mathcal{A}(E_\rho), d)$ are isomorphic.

Given $\varphi \in A^q(\Gamma, G, \rho)$, we define the V_ρ -valued form φ_0 on G , by $\varphi_0(g) = \rho(g)\varphi(g)$, $g \in G$. Then, under the map $\varphi \rightarrow \varphi_0$, $A^q(\Gamma, G, \rho)$ corresponds to the space $A_0^q(\Gamma, G, \rho)$, of all C^∞ , V_ρ -valued forms, φ on G , such that

$$(4.4) \quad \varphi \circ R_\gamma = \varphi, \quad \gamma \in \Gamma;$$

and $\mathcal{A}^q(\Gamma, \mathbf{M}, \rho)$ corresponds to the subspace, $\mathcal{A}_0^q(\Gamma, \mathbf{M}, \rho)$, of all φ in $A^q(\Gamma, G, \rho)$, such that

$$(4.5) \quad \varphi \circ L_k = \rho(k)\varphi, \quad k \in K; \quad i(X_\alpha)\varphi = 0.$$

Under the correspondence $\varphi \rightarrow \varphi_0$, \hat{d} and d correspond to operators \hat{d}^0 on $A_0(\Gamma, G, \rho)$ and d^0 on $\mathcal{A}_0(\Gamma, \mathbf{M}, \rho)$, respectively. Here $A_0(\Gamma, G, \rho)$ and $\mathcal{A}_0(\Gamma, \mathbf{M}, \rho)$ denote the direct sums of the $A_0^q(\Gamma, G, \rho)$ and the $\mathcal{A}_0^q(\Gamma, \mathbf{M}, \rho)$, respectively. We have remarked that the complexes $(\mathcal{A}(\Gamma, \mathbf{M}, \rho), d)$ and $(\mathcal{A}(E_\rho), d)$ are isomorphic, and hence $(\mathcal{A}(E_\rho), d)$ is isomorphic to the complex $(\mathcal{A}_0(\Gamma, \mathbf{M}, \rho), d^0)$. Our next goal will be to describe the operators d^0 , δ^0 , and Δ^0 on $\mathcal{A}_0(\Gamma, \mathbf{M}, \rho)$, corresponding to the operators, d , δ , and Δ , respectively, on $\mathcal{A}(E_\rho)$. We shall also describe \hat{d}^0 (see (4.20), below).

First, we can identify $A_0^q(\Gamma, G, \rho)$ with the space of all C^∞ functions, φ , on G , with values in $V_\rho \otimes \bigwedge^q \mathfrak{G}^*$, such that

$$(4.4') \quad \varphi(g\gamma) = \varphi(g), \quad g \in G, \quad \gamma \in \Gamma.$$

We fix a basis Y_1, \dots, Y_m of V_ρ , which is orthonormal with respect to $(\cdot, \cdot)_\rho$. Then $\varphi \in A_0^q(\Gamma, G, \rho)$ has a representation $\varphi = f_{\lambda_1 \dots \lambda_q}^s Y_s \otimes \omega^{\lambda_1} \wedge \dots \wedge \omega^{\lambda_q}$, where the $f_{\lambda_1 \dots \lambda_q}^s$ are C^∞ functions on G (alternating in $\lambda_1, \dots, \lambda_q$), such that

$$f_{\lambda_1 \dots \lambda_q}^s(g\gamma) = f_{\lambda_1 \dots \lambda_q}^s(g), \quad g \in G, \quad \gamma \in \Gamma.$$

With regard to the above expression for φ , recall that we are using the Einstein summation convention.

We now define operators $\Theta'(X_\lambda)$, $\rho(X_\lambda)$, $\Theta^*(X_\lambda)$, $\varepsilon(\omega^\lambda)$, and $i(X_\lambda)$ on $A_0^q(\Gamma, G, \rho)$. First, let $i(X_\lambda): \bigwedge^q \mathfrak{G}^* \rightarrow \bigwedge^{q-1} \mathfrak{G}^*$ denote the adjoint of the left multiplication operator $\varepsilon(X_\lambda): \bigwedge^q \mathfrak{G} \rightarrow \bigwedge^{q-1} \mathfrak{G}$. Then, for $\varphi \in A_0^q(\Gamma, G, \rho)$,

$$\Theta'(X_\lambda)\varphi = (X_\lambda f_{\lambda_1 \dots \lambda_q}^s) Y_s \otimes \omega^{\lambda_1} \wedge \dots \wedge \omega^{\lambda_q},$$

$$\rho(X_\lambda)\varphi = f_{\lambda_1 \dots \lambda_q}^s (\rho(X_\lambda) Y_s) \otimes \omega^{\lambda_1} \wedge \dots \wedge \omega^{\lambda_q},$$

$$\Theta^*(X_\lambda)\varphi = \sum_{j=1}^q f_{\lambda_1 \dots \lambda_q}^s Y_s \otimes \omega^{\lambda_1} \wedge \dots \wedge \Theta^*(X_\lambda) \omega^{\lambda_j} \wedge \dots \wedge \omega^{\lambda_q},$$

