

ON A PIECE OF HYPERSURFACE IN A RIEMANNIAN MANIFOLD WITH MEAN CURVATURE BOUNDED AWAY FROM ZERO

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Introduction. In 1965, S. S. Chern proved the following theorem:

THEOREM 0.1. *Let M be a compact piece of an oriented hypersurface of dimension m with smooth boundary ∂M , which is immersed in an euclidean space E of dimension $m+1$. Suppose the mean curvature $H_1 \geq c > 0$. Let a be a fixed unit vector which makes an angle $\leq \pi/2$ with the normals of M . Then*

$$mcV_a \leq L_a,$$

where V_a is the volume of the orthogonal projection of M and L_a that of ∂M in the hyperplane perpendicular to a . If M is defined by the equation

$$z = z(x_1, \dots, x_m), \quad x_1^2 + \dots + x_m^2 \leq R^2,$$

where x_1, \dots, x_m, z are rectangular coordinates in the space E and the z -axis is chosen in the direction of a , then $cR \leq 1$ [1, p. 82]⁽²⁾.

This theorem is a generalization of Heinz's theorem for the C^2 -surface defined by an equation $z = z(x, y)$ in euclidean three space [2].

The purpose of this paper is to extend this theorem to a hypersurface in a Riemann manifold.

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1. A certain generalization of Chern's theorem. We consider a Riemann manifold R^{m+1} ($m+1 \geq 3$) of class C^ν ($\nu \geq 3$) which admits a one-parameter group G of transformations generated by an infinitesimal transformation:

$$(1.1) \quad \bar{x}^i = x^i + \xi^i(x) \delta\tau$$

(where x^i are local coordinates in R^{m+1} and ξ^i are the components of a contra-variant vector ξ . If ξ is a Killing vector, a homothetic Killing vector, a conformal Killing vector, etc. [3, p. 32], then the group G is called isometric, homothetic, conformal, etc.

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⁽²⁾ Numbers in brackets refer to the references at the end of the paper.

In R^{m+1} , we consider a domain D . If the domain D is simply covered by the paths of the transformations generated by ξ , and ξ is everywhere of class C^v and $\neq 0$ in D , then we call D a regular domain with respect to the vector field ξ .

In a regular domain D , we choose a coordinate system such that the paths of the transformations generated by ξ are new x^1 -coordinate curves, that is, a coordinate system in which the vector ξ has the components δ_1^i (where δ_j^i denotes the Kronecker delta). Then (1.1) becomes $\bar{x}^i = x^i + \delta_1^i \delta\tau$ and the domain D admits the transformation given by

$$(1.1)' \quad \bar{x}^i = x^i + \delta_1^i \tau.$$

Now we consider a compact piece M of an oriented hypersurface of dimension m with smooth boundary ∂M , which is immersed in a regular domain D with respect to the vector field ξ . The immersion $x: M \rightarrow R^{m+1}$ is locally given by

$$(1.2) \quad x^i = x^i(u^\alpha), \quad i = 1, \dots, m+1, \quad \alpha = 1, \dots, m.$$

Here and henceforth, Latin indices run from 1 to $m+1$ and Greek indices from 1 to m . Let us consider a hypersurface which is orthogonal to the new x^1 -coordinate curves and project $x(M)$ along the x^1 -coordinate curve into the orthogonal hypersurface, and let M' be the image of M . If $x'(p)$ is the image of $x(p)$, $p \in M$ under this projection, we have

$$(1.3) \quad x'^i = x^i(u^\alpha) - \delta_1^i \tau(u^\alpha),$$

where τ is a function defined on M .

