# A STUDY OF METRIC-DEPENDENT DIMENSION FUNCTIONS([1])

BY

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1. **Introduction.** This paper is a study of metric-dependent dimension functions for metric spaces. Let $X$ be a metric space with metric $\rho$. We introduced in a previous paper [15] two dimension functions $d_1$ and $d_2$ of $(X, \rho)$ which by definition appear to depend on $\rho$. We showed however, that $d_1(X, \rho) = \dim X$ (the covering dimension of $X$). On the other hand $d_2$ does depend on the particular metric $\rho$, and there exists $(X, \rho)$ with $d_2(X, \rho) < \dim X$.

Definition 1. The empty set $\varnothing$ has $d_2 \varnothing = -1$. $d_2(X, \rho) \leqq n$ if $(X, \rho)$ satisfies the condition:

($D_2$) For any $n+1$ pairs of closed sets $C_1, C_1'; \dots; C_{n+1}, C_{n+1}'$ with $\rho(C_i, C_i') > 0$, $i = 1, \dots, n+1$, there exist closed sets $B_i$, $i = 1, \dots, n+1$, such that (i) $B_i$ separates $C_i$ and $C_i'$ for each $i$ and (ii) $\bigcap_{i=1}^{n+1} B_i = \varnothing$.

If $d_2(X, \rho) \leqq n$ and the statement $d_2(X, \rho) \leqq n-1$ is false, we set $d_2(X, \rho) = n$.

This definition stems from Eilenberg-Otto's characterization of dimension [3]:

A metric space $X$ has $\dim X \leqq n$ if and only if the following condition is satisfied:

($D_2'$) For any $n+1$ pairs of closed sets $C_1, C_1'; \dots; C_{n+1}, C_{n+1}'$ with $C_i \cap C_i' = \varnothing$, $i = 1, \dots, n+1$, there exist closed sets $B_i$, $i = 1, \dots, n+1$, such that (i) $B_i$ separates $C_i$ and $C_i'$ for each $i$ and (ii) $\bigcap_{i=1}^{n+1} B_i = \varnothing$.

This characterization of (covering) dimension is still true even when $X$ is only a normal space (cf. Hemmingsen [5, Theorem 6.1] or Morita [10, Theorem 3.1]). All spaces considered in this paper are $T_1$. To clarify the situation of $d_2$ we introduce the following two apparently metric-dependent dimension functions which are similar to $d_2$.

Definition 2. First we set $d_3 \varnothing = -1$. $d_3(X, \rho) \leqq n$ if $(X, \rho)$ satisfies the condition:

($D_3$) For any finite number $m$ of pairs of closed sets $C_1, C_1'; \dots; C_m, C_m'$ with $\rho(C_i, C_i') > 0$, $i = 1, \dots, m$, there exist closed sets $B_i$, $i = 1, \dots, m$, such that (i) $B_i$ separates $C_i$ and $C_i'$ for each $i$ and (ii) the order of $\{B_i : i = 1, \dots, m\}$, ord $\{B_i\}$, is at most $n$.

If $d_3(X, \rho) \leqq n$ and the statement $d_3(X, \rho) \leqq n-1$ is false, then we set $d_3(X, \rho) = n$.

Definition 3. First we set $d_4 \varnothing = -1$. $d_4(X, \rho) \leqq n$ if $(X, \rho)$ satisfies the condition:

($D_4$) For any countable number of pairs of closed sets $C_1, C_1'; C_2, C_2'; \dots$ with $\rho(C_i, C_i') > 0$, $i = 1, 2, \dots$, there exist closed sets $B_i$, $i = 1, 2, \dots$, such that (i) $B_i$ separates $C_i$ and $C_i'$ for each $i$ and (ii) ord $\{B_i : i = 1, 2, \cdots\} \leqq n$.

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If $d_4(X, \rho) \leqq n$ and the statement $d_4(X, \rho) \leqq n-1$ is false, then we set $d_4(X, \rho)=n$.

Let $(D_3')$ (respectively $(D_4')$) be the condition which is obtained from $(D_3)$ (resp. from $(D_4)$) when "$\rho(C_i, C_i')>0$" is replaced by "$C_i \cap C_i' = \varnothing$". It is evident that $(D_4')$ implies $(D_3')$, say $(D_4') \to (D_3')$, and $(D_3') \to (D_2')$. It is also true that $(D_4) \to (D_3) \to (D_2)$. Morita [10] proved that $(D_2') \to (D_3') \to (D_4')$ even when $X$ is only a normal space. Then it is natural to ask whether or not $(D_2) \to (D_3) \to (D_4)$. The answer is no for each implication. It will be shown that $d_4(X, \rho) = \dim X$ for any $(X, \rho)$ (Theorem 2 below). Moreover we shall construct in this paper a space $(R, \rho)$ such that $d_2(R, \rho)=2$, $d_3(R, \rho)=3$ and $d_4(R, \rho)=4$. It is to be noticed that $(R, \rho)$ is topologically complete and $\rho$ is totally bounded. Our dimension functions are closely related to so-called *metric dimension* which is defined as follows:

Definition 4. First we set $\mu \dim \varnothing = -1$. $\mu \dim (X, \rho) \leqq n$ if $(X, \rho)$ satisfies the condition:

$(D_0)$ There exists a sequence of open coverings $\mathcal{U}_i$ of $X$ such that (i) ord $\mathcal{U}_i \leqq n+1$ for each $i$ and (ii) mesh $\mathcal{U}_i$ $(=\sup\{\rho(U) : U \in \mathcal{U}_i\})$ converges to zero.

If $\mu \dim (X, \rho) \leqq n$ and the statement $\mu \dim (X, \rho) \leqq n-1$ is false, then we set $\mu \dim (X, \rho) = n$.

Here we note that whether $\mu \dim (X, \rho) = \dim X$ or not had been a serious problem in dimension theory and that the gap between $\mu \dim$ and dim played an important role when the study of dimension theory moved to general metric spaces from separable metric spaces (cf. Sitnikov [19], Nagata [17], [18], Nagami [13], Vopěnka [22], Dowker-Hurewicz [2] and Katětov [8]).

We prove that $d_3(X, \rho) \leqq \mu \dim (X, \rho)$ for any $(X, \rho)$ and that $d_3(X, \rho) = \mu \dim (X, \rho)$ when $\rho$ is totally bounded (Theorems 4 and 5 below). Thus the space $(R, \rho)$ mentioned before offers an example such that $d_2(R, \rho) < \mu \dim (R, \rho) < \dim R$. Sitnikov [19] was the first to construct a space $(Y, \rho)$ such that $\mu \dim (Y, \rho) < \dim Y$.

In every Cantor $n$-manifold $(K_n, \rho)$, $n \geqq 3$, we shall construct subspaces $(X_n, \rho)$ and $(Y_n, \rho)$ such that

(i) $\dim X_n = \dim Y_n \geqq n-1$ and

(ii) $d_2(X_n, \rho) = \mu \dim (Y_n, \rho) = [n/2]$.

To prove $\dim X_n$ or $\dim Y_n \geqq n-1$ we need the following theorem (Theorem 1 below) which is interesting in itself:

If $A_i$, $i=1, 2, \ldots$, are disjoint closed sets of $K_n$ with $\dim A_i \leqq n-1$ for every $i$, then $\dim (K_n - \bigcup A_i) \geqq n-1$.

Sitnikov [20] proved that $\dim (K_n - \bigcup A_i) \geqq n-1$ if $K_n = I^n$ ($n$-cube) without the condition $\dim A_i \leqq n-1$ and with $A_i \neq I^n$ for $i=1, 2, \ldots$. Then it is natural to ask whether our present theorem for $K_n$ is still true without any hypothesis on $\dim A_i$, and with $A_i \neq K_n$ for $i=1, 2, \ldots$. We give a negative answer for this question. (See Figure 2.)

We give for each $n \geqq 2$ a metric space $Z_n$ which allows equivalent metrics $\rho_m$, $m = [(n+1)/2], [(n+1)/2]+1, \ldots, n$, such that $d_2(Z_n, \rho_m) = \mu \dim (Z_n, \rho_m) = m$. This space not only illustrates the dependence of $\mu$ dim and $d_2$ on the metric but plays a role in the construction of our final example $R$ which is mentioned above.

The final section lists four unsolved problems.

## 2. Dimension of the complement of a disjoint collection of sets.

LEMMA 1. *Let $X$ be a hereditarily normal space and $Y$ a subset of $X$ with* $\dim (X-Y)<n$. *Then for any n pairs of disjoint closed sets of $X$, $C_1, C_1'; \ldots; C_n, C_n'$, there exist closed sets of $X$, $B_1, \ldots, B_n$, such that $\bigcap B_i \subset Y$ and $B_i$ separates $C_i$ and $C_i'$ for each $i$.*

**Proof.** Let $D_1, D_1'; \ldots; D_n, D_n'$ be open sets of $X$ such that $C_i \subset D_i$, $C_i' \subset D_i'$ and $\bar{D}_i \cap \bar{D}_i' = \varnothing$ for each $i$. By Hemmingsen [5, Theorem 6.1] or Morita [10, Theorem 3.1] there exist relatively open sets $U_1, \ldots, U_n$ of $X-Y$ such that

(i) $\bar{D}_i - Y \subset U_i \subset \bar{U}_i - Y \subset (X-\bar{D}_i') - Y$, $i=1, \ldots, n$,

(ii) $\bigcap (\bar{U}_i - U_i) \subset Y$.

If we set $G_i = C_i \cup U_i$ and $H_i = C_i' \cup ((X-Y)-\bar{U}_i)$, then $\bar{G}_i \cap H_i = G_i \cap \bar{H}_i = \varnothing$. By the hereditary normality of $X$ there exists an open set $V_i$ of $X$ such that $G_i \subset V_i \subset \bar{V}_i \subset X - H_i$. Set $B_i = \bar{V}_i - V_i$. Then $B_i$, $i=1, \ldots, n$, satisfy the required condition.

LEMMA 2. *Let $X$ be a compact Hausdorff space and let $H$ and $K$ be disjoint closed sets of $X$ such that no connected set meets both $H$ and $K$. Then there exist disjoint open sets $H_1$ and $K_1$ such that $H \subset H_1$, $K \subset K_1$ and $H_1 \cup K_1 = X$.*

This can be proved by a method analogous to the one in Moore [9, Theorem 44, p. 15] with the consideration of Hocking-Young [6, Theorem 2-9, p. 44].