$$\varepsilon(\omega^\lambda)\varphi = f_{\lambda_1 \dots \lambda_q}^s Y_s \otimes \omega^\lambda \wedge \omega^{\lambda_1} \wedge \dots \wedge \omega^{\lambda_q},$$

$$i(X_\lambda)\varphi = f_{\lambda_1 \dots \lambda_q}^s Y_s \otimes i(X_\lambda)(\omega^{\lambda_1} \wedge \dots \wedge \omega^{\lambda_q}),$$

where in the first equation $X_\lambda f_{\lambda_1}^{s_1} \dots f_{\lambda_q}^{s_q}$ denotes the Lie derivative of $f_{\lambda_1}^{s_1} \dots f_{\lambda_q}^{s_q}$ by the right-invariant vector field X_λ , and in the third equation, $\Theta^*(X_\lambda)\omega^\mu$ is defined by $\langle \Theta^*(X_\lambda)\omega^\mu, X_\nu \rangle = -\langle \omega^\mu, [X_\lambda, X_\nu] \rangle$, with $\langle \cdot, \cdot \rangle$ here denoting the dual pairing between \mathfrak{G} and \mathfrak{G}^* , and $[\cdot, \cdot]$ the Lie product on \mathfrak{G} .

$\mathcal{A}_0^q(\Gamma, \mathbf{M}, \rho)$ can now be described as the space of all φ in $\mathcal{A}_0^q(\Gamma, G, \rho)$, such that for all α

$$(4.6) \quad i(X_\alpha)\varphi = \Theta(X_\alpha)\varphi = 0,$$

where

$$\Theta(X_\lambda) = \rho(X_\lambda) + \Theta'(X_\lambda) + \Theta^*(X_\lambda).$$

We can now describe the operators d^0 , δ^0 , and Δ^0 on $\mathcal{A}_0(\Gamma, \mathbf{M}, \rho)$. Namely,

$$(4.7) \quad d^0 = D + d_\rho, \quad \delta^0 = D_* + \delta_\rho,$$

where we set

$$\begin{aligned} d_\rho &= \sum_j \rho(X_j)\varepsilon(\omega^j), & \delta_\rho &= \sum_j \rho(X_j)i(X_j), \\ D &= \sum_j \Theta'(X_j)\varepsilon(\omega^j), & D_* &= -\sum_j \Theta'(X_j)i(X_j). \end{aligned}$$

Of course, we have $\Delta^0 = d^0\delta^0 + \delta^0d^0$. However, it turns out that Δ^0 has the following useful decomposition [10, Theorem 1]:

$$(4.8) \quad \Delta^0 = \Delta_D + \Delta_\rho,$$

where

$$(4.9) \quad \Delta_D = DD_* + D_*D, \quad \Delta_\rho = d_\rho\delta_\rho + \delta_\rho d_\rho.$$

In fact, (4.8) follows from a direct calculation, using (3.6) and (4.7). Another direct computation using (3.6), gives us

$$(4.10) \quad \Delta_\rho = \sum_{i,j} [\rho(X_i), \rho(X_j)]\varepsilon(\omega^i)i(X_j) + \sum_i \rho(X_i)^2.$$

By (4.4'), $\varphi \in \mathcal{A}_0^q(\Gamma, \mathbf{M}, \rho)$ induces a function on G/Γ : $\mathcal{A}_c^q(E_\rho)$ then corresponds to the space $\mathcal{A}_{0c}^q(\Gamma, \mathbf{M}, \rho)$ of all $\varphi \in \mathcal{A}_0^q(\Gamma, \mathbf{M}, \rho)$, such that φ induces a function on G/Γ with compact support. The positive-definite inner product (\cdot, \cdot) (see (2.4) for the definition), which is at least defined on $\mathcal{L}^q \cap \mathcal{A}^q(E_\rho)$, induces a positive-definite inner product (still denoted by (\cdot, \cdot)), which is at least defined on the subspace of $\mathcal{A}_0^q(\Gamma, \mathbf{M}, \rho)$ corresponding to $\mathcal{L}^q \cap \mathcal{A}^q(E_\rho)$.

In order to give an explicit description of this induced inner product, let $dv = \omega^1 \wedge \dots \wedge \omega^n$, where the ω^λ are here considered as one-forms on G/Γ ; also, let

$$\varphi = f_{i_1 \dots i_q}^{s_1 \dots s_q} Y_s \otimes \omega^{i_1} \wedge \dots \wedge \omega^{i_q},$$

$$\psi = g_{i_1 \dots i_q}^{s_1 \dots s_q} Y_s \otimes \omega^{i_1} \wedge \dots \wedge \omega^{i_q},$$

be forms in $\mathcal{A}_0^q(\Gamma, \mathbf{M}, \rho)$ (by virtue of (4.6), no α 's appear in these expressions for φ and ψ); then

$$(\varphi, \psi) = eq! \int_{G/\Gamma} \sum_{i_1 \dots i_q} f_{i_1 \dots i_q}^s g_{i_1 \dots i_q}^s dv,$$

provided the right-hand integral is defined. Here, $e > 0$ is a constant independent of q , φ , and ψ . Also, we note that $\mathcal{L}^q \cap \mathcal{A}^q(E_\rho)$ corresponds to the space of all φ in $\mathcal{A}_0^q(\Gamma, \mathbf{M}, \rho)$, such that the integrals $\int_{G/\Gamma} (f_{i_1 \dots i_q}^s)^2 dv$ are convergent for all s, i_1, \dots, i_q .

