INVARIANT MEANS AND FIXED POINTS; A SEQUEL TO MITCHELL'S PAPER

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The purpose of this note is to present a new proof of a generalized form of Day's fixed point theorem. The proof we give is suggested by the work of T. Mitchell in his paper, *Function algebras, means, and fixed points*, [2]. The version of Day's theorem which we present here has not appeared explicitly in the literature before, and seems especially well suited for application to questions concerning fixed point properties of topological semigroups.

1. **Preliminaries.**

We adopt the terminology and notation of [2] except where otherwise specified. New terminology will be introduced as needed.

Let Y be a convex compactum (compact convex set in a real locally convex linear topological space E), and let $A(Y)$ denote the Banach space of all (real) continuous affine functions on Y under the supremum norm. Observe that $A(Y)$ contains every function of the form $h=f|_Y+r$ where $f\in E^*$ and r is real; thus $A(Y)$ separates points of Y.

LEMMA 1. *The subspace $M=E^*|_Y+R$ is uniformly dense in $A(Y)$.*

Proof. See [3, Proposition 4.5, pp. 31–32].

LEMMA 2. *If φ is a mean on $A(Y)$ then there exists a unique point $y_0\in Y$ such that $\varphi(h)=h(y_0)$ for all $h\in A(Y)$.*

Proof. For each $f\in E^*$, let $H_f=\{x\in E : f(x)=\varphi(f|_Y)\}$. These sets are closed hyperplanes, and we want to show that $\bigcap\{H_f : f\in E^*\}\cap Y$ is nonvoid. Since Y is compact, it suffices to show that $\bigcap_{i=1}^m H_{f_i}\cap Y\neq\varnothing$ for any finite set $f_1,\ldots, f_m\in E^*$. To this end, define T: $E\to R^m$ by $T_x=(f_1(x),f_2(x),\ldots,f_m(x))$. Then T is linear and continuous, so $T(Y)$ is compact and convex. Let $p=(\varphi(f_1|_Y),\ldots,\varphi(f_m|_Y))$. If $p\notin T(Y)$, then there exists a linear functional on R^m which strictly separates p and $T(Y)$; hence there exists a point $a=(a_1,\ldots,a_m)\in R^m$ such that

$$\sum_{i=1}^m a_i\varphi(f_i|_Y) > \sup\left\{\sum_{i=1}^m a_if_i(y) : y\in Y\right\}.$$

If we let $g=\sum_{i=1}^m a_if_i$, then $g\in E^*$ and the above inequality implies that $\varphi(g|_Y)>\sup\{g(y) : y\in Y\}$. This last inequality is impossible since φ is a mean on $A(Y)$. Thus $p\in T(Y)$, and so $\bigcap_{i=1}^m H_{f_i}\cap Y\neq\varnothing$.

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We have shown that there exists a point $y_0 \in Y$ such that $\varphi(f|_Y)=f(y_0)$ for all $f \in E^*$. It follows from Lemma 1 that $\varphi(h)=h(y_0)$ for all $h \in A(Y)$. Uniqueness of y_0 is a consequence of the fact that $A(Y)$ separates points of Y.

2. Day's fixed point theorem.

Let S be a semigroup, X a subset of $m(S)$, and Y a convex compactum. Let $s \to \eta s$ be a representation of S by continuous *affine* maps of Y into Y. For each $y \in Y$, let $(Tyh)s=h((\eta s)y)$ for all $s \in S$, $h \in A(Y)$; then Ty is a norm-decreasing linear map of $A(Y)$ into $m(S)$. We say that η is an E, D, or *A-representation of S, X by continuous affine maps* on Y if $\{y \in Y : Ty(A(Y)) \subset X\}$ is nonempty, dense in Y, or equal to Y respectively. The pair S, X has the *common fixed point property on convex compacta with respect to i-representations by affine maps* (where $i=E$, D, or A) if, for each convex compactum Y and each i-representation η of S, X by continuous affine maps on Y, there is a point $y_0 \in Y$ such that $\eta s(y_0)=y_0$ for all $s \in S$.