LEMMA 3. *A connected compact Hausdorff space cannot be decomposed into a countably infinite or finite* (*but more than one*) *union of disjoint closed subsets.*

This can be proved by a method analogous to the one in Moore [9, Theorem 56, p. 23] with the aid of Lemma 2.

DEFINITION 5. Let $X$ be a normal space. A system of pairs $C_1, C_1'; \ldots; C_n, C_n'$ is called a *defining* system of $X$ if (i) $C_i$ and $C_i'$ are disjoint closed sets of $X$ for each $i$ and (ii) for arbitrary closed sets $B_i$, $i=1, \ldots, n$, separating $C_i$ and $C_i'$ we have $\bigcap B_i \neq \varnothing$.

LEMMA 4. *Let $X$ be a compact Hausdorff space, $F$ a closed set of $X$ and $f$ a mapping* (*continuous transformation*) *of $F$ into the* $(n-1)$*-sphere $S^{n-1}$. Consider $S^{n-1}$ as the surface of the n-cube $I^n = \{(x_1, \ldots, x_n) : -1 \leqq x_i \leqq 1\}$. Let $C_1, C_1'; \ldots; C_n, C_n'$ be n pairs of opposite faces of $I^n$ defined by*:

$$C_i = \{(x_1, \ldots, x_n) : x_i = -1\}, \qquad C_i' = \{(x_1, \ldots, x_n) : x_i = 1\},$$

$i=1, \ldots, n$. *If the system $f^{-1}(C_1), f^{-1}(C_1'); \ldots; f^{-1}(C_n), f^{-1}(C_n')$ is not defining, then $f$ has an extension $f^*: X \to S^{n-1}$.*

**Proof.** Let $B_1, \ldots, B_n$ be closed sets of $X$ such that $B_i$ separates $f^{-1}(C_i)$ and $f^{-1}(C_i')$ for every $i$ and such that $\bigcap B_i = \varnothing$. By Morita [10, Lemma 1.2] we can assume that every $B_i$ is a $G_\delta$. Let $f(x) = (f_1(x), \ldots, f_n(x))$, where each $f_i$ is a mapping into $[-1, 1]$. Let $g_i: X \to [-1, 1]$ be an extension of $f_i | f^{-1}(C_i) \cup f^{-1}(C_i')$ such that $g_i(x) = 0$ if and only if $x \in B_i$ and such that $|g_i(x)| = 1$ if and only if $x \in f^{-1}(C_i) \cup f^{-1}(C_i')$. Let $g(x) = (g_1(x), \ldots, g_n(x))$. Then $g$ is a mapping of $X$ into $I^n$ and $g(F) \subset S^{n-1}$. If $x \in F$, then $f(x)$ and $g(x)$ cannot be a pair of opposite points on $S^{n-1}$. Hence $f$ is homotopic to $g|F$. Let $p$ be the original point $(0, \ldots, 0)$ of $I^n$. Then $p \notin g(X)$. Let $r: I^n - p \to S^{n-1}$ be a retraction. Then $rg$ maps $X$ into $S^{n-1}$. By the same argument as in Hurewicz-Wallman [7, Chapter VI], $f$ has an extension $f^*$ over $X$ whose values are still in $S^{n-1}$.

THEOREM 1. *Let $X$ be a compact hereditarily normal space with* $\dim X = n$, $n \geqq 1$, *and $A_1, A_2, \cdots$ be a sequence of disjoint closed sets of $X$ such that* $\dim A_i \leqq n-1$ *for each $i$. Then*

$$\dim (X - \bigcup A_i) \geqq n-1.$$

**Proof.** *First step.* Since $\dim X = n$ there exist by Morita [10, Theorem 5.1] a closed set $F$ of $X$, a mapping $f$ of $F$ into $S^{n-1}$ and a closed set $Y \supset F$ such that (i) $f$ cannot be extended over $Y$ and (ii) if $Z$ is any proper closed subset of $Y$ with $F \subset Z$, then $f$ is extendable over $Z$. We may even assume that $Y$ is actually a Cantor $n$-manifold (cf. Hurewicz-Wallman [7, pp. 99–100]). Since $\dim (Y - \bigcup A_i) \geqq n-1$ implies $\dim (X - \bigcup A_i) \geqq n-1$, we assume hereafter $X$ is $Y$ itself with the above minimal property.

*Second step.* Since $\dim A_1 \leqq n-1$, there exists a mapping $f': F \cup A_1 \to S^{n-1}$ with $f'|F = f$. Since $S^{n-1}$ is a neighborhood extensor for normal spaces by Hanner [4, Theorem 13.2], there exist an open set $U_1$ with $U_1 \supset F \cup A_1$ and a mapping $f_1: \overline{U}_1 \to S^{n-1}$ with $f_1 | F \cup A_1 = f'$. Continuing such procedure, we have a sequence of open sets $U_1, U_2, \cdots$ and a sequence of mappings $f_i: \overline{U}_i \to S^{n-1}$ such that (i) $F \cup (\bigcup_{j \leqq i} A_j) \subset U_i$ and $\overline{U}_i \subset U_{i+1}$, for every $i$ and (ii) $f_i$ is an extension of $f_{i-1}$ for every $i$, where $f_0 = f$.

Define $g: \bigcup U_i \to S^{n-1}$ in such a way that $g | \overline{U}_i = f_i$ for each $i$. Then $g$ is an extension of $f$ over $\bigcup U_i$. Let $\varphi: X \to [0, 1]$ be a mapping such that

(i) $\varphi(x) = 0$ if and only if $x \notin \bigcup U_i$,

(ii) $\varphi(x) = 1$ if $x \in F$,

(iii) $\overline{U}_i \subset \{x : \varphi(x) > 2^{-i}\} \subset U_{i+1}$ for every $i$.

Consider $S^{n-1}$ as the surface of the solid $n$-ball $I^n$ of radius 1 whose center is the origin $p$. We define $h: X \to I^n$ as follows:

(i) $h(x) = p$ if $x \notin \bigcup U_i$,

(ii) $h(x) = \varphi(x) g(x)$ if $x \in \bigcup U_i$, where $g(x)$ is considered as a vector from $p$ to $g(x)$.

Then $h$ is continuous and $h|F = f$. Moreover $h^{-1}(p) \cap (\bigcup A_i) = \varnothing$, which will be a meaningful fact later.

*Third step.* Here we reconsider that $I^n$ is the $n$-cube expressed as

$$\{(x_1, \ldots, x_n) : -1 \leqq x_i \leqq 1, i = 1, \ldots, n\}$$

whose surface is $S^{n-1}$ and whose origin $(0, \ldots, 0)$ is $p$. Consider the solid pyramid $P$ in $I^n$ whose base is $B = \{(x_1, \ldots, x_n) : x_n = -1\}$ and whose apex is $p$. The $n-1$ pairs of opposite sides of $P$ may be denoted by $(S_i, T_i)$, $i = 1, \ldots, n-1$, where $S_i$ is spanned by

$$S_i' = \{(x_1, \ldots, x_n) : x_i = x_n = -1\}$$

and $p$, and $T_i$ is spanned by

$$T_i' = \{(x_1, \ldots, x_n) : x_i = 1, x_n = -1\}$$

and $p$. Then

$$C_i = h^{-1}(S_i) - h^{-1}(p), \quad C_i' = h^{-1}(T_i) - h^{-1}(p), \qquad i = 1, \ldots, n-1,$$

are $n-1$ pairs of disjoint closed sets of $X' = X - h^{-1}(p)$. Figure 1 will help us to treat the situation.

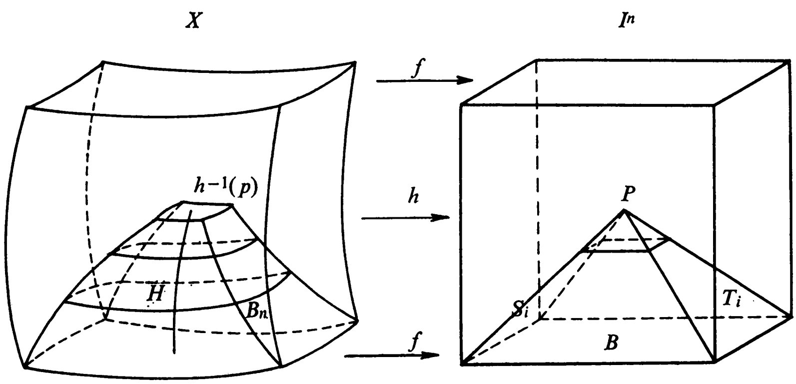


FIGURE 1

*Fourth step.* Assume that $\dim (X - \bigcup A_i) < n-1$. Then

$$\dim (X - ((\bigcup A_i) \cup h^{-1}(p))) < n-1,$$

since $h^{-1}(p)$ is a $G_\delta$. By Lemma 1 there exist closed sets $B_i$ of $X - h^{-1}(p)$, $i = 1, \ldots, n-1$, such that (i) $B_i$ separates $C_i$ and $C_i'$ for each $i$ and (ii)

$$\left(\bigcap_{i=1}^{n-1} B_i\right) \cap (X - \bigcup A_i) = \varnothing.$$

The latter condition implies $H = \bigcap_{i=1}^{n-1} B_i \subset \bigcup A_i$. Let us consider the compact set $H \cup h^{-1}(p)$ and the two disjoint subsets $H \cap h^{-1}(B)$ and $h^{-1}(p)$. Suppose that $H \cap h^{-1}(B) \neq \varnothing$ and there exists a connected closed set $K \subset H \cup h^{-1}(p)$ such that

$K \cap H \cap h^{-1}(B) \neq \varnothing$ and $h^{-1}(p) \cap K \neq \varnothing$. Then for some $i$, $K \cap A_i \neq \varnothing$. Since $K \subset h^{-1}(p) \cup A_1 \cup A_2 \cup \cdots$, we have a contradiction by Lemma 3.