Let us consider a differential form of degree $m-1$ at a point p of M , defined by

$$(1.4) \quad ((n, (\rho'/\rho)^{m-1}(\delta_1/\rho), dx, \dots, dx)) \stackrel{\text{def.}}{=} g^{1/2}(n, (\rho'/\rho)^{m-1}(\delta_1/\rho), dx, \dots, dx),$$

where the symbol $(\)$ means a determinant of order $m+1$ whose columns are the components of respective vectors or vector-valued differential forms, n is a unit normal vector at a point p of M , dx is a displacement along M , g is the determinant of the metric tensor g_{ij} of R^{m+1} , and the functions ρ and ρ' are defined by

$$\begin{aligned} \rho(p) &= \text{length of the vector } \delta_1 \text{ at } x(p) = (g_{11}(x(p)))^{1/2}, \\ \rho'(p) &= \text{length of the vector } \delta_1 \text{ at } x'(p) = (g_{11}(x'(p)))^{1/2}. \end{aligned}$$

Then the exterior differential of the differential form (1.4) divided by $m!$ becomes

$$\begin{aligned} (1.5) \quad & \frac{1}{m!} d((n, (\rho'/\rho)^{m-1} \delta_1/\rho, dx, \dots, dx)) \\ &= \frac{1}{m!} \{((Dn, (\rho'/\rho)^{m-1} \delta_1/\rho, dx, \dots, dx)) + ((n, D(\rho'^{m-1}/\rho_m \delta_1), dx, \dots, dx))\}, \end{aligned}$$

where the symbol D means the covariant differential.

The first term of the right-hand member of the above expression becomes

$$(1.6) \quad \frac{1}{m!} ((Dn, (\rho'^{m-1}/\rho^m) \delta_1, dx, \dots, dx)) = (-1)^m H_1 (\rho'^{m-1}/\rho^m) n_i \delta_1^i dV,$$

where H_1 is the first mean curvature of M and dV is the volume element of M . On the other hand, we easily obtain the following relation:

$$(1.7) \quad n_i \delta_1^i dV = \frac{(-1)^m}{m!} ((\delta_1, dx, \dots, dx)),$$

and from (1.3) we have

$$(1.8) \quad \begin{aligned} \frac{(-1)^m}{m!} ((\delta_1, dx, \dots, dx)) &= \frac{g^{1/2}}{g'^{1/2}} \frac{(-1)^m}{m!} g'^{1/2} (\delta_1, dx', \dots, dx') \\ &= \frac{g^{1/2}}{g'^{1/2}} n'_i \delta_1^i dV'_\xi, \end{aligned}$$

where n' and dV'_ξ mean the unit normal vector and the volume element at the corresponding point $x'(p)$ of M' respectively and $g' = g(x')$. Consequently from (1.7) and (1.8), we have

$$(1.9) \quad \begin{aligned} \frac{\rho'^{m-1}}{\rho^m} n_i \delta_1^i dV &= \frac{g^{1/2}}{(g')^{1/2}} \left(\frac{\rho'}{\rho}\right)^m \frac{\delta_1^i}{\rho} n'_i dV'_\xi \\ &= \frac{g^{1/2}}{(g')^{1/2}} \left(\frac{\rho'}{\rho}\right)^m dV'_\xi, \end{aligned}$$

because δ_1/ρ' is nothing but the unit normal vector of M' .

Now we suppose that the transformation group G is conformal. Then R^{m+1} admits the infinitesimal transformation given by (1.1) with the additional condition

$$\xi_{i;j} + \xi_{j;i} = 2\phi(x)g_{ij},$$

where the symbol “;” means the covariant derivative. In the new coordinate system of the domain D , the additional condition becomes

$$\frac{\partial g_{ij}}{\partial x^1} = 2\phi(x)g_{ij}.$$

Therefore the metric tensor g_{ij} with respect to the new coordinate system has the form

$$(1.10) \quad g_{ij} = \exp \left(2 \int \phi(x) dx^1 \right) f_{ij}(x^2, \dots, x^{m+1}).$$