For $\varphi \in \mathcal{A}_{0c}^q(\Gamma, \mathbf{M}, \rho)$ and $\psi \in \mathcal{A}_{0c}^{q+1}(\Gamma, \mathbf{M}, \rho)$, we have

$$(4.11) \quad (d^0 \varphi, \psi) = (\varphi, \delta^0 \psi), \quad (d_\rho \varphi, \psi) = (\varphi, \delta_\rho \psi), \quad (D\varphi, \psi) = (\varphi, D_* \psi),$$

and hence, applying (4.9), we have for $\varphi \in \mathcal{A}_{0c}^q(\Gamma, \mathbf{M}, \rho)$

$$(4.12) \quad (\varphi, \Delta^0 \varphi) \geq 0, \quad (\varphi, \Delta_D \varphi) \geq 0, \quad (\varphi, \Delta_\rho \varphi) \geq 0.$$

PROPOSITION 4.13. *If there exists a constant $c > 0$, such that for all*

$$\varphi \in \mathcal{A}_{0c}^q(\Gamma, \mathbf{M}, \rho),$$

we have

$$(4.13) \quad (\Delta_\rho \varphi, \varphi) \geq c(\varphi, \varphi),$$

then E_ρ is W^q -elliptic.

Proof. From (4.8), $(\Delta^0 \varphi, \varphi) = (\Delta_\rho \varphi, \varphi) + (\Delta_D \varphi, \varphi)$. However, (4.12) then implies $(\Delta^0 \varphi, \varphi) \geq (\Delta_\rho \varphi, \varphi)$, and combining this inequality with (4.13), we obtain the desired conclusion. Q.E.D.

We now restrict ourselves to the case when $q=1$. In this case, (4.4') and (4.6) imply that $\varphi \in \mathcal{A}_0^1(\Gamma, \mathbf{M}, \rho)$ may be regarded as a function on G/Γ , with values in $\text{Hom}(\mathfrak{P}, V_\rho)$. More explicitly, if $\varphi = f_k^s Y_s \otimes \omega^k$, then for $\mathbf{p} \in G/\Gamma$, we identify $\varphi(\mathbf{p})$ with the element in $\text{Hom}(\mathfrak{P}, V_\rho)$ defined by

$$(4.14) \quad \varphi(\mathbf{p})(X_i) = f_i^s(\mathbf{p}) Y_s$$

(where, using (4.4'), we identify the f_i^s with functions on G/Γ).

We now define a positive-definite inner product $I(\cdot, \cdot)$ on $\text{Hom}(\mathfrak{P}, V_\rho)$, by setting $I(\zeta, \xi) = \sum_i (\zeta(X_i), \xi(X_i))_\rho$, $\zeta, \xi \in \text{Hom}(\mathfrak{P}, V_\rho)$. Next, we define

$$H: \text{Hom}(\mathfrak{P}, V_\rho) \rightarrow \text{Hom}(\mathfrak{P}, V_\rho)$$

by

$$(4.15) \quad H(\zeta)(Y) = \sum_k \rho(X_k)^2 \zeta(Y) + \sum_k \rho([Y, X_k]) \zeta(X_k), \quad \zeta \in \text{Hom}(\mathfrak{P}, V_\rho), \quad Y \in \mathfrak{P}.$$

Using the fact that $(\cdot, \cdot)_\rho$ is admissible, one can show directly that H is symmetric with respect to $I(\cdot, \cdot)$. Also, a direct computation using (4.10) and (4.14), shows that for $\varphi \in \mathcal{A}_{0c}^1(\Gamma, \mathbf{M}, \rho)$

$$(4.16) \quad (\Delta_\rho \varphi, \varphi) = e \int_{G/\Gamma} I(H\varphi(\mathbf{p}), \varphi(\mathbf{p})) dv.$$

THEOREM 4.17. *Let G be an admissible analytic group, $\Gamma \subset G$ a lattice containing no nontrivial elements of finite order, and $\rho: G \rightarrow \mathrm{GL}(V_\rho)$ a representation of G in the finite-dimensional, real vector space V_ρ . Assume that the transformation $H: \mathrm{Hom}(\mathfrak{P}, V_\rho) \rightarrow \mathrm{Hom}(\mathfrak{P}, V_\rho)$, is positive-definite with respect to $I(\ , \)$. Then, if $\varphi \in \mathcal{A}_0^1(\Gamma, \mathbf{M}, \rho)$ satisfies $(\varphi, \varphi) < \infty$ and $d^0\varphi = 0$, there must then exist*

$$\psi \in \mathcal{A}_0^0(\Gamma, \mathbf{M}, \rho)$$

such that $d^0\psi = \varphi$.