*Fifth step.* By Lemma 2 we can now conclude that there exist disjoint compact sets $H_1$ and $H_2$ such that (i) $H_1 \cup H_2 = h^{-1}(B) \cup H \cup h^{-1}(p)$, (ii) $h^{-1}(p) \subset H_1$ and (iii) $h^{-1}(B) \subset H_2$, whether $H \cap h^{-1}(B) = \varnothing$ or not. Hence there exists a closed set $B_n$ of $X$ separating $h^{-1}(p)$ and $h^{-1}(B)$ without touching $H$. Let $c$ be a number with $0 < c < 1$, $Q_c$ the intersection of $P$ and the hyperplane $\{(x_1, \ldots, x_n) : x_n = -c\}$, $P_c$ the intersection of $P$ and $\{(x_1, \ldots, x_n) : x_n \leqq -c\}$ and $R_c$ the surface of $P_c$. Then there exists a number $b$ with $0 < b < 1$ such that

$$h^{-1}(\overline{P - P_b}) \cap B_n = \varnothing.$$

If we confine our attention to the set $h^{-1}(P_b)$, there are closed sets

$$B_1 \cap h^{-1}(P_b), \ldots, B_n \cap h^{-1}(P_b)$$

which separate pairs

$$h^{-1}(S_1 \cap P_b),\ h^{-1}(T_1 \cap P_b); \ldots; h^{-1}(S_{n-1} \cap P_b),\ h^{-1}(T_{n-1} \cap P_b);$$
$$h^{-1}(B),\ h^{-1}(Q_b),$$

respectively. Denote this system of pairs by $\alpha$. Since

$$\bigcap_{i=1}^{n} (B_i \cap h^{-1}(P_b)) \subset \bigcap_{i=1}^{n} B_i = \varnothing,$$

$\alpha$ is not defining. Then by Lemma 4 there exists a mapping $k_1 : h^{-1}(P_b) \to R_b$ such that $k_1 | h^{-1}(R_b) = h | h^{-1}(R_b)$. Let $k: X \to I^n$ be a mapping such that

(i) $k | X - h^{-1}(P_b) = h | X - h^{-1}(P_b)$,

(ii) $k | h^{-1}(P_b) = k_1$.

Let $s$ be an inner point of $P_b$ and $r$ a retraction of $I^n - \{s\}$ onto $S^{n-1}$. Then $rk: X \to S^{n-1}$ is an extension of $f$, a contradiction. Thus we have $\dim (X - \bigcup A_i) \geqq n-1$ and the proof is completed.

Corollary 1 (Sitnikov [20]). *Let $n \geqq 1$. Let $A_i$, $i = 1, 2, \ldots$, be a disjoint sequence of closed sets of $I^n$ at least two of which are not empty. Then*

$$\dim (I^n - \bigcup A_i) \geqq n-1.$$

**Proof.** Let $S^{n-1}$ be the surface of $I^n = \{(x_1, \ldots, x_n) : -1 \leqq x_i \leqq 1, i = 1, \ldots, n\}$. Let $f: S^{n-1} \to S^{n-1}$ be the identity mapping. Since it is impossible that $I^n - S^{n-1}$ is contained in one $A_i$, we have one of the following two cases:

(i) There exists $i$ such that $(I^n - S^{n-1}) - A_i \neq \varnothing$ and $A_j \subset S^{n-1}$ for any $j \neq i$.

(ii) There exist $i$ and $j$ with $i \neq j$ such that

$$(I^n - S^{n-1}) \cap A_i \neq \varnothing \quad \text{and} \quad (I^n - S^{n-1}) \cap A_j \neq \varnothing.$$

The first case yields $\dim(I^n - \bigcup A_i) = n$. If the second case happens, then there exists a number $\varepsilon$ with $0 < \varepsilon < 1$ such that

$$I^n_\varepsilon = \{(x_1, \ldots, x_n) : |x_i| \leqq \varepsilon,\ i = 1, \ldots, n\}$$

meets $A_i$ and $A_j$. Then by Lemma 3 there exists a point $q$ in $I^n_\varepsilon - \bigcup A_i$. Then we can apply the same argument on $f$ and $q$ as in the proof of Theorem 1 and we get $\dim(I^n - \bigcup A_i) \geqq n-1$.

COROLLARY 2. *Let $X$ be a connected metric space such that every point has a neighborhood homeomorphic to $I^n$, $n \geqq 1$. Let $A_i$, $i = 1, 2, \ldots$, be a disjoint sequence of closed sets of $X$ at least two of which are not empty. Then*

$$\dim(X - \bigcup A_i) \geqq n-1.$$

**Proof.** Consider a closed covering $\{F_\alpha\}$ of $X$ such that each $F_\alpha$ is homeomorphic to $I^n$. If each $F_\alpha$ is contained in some $A_i$, then each $A_i$ has to be open, which contradicts the fact that $X$ is connected. Hence (i) there exists $F_\alpha$ which meets at least two of the $A_i$'s, (ii) there exists $F_\beta$ such that $F_\beta \cap A_i \neq \varnothing$, $F_\beta - A_i \neq \varnothing$ and $F_\beta \cap A_j = \varnothing$ for $j \neq i$, or (iii) there exists $F_\gamma$ such that $F_\gamma \cap A_i = \varnothing$ for every $i$. The first case yields $\dim(X - \bigcup A_i) \geqq \dim(F_\alpha - \bigcup A_i) \geqq n-1$. The second case yields $\dim(X - \bigcup A_i) \geqq \dim(F_\beta - A_i) \geqq n-1$. The third case yields $\dim(X - \bigcup A_i) \geqq \dim F_\gamma \geqq n-1$.

Figure 2 gives a Cantor 2-manifold $X$ such that a proposition for $X$ analogous to Corollary 1 fails. In fact $\dim X = 2$, yet $\dim(X - \bigcup A_{ij}) = 0$ since $X - \bigcup A_{ij}$ is a subset of the Cantor discontinuum.

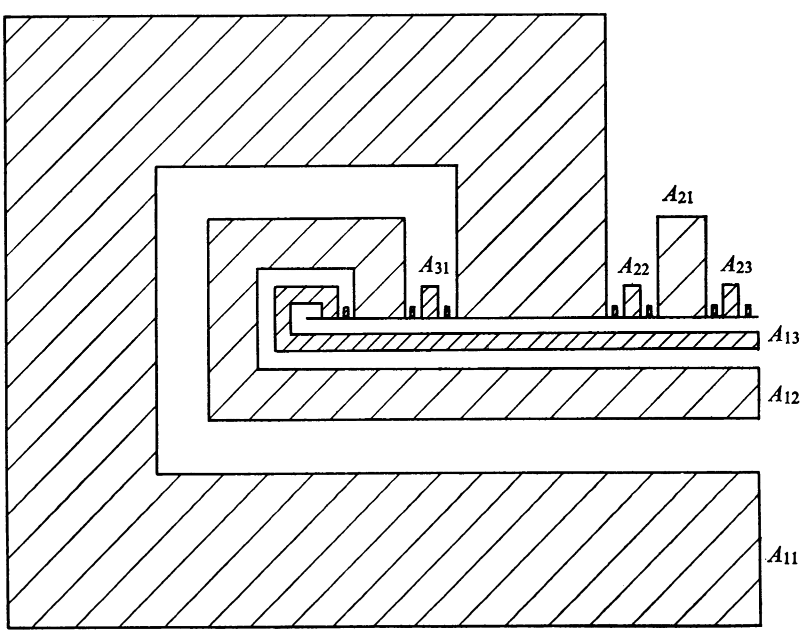


FIGURE 2

## 3. Relations among the various functions.

LEMMA 5 (MORITA [10, THEOREM 3.4]). *Let $X$ be a normal space with* $\dim X \leqq n$. *Then $X$ satisfies the condition* ($D_4'$).

LEMMA 6. *If a metric space $X$ has a σ-locally finite open base $\mathcal{U}$ such that* $\operatorname{ord}\{\bar{U}-U : U \in \mathcal{U}\} \leqq n$, *then* $\dim X \leqq n$. *($\mathcal{U}$ is called σ-locally finite if $\mathcal{U}$ can be decomposed into a countable number of locally finite subcollections.)*

This is a slight modification of Morita [12, Theorem 8.7].

THEOREM 2. $\dim X = d_4(X, \rho)$ *for any* $(X, \rho)$.

**Proof.** By Lemma 5 $\dim X \geqq d_4(X, \rho)$. To prove $\dim X \leqq d_4(X, \rho)$ assume $d_4(X, \rho) \leqq n$. Let us show the existence of such $\mathcal{U}$ as in Lemma 6. By Stone [21] there exists an open base $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ of $X$, $\mathcal{V}_i = \{V_\alpha : \alpha \in \Lambda_i\}$, $i = 1, 2, \ldots$, such that each $\mathcal{V}_i$ is discrete. $\mathcal{V}_i$ is called discrete if $\bar{\mathcal{V}}_i = \{\bar{V}_\alpha : \alpha \in \Lambda_i\}$ is a locally finite disjoint collection. Set $V_i = \bigcup \{V_\alpha : \alpha \in \Lambda_i\}$, $i = 1, 2, \ldots$, and

$$F_{ij} = \{x : \rho(x, X - V_i) \geqq 1/j\}, \qquad j = 1, 2, \ldots.$$

Then by $d_4(X, \rho) \leqq n$ there exist open sets $U_{ij}$, $i, j = 1, 2, \ldots$, such that

(i) $V_i \supset \bar{U}_{ij} \supset U_{ij} \supset F_{ij}$ for each $i$ and $j$ and

(ii) $\operatorname{ord}\{\bar{U}_{ij} - U_{ij} : i, j = 1, 2, \cdots\} \leqq n$.

Set

$$\mathcal{U}_{ij} = \{V_\alpha \cap U_{ij} : \alpha \in \Lambda_i\}.$$

Then $\mathcal{U}_{ij}$ is discrete and hence locally finite. $\mathcal{U} = \bigcup_{i,j=1}^{\infty} \mathcal{U}_{ij}$ is an open base for $X$ such that $\operatorname{ord}\{\bar{U} - U : U \in \mathcal{U}\} \leqq n$. Hence by Lemma 6 we have $\dim X \leqq n$ and the theorem is proved.

LEMMA 7. *If $\{F_\alpha\}$ is a locally finite closed collection of a paracompact Hausdorff space, then there exists an open collection $\{G_\alpha\}$ such that* (i) $G_\alpha \supset F_\alpha$ *for each α and* (ii) $\operatorname{ord}\{F_\alpha\} = \operatorname{ord}\{G_\alpha\}$.

This can easily be seen with the aid of Morita [11, Theorem 1.3].

THEOREM 3 (KATĚTOV [8]). $\dim X \leqq 2\mu \dim(X, \rho)$ *for any* $(X, \rho)$.