From this, we can easily deduce the following result:

$$(1.11) \quad \frac{g^{1/2}}{(g')^{1/2}} \left(\frac{\rho'}{\rho}\right)^m = \exp \left(\int_{x^1}^{x^1} \phi(x) dx^1 \right)$$

since $\rho = (g_{11})^{1/2}$ and $\rho' = (g'_{11})^{1/2}$. Therefore from (1.9) and (1.11), (1.6) becomes

$$(1.12) \quad \frac{1}{m!} ((Dn, (\rho'^{m-1}/\rho^m)\delta_1, dx, \dots, dx)) = (-1)^m H_1 \exp \left(\int_{x'^1}^{x^1} \phi dx^1 \right) dV'_\xi.$$

Put

$$(1.13) \quad v_i = \frac{\rho'^{m-1}}{\rho^m} \delta_1^i g_{ij}.$$

Then the second term of the right-hand member of (1.5) may be written as follows:

$$(1.14) \quad \frac{1}{m!} ((n, D((\rho'^{m-1}/\rho^m)\delta_1), dx, \dots, dx)) = \frac{(-1)^m}{2m} L_v g_{ij} g^{*ij} dV,$$

where $L_v g_{ij}$ is the Lie derivative of the tensor g_{ij} with respect to v^i , g^{*ij} means $(\partial x^i / \partial u^\alpha)(\partial x^j / \partial u^\beta) g^{\alpha\beta}$, and $g_{\alpha\beta}$ is the metric tensor of M .

Accordingly from (1.12) and (1.14), we can rewrite (1.5) as follows:

$$(1.15) \quad \begin{aligned} & \frac{(-1)^m}{m!} d((n, (\rho'/\rho)^{m-1}\delta_1/\rho, dx, \dots, dx)) \\ &= \exp \left(\int_{x'^1}^{x^1} \phi dx^1 \right) H_1 dV'_\xi + \frac{1}{2m} L_v g_{ij} g^{*ij} dV. \end{aligned}$$

Integrating both members of (1.15) over M and applying Stokes' theorem we have

$$(1.16) \quad \begin{aligned} & \frac{(-1)^m}{m!} \int_{\partial M} ((n, (\rho'/\rho)^{m-1}\delta_1/\rho, dx, \dots, dx)) \\ &= \int_M \exp \left(\int_{x'^1}^{x^1} \phi dx^1 \right) H_1 dV'_\xi + \frac{1}{2m} \int_M L_v g_{ij} g^{*ij} dV. \end{aligned}$$

On the other hand, from (1.3) and (1.11) we can see that

$$(1.17) \quad \begin{aligned} & \frac{(-1)^m}{m!} ((n, (\rho'/\rho)^{m-1}\delta_1/\rho, dx, \dots, dx)) \\ &= \frac{(-1)^m}{m!} \frac{g^{1/2}}{(g')^{1/2}} \left(\frac{\rho'}{\rho} \right)^m (g')^{1/2} (n, \delta_1/\rho', dx', \dots, dx') \\ &= \frac{(-1)^m}{m!} \exp \left(\int_{x'^1}^{x^1} \phi dx^1 \right) (g')^{1/2} (n, \delta_1/\rho', dx', \dots, dx'). \end{aligned}$$

Now for each point $p \in M$, let us take the small piece \tilde{M}_p of the hypersurface passing through the corresponding point $x'(p)$ of M' , given by the following expression

$$\tilde{x}^i = x^i(u^\alpha) - \delta_1^i \tau_p, \quad \tau_p = \tau(p) = \text{const},$$

where τ_p means $x^1(p) - x'^1(p)$. Then we have

$$\frac{\partial \tilde{x}^i}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha}$$

and we can obtain the following relation:

$$n_i = \frac{|\tilde{g}_{\alpha\beta}|^{1/2}}{|g_{\alpha\beta}|^{1/2}} \frac{g^{1/2}}{(g')^{1/2}} \tilde{n}_i,$$

where $\tilde{g}_{\alpha\beta}$ is the metric tensor of \tilde{M}_p and \tilde{n} is the unit normal vector of \tilde{M}_p at $x'(p)$, $|g_{\alpha\beta}|$ and $|\tilde{g}_{\alpha\beta}|$ mean the determinants of $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$. Since

$$g_{\alpha\beta} = g_{ij}(x) \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta}, \quad \tilde{g}_{\alpha\beta} = g_{ij}(x') \frac{\partial \tilde{x}^i}{\partial u^\alpha} \frac{\partial \tilde{x}^j}{\partial u^\beta},$$

making use of (1.10) we have

$$|g_{\alpha\beta}| = \exp \left(2m \int_{x'^1}^{x^1} \phi \, dx^1 \right) |\tilde{g}_{\alpha\beta}|.$$