Proof. Proposition 4.13, our assumption that H is positive-definite with respect to $I(\ , \)$, and (4.16) imply that E_ρ is W^1 -elliptic. The assumption that $(\varphi, \varphi) < \infty$ implies that φ corresponds to an element in $\mathcal{L}^1 \cap \mathcal{A}^1(E_\rho)$, under our identification of $\mathcal{A}_0^1(\Gamma, \mathbf{M}, \rho)$ with $\mathcal{A}^1(E_\rho)$. Thus the existence of ψ follows from Theorem 3.22 and W^1 -ellipticity. Q.E.D.

In fact, it is shown in [11], that most often H is indeed positive-definite with respect to I . In particular, we have from [11] that H is positive-definite for $\rho = \mathrm{ad}$. Thus we have

COROLLARY. *Let G and Γ be as in Theorem 4.17; then if $\varphi \in \mathcal{A}_0^1(\Gamma, \mathbf{M}, \mathrm{ad})$ satisfies $(\varphi, \varphi) < \infty$ and $d^0\varphi = 0$, there must exist $\psi \in \mathcal{A}_0^0(\Gamma, \mathbf{M}, \mathrm{ad})$ such that $d^0\psi = \varphi$.*

We now give a brief discussion of the operator d^0 . First, we note that $A^q(\Gamma, G, \rho)$ may be identified with the space of all C^∞ functions φ on G , with values in $V_\rho \otimes \bigwedge^q \mathfrak{G}^*$, and such that

$$(4.18) \quad \varphi(g\gamma) = (\rho(\gamma)^{-1} \otimes \mathrm{Id})\varphi(g),$$

where $g \in G$, $\gamma \in \Gamma$, and Id denotes the identity map on $\bigwedge^q \mathfrak{G}^*$. Using this identification, we define the operators $i(X_\lambda)$, $\Theta'(X_\lambda)$, and $\Theta^*(X_\lambda)$ on $A^q(\Gamma, G, \rho)$, just as we defined them on $A_0^q(\Gamma, G, \rho)$. We let $\tilde{\Theta}(X_\lambda) = \Theta'(X_\lambda) + \Theta^*(X_\lambda)$. Regarding $\varphi \in A^q(\Gamma, G, \rho)$ again as a V_ρ -valued q -form on G , and regarding X_λ as a right-invariant vector field on G , we note that $\tilde{\Theta}(X_\lambda)$ is simply the componentwise Lie derivative of φ , with respect to X_λ . We thus have the Cartan identity

$$(4.19) \quad i(X_\lambda)d + d i(X_\lambda) = \tilde{\Theta}(X_\lambda).$$

Now a direct computation shows that under the correspondence $\varphi \rightarrow \varphi^0$, the operator $\Theta'(X_\lambda)$ on $A^q(\Gamma, G, \rho)$ corresponds to $\rho(X_\lambda) + \Theta'(X_\lambda)$ on $A_0^q(\Gamma, G, \rho)$. From this we obtain

$$(4.20) \quad d^0 = \sum_\lambda \rho(X_\lambda)\varepsilon(\omega^\lambda) + \sum_\lambda \Theta'(X_\lambda)\varepsilon(\omega^\lambda) + (1/2) \sum_\lambda \varepsilon(\omega^\lambda)\Theta^*(X_\lambda),$$

where we recall that d^0 is the coboundary operator on $A_0(\Gamma, G, \rho)$, corresponding to d on $A(\Gamma, G, \rho)$, under the map $\varphi \rightarrow \varphi^0$. Also, we have from (4.19), that on $A_0(\Gamma, G, \rho)$

$$(4.21) \quad i(X_\lambda)d^0 + d^0 i(X_\lambda) = \Theta(X_\lambda).$$

We then have from (4.21)

PROPOSITION 4.22. *If $\varphi \in A_0^3(\Gamma, G, \rho)$ satisfies $i(X_\alpha)\varphi = 0$ for all α , and if $\hat{d}^0\varphi = 0$, then $\Theta(X_\alpha)\varphi = 0$ for all α , and hence $\varphi \in \mathcal{A}_0^3(\Gamma, \mathbf{M}, \rho)$.*

We conclude this section with some formulae for the case $\rho = \text{ad}$. Thus, if $\psi \in A_0^0(\Gamma, G, \text{ad})$, then ψ has an expression $\psi = h^\lambda X_\lambda$, where the h^λ are C^∞ functions on G such that $h^\lambda(g\gamma) = h(g)$, for $g \in G$ and $\gamma \in \Gamma$. We note that ψ may be regarded as a vector field on G . If $\varphi \in A_0^1(\Gamma, G, \text{ad})$, then φ has an expression $\varphi = f_\mu^\lambda X_\lambda \otimes \omega^\mu$, where the f_μ^λ are again Γ -invariant, C^∞ functions on G , and we may regard $\sum_\lambda f_\mu^\lambda X_\lambda$ as a vector field on G . We then have

$$(4.23) \quad \hat{d}^0\psi = [X_\lambda, \psi] \otimes \omega^\lambda,$$

where $[,]$ here denotes the Poisson bracket of vector fields. Also, we have that $\hat{d}^0\varphi = 0$, if and only if we have

$$(4.24) \quad X_\nu f_\mu^\lambda - X_\mu f_\nu^\lambda = f_\nu^\rho c_{\mu\rho}^\lambda + f_\rho^\lambda c_{\nu\mu}^\rho - f_\mu^\rho c_{\nu\rho}^\lambda,$$

where we define the $c_{\lambda\mu}^\rho$ by $[X_\lambda, X_\mu] = c_{\lambda\mu}^\rho X_\rho$.