**Proof.** Suppose $\mu \dim(X, \rho) \leqq n$. Let $\mathcal{U}_i = \{U_\alpha : \alpha \in \Lambda_i\}$, $i = 1, 2, \ldots$, be a sequence of open coverings of $X$ such that

(i) mesh $\mathcal{U}_i < 2^{-i}$ for each $i$ and

(ii) $\operatorname{ord} \mathcal{U}_i \leqq n+1$ for each $i$.

By an easy observation we can assume that each $\mathcal{U}_i$ is locally finite. Let $\mathcal{G} = \{G_1, \ldots, G_m\}$ be an arbitrary finite open covering of $X$. Set

$$D_i = \bigcup \{\{x : \rho(x, X - G_j) > 2^{-i+1}\} : j = 1, \ldots, m\}, \qquad i = 1, 2, \ldots.$$

Then (i) $D_1 \subset \bar{D}_1 \subset D_2 \subset \bar{D}_2 \subset D_3 \cdots$, (ii) each $D_i$ is open and (iii) $\bigcup_{i=1}^{\infty} D_i = X$. Let

$\mathscr{F}_i=\{F_\alpha : \alpha \in \Lambda_i\}$ be a closed covering of $X$ such that $F_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda_i$. Set

$$\Lambda_i' = \{\alpha : F_\alpha \cap \bar{D}_i = \varnothing\},$$
$$W_i = X - \bigcup \{F_\alpha : \alpha \in \Lambda_i'\}.$$

Then $W_i$ is an open set with $\bar{D}_i \subset W_i \subset \overline{W}_i \subset D_{i+1}$ for every $i$. Since every point of $\overline{W}_i - W_i$ is contained in some $F_\alpha \in \mathscr{F}_i$ with $\alpha \in \Lambda_i'$, ord $\{F_\alpha \cap (\overline{W}_i - W_i) : \alpha \notin \Lambda_i'\} \leqq n$. (This type of argument comes from Morita [10, Theorem 3.3].) Since $\{F_\alpha : \alpha \in \Lambda_i - \Lambda_i'\}$ covers $\overline{W}_i - W_i$, there exists by Lemma 7 an open collection $\mathscr{V}_i=\{V_\alpha : \alpha \in \Lambda_i - \Lambda_i'\}$ of $X$ such that

(i) $F_\alpha \cap (\overline{W}_i - W_i) \subset V_\alpha \subset (D_{i+1} - \bar{D}_i) \cap U_\alpha$ for each $\alpha \in \Lambda_i - \Lambda_i'$ and

(ii) ord $\{V_\alpha : \alpha \in \Lambda_i - \Lambda_i'\} \leqq n$.

Then $\mathscr{V}_i$ refines $\mathscr{G}$. Moreover $V \in \mathscr{V}_i$, $V' \in \mathscr{V}_j$, $i \neq j$, imply $V \cap V' = \varnothing$. We set

$$\mathscr{H}_i = \{U_\alpha \cap (W_i - \overline{W}_{i-1}) : \alpha \in \Lambda_i\}, \quad \text{where } W_{-1} = \varnothing.$$

Then $H \in \mathscr{H}_i$, $H' \in \mathscr{H}_j$, $i \neq j$, imply $H \cap H' = \varnothing$. If we set

$$\mathscr{H} = \left(\bigcup_{i=1}^{\infty} \mathscr{H}_i\right) \cup \left(\bigcup_{i=1}^{\infty} \mathscr{V}_i\right),$$

then it is easy to see that $\mathscr{H}$ is an open covering of $X$ such that (i) $\mathscr{H}$ refines $\mathscr{G}$ and (ii) ord $\mathscr{H} \leqq 2n+1$. Thus we have dim $X \leqq 2n$ and the theorem is proved.

THEOREM 4. *$d_3(X, \rho) \leqq \mu$ dim $(X, \rho)$ for any $(X, \rho)$.*

**Proof.** Suppose $\mu$ dim $(X, \rho) \leqq n$. Let $C_1, C_1'; \cdots; C_m, C_m'$ be a finite number of pairs of closed sets of $X$ such that there exists a positive number $\varepsilon$ such that $\rho(C_i, C_i') > \varepsilon$ for each $i$. By $\mu$ dim $(X, \rho) \leqq n$ there exists a locally finite open covering $\mathscr{U}=\{U_\alpha : \alpha \in \Lambda\}$ of $X$ such that (i) ord $\mathscr{U} \leqq n+1$ and (ii) mesh $\mathscr{U} < \varepsilon/2$. Let $\mathscr{F}=\{F_\alpha : \alpha \in \Lambda\}$ be a locally finite closed covering of $X$ such that $F_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda$. Set

$$\Lambda' = \{\alpha : F_\alpha \cap C_1 = \varnothing\},$$
$$W = X - \bigcup \{F_\alpha : \alpha \in \Lambda'\},$$
$$B_1 = \overline{W} - W.$$

Then $B_1$ separates $C_1$ and $C_1'$ and

$$\mathscr{U}_1 = \{U_\alpha : \alpha \in \Lambda - \Lambda'\} \cup \{U_\alpha - B_1 : \alpha \in \Lambda'\}$$

is an open covering of $X$ such that

(i) $\mathscr{U}_1$ refines $\mathscr{U}$,

(ii) $\mathscr{U} | X - B_1 = \mathscr{U}_1 | X - B_1$,

(iii) ord $(x, \mathscr{U}_1) \leqq$ ord $(x, \mathscr{U}) - 1$ for each $x \in B_1$, where ord $(x, \mathscr{U}_1)$ is the order of $\mathscr{U}_1$ at $x$.

This can be verified by the same argument as in the proof of the previous theorem. Continuing this procedure, we get closed sets $B_i$, $i=2,\dots,m$, separating $C_i$ and $C_i'$ respectively and open coverings $\mathscr{U}_2,\dots,\mathscr{U}_m$ such that

(i) $\mathscr{U}_{i+1}$ refines $\mathscr{U}_i$ for $i=1,\dots,m-1$,

(ii) $\mathscr{U}_i|X-B^i=\mathscr{U}_{i-1}|X-B_i$ for each $i$,

(iii) $\operatorname{ord}(x,\mathscr{U}_i)\leqq\operatorname{ord}(x,\mathscr{U}_{i-1})-1$ for each $x\in B_i$, $i=2,\dots,m$.

If $x\in B_{i_1}\cap\cdots\cap B_{i_{n+1}}$, $i_1<i_2<\cdots<i_{n+1}$, then

$$\begin{aligned}1 &\leqq \operatorname{ord}(x,\mathscr{U}_{i_{n+1}}) \leqq \operatorname{ord}(x,\mathscr{U}_{i_{n+1}-1})-1\\ &\leqq \operatorname{ord}(x,\mathscr{U}_{i_n})-1 \leqq \operatorname{ord}(x,\mathscr{U}_{i_n-1})-2\\ &\leqq\cdots\leqq \operatorname{ord}(x,\mathscr{U}_{i_1})-n \leqq \operatorname{ord}(x,\mathscr{U})-(n+1).\end{aligned}$$

Thus we have $\operatorname{ord}(x,\mathscr{U})\geqq n+2$, a contradiction. We have therefore

$$\operatorname{ord}\{B_i : i=1,\dots,m\}\leqq n,$$

and the theorem is proved.

THEOREM 5. *$d_3(X,\rho)=\mu\dim(X,\rho)$ for a totally bounded metric space $(X,\rho)$.*

**Proof.** Suppose that $d_3(X,\rho)\leqq n$. For an arbitrary positive number $\varepsilon$ there exists a finite set of points $x_1,\dots,x_m$ such that

$$\{U_i = S_\varepsilon(x_i) : i=1,\dots,m\}$$

covers $X$. Set

$$V_i = S_{2\varepsilon}(x_i), \qquad i=1,\dots,m.$$

Then there exist open sets $W_1,\dots,W_m$ such that

(i) $\bar{U}_i\subset W_i\subset\bar{W}_i\subset V_i$ for each $i$,

(ii) $B_i=\bar{W}_i-W_i$ separates $X-V_i$ and $\bar{U}_i$ for each $i$,

(iii) $\operatorname{ord}\{B_i : i=1,\dots,m\}\leqq n$.

By Lemma 7 there exist open sets $G_1,\dots,G_m$ of $X$ such that (i) $B_i\subset G_i\subset V_i$ for each $i$ and (ii) $\operatorname{ord}\{G_i\}\leqq n$. Set

$$\mathscr{W}_i = \{W_{i1}=W_i,\ W_{i2}=X-\bar{W}_i\},$$

$$\mathscr{W} = \bigwedge_{i=1}^m \mathscr{W}_i = \{W_{1i_1}\cap\cdots\cap W_{mi_m} : i_1,\dots,i_m = 1,2\}.$$

Since

$$\bigcap_{i=1}^m W_{i2} = \bigcap_{i=1}^m (X-\bar{W}_i) = X-\bigcup_{i=1}^m \bar{W}_i = X-X=\varnothing,$$

$\mathscr{W}$ refines $\{W_1,\dots,W_m\}$ and hence $\mathscr{W}$ refines $\{V_1,\dots,V_m\}$. Moreover

$$\bigcup\{W : W\in\mathscr{W}\} = X-\bigcup_{i=1}^m B_i$$

and $\operatorname{ord}\mathscr{W}\leqq 1$. If we set

$$\mathscr{G} = \mathscr{W}\cup\{G_1,\dots,G_m\},$$

then $\mathscr{G}$ is an open covering of $X$ such that (i) mesh $\mathscr{G} \leqq 4\varepsilon$ and (ii) ord $\mathscr{G} \leqq n+1$. Thus we have $\mu$ dim $(X, \rho) \leqq n$. If we combine $d_3(X, \rho) \geqq \mu$ dim $(X, \rho)$ just proved with Theorem 4, we get $d_3(X, \rho) = \mu$ dim $(X, \rho)$ and the theorem is proved.

Since $d_2(X, \rho) \leqq d_3(X, \rho) \leqq d_4(X, \rho)$ are trivially true, we have now $d_2 \leqq d_3 \leqq \mu$ dim $\leqq d_4 =$ dim $\leqq 2\mu$ dim, symbolically.