Therefore we have the following relation:

$$n_i = \exp \left(\int_{x'^1}^{x^1} \phi \, dx^1 \right) \tilde{n}_i.$$

Since

$$g_{ij}(x) = \exp \left(2 \int_{x'^1}^{x^1} \phi \, dx^1 \right) g_{ij}(x'),$$

it follows that

$$g^{ij}(x) = \exp \left(-2 \int_{x'^1}^{x^1} \phi \, dx^1 \right) g^{ij}(x').$$

Thus we obtain

$$n^i = \exp \left(- \int_{x'^1}^{x^1} \phi \, dx^1 \right) \tilde{n}^i.$$

Substituting the above result in (1.17), we get

$$(1.18) \quad \frac{(-1)^m}{m!} ((n, (\rho^1/\rho)^{m-1} \delta_1/\rho, dx, \dots, dx)) = \frac{(-1)^m}{m!} (g')^{1/2} (\tilde{n}, \delta_1/\rho', dx', \dots, dx').$$

Let e_A , $A=1, \dots, m-1$, be an oriented basis for the tangent plane of $\partial M'$ at a point $x'(p)$ on the image $\partial M'$ of ∂M , and let n'' be the unit vector of the tangent plane of M' at the same point $x'(p)$ which is orthogonal to e_A , $A=1, \dots, m-1$, n' and such that

$$((n'', n', e_1, \dots, e_{m-1})) > 0.$$

If dx' is a displacement along $\partial M'$, then

$$\frac{(-1)^m}{(m-1)!} ((n'', n', dx', \dots, dx'))$$

is the volume element of $\partial M'$. The unit normal vector \tilde{n} of \tilde{M}_p at $x'(p)$ can be expressed as follows:

$$\tilde{n} = \alpha n'' + \beta n' + \sum_{A=1}^{m-1} \gamma^A e_A,$$

where α, β, γ^A satisfy

$$\alpha^2 + \beta^2 + g_{ij}(x')e_A^i e_B^j \gamma^A \gamma^B = 1.$$

Therefore we have $|\alpha| \leq 1$. Since δ_1/ρ' is the unit normal vector n' , it follows that

$$(1.19) \quad \frac{(-1)^m}{(m-1)!} ((\tilde{n}, \delta_1/\rho', dx', \dots, dx')) = \frac{(-1)^m}{(m-1)!} \alpha((n'', n', dx', \dots, dx')).$$

By making use of a notation dL'_ξ for the volume element of $\partial M'$, from (1.18) and (1.19), (1.17) becomes

$$(1.20) \quad \frac{(-1)^m}{m!} ((n, (\rho'/\rho)^{m-1} \delta_1/\rho, dx, \dots, dx)) = \frac{1}{m} \alpha dL'_\xi, \quad |\alpha| \leq 1.$$

Substituting (1.20) in (1.16), we have

$$\frac{1}{m} \int_{\partial M} \alpha dL'_\xi = \int_M \exp \left(\int_{x'^1}^{x^1} \phi dx^1 \right) H_1 dV'_\xi + \frac{1}{2m} \int_M L_v g_{ij} g^{*ij} dV.$$

Consequently it follows that

$$(1.21) \quad \frac{1}{m} \int_{\partial M} dL'_\xi \geq \int_M \exp \left(\int_{x'^1}^{x^1} \phi dx^1 \right) H_1 dV'_\xi + \frac{1}{2m} \int_M L_v g_{ij} g^{*ij} dV.$$