REMARK. By (4.4) we may regard the elements of $A_0(\Gamma, G, \rho)$, as differential forms on G/Γ . In particular, we may regard the h^λ and f_μ^λ , defined above, as functions on G/Γ , and we may then interpret (4.23) and (4.24) as holding on G/Γ .

5. Proof of the Main Theorem. Let Z_1 denote a central subgroup of G , and let $G' = G/Z_1$. Let $p: G \rightarrow G'$ denote the natural projection, and let $\Gamma' = p(\Gamma)$. It then follows from Borel's density theorem (see [4]) that Γ' is a lattice in G' . Moreover, a deformation $(I, 0, \chi)$ of Γ induces a deformation $(I, 0, \chi')$ of Γ' , with $p \circ \chi_t = \chi'_t \circ p$. In fact, this is a consequence of Borel's density theorem and the arguments in [14, §16]. We note that $(I, 0, \chi)$ is trivial if and only if $(I, 0, \chi')$ is, and it follows that we can assume without loss of generality, that G has finite center, and hence that K is compact.

We let K act on $G \times I$ by

$$(5.1) \quad k(g, t) = (kg, t), \quad k \in K, \quad g \in G, \quad t \in I.$$

Since this action of K obviously commutes with the action of Γ defined by (1.6), we have an induced action of K on \mathfrak{B} , which we will refer to as left translation.

Now assume Y is a C^∞ vector field on \mathfrak{B} such that (1.8) and (1.10) hold. We first note that (1.10) is independent of which basis of right-invariant vector fields X_λ we choose. Thus we assume the X_λ are normalized so that (4.1) holds (where we here use the identification of \mathfrak{G} with right-invariant vector fields). For $k \in K$, we let L_k denote the left translation by k on \mathfrak{B} . We also let L_k denote the induced maps on forms and vector fields on \mathfrak{B} . We then have

LEMMA 5.2. *Let Y be a C^∞ vector field on \mathfrak{B} satisfying (1.8) and (1.10); also, let*

$\tilde{Y} = \int_K L_k \cdot Y \, dk$. We define C^∞ functions \tilde{f}_λ^μ on \mathfrak{B} by $[X_\lambda, \tilde{Y}] = \tilde{f}_\lambda^\mu X_\mu$ and we let $\tilde{f}_\lambda^{t\mu}$ denote the restriction of \tilde{f}_λ^μ to G/Γ_t . We then have

$$(5.3) \quad \int_{G/\Gamma_t} |\tilde{f}_\lambda^{t\mu}|^2 \, dv_t < \infty,$$

for all λ, μ , and for all $t \in I$.

Proof. For $g \in G$, we let $(\text{Ad } g)_\mu^\lambda$ denote the matrix coefficients of $\text{Ad } g$, relative to the basis X_λ . We then have

$$\tilde{f}_\lambda^\mu = \int_K (\text{Ad } k^{-1})_\lambda^\alpha (\text{Ad } k)_\alpha^\mu L_k \cdot (f_\rho^\sigma) \, dk,$$

and (5.3) now follows from a direct calculation, using (1.10) and the compactness of K . Q.E.D.

Using Lemma 5.2, we see that the assumptions of the Main Theorem in §1 imply that there exists a C^∞ vector field Y on \mathfrak{B} , such that $L_k \cdot Y = Y$ for all $k \in K$, and such that (1.8) and (1.10) hold. From now on Y will denote such a vector field. We let $\varphi^t \in A_0^1(\Gamma, G, \text{ad})$ be defined by

$$\varphi^t = f_\mu^{t\lambda} X_\lambda \otimes \omega^\mu.$$

From the Jacobi identity, we have

$$[X_\nu, [X_\mu, Y]] + [Y, [X_\nu, X_\mu]] + [X_\mu, [Y, X_\nu]] = 0,$$

and this implies directly, that the $f_\mu^{t\lambda}$ satisfy (4.24). Hence $d^0 \varphi^t = 0$.

On the other hand $[X_\alpha, Y] = 0$ for all α , since Y is invariant under left translations by elements of K . Thus the $f_\alpha^{t\lambda}$ are zero; that is, $i(X_\alpha)\varphi^t = 0$ for all α . Hence, by Proposition 4.22, $\varphi_t \in \mathcal{A}_0^1(\Gamma, \mathbf{M}, \text{ad})$. However, since φ_t is a cocycle, we have from (1.10) and from the Corollary to Theorem 4.17, that there exists $\psi^t \in A_0^0(\Gamma, \mathbf{M}, \text{ad})$ such that $d^0 \psi^t = \varphi^t$. From this, and (4.23) we have

$$(5.4) \quad [X_\lambda, \psi^t] = f_\lambda^{t\mu} X_\mu,$$

where we can either regard ψ^t as a vector field on G/Γ_t , or as a vector field on G .