Remark 1. When $(X, \rho)$ is locally compact, all of these dimension functions coincide with each other, of course. On the other hand there exists a space $(X, \rho)$ which is not locally compact at any point yet $d_2(X, \rho) = d_3(X, \rho) = \mu$ dim $(X, \rho)$ $=$ dim $X$. Let us consider $I^3 = \{(x_1, x_2, x_3) : 0 \leqq x_i \leqq 1, i = 1, 2, 3\}$. Let $\rho$ be a metric of $I^3$. Set $B = \{(x_1, x_2, x_3) : x_1 = 0\}$. Let $C$ be the set of all points in $I^3$ whose coordinates are all rational. If we set $X = B \cup C$, then $(X, \rho)$ satisfies the condition as follows: $2 = d_2(B, \rho) \leqq d_2(X, \rho) \leqq d_3(X, \rho) \leqq \mu$ dim $(X, \rho) \leqq$ dim $X = 2$.

Remark 2. It is to be noticed that even if $(X, \rho)$ is any metric space, (i) dim $X = 1$ implies $d_2(X, \rho) = 1$ and (ii) $d_2(X, \rho) = 0$ implies dim $X = 0$ (see Nagami-Roberts [15]).

## 4. Spaces $(X_n, \rho)$ with $d_2 = [n/2]$ and dim $\geqq n-1$.

Lemma 8. *Let $F_1, F_2, \ldots$ be a sequence of closed sets of a metric space $X$ with* dim $F_i = n_i$. *Let $C_1, C_1'; \ldots; C_m, C_m'$ be $m$ disjoint pairs of closed sets of $X$. Then there exist closed sets $B_1, \ldots, B_m$ such that*

(i) *for each $i$ $B_i$ separates $C_i$ and $C_i'$,*

(ii) *for each $j$ and for each sequence* $1 \leqq i_1 < i_2 < \cdots < i_t \leqq \min\{n+1, m\}$,

$$\dim (B_{i_1} \cap \cdots \cap B_{i_t} \cap F_j) \leqq n_j - t.$$

See Morita [12, Theorem 9.1].

Construction of $(X_n, \rho)$. Let $(K_n, \rho)$ be a Cantor $n$-manifold with $n \geqq 3$. Put $m = [n/2]+1$. By compactness of $K_n$ there exists a sequence of $m$ disjoint pairs of closed sets of $K_n$, say $C_{11}, C_{11}'; \ldots; C_{1m}, C_{1m}' : C_{21}, C_{21}'; \ldots; C_{2m}, C_{2m}' : \ldots$, such that for any $m$ disjoint pairs of closed sets $C_1, C_1'; \ldots; C_m, C_m'$ there exists $i$ with $C_j \subset C_{ij}$ and $C_j' \subset C_{ij}'$ for $j = 1, \ldots, m$. By Lemma 8 there exist closed sets $B_{11}, \ldots, B_{1m}$ such that

(i) for each $i$ $B_{1i}$ separates $C_{1i}$ and $C_{1i}'$,

(ii) dim $B_1 \leqq n-m$ where $B_1 = \bigcap_{j=1}^m B_{1j}$.

By repeated application of Lemma 8 there exist closed sets $B_{21}, \ldots, B_{2m}$ such that

(i) for each $i$ $B_{2i}$ separates $C_{2i}$ and $C_{2i}'$,

(ii) dim $B_2 \leqq n-m$ where $B_2 = \bigcap_{j=1}^m B_{2j}$,

(iii) $B_1 \cap B_2 = \varnothing$.

Continuing such process we get finally a sequence of closed sets $B_{ij}$, $i = 1, 2, \ldots$, $j = 1, \ldots, m$, which have the following property:

(i) $B_{ij}$ separates $C_{ij}$ and $C_{ij}'$ for each $i$ and $j$.

(ii) dim $B_i \leqq n-m$ for $i = 1, 2, \ldots$, where $B_i = \bigcap_{j=1}^m B_{ij}$.

(iii) $B_i \cap B_j = \varnothing$ if $i \neq j$.

If we set

$$X_n = K_n - \bigcup_{i=1}^{\infty} B_i,$$

then we have the space $(X_n, \rho)$.

ASSERTION 1. $d_2(X_n, \rho) \leqq [n/2]$.

**Proof.** Let $C_1, C_1'; \dots; C_m, C_m'$ be $m$ pairs of closed sets of $X_n$ such that $\rho(C_i, C_i') > 0$ for $i = 1, \dots, m$. Since their closures $\bar{C}_1, \bar{C}_1'; \dots; \bar{C}_m, \bar{C}_m'$ in $K_n$ constitute $m$ disjoint pairs of closed sets of $K_n$, there exists $i$ such that $\bar{C}_j \subset C_{ij}$ and $\bar{C}_j' \subset C_{ij}'$ for $j = 1, \dots, m$. Then $B_{i1} \cap X_n, \dots, B_{im} \cap X_n$ are closed sets of $X_n$ such that $B_{ij} \cap X_n$ separates $C_j$ and $C_j'$ for $j = 1, \dots, m$.

$$\bigcap_{j=1}^{m} (B_{ij} \cap X_n) = B_i \cap X_n = \varnothing,$$

and hence we have $d_2(X_n, \rho) \leqq m - 1 = [n/2]$.

ASSERTION 2. $d_2(X_n, \rho) \geqq [n/2]$.

**Proof.** If $G$ is a nonempty open set of $K_n$, then $\dim G = n$. Since

$$\dim \left(\bigcup B_i\right) \leqq n - m < n, \qquad G - \left(\bigcup B_i\right) \neq \varnothing$$

and hence $G \cap X_n \neq \varnothing$. Thus $X_n$ is dense in $K_n$. Assume $d_2(X_n, \rho) = t < [n/2]$. Take a defining system $\bar{D}_1, \bar{D}_1'; \dots; \bar{D}_{t+1}, \bar{D}_{t+1}'$ of $K_n$ such that

(i) each $D_i$ and $D_i'$ are open in $K_n$,

(ii) for any closed sets $A_i$, $i = 1, \dots, t+1$, separating $\bar{D}_i$ and $\bar{D}_i'$,

$$\dim \left(\bigcap_{i=1}^{t+1} A_i\right) \geqq n - (t+1).$$

Set $C_i = \bar{D}_i \cap X_n$ and $C_i' = \bar{D}_i' \cap X_n$. Then it is easy to see that $\bar{C}_i = \bar{D}_i$ and $\bar{C}_i' = \bar{D}_i'$, since $X_n$ is dense in $K_n$. Set

$$\varepsilon = \min\{\rho(C_i, C_i') : i = 1, \dots, t+1\}.$$

Take open sets $U_i$, $i = 1, \dots, t+1$, of $K_n$ such that

$$\text{(i)} \quad \begin{aligned} \{x : x \in X_n, \rho(x, C_i) < \varepsilon/4\} &\subset U_i \cap X_n \subset \bar{U}_i \cap X_n \\ &\subset X_n - \{x : x \in X_n, \rho(x, C_i') < \varepsilon/4\} \quad \text{for each } i, \end{aligned}$$

$$\text{(ii)} \quad \bigcap_{i=1}^{t+1} ((\bar{U}_i - U_i) \cap X_n) = \varnothing.$$

It is easy to see that $\bar{U}_i - U_i$ thus chosen separates $\bar{C}_i = \bar{D}_i$ and $\bar{C}_i' = \bar{D}_i'$ for each $i$. Hence

$$\dim B \geqq n - (t+1) \geqq n - [n/2] \quad \text{where} \quad B = \bigcap_{i=1}^{t+1} (\bar{U}_i - U_i).$$

On the other hand

$$\dim B \leqq \dim (\bigcup B_i) \leqq n-m = n-[n/2]-1$$

because $B \cap X_n = \varnothing$, which is a contradiction.

ASSERTION 3. $\dim X_n \geqq n-1$.

**Proof.** Since $\dim B_i \leqq n-m = n-[n/2]-1 \leqq n-1$, we have $\dim X_n \geqq n-1$ at once by Theorem 1.

5. **Spaces $(Y_n, \rho)$ with $\mu$ dim $=[n/2]$ and dim $\geqq n-1$.**

LEMMA 9. *$(X, \rho)$ has $\mu \dim (X, \rho) \leqq n$ if and only if there exists a sequence of locally finite closed coverings $\mathscr{F}_i$, $i=1, 2, \ldots$, such that*

(i) $\operatorname{mesh} \mathscr{F}_i < 1/i$ *for any $i$,*

(ii) $\operatorname{ord} \mathscr{F}_i \leqq n+1$ *for any $i$.*

This is verified at once by Lemma 7.

LEMMA 10. *Let $X$ be a metric space with* $\dim X \leqq n$ *and $B_1, B_2, \ldots$ a sequence of closed sets of $X$ with* $\dim B_i = n_i$, *where $B_1 = X$. Let $\varepsilon$ be an arbitrary positive number. Then there exists a locally finite closed covering $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ which satisfies the following conditions*:

(i) $\operatorname{mesh} \mathscr{F} < \varepsilon$.

(ii) *For any $i$* $\operatorname{ord} \mathscr{F} | B_i \leqq n_i + 1$.

(iii) *For any $i$, any $j \leqq n_i + 2$ and any $j$ different indices $\alpha(1), \ldots, \alpha(j)$ of $\Lambda$,*

$$\dim \bigcap_{k=1}^{j} (F_{\alpha(k)} \cap B_i) \leqq n_i - j + 1.$$

This is proved essentially in Nagami [13, Theorem 3.6].

CONSTRUCTION OF $(Y_n, \rho)$. Let $(K_n, \rho)$ be a Cantor $n$-manifold, $n \geqq 3$. Set $m = [n/2] + 2$. By Lemma 10 there exists a locally finite closed covering $\mathscr{F}_1 = \{F_\alpha : \alpha \in \Lambda_1\}$ of $K_n$ such that (i) $\operatorname{mesh} \mathscr{F}_1 < 1$, (ii) $\operatorname{ord} \mathscr{F}_1 \leqq n+1$, and (iii) $\dim B_1 \leqq n-m+1$ where $B_1 = \{x : \operatorname{ord}(x, \mathscr{F}_1) \geqq m\}$ which is closed by the local finiteness of $\mathscr{F}_1$. Then $\operatorname{ord} \mathscr{F}_1 | K_n - B_1 < m$.