From the above inequality, we can obtain the following

THEOREM 1.1 (MAIN THEOREM). *Let M be a compact piece of an oriented hypersurface of dimension m with smooth boundary ∂M , which is immersed in a regular domain of a Riemann manifold of dimension $m+1$ admitting a conformal Killing vector field ξ (i.e., $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$). Suppose the mean curvature $H_1 \geq c > 0$ ($c = \text{const}$). Let us suppose that the $\xi(x)$ makes an angle $\leq \pi/2$ with the normals of M , and let*

$$(I) \quad \int_{x'^1}^{x^1} \phi dx^1 \geq 0 \quad \text{everywhere on } M,$$

$$(II) \quad \int_M L_v g_{ij} g^{*ij} dV \geq 0.$$

Then

$$mcV'_\xi \leq L'_\xi,$$

where V'_ξ and L'_ξ are the volumes of the images M' of M and $\partial M'$ of ∂M , by the projection along the paths of the vector field ξ into a hypersurface perpendicular to the paths of ξ .

2. Some examples of the main theorem. In this section, we shall give some examples which satisfy the hypotheses (I) and (II) in the main theorem.

The hypothesis (I) means that either ξ is a Killing vector or $\rho \geq \rho'$ everywhere on M .

As an example in the case satisfying the hypothesis (II), we have the following

THEOREM 2.1. *Let R^{m+1} admit a conformal Killing vector field (i.e., $\xi_{i;j} + \xi_{j;i} = 2\phi_{ij}$) and if the paths of the transformations generated by ξ are geodesics (i.e., the new x^1 -coordinate curves are geodesics), and $\phi \geq 0$ everywhere on M , then it follows that*

$$L_v g_{ij} g^{*ij} \geq 0$$

everywhere on M .

Proof. Let us choose a coordinate system such that the family of hypersurfaces perpendicular to the new x^1 -coordinate curves are new $x^2 x^3 \dots x^{m+1}$ -coordinate hypersurfaces. Then $g_{1j} = 0$ for $j = 2, \dots, m+1$.

Since the x^1 -coordinate curve is a geodesic, substituting

$$dx^1/ds = 1/(g_{11})^{1/2}, \quad dx^2/ds = \dots = dx^{m+1}/ds = 0$$

in the differential equation of a geodesic:

$$\frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \quad i = 1, 2, \dots, m+1,$$

we have

$$-\frac{1}{2} \frac{1}{g_{11}} \frac{\partial g_{11}}{\partial x^1} + \Gamma_{11}^1 = 0, \quad \Gamma_{11}^j = 0, \quad j = 2, \dots, m+1.$$

Substituting $\partial g_{ik}/\partial x^1 = 2\phi g_{ik}$ in Γ_{11}^j ($j = 2, \dots, m+1$), we obtain

$$\Gamma_{11}^j = \frac{1}{2} g^{jk} \frac{\partial g_{11}}{\partial x^k}, \quad j = 2, \dots, m+1.$$

And also from $g^{j1} = 0$, $j = 2, \dots, m+1$, the conditions $\Gamma_{11}^j = 0$, $j = 2, \dots, m+1$, become

$$\sum_{k=2}^{m+1} g^{jk} \frac{\partial g_{11}}{\partial x^k} = 0, \quad j = 2, \dots, m+1.$$

Since the determinant of g^{jk} ($j, k = 2, \dots, m+1$) is not equal to zero, the conditions $\Gamma_{11}^j = 0$ ($j = 2, \dots, m+1$) are rewritten as follows:

$$\frac{\partial g_{11}}{\partial x^k} = 0, \quad k = 2, \dots, m+1.$$

And the condition

$$-\frac{1}{2} \frac{1}{g_{11}} \frac{\partial g_{11}}{\partial x^1} + \Gamma_{11}^1 = 0$$

follows clearly from $g_{1j} = 0$, $j = 2, \dots, m+1$. Therefore in this case, a necessary and sufficient condition that the x^1 -coordinate curve be a geodesic is that $\partial g_{11}/\partial x^k = 0$, $k = 2, \dots, m+1$.