For the moment, rather than considering the fibering $\varpi: \mathfrak{B} \rightarrow I$, we consider only the curve of homomorphisms χ_t . Thus for $\gamma \in \Gamma$ and $s \in I$, let $\chi'_s(\gamma)$ denote the tangent vector to the curve $\chi_t(\gamma)$ at $t=s$. We set $\gamma_s = \chi_s(\gamma)$ and let $f_s(\gamma_s) = \chi'_s(\gamma)\gamma_s^{-1}$, the right translate of $\chi'_s(\gamma)$ by γ_s^{-1} ; then $f_s(\gamma_s) \in \mathfrak{G}$, and a direct computation shows

$$(5.5) \quad f_s(\gamma_s \gamma'_s) = f_s(\gamma_s) + \text{Ad } (\gamma_s) f_s(\gamma'_s), \quad \gamma, \gamma' \in \Gamma.$$

Thus $f_s: \Gamma_s \rightarrow \mathfrak{G}$ is an Eilenberg-MacLane one-cocycle with respect to adjoint action. We will now use (5.4) to show

LEMMA 5.6. *Under the hypotheses of the Main Theorem in §1, we have that for*

$s \in I$ there exists $b_s \in \mathfrak{G}$, such that

$$(5.7) \quad f_s(\gamma_s) = \text{Ad } \gamma_s(b_s) - b_s, \quad \gamma_s \in \Gamma_s;$$

that is, f_s is a coboundary.

Proof. We can assume without loss of generality, that $s=0$; then $\Gamma_s = \Gamma$, and $\gamma_s = \gamma$. We let $\pi_0: G \rightarrow G/\Gamma$ denote the natural projection, and for $\varepsilon > 0$, we let I_ε denote the open interval of radius ε about 0. Let A' be an open, relatively compact subset of G/Γ ; we can then find $\varepsilon' > 0$ so that $I_{\varepsilon'} \subset I$, and so that we have a C^∞ diffeomorphism Λ , which maps $A' \times I_{\varepsilon'}$, onto an open subset of \mathfrak{B} , so that the diagram

$$(5.8) \quad \begin{array}{ccc} A' \times I_{\varepsilon'} & \xrightarrow{\Lambda} & \mathfrak{B} \\ \text{pr}_2 \searrow & & \swarrow \varpi \\ & I & \end{array}$$

is commutative, and so that

$$(5.9) \quad \Lambda(a, 0) = a, \quad a \in A'.$$

Now ψ^0 induces a vector field on A' , and hence a vector field (still denoted by ψ^0) on $A' \times I_{\varepsilon'}$. We let $\hat{\psi}^0 = \Lambda_*(\psi^0)$, so $\hat{\psi}^0$ is a vector field on $\Lambda(A' \times I_{\varepsilon'})$. Let $U = \pi^{-1}(\Lambda(A' \times I_{\varepsilon'}))$, and let Ψ denote the lift of $\hat{\psi}^0$ to U (so $\pi_* \Psi = \hat{\psi}^0$). Since (5.8) is commutative, we have that $\hat{\psi}^0$ and Ψ are vertical in the sense that

$$(5.10) \quad \varpi_* \hat{\psi}^0 = 0, \quad (\text{pr}_2)_* \Psi = 0.$$

We let Y' be the lift of Y to $G \times I$, and then define the vector field Y'' on U by $Y'' = Y' - \Psi$. We then have from (1.8) and (5.10), that

$$(5.11) \quad (\text{pr}_2)_* Y'' = (\text{pr}_2)_* Y' = \partial/\partial t.$$

Hence on U we have $Y'' = Z + \partial/\partial t$, where $(\text{pr}_2)_* Z = 0$.

From (5.9) and the definition of Ψ , we have that Ψ restricted to $\pi_0^{-1}(A') \times 0$ is equal to ψ^0 . But then (5.4) implies

$$(5.12) \quad [X_\lambda, Y''] = 0, \quad \text{on } \pi_0^{-1}(A') \times 0.$$

Since Z is vertical, we can define a vector field Z' on $\pi_0^{-1}(A')$ by $Z'(x) = Z(x, 0)$ for $x \in \pi_0^{-1}(A')$. Then (5.12) implies that Z' is left invariant; that is, we have

$$(5.13) \quad Z'(gx) = gZ'(x), \quad g \in G, \quad x, gx \in \pi_0^{-1}(A').$$

Let e denote the identity element of G , and set $e_0 = \pi_0(e)$. Now assume that A' was chosen so as to contain e_0 ; then $\gamma \in \pi_0^{-1}(A')$ for all $\gamma \in \Gamma$. In order to prove the lemma, it will suffice to show

$$(5.14) \quad f_0(\gamma) = \text{Ad } \gamma(Z'(e)) - Z'(e), \quad \gamma \in \Gamma,$$

for we can then set $b_0 = Z'(e)$.