By Lemma 10 again there exists a locally finite closed covering

$$\mathscr{F}_2 = \{F_\alpha : \alpha \in \Lambda_2\}$$

of $K_n$ such that (i) $\operatorname{mesh} \mathscr{F}_2 < 1/2$, (ii) $\operatorname{ord} \mathscr{F}_2 \leqq n+1$, (iii) $\dim B_2 \leqq n-m+1$ where $B_2 = \{x : \operatorname{ord}(x, \mathscr{F}_2) \geqq m\}$, and (iv) $\dim \bigcap_{k=1}^{j} (F_{\alpha(k)} \cap B_1) \leqq \dim B_1 - j + 1$ for any $j \leqq \dim B_1 + 2$ and any $j$ different indices $\alpha(1), \ldots, \alpha(j)$ of $\Lambda_2$. To show that the last condition (iv) implies $B_1 \cap B_2 = \varnothing$, set $\dim B_1 = n_1$. Take $n_1 + 2$ different indices $\alpha(1), \ldots, \alpha(n_1+2)$ of $\Lambda_2$. Then

$$\dim \bigcap_{k=1}^{n_1+2} (F_{\alpha(k)} \cap B_1) \leqq n_1 - (n_1+2) + 1 = -1.$$

Hence we have

$$B_1 \cap \{x : \operatorname{ord}(x, \mathscr{F}_2) \geqq n_1 + 2\} = \varnothing.$$

Since

$$\begin{aligned} n_1+2 \leqq (n-m+1)+2 &\leqq n-([n/2]+2)+3 \\ &= n-[n/2]+1 \leqq (2[n/2]+1)-[n/2]+1 \\ &= [n/2]+2 = m, \end{aligned}$$

we have $B_1 \cap B_2 = \varnothing$.

Repeating such procedure we have a sequence of locally finite closed coverings $\mathscr{F}_i$, $i=1, 2, \ldots$, which satisfy the following conditions:

(i) For each $i$, mesh $\mathscr{F}_i < 1/i$.

(ii) For each $i$, $\dim B_i \leqq n-m+1$ where $B_i$ is a closed set defined by

$$B_i = \{x : \operatorname{ord}(x, \mathscr{F}_i) \geqq m\}.$$

(iii) $B_i$, $i=1, 2, \ldots$, are mutually disjoint.

We set $Y_n = K_n - \bigcup B_i$. Then $(Y_n, \rho)$ is the desired space.

ASSERTION 1. $\dim Y_n \geqq n-1$.

**Proof.** Since

$$\dim B_i \leqq n-m+1 = n-[n/2]-1 \leqq n-1,$$

the assertion is true by Theorem 1.

ASSERTION 2. $\mu \dim (Y_n, \rho) \leqq [n/2]$.

**Proof.** Since $\operatorname{ord} \mathscr{F}_i | Y_n \leqq \operatorname{ord} \mathscr{F}_i | K_n - B_i \leqq m-1 = ([n/2]+2)-1 = [n/2]+1$, the assertion is true by Lemma 9.

ASSERTION 3. $\dim Y_n \leqq n-1$ when $n$ is odd.

**Proof.** Since $\dim Y_n \leqq 2\mu \dim (Y_n, \rho)$ by Theorem 3, we have

$$\dim Y_n \leqq 2[n/2] = 2((n-1)/2) = n-1.$$

ASSERTION 4. $\mu \dim (Y_n, \rho) \geqq [n/2]$.

**Proof.** Assume the contrary. Then

$$\dim Y_n \leqq 2\mu \dim (Y_n, \rho) \leqq 2([n/2]-1) \leqq n-2,$$

a contradiction.

Thus $(Y_n, \rho)$ satisfies (i) $\dim Y_n \geqq n-1$ and (ii) $\mu \dim (Y_n, \rho) = [n/2]$. Furthermore when $n$ is odd, $\dim Y_n = n-1$.

REMARK 3. It is to be noted that for $X_n$ and $Y_n$ obtained by replacing $K_n$ with $I^n$, $\dim X_n = \dim Y_n = n-1$ for any $n$, because of the fact that $I^n - X_n$ and $I^n - Y_n$ are dense in $I^n$, and the invariance theorem of domain.

REMARK 4. Note that the existence of a sequence of open coverings $\mathscr{U}_i$, $i=1, 2, \ldots$, with $\operatorname{ord} \mathscr{U}_i \leqq n+1$ and $\lim \operatorname{mesh} \mathscr{U}_i = 0$ does not characterize dimension. Thus it is natural to seek an additional condition upon $\mathscr{U}_i$ with which the existence of the sequence does characterize dimension. Dowker-Hurewicz [2], Nagata [17] and Nagami [14] considered such a condition. This type of characterization theorem is one of the main foundations on which modern dimension theory has been built up. Vopěnka [22] gave a simple condition: "$\mathscr{U}_{i+1} <$ (refines) $\mathscr{U}_i$ for each $i$". Recently Nagami-Roberts [16, Theorem 3] refined Vopěnka's theorem, weakening the mesh condition. But our proof contains an error. The definition of $V_\alpha$ in [16, line 15, p. 157] is not adequate. Let us take this opportunity to give a correct proof as follows:

THEOREM 6. *A metric space $X$ has* $\dim X \leqq n$ *if there exists a sequence $\mathscr{U}_1 > \mathscr{U}_2 > \cdots$ of open coverings $\mathscr{U}_i$ of $X$ such that*

(i) *for each $x \in X$, $\{\mathrm{St}(x, \mathscr{U}_i^\Delta) : i = 1, 2, \ldots\}$ is a local base of $x$,*

(ii) $\mathrm{ord}\, \mathscr{U}_i \leqq n+1$.

**Proof.** Set

$$\mathscr{U}_i = \{U(\alpha_i) : \alpha_i \in A_i\}, \qquad i = 1, 2, \ldots.$$

Let $f_i^{i+1} : A_{i+1} \to A_i$ be a function such that $f_i^{i+1}(\alpha_{i+1}) = \alpha_i$ yields $U(\alpha_{i+1}) \subset U(\alpha_i)$. For each pair $i < j$ let $f_i^j = f_i^{i+1} \cdots f_{j-1}^j$ and $f_i^i$ be the identity mapping. Let $\mathscr{G}$ be an arbitrary finite open covering of $X$. Set

$$X_i = \bigcup \{U(\alpha_i) : \mathrm{St}(U(\alpha_i), \mathscr{U}_i) \text{ refines } \mathscr{G}\}.$$

Then by the condition (i) $\{X_1, X_2, \ldots\}$ is an open covering of $X$. Set $X_0 = \varnothing$. Set

$$B_i = \{\alpha_i : U(\alpha_i) \cap X_i \neq \varnothing\},$$

$$C_i = \left\{\alpha_i : \alpha_i \in B_i,\ U(\alpha_i) \cap \left(\bigcup_{j<i} X_j\right) = \varnothing\right\},$$

$$D_i = \left\{\alpha_i : \alpha_i \in B_i,\ U(\alpha_i) \cap \left(\bigcup_{j<i} X_j\right) \neq \varnothing\right\}.$$

Then $B_i \subset A_i$, $B_1 = C_1$, $B_i = C_i \cup D_i$ and $C_i \cap D_i = \varnothing$.

For every $i < j$ and every $\alpha_i \in C_i$ set

$$D_j(\alpha_i) = \left(\bigcap_{k=i+1}^{j} (f_k^j)^{-1}(D_k)\right) \cap (f_i^j)^{-1}(\alpha_i).$$

Then

(i) $f_k^j(D_j(\alpha_i)) \subset D_k(\alpha_i)$, $i < k \leqq j$,

(ii) $D_j = \bigcup \{D_j(\alpha_i) : \alpha_i \in C_i, i < j\}$.

For every $\alpha_i \in C_i$ let

$$V(\alpha_i) = (U(\alpha_i) \cap X_i) \cup \left(\bigcup \{U(\alpha_j) \cap X_j : \alpha_j \in D_j(\alpha_i), i < j\}\right).$$

Let us show that

$$\mathscr{V} = \{V(\alpha_i) : \alpha_i \in C_i, i = 1, 2, \ldots\}$$

is an open covering of $X$ such that $\mathscr{V}$ refines $\mathscr{G}$ and $\mathrm{ord}\, \mathscr{V} \leqq n+1$, which will prove $\dim X \leqq n$.

Let $x$ be an arbitrary point of $X$. Since $X_0 = \varnothing$, there exists $i$ with $x \in X_i - \bigcup_{j<i} X_j$. Take $\alpha_i \in B_i$ with $x \in U(\alpha_i)$. When $\alpha_i \in C_i$, $x \in U(\alpha_i) \cap X_i \subset V(\alpha_i)$. When $\alpha_i \in D_i$, there exist $j < i$ and $\alpha_j \in C_j$ such that $\alpha_i \in D_i(\alpha_j)$. Then $x \in U(\alpha_i) \cap X_i \subset V(\alpha_j)$. Thus $\mathscr{V}$ is an open covering of $X$.

Let $i$ be an arbitrary positive integer and $\alpha_i$ an arbitrary index in $C_i$. Since $\varnothing \neq U(\alpha_i) \cap X_i \subset V(\alpha_i) \subset U(\alpha_i)$, there exists $\beta_i \in A_i$ such that $U(\beta_i) \cap U(\alpha_i) \cap X_i \neq \varnothing$ and St $(U(\beta_i), \mathscr{U}_i)$ refines $\mathscr{G}$. Thus $V(\alpha_i)$ refines $\mathscr{G}$ and hence $\mathscr{V}$ refines $\mathscr{G}$.

To prove ord $\mathscr{V} \leqq n+1$ assume the contrary. Then there exist a point $x$ and $n+2$ indices $\alpha^1, \ldots, \alpha^{n+2}$ such that

(i) $\alpha^i \in C_{m_i}$, $i=1, \ldots, n+2$,

(ii) $x \in V(\alpha^i)$, $i=1, \ldots, n+2$.

Let $k$ be the smallest integer such that $x \in X_k - \bigcup_{j<k} X_j$. Every $m_i$ is less than or equal to $k$. For every $\alpha^i$ there exist $j(i)$ with $j(i) \geqq k$ and $\beta^i \in D_{j(i)}(\alpha^i)$ such that $x \in U(\beta^i)$. Set $\gamma^i = f_k^{j(i)}(\beta^i)$. Then (i) $x \in U(\gamma^i)$, (ii) $\gamma^i \in D_k(\alpha^i)$ if $m_i < k$ and (iii) $\gamma^i = \alpha^i$ if $m_i = k$. Since $\gamma^i$, $i=1, \ldots, n+2$, are all different from one another by our construction, ord $(x, \mathscr{U}_k) \geqq n+2$, a contradiction. Hence ord $\mathscr{V} \leqq n+1$ and the proof is finished.