To calculate $L_v g_{ij} g^{*ij}$, we go back to the definition (1.13):

$$v_i = \frac{\rho'^{m-1}}{\rho^m} \delta_1^i g_{ij},$$

where $\rho = (g_{11}(x))^{1/2}$ and $\rho' = (g_{11}(x'))^{1/2}$. ρ' is constant on the x^1 -coordinate curve. Then we have

$$\begin{aligned} v_{i;j} &= \frac{1}{2} \frac{\partial g_{ij}}{\partial x^1} \frac{(g'_{11})^{(m-1)/2}}{(g_{11})^{m/2}} - \frac{m}{2} \frac{(g'_{11})^{(m-1)/2}}{(g_{11})^{(m+2)/2}} \frac{\partial g_{11}}{\partial x^j} \delta_1^i g_{ij} \\ &\quad + \frac{m-1}{2} \frac{(g'_{11})^{(m-3)/2}}{(g_{11})^{m/2}} \frac{\partial g'_{11}}{\partial x^j} \delta_1^i g_{ij}, \end{aligned}$$

$$L_v g_{ij} = v_{i;j} + v_{j;i}$$

$$\begin{aligned} &= 2\phi g_{ij} \frac{(g'_{11})^{(m-1)/2}}{(g_{11})^{m/2}} - \frac{m}{2} \frac{(g'_{11})^{(m-1)/2}}{(g_{11})^{(m+2)/2}} ((\partial g_{11}/\partial x^j) g_{1i} + (\partial g_{11}/\partial x^i) g_{1j}) \\ &\quad + \frac{m-1}{2} \frac{(g'_{11})^{(m-3)/2}}{(g_{11})^{m/2}} ((\partial g'_{11}/\partial x^j) g_{1i} + (\partial g'_{11}/\partial x^i) g_{1j}) \end{aligned}$$

because $\partial g_{ij}/\partial x^1 = 2\phi g_{ij}$. From the fact that the x^1 -coordinate curves are geodesics and $(g'_{11})^{1/2}$ is constant on the x^1 -coordinate curve, we have the following relations:

$$\frac{\partial g_{11}}{\partial x^j} = 0, \quad j = 2, \dots, m+1,$$

$$\frac{\partial g'_{11}}{\partial x^1} = 0, \quad \frac{\partial g'_{11}}{\partial x^j} = \frac{\partial g'_{11}}{\partial x'^j} = 0, \quad j = 2, \dots, m+1,$$

since $x'^j = x^j$, $j = 2, \dots, m+1$. By making use of the above relations, we get

$$\begin{aligned} L_v g_{ij} g^{*ij} &= 2m\phi \frac{(g'_{11})^{(m-1)/2}}{(g_{11})^{m/2}} - m \frac{(g'_{11})^{(m-1)/2}}{(g_{11})^{(m+2)/2}} \frac{\partial g_{11}}{\partial x^1} g_{11} \frac{\partial x^1}{\partial u^\alpha} \frac{\partial x^1}{\partial u^\beta} g^{\alpha\beta} \\ &= 2m\phi \frac{(g'_{11})^{(m-1)/2}}{(g_{11})^{m/2}} \left(1 - g_{11} \frac{\partial x^1}{\partial u^\alpha} \frac{\partial x^1}{\partial u^\beta} g^{\alpha\beta} \right), \end{aligned}$$

and also we can see that

$$g_{jk} \frac{\delta_1^i}{(g_{11})^{1/2}} \frac{\partial x^k}{\partial u^\alpha} = (g_{11})^{1/2} \frac{\partial x^1}{\partial u^\alpha},$$

since the unit vector $\delta_1^i/(g_{11})^{1/2}$ at a point on M is expressed as follows:

$$\frac{\delta_1^i}{(g_{11})^{1/2}} = \lambda n^i + \mu^\alpha \frac{\partial x^i}{\partial u^\alpha},$$

where

$$\lambda^2 + g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \mu^\alpha \mu^\beta = \lambda^2 + g_{\alpha\beta} \mu^\alpha \mu^\beta = 1,$$

we have

$$g_{ij} \frac{\delta_1^i}{(g_{11})^{1/2}} \frac{\partial x^j}{\partial u^\beta} = g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \mu^\alpha = g_{\alpha\beta} \mu^\alpha.$$