To prove (5.14), we first choose an open, relatively compact subset A of A' , such that $e_0 \in A \subset \bar{A} \subset A'$. The vector field $Y - \hat{\psi}^0$ is defined on $\Lambda(A' \times I_{\varepsilon'})$, and we will now consider the integral curves of this vector field. Thus, if we choose a sufficiently small positive number ε , we can assume that $0 < \varepsilon < \varepsilon'$, and that we have a C^∞ map

$$\kappa: A \times I_\varepsilon \rightarrow \Lambda(A' \times I_{\varepsilon'}),$$

so that

$$(5.15) \quad \kappa'_s(x) = (Y - \hat{\psi}^0)(\kappa_s(x)), \quad x \in A, \quad s \in I_\varepsilon,$$

where we set $\kappa_s(x) = \kappa(x, s)$ and let $\kappa'_s(x)$ denote the tangent to the curve $\kappa_t(x)$ at $t = s$. We let $\tilde{\kappa}: \pi_0^{-1}(A) \times I_\varepsilon \rightarrow \mathfrak{B}$ be the composition of κ and of $\pi_0 \times \text{Id}$, where Id denotes the identity on I_ε .

By the covering homotopy theorem, there exists a map

$$K: \pi_0^{-1}(A) \times I_\varepsilon \rightarrow G \times I,$$

such that $K(x, 0) = (x, 0)$ for $x \in \pi_0^{-1}(A)$, and such that

$$(5.16) \quad \pi(K(x, t)) = \kappa(\pi_0(x), t), \quad x \in \pi_0^{-1}(A), \quad t \in I_\varepsilon.$$

We observe that

$$(5.17) \quad \text{pr}_2(K_t(x)) = t, \quad x \in \pi_0^{-1}(A), \quad t \in I_\varepsilon,$$

where $K_t(x) = K(x, t)$. In fact, (5.17) follows from (5.11) and from

$$(5.18) \quad K'_s(x) = Y''(K_s(x)), \quad x \in \pi_0^{-1}(A), \quad s \in I_\varepsilon,$$

where $K'_s(x)$ is the tangent to the curve $K_t(x)$ at $t = s$. On the other hand, (5.18) follows from (5.15).

From (5.17) we have $\pi(K_t(e)\gamma) = \pi(K_t(e))$, where we use the right action of Γ on $G \times I$ defined in (1.6). Then applying (5.16) to $\pi(K_t(e))$ and to $\pi(K_t(\gamma))$ and using the above equation, we get $\pi(K_t(e)\gamma) = \pi(K_t(\gamma))$. But since $K_0(e)\gamma = K_0(\gamma) = \gamma$, we have by the unique lifting theorem

$$(5.19) \quad K_t(\gamma) = K_t(e)\gamma, \quad \gamma \in \Gamma, \quad t \in I_\varepsilon.$$

Using (5.17), we define the one-parameter family of maps $K_t^\#: \pi_0^{-1}(A) \rightarrow G$ by $K_t(x) = (K_t^\#(x), t)$. From (5.19) we then have

$$(5.20) \quad K_t^\#(\gamma) = K_t^\#(e)\chi_t(\gamma), \quad \gamma \in \Gamma, \quad t \in I_\varepsilon.$$

From (5.18) we have that $Z'(x)$ is the tangent vector to $K_t^\#(x)$ at $t = 0$. Hence, differentiating both sides of (5.20) at $t = 0$, we get $Z'(\gamma) = Z'(e)\gamma + \chi'_0(\gamma)$. But (5.14), and hence the lemma, follows from this last equation by applying (5.13) to $Z'(\gamma)$, and right-translating both sides of the equation by γ^{-1} . Q.E.D.

LEMMA 5.21. *Given $s \in I$, we can find $\gamma^1, \dots, \gamma^\ell \in \Gamma$, and real numbers a_1, \dots, a_ℓ , so*

that the linear transformation $\sum_{i=1}^{\ell} a_i(\text{Id} - \text{Ad } \gamma_s^i)$ is nonsingular (here Id denotes the identity on \mathfrak{G}).

Proof. Given real numbers a_1, \dots, a_{ℓ} , we define a polynomial function on $\text{GL}(\mathfrak{G}) \times \dots \times \text{GL}(\mathfrak{G})$ (ℓ times) by

$$P(T_1, \dots, T_{\ell}) = \det \left(\sum_{i=1}^{\ell} a_i (\text{Id} - T_i) \right), \quad T_1, \dots, T_{\ell} \in \text{GL}(\mathfrak{G}).$$

Now G/Γ_s has finite invariant volume, so by Borel's density theorem (see [4]), the Zariski closure of $\text{Ad } \Gamma_s$ contains $\text{Ad } G$. Hence, if $P=0$ on $\text{Ad } \Gamma_s \times \dots \times \text{Ad } \Gamma_s$ (ℓ times), $P=0$ on $\text{Ad } G \times \dots \times \text{Ad } G$ (ℓ times). Hence, the lemma will follow from

$$(5.22) \quad \begin{array}{l} \text{There are elements } g^1, \dots, g^{\ell} \text{ in } G \text{ and real numbers} \\ a_1, \dots, a_{\ell}, \text{ such that } \sum_{i=1}^{\ell} a_i (\text{Id} - \text{Ad } g^i) \text{ is nonsingular.} \end{array}$$