## 6. Spaces $(Z_n, \sigma_i)$ illustrating the dependence of $\mu$ dim and $d_2$ on the metric.

LEMMA 11. *If $(X, \rho)$ is a metric space with* dim $X=n$, *then there exists an equivalent metric $\rho'$ to $\rho$ such that $d_2(X, \rho')=n$.*

**Proof.** Since dim $X=n$, there exists a defining system of $n$ pairs $C_1, C_1'; \cdots; C_n, C_n'$. Let $f_1, \ldots, f_n$ be real-valued mappings of $X$ such that

(i) $0 \leqq f_i(x) \leqq 1$ for any $i$ and any $x \in X$,

(ii) $f_i(x)=0$ for any $i$ and any $x \in C_i$,

(iii) $f_i(x)=1$ for any $i$ and any $x \in C_i'$.

Set

$$\rho'(x, y) = \rho(x, y) + \sum_{i=1}^{n} |f_i(x) - f_i(y)|.$$

Then $\rho'$ is an equivalent metric to $\rho$ and $\rho'(C_i, C_i') > 0$ for each $i$. Thus we have $d_2(X, \rho') \geqq n$ and hence $d_2(X, \rho')=n$.

CONSTRUCTION OF $Z_n$, $n \geqq 2$. Set $m=[(n+1)/2]+1$. In every $(I^i, \rho_i)$, $i=m, m+1, \ldots, n+1$, we construct $(Y_i, \rho_i)$ as in the preceding section. Then $\mu$ dim $(Y_i, \rho_i) \leqq [i/2] \leqq [(n+1)/2]$ and dim $Y_i = i-1$ for $i=m, \ldots, n+1$. We assume here that $\rho_i(I^i) \leqq 1$ for $i=m, \ldots, n+1$. Take a metric $\rho_i'$ equivalent to $\rho_i$ as in Lemma 11 such that $d_2(Y_i, \rho_i')=i-1$. Then $\mu$ dim $(Y_i, \rho_i') = d_3(Y_i, \rho_i') = i-1$ are automatically true for $i=m, \ldots, n+1$. By the construction of $\rho_i'$ in Lemma 11 $\rho_i'$ satisfies $\rho_i'(Y_i) \leqq i+1$.

$Z_n$ is merely the disjoint sum of $Y_m, Y_{m+1}, \ldots, Y_{n+1}$. The topology of $Z_n$ is defined in such a way that a subset $G$ of $Z_n$ is open if and only if $G \cap Y_i$ is open in $Y_i$ for $i=m, \ldots, n+1$. Then $Z_n$ is a metric space. Define for $i=m, \ldots, n+1$ the metrics $\sigma_i$ of $Z_n$ as follows:

(i) $\sigma_i | Y_j = \rho_j$ if $i \neq j$.

(ii) $\sigma_i | Y_i = \rho_i'$.

(iii) $\sigma_i(x, y) = n+2$ if for any $j=m, \ldots, n+1$, $x$ and $y$ are not in the same $Y_j$.

$\sigma_m, \ldots, \sigma_{n+1}$ are equivalent metrics which give the preassigned topology of $Z_n$.

ASSERTION 1. $\dim Z_n = n$.

**Proof.** $\dim Z_n = \max\{\dim Y_i : i=m,\dots,n+1\} = n$.

ASSERTION 2. $\mu \dim (Z_n, \sigma_i) = d_2(Z_n, \sigma_i) = d_3(Z_n, \sigma_i) = i-1$ for $i = [(n+1)/2]+1, \dots, n+1$.

**Proof.** If $j \neq i$, then $\mu \dim (Y_j, \sigma_i) = \mu \dim (Y_j, \rho_j) \leqq [(n+1)/2]$. Since

$$\mu \dim (Y_i, \sigma_i) = d_2(Y_i, \sigma_i) = d_3(Y_i, \sigma_i) = \mu \dim (Y_i, \rho_i') = i-1 \geqq [(n+1)/2],$$

we have

$$\begin{aligned} i-1 &= d_2(Z_n, \sigma_i) \leqq d_3(Z_n, \sigma_i) \leqq \mu \dim (Z_n, \sigma_i) \\ &= \max\{\mu \dim (Y_m, \rho_m), \dots, \mu \dim (Y_{i-1}, \rho_{i-1}), \mu \dim (Y_i, \rho_i'), \\ &\qquad \mu \dim (Y_{i+1}, \rho_{i+1}), \dots, \mu \dim (Y_{n+1}, \rho_{n+1})\} = i-1. \end{aligned}$$

Thus the assertion is proved.

7. **A space $(R, \rho)$ with $d_2=2$, $\mu$ dim$=3$, dim$=4$.**

First let us construct a space $(S, \sigma)$ with $d_2(S, \sigma)=2$ and $\mu \dim (S, \sigma) = \dim S = 3$.

CONSTRUCTION OF $(S, \sigma)$. $(S, \sigma)$ will be a subset of

$$(I^4 = \{(x_1, \dots, x_4) : 0 \leqq x_i \leqq 1, i = 1, \dots, 4\}, \sigma),$$

where $\sigma$ is Euclidean metric on $I^4$. Let $C_{ij}$, $C_{ij}'$, $i=1,2,\dots, j=1,2,3$, be disjoint pairs of closed sets of $I^4$ such that for any three disjoint pairs of closed sets $C_1, C_1'; C_2, C_2'; C_3, C_3'$, there exists $i$ with $C_j \subset C_{ij}$ and $C_j' \subset C_{ij}'$ for $j=1,2,3$. Let $\pi$ be a prime number with $5 \leqq \pi$. Consider an open covering $\mathscr{D}(\pi)$ of the unit interval $[0,1]$ consisting of overlapping intervals $[0, 2/\pi)$, $((\pi-2)/\pi, 1]$ and $((2k-1)/\pi, (2k+2)/\pi)$, $k=1,\dots,(\pi-3)/2$. Define an open covering $\mathscr{E}(\pi)$ of $I^4$ as follows:

$$\begin{aligned} \mathscr{E}(\pi) &= \{D_1 \times D_2 \times D_3 \times D_4 : D_1, \dots, D_4 \in \mathscr{D}(\pi)\} \\ &= \{E_\lambda : \lambda \in \Lambda(\pi)\}. \end{aligned}$$

Let $\pi_{ij}$, $i=1,2,\dots, j=1,2,3$, be prime numbers which are different from each other and satisfy the following conditions:

(i) $5 \leqq \pi_{ij}$ for every $i$ and $j$.

(ii) $\max\{\text{mesh } \mathscr{E}(\pi_{ij}) : j=1,2,3\} < \min\{\sigma(C_{ij}, C_{ij}') : j=1,2,3\}$ for every $i$.

Let $U_{ij}$ be the sum of all elements of $\mathscr{E}(\pi_{ij})$ which meet $C_{ij}$. Set $B_{ij} = \overline{U}_{ij} - U_{ij}$ and $B_i = \bigcap_{j=1}^3 B_{ij}$. Then $B_{ij}$ separates $C_{ij}$ and $C_{ij}'$. Set

$$S = I^4 - \bigcup B_i.$$

Then $(S, \sigma)$ satisfies the required equalities.

ASSERTION 1. $B_i \cap B_k = \varnothing$ if $i \neq k$.

**Proof.** Set

$$L_{ij} = \{a/\pi_{ij} : a = 1, \dots, \pi_{ij}-1\}.$$

Then $L_{ij} \cap L_{kl} \neq \varnothing$ if and only if $i=k$ and $j=l$. If $x=(x_1,\dots,x_4)$ is a point of $B_{ij}$, then for some $t$, $x_t \in L_{ij}$. Hence $B_i \cap B_k = \varnothing$ if $i \neq k$.

ASSERTION 2. $B_i$ does not meet the 2-dimensional edge of $I^4$. $B_i$ meets the surface of $I^4$ at only a finite number of points. $B_i$ is the sum of a finite number of segments.

This is evident from the above observation.

ASSERTION 3. $B_i$ is the disjoint sum of a finite number of simple closed curves and a finite number of simple arcs.

**Proof.** If three different lines $l_1, l_2, l_3$ lying in $B_i$ have a common point, then they lie in some hyperplane $H : x_j = \text{constant}$. Since $H$ is 3-dimensional, it is now easy to see that $H \cap B_i$ cannot contain $l_1, l_2, l_3$ at the same time because (i) $\mathscr{E}(\pi_{ij})|H$, $=1, 2, 3$, are collections of bordered blicks and (ii) $\pi_{ij}$, $j=1, 2, 3$, are different from each other.

ASSERTION 4. $d_2(S, \sigma)=2$ and $\dim S=3$.

The first equality was proved in §4. As for the second equality see Remark 3.

ASSERTION 5. $\mu \dim (S, \sigma)=3$.

**Proof.** To show $\mu \dim (S, \sigma)>2$, assume that $\mu \dim (S, \sigma) \leqq 2$. Then there exists a finite closed (in $S$) covering $\mathscr{F}=\{F\}$ of $S$ which satisfies the following conditions:

(i) $\{G(F)=\text{interior of } F \text{ with respect to } S : F \in \mathscr{F}\}$ covers $S$.

(ii) $\text{mesh}\, \mathscr{F}<1$.

(iii) $\text{ord}\, \mathscr{F} \leqq 3$.