Therefore it follows that

$$g_{11} \frac{\partial x^1}{\partial u^\alpha} \frac{\partial x^1}{\partial u^\beta} g^{\alpha\beta} = g_{\alpha\gamma} \mu^\gamma g_{\beta\theta} \mu^\theta g^{\alpha\beta} = g_{\gamma\theta} \mu^\gamma \mu^\theta \leq 1.$$

Consequently if $\phi \geq 0$ everywhere on M , then it follows that

$$L_v g_{ij} g^{*ij} = 2m\phi \frac{(g'_{11})^{(m-1)/2}}{(g_{11})^{m/2}} \left(1 - g_{11} \frac{\partial x^1}{\partial u^\alpha} \frac{\partial x^1}{\partial u^\beta} g^{\alpha\beta} \right) \geq 0$$

everywhere on M .

From Theorem 1.1 and Theorem 2.1, we can see

THEOREM 2.2. *Let M be a compact piece of an oriented hypersurface of dimension m with smooth boundary ∂M , which is immersed in a regular domain of a Riemann manifold of dimension $m+1$ admitting a conformal Killing vector field ξ with $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$, $\phi \geq 0$. Suppose the mean curvature $H_1 \geq c > 0$ ($c = \text{const}$) and suppose that $\xi(x)$ makes an angle $\leq \pi/2$ with the normal at each point of M . If the paths of the transformations generated by ξ are geodesics, then*

$$mcV'_\xi \leq L'_\xi,$$

where V'_ξ and L'_ξ are the volumes of the image M' of M and the image $\partial M'$ of ∂M , by the projection along the paths into a hypersurface perpendicular to the paths, which never intersects M .

COROLLARY 2.1. *Let M be a compact piece of an oriented hypersurface of dimension m with smooth boundary ∂M , which is immersed in a regular domain of a Riemann manifold of dimension $m+1$ admitting a homothetic Killing vector field $\xi(x)$ with $\xi_{i;j} + \xi_{j;i} = 2c_0 g_{ij}$, $c_0 > 0$, or a Killing vector field. Suppose the mean curvature $H_1 \geq c > 0$ ($c = \text{const}$) and that $\xi(x)$ makes an angle $\leq \pi/2$ with the normal at each point of M . If the paths of the transformations generated by ξ are geodesics, then*

$$mcV'_\xi \leq L'_\xi,$$

where V'_ξ and L'_ξ are the volumes of the image M' of M and the image $\partial M'$ of ∂M , by the projection along the paths into a hypersurface perpendicular to the paths, which never intersects M .

Especially if our manifold R^{m+1} is an euclidean space E^{m+1} and if ξ is the homothetic Killing vector field on E^{m+1} with components $\xi^i = x^i$, x^i being rectangular coordinates in the space E^{m+1} , then the paths of the transformations generated by

ξ are the lines through O and a hypersurface perpendicular to these paths is a hypersphere of radius, say R_0 , around O . The functions ρ and ρ' are given by

$$\rho = \left(\sum_{i=1}^{m+1} (x^i)^2 \right)^{1/2} \quad \text{and} \quad \rho' = R_0.$$

Thus, as a special case of Corollary 2.1, we have the following

COROLLARY 2.2. *Let M be a compact piece of an oriented hypersurface of dimension m with smooth boundary ∂M , which is immersed in an euclidean space of dimension $m+1$. Suppose that the position vector $x(u)$ makes an angle $\leq \pi/2$ with the normal of $x(M)$ at $x(u)$ for every $u \in M$ and that the mean curvature H_1 satisfies $H_1 \geq c > 0$ ($c = \text{const}$). Then*

$$mcV'_\xi \leq L'_\xi$$

where V'_ξ and L'_ξ are the volumes of the image M' of M and the image $\partial M'$ of ∂M , by the projection from the origin O into a hypersphere around O not intersecting M .