However, to prove (5.22) it suffices to assume G is simple. In that case the commuting algebra of $\text{Ad } G$ is either \mathbf{R} or \mathbf{C} . If $\dim G$ is even, then it follows that $\mathcal{E}(\text{Ad } G)$, the enveloping algebra of $\text{Ad } G$, must contain an element with no real eigenvalues. Hence there exist real numbers a_i , and elements g^i in G , such that $\sum a_i \text{Ad } g^i$ is nonsingular, and has no real eigenvalues. But then $\sum a_i (\text{Id} - \text{Ad } g^i)$ is nonsingular.

If G is simple and $\dim G$ is odd (say $\dim G = 2k + 1$), then the commuting algebra of $\text{Ad } G$ is \mathbf{R} , and $\mathcal{E}(\text{Ad } G)$ is the full endomorphism algebra of \mathfrak{G} . We fix a basis of \mathfrak{G} , and we let

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T_1 = \begin{pmatrix} T_1 & & 0 \\ & S & \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ 0 & & S \end{pmatrix}, \quad \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \left. \vphantom{\begin{matrix} S \\ \cdot \\ \cdot \\ \cdot \end{matrix}} \right\} k-1 \text{ times}$$

$$T_2 = \begin{pmatrix} T_2 & & 0 \\ & S & \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ 0 & & S \end{pmatrix}, \quad \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \left. \vphantom{\begin{matrix} S \\ \cdot \\ \cdot \\ \cdot \end{matrix}} \right\} k-1 \text{ times}$$

We then choose real numbers $b_1, \dots, b_q, c_1, \dots, c_r$, and $g^1, \dots, g^q, h^1, \dots, h^r$, in G , so that

$$\sum_{i=1}^q b_i \operatorname{Ad} g^i = T_1, \quad \sum_{j=1}^r c_j \operatorname{Ad} h^j = T_2.$$

If $\sum b_i \neq 0$ (resp. if $\sum c_j \neq 0$), then $\sum b_i (\operatorname{Id} - \operatorname{Ad} g^i)$ (resp. $\sum c_j (\operatorname{Id} - \operatorname{Ad} h^j)$) is nonsingular. Hence we may suppose $\sum b_i = \sum c_j = 0$. But then $\sum b_i (\operatorname{Id} - \operatorname{Ad} g^i) + \sum c_j (\operatorname{Id} - \operatorname{Ad} h^j) = \sum b_i \operatorname{Ad} g^i + \sum c_j \operatorname{Ad} h^j = T_1 + T_2$, is nonsingular, so (5.22) (and hence the lemma) follows. Q.E.D.

LEMMA 5.23. *The b_t ($t \in I$) in (5.7) are C^∞ functions in t .*

Proof. Fix $s \in I$, and then choose γ^i and a_i ($i=1, \dots, \ell$) as in Lemma 5.21. Then $\sum_{i=1}^\ell a_i (\operatorname{Id} - \operatorname{Ad} \gamma^i)$ is nonsingular for t in a sufficiently small interval I' containing s . But from (5.7), we have for $t \in I$

$$\sum_{i=1}^\ell a_i f_i(\gamma^i) = \sum_{i=1}^\ell a_i (\operatorname{Ad} \gamma^i - \operatorname{Id}) b_t,$$

so that for $t \in I'$, we have

$$b_t = \left(\sum_{i=1}^\ell a_i (\operatorname{Ad} \gamma^i - \operatorname{Id}) \right)^{-1} \sum_{i=1}^\ell a_i f_i(\gamma^i).$$

But on I' , the right-hand side of the above equation is a C^∞ function of t . Hence b_t is a C^∞ function of t , on I' . Since s was chosen arbitrarily, b_t is a C^∞ function of t , on I . Q.E.D.

The lemma in [8, p. 69] implies that there exists a C^1 curve c_t in G ($t \in I$) such that

$$(5.24) \quad c'_s c_s^{-1} = -b_s, \quad s \in I,$$

where c'_s is the tangent to c_t at $t=s$, and $c'_s c_s^{-1}$ is the right translate of c'_s by c_s^{-1} ; moreover, we can choose this curve c_t so that $c_0 = e$. An inspection of the proof of the Lemma in [8] mentioned above, shows that c_t is in fact a C^∞ curve. A direct computation, using (5.7) and (5.24), shows that the tangent vector to the curve $c_t^{-1} \chi_t(\gamma) c_t$, $\gamma \in \Gamma$, is zero for all $t \in I$. Therefore

$$\chi_t(\gamma) = c_t \gamma c_t^{-1}, \quad \gamma \in \Gamma, \quad t \in I;$$

that is, $(I, 0, \chi)$ is a trivial deformation. The proof of the Main Theorem is now complete.

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