The proof for the existence of such $\mathscr{F}$ is left to the reader. Cf. Lemma 7 and also use the total boundedness of $(S, \sigma)$. Set

$$\mathscr{F}_1 = \{F : F \in \mathscr{F}, \bar{F} \cap \{x : x_1 = 0\} \neq \varnothing\},$$
$$M_1 = \text{boundary in } I^4 \text{ of } \bigcup \{\bar{F} : F \in \mathscr{F}_1\}.$$

Let $F$ be an arbitrary element of $\mathscr{F}_1$. Let $G'$ be an open set of $I^4$ with $G' \cap S = G(F)$. Since $\dim \bigcup B_i=1$, $S$ is dense in $I^4$. Hence $G'-\bar{F} \neq \varnothing$ yields $(G'-\bar{F}) \cap S \neq \varnothing$, a contradiction. Thus $G' \subset \bar{F}$, which implies $G(F) \cap M_1=\varnothing$. Take an arbitrary point $x$ from $M_1 \cap S$. Since $x \notin G(F)$ for any $F$ in $\mathscr{F}_1$, there exists an element $F_0 \in \mathscr{F}-\mathscr{F}_1$ such that $x \in G(F_0)$ by the condition (i) imposed upon $\mathscr{F}$. Hence

$$\text{ord}\, \mathscr{F}_1 | M_1 \cap S \leqq \text{ord}\, \mathscr{F}-1 \leqq 2.$$

Set

$$\mathscr{F}_2 = \{F : F \in \mathscr{F}_1, \bar{F} \cap \{x : x_2 = 0\} \neq \varnothing\},$$
$$M_2 = \text{boundary in } M_1 \text{ of } \bigcup \{\bar{F} \cap M_1 : F \in \mathscr{F}_2\}.$$

Take an arbitrary point $x'$ from $M_2 \cap S$. Let $y^1, y^2, \ldots$ be a sequence of points of $M_1 - \bigcup \{\bar{F} \cap M_1 : F \in \mathscr{F}_2\}$ with $\lim y^i = x'$. Since $\mathscr{F}_1$ is finite and $\bar{\mathscr{F}}_1 = \{\bar{F} : F \in \mathscr{F}_1\}$ covers $M_1$, we assume here without loss of generality that the sequence $\{y^i\}$ is contained in one $\bar{F}_1$ with $F_1 \in \mathscr{F}_1 - \mathscr{F}_2$. For any $i$ let $z^i$ be a point of $F_1$ with $\sigma(y^i, z^i) < \sigma(y^i, x')$. Since $\lim z^i = x'$, $x' \in F_1$. Therefore

$$\text{ord}\, \mathscr{F}_2 | M_2 \cap S \leqq \text{ord}\, \mathscr{F}_1 | M_1 \cap S - 1 \leqq 1.$$

Set

$$\mathscr{F}_3 = \{F : F \in \mathscr{F}_2, \bar{F} \cap \{x : x_3 = 0\} \neq \varnothing\},$$
$$M_3 = \text{boundary in } M_2 \text{ of } \bigcup \{\bar{F} \cap M_2 : F \in \mathscr{F}_3\}.$$

Since ord $\overline{\mathscr{F}}_2|M_2 \cap S = \text{ord } \mathscr{F}_2|M_2 \cap S \leqq 1$,

$$M_3 \cap S = \varnothing.$$

Set

$$T = \{x : x \in M_2, \text{ ord }(x, \overline{\mathscr{F}}_2) \geqq 2\}.$$

Then $T$ is a closed set of $I^4$ such that

$$M_3 \subset T \subset M_2 \cap (\bigcup B_i).$$

Let $K_1$ and $K_2$ be mutually separated relatively open sets of $M_2$ such that

$$M_2 - M_3 = K_1 \cup K_2,$$
$$K_1 \supset M_2 \cap \{x : x_3 = 0\},$$
$$K_2 \supset M_2 \cap \{x : x_3 = 1\}.$$

Let $P$, $P'$, $Q$ or $Q'$ be the union of all components of $T$ which meet $\{x : x_3 = 0\}$, $\{x : x_3 = 1\}$, $\{x : x_4 = 0\}$ or $\{x : x_4 = 1\}$, respectively. Then these four sets are closed. Let us show for instance $P$ is closed. Let $x^0$ be an arbitrary point of the closure of $P$ and $C_1, C_2, \ldots$ a sequence of components of $T$ such that

(i) each $C_i$ intersects $\{x : x_3 = 0\}$,

(ii) each $C_i$ contains a point $z^i$ with $\lim z^i = x^0$.

Since $x^0 \in \liminf C_i$, $\limsup C_i$ is connected by [6, Theorem 2-101]. Since $\limsup C_i$ intersects $\{x : x_3 = 0\}$ and $\limsup C_i \subset T$, $\limsup C_i \subset P$. Especially $x^0 \in P$ and hence $P$ is closed.

By Assertions 2 and 3 $P \cup P'$ and $Q \cap Q'$ are disjoint closed sets of $T$ such that there is no continuum in $T$ between them. Hence by Lemma 2 there exists a subset $V$ of $T$ such that

(i) $V$ is open and closed in $T$,

(ii) $Q \cap Q' \subset V$,

(iii) $V \cap (P \cup P') = \varnothing$.

Since $Q \cap Q' \cap \{x : x_3 = 0, 1\} = \varnothing$, there exists a subset $W$ of $M_2$ such that

(i) $W$ is open in $M_2$,

(ii) $W \cap T = V$,

(iii) $\overline{W} \cap \{x : x_3 = 0, 1\} = \varnothing$.

Then

$$(\overline{W} - W) \cap T = \varnothing,$$
$$Q \cap Q' \subset W,$$
$$\overline{W} \cap (P \cup P' \cup \{x : x_3 = 0, 1\}) = \varnothing.$$

Set

$$M = (M_3 - W) \cup (\overline{W} - W),$$
$$G_1 = K_1 - \overline{W},$$
$$G_2 = (K_2 \cup W) - (\overline{W} - W).$$

Then

$$M_2 - M = G_1 \cup G_2,$$
$$G_1 \cap G_2 = \varnothing,$$
$$G_1 \supset M_2 \cap \{x : x_3 = 0\},$$
$$G_2 \supset M_2 \cap \{x : x_3 = 1\}.$$

Since $G_1$ and $G_2$ are open in $M_2$, $M$ separates $M_2 \cap \{x : x_3 = 0\}$ and $M_2 \cap \{x : x_3 = 1\}$ in $M_2$.

Let us show that no component of $M$ meets both $\{x : x_4 = 0\}$ and $\{x : x_4 = 1\}$. Take an arbitrary element $F$ from $\mathscr{F}_2$. Set

$$U(F) = M_2 - \bigcup \{\bar{F}' : F' \in \mathscr{F}_2, F' \neq F\}.$$

Then $\{U(F) : F \in \mathscr{F}_2\}$ is a disjoint collection of open sets of $M_2$. Since

$$M_2 - T = \bigcup \{U(F) : F \in \mathscr{F}_2\}$$

and $(\overline{W} - W) \cap T = \varnothing$, $\overline{W} - W$ is the sum of the disjoint collection:

$$\mathscr{H} = \{(\overline{W} - W) \cap U(F) = H(F) : F \in \mathscr{F}_2\}.$$

Since

$$H(F) = (\overline{W} - W) - \bigcup \{U(F') : F' \neq F, F' \in \mathscr{F}_2\},$$

$H(F)$ is closed and hence $\mathscr{H}$ is a disjoint collection of closed sets. Since

$$\text{mesh } \mathscr{H} \leqq \text{mesh } \mathscr{F}_2 < 1,$$

no $H(F)$ meets both $\{x : x_4 = 0\}$ and $\{x : x_4 = 1\}$. Now $M$ is the sum of the disjoint closed sets:

$$M \cap B_i, \quad i = 1, 2, \ldots, H(F) \in \mathscr{H}.$$

By our construction no $M \cap B_i$ meets both $\{x : x_4 = 0\}$ and $\{x : x_4 = 1\}$ since $Q \cap Q' \cap M = \varnothing$. Therefore no component of $M$ meets both $\{x : x_4 = 0\}$ and $\{x : x_4 = 1\}$ by Lemma 3.

Consider the closed set:

$$X = \{x : x_4 = 0, 1\} \cup M.$$

Let $X_1$ be the sum of $\{x : x_4 = 0\}$ and all components of $M$ which meet $\{x : x_4 = 0\}$. Let $X_2$ be the sum of $\{x : x_4 = 1\}$ and all components of $M$ which meet $\{x : x_4 = 1\}$. Then $X_1$ and $X_2$ are closed by the same argument as in the proof for the closedness of $P$. With the aid of Lemma 2 we can find a closed set $N$ of $M_2$ which separates $\{x : x_4 = 0\}$ and $\{x : x_4 = 1\}$ such that $N \cap M = \varnothing$. Thus two pairs of opposite sides of $M_2$ are not defining, which shows in turn three pairs of opposite sides of $M_1$ are not defining as can easily be seen. At last four pairs of opposite faces of $I^4$ are not defining, a contradiction. Hence $2 < \mu \dim (S, \sigma)$. Since

$$\mu \dim (S, \sigma) \leqq \dim S = 3, \qquad \mu \dim (S, \sigma) = 3.$$

ASSERTION 6. $d_3(S, \sigma)=3$.

**Proof.** Since $\sigma$ is totally bounded, $d_3(S, \sigma)=\mu \dim (S, \sigma)=3$ by Theorem 5.

CONSTRUCTION OF $(R, \rho)$. Take the space $(Z_4, \sigma_3)$ constructed in the preceding section. Then $\dim Z_4=4$ and $d_2(Z_4, \sigma_3)=d_3(Z_4, \sigma_3)=\mu \dim (Z_4, \sigma_3)=2$. $R$ is the disjoint union of $Z_4$ and $S$ just constructed. The metric $\rho$ on $R$ is defined as follows:

$$\rho|Z_4 = \sigma_3,$$
$$\rho|S = \sigma,$$
$$\rho(x, y) = \max\{\sigma_3(Z_4), \sigma(S)\} \ (\leqq 6) \text{ if } \{x, y\} \text{ is contained in neither } Z_4 \text{ nor } S.$$

Then it is evident that $d_2(R, \rho)=2$, $d_3(R, \rho)=\mu \dim (R, \rho)=3$ and $\dim R=4$.

## 8. Problems.

*Problem* 1. Is it true that $\dim X \leqq 2d_2(X, \rho)$ for all (separable) metric spaces $(X, \rho)$?

*Problem* 2. Let $(X, \rho)$ be a metric space with $d_2(X, \rho)<\dim X$ and $k$ an arbitrary integer with

$$d_2(X, \rho) \leqq k \leqq \dim X.$$

Can $X$ allow an equivalent metric $\sigma$ with $d_2(X, \sigma)=k$?

REMARK 5. Recently Roberts and his student Slaughter solved a problem analogous to Problem 2 for the case when $d_2$ is replaced by $\mu$ dim. (**Added in proof.** This paper has been accepted for publication in Fundamenta Mathematicae.)

*Problem* 3. Find a necessary and sufficient condition on $X$ with which $d_2(X, \rho)$ (or $\mu \dim (X, \rho))=\dim X$ for any metric $\rho$ agreeing with the preassigned topology of $X$.

REMARK 6. It is reported by Alexandroff [1] that K. Sitnikov got a sufficient condition: If $X$ is a subset of the $n$-dimensional Euclidean space $(R^n, \rho)$ such that $\dim X=\dim \bar{X}$, then $\mu \dim (X, \rho)=\dim X$.

*Problem* 4. Is there a space $(X, \rho)$ with $d_3(X, \rho)<\mu \dim (X, \rho)$?

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