From this result we have (under the same assumptions as Corollary 2.2)

COROLLARY 2.3. *Let $\rho, \theta_1, \dots, \theta_m$ be polar coordinates in E^{m+1} , i.e.,*

$$\begin{aligned} x^1 &= \rho \sin \theta_m \sin \theta_{m-1} \cdots \sin \theta_2 \sin \theta_1, \\ x^2 &= \rho \sin \theta_m \sin \theta_{m-1} \cdots \sin \theta_2 \cos \theta_1, \\ x^3 &= \rho \sin \theta_m \sin \theta_{m-1} \cdots \sin \theta_3 \cos \theta_2, \\ &\vdots \\ x^m &= \rho \sin \theta_m \cos \theta_{m-1}, \\ x^{m+1} &= \rho \cos \theta_m. \end{aligned}$$

If M is defined by the equation

$$\rho = \rho(\theta_1, \dots, \theta_m), \quad 0 \leq \theta_m \leq \phi_0 (= \text{const}),$$

then

$$cC'_0 \leq 1,$$

where C'_0 is a constant which depends only on R_0 and ϕ_0 (R_0 is the radius of the hypersphere into which M is projected).

Proof. For simplicity, we shall prove in case $m=2$. Since M is written as follows:

$$\begin{aligned} x^1 &= \rho(\theta_1, \theta_2) \sin \theta_2 \sin \theta_1, \\ x^2 &= \rho(\theta_1, \theta_2) \sin \theta_2 \cos \theta_1, \quad 0 \leq \theta_2 \leq \phi_0, \\ x^3 &= \rho(\theta_1, \theta_2) \cos \theta_2, \end{aligned}$$

in this case, M' is defined by

$$\begin{aligned} x^1 &= R_0 \sin \theta_2 \sin \theta_1, \\ x^2 &= R_0 \sin \theta_2 \cos \theta_1, \quad 0 \leq \theta_2 \leq \phi_0, \\ x^3 &= R_0 \cos \theta_2, \end{aligned}$$

and $\partial M'$ is the circle on the sphere S_0 with the center O and radius R_0 , given by

$$x^1 = R_0 \sin \phi_0 \sin \theta_1,$$

$$x^2 = R_0 \sin \phi_0 \cos \theta_1,$$

$$x^3 = R_0 \cos \phi_0.$$

Let g^* be the determinant of the metric tensor $g_{\alpha\beta}^*$ on S_0 with respect to the coordinates θ_1, θ_2 . Then $g^* = R_0^4 \sin^2 \theta_2$. Therefore the area of M' is given by

$$V'_\xi = \int_0^{2\pi} \int_0^{\phi_0} R_0^2 \sin \theta_2 d\theta_2 d\theta_1 = 2R_0^2 \pi (1 - \cos \phi_0),$$

and the length of $\partial M'$, which is a circle of radius $R_0 \sin \phi_0$, is given by

$$L'_\xi = 2\pi R_0 \sin \phi_0.$$

From the results above, we have the following relation:

$$2V'_\xi = 2R_0 \frac{1 - \cos \phi_0}{\sin \phi_0} L'_\xi.$$

Putting

$$C'_0 = 2R_0 \frac{1 - \cos \phi_0}{\sin \phi_0},$$

and making use of Corollary 2.2, we obtain

$$cC'_0 L'_\xi \leq L'_\xi.$$

Thus $cC'_0 \leq 1$.

Finally we consider a Killing vector field ξ on E^{m+1} which generates translations so that the paths of the transformations generated by ξ are parallel lines in the direction of ξ and a hypersurface perpendicular to these paths is a hyperplane perpendicular to ξ . In this case, Corollary 2.1 is nothing but Theorem 0.1. Thus Chern's theorem is included as a special case in Corollary 2.1.

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