THEOREM 1. *Let S be a semigroup, and let X be a closed translation-invariant subspace of m(S) containing the constant functions. Then the following are equivalent:*

(1) *X has a left invariant mean.*

(2) *S, X has the common fixed point property on convex compacta with respect to E-representations by affine maps.*

(3) *S, X has the common fixed point property on convex compacta with respect to D-representations by affine maps.*

Furthermore, each of the conditions (1), (2), (3) *above implies*:

(4) *S, X has the common fixed point property on convex compacta with respect to A-representation by affine maps.*

REMARK. The equivalence (1) $\Leftrightarrow$ (2) is originally due to M. M. Day [1]. Day worked with the space $M=E^*|_Y+R$ in place of $A(Y)$, but the present version follows easily from his by an application of Lemma 1. The following proof is suggested by the work of T. Mitchell (see [2, Theorem 1]).

Proof of Theorem 1. Obviously (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4). We will prove (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Let η be an E-representation of S by continuous affine self-maps of a convex compactum Y. Then there exists an element $z \in Y$ such that $Tz(A(Y)) \subset X$. Let $Tz^*: X^* \to A(Y)^*$ denote the adjoint map of Tz, and note that Tz^* takes means on X into means on $A(Y)$. Thus, if μ is a left invariant mean on X, then there exists a point $y_0 \in Y$ such that $Tz^*\mu(h)=h(y_0)$ for all $h \in A(Y)$. We will show that y_0 is the desired fixed point.

For each $s \in S$, define θ_s: $A(Y) \to A(Y)$ by $(\theta_s h)y=h((\eta s)y)$. Then for each $s' \in S$ we have

$$\begin{aligned}(Tz(\theta_s h))s' &= \theta_s h((\eta s')z) = h((\eta s)(\eta s')z) \\ &= h((\eta ss')z) = (Tzh)(ss') = (l_s(Tzh))s'\end{aligned}$$

and so

$$\begin{aligned} h((\eta s)y_0) &= \theta_s h(y_0) = Tz^*\mu(\theta_s h) \\ &= \mu(Tz(\theta_s h)) = \mu(l_s(Tzh)) \\ &= \mu(Tzh) = Tz^*\mu(h) = h(y_0) \end{aligned}$$

for all $h \in A(Y)$. But $A(Y)$ separates points of Y, and so $(\eta s)y_0 = y_0$ for all $s \in S$.

(3) $\Rightarrow$ (1). Let $Y \subset X^*$ be the set of all means on X. Then Y is a weak*-compact convex set in X^*, and a mean μ on X is left invariant if and only if $l_s^*\mu = \mu$ for all $s \in S$ (here l_s denotes the left translation operator in X). The mapping $s \to l_s^*$ is clearly a representation of S by weak*-continuous affine mappings of Y into Y; thus it suffices to show that this is a D-representation of S, X on Y.

For each $s \in S$, let $Qs(g) = g(s)$ for all $g \in X$. A functional on X of the form $\phi = \sum_{i=1}^m \lambda_i Qs_i$, $\lambda_i > 0$, $\sum_{i=1}^m \lambda_i = 1$, is called a finite mean on X. It is a well-known fact that the set of all finite means on X is weak* dense in Y; thus it suffices to show that $T\phi(A(Y)) \subset X$ for all finite means ϕ.

Let $E = X^*$ with the weak* topology. If $h \in E^*$, then there exists a function $f \in X$ such that $h(\mu) = \mu(f)$ for all $\mu \in X^*$. Thus if $\phi = \sum_{i=1}^m \lambda_i Qs_i$ is a finite mean on X, then

$$\begin{aligned} (T\phi h)s &= h(l_s^*\phi) = l_s^*\phi(f) \\ &= \phi(l_s f) = \sum_{i=1}^m \lambda_i f(ss_i) \\ &= \sum_{i=1}^m \lambda_i r_{s_i} f(s) \end{aligned}$$

and, since X is right translation-invariant, it follows that $T\phi(h|_Y) \in X$ for all $h \in E^*$. It is clear that $T\phi(r) = r \in X$ for all $r \in R$ since X contains the constant functions. Thus $T\phi(E^*|_Y + R) \subset X$ and it follows from Lemma 1 that $T\phi(A(Y)) \subset \bar{X} = X$. This completes the proof.

A closed left translation-invariant subspace X of $m(S)$ is *left introverted* if $\mu_l X \subset X$ for all $\mu \in X^*$ (where $\mu_l f(s) = \mu(l_s f)$ for all $f \in X$, $s \in S$). Note that $(Qs')_l f(s) = r_{s'} f(s)$, and hence a left introverted subspace is necessarily right translation-invariant.

THEOREM 2. *Let S be a semigroup, and let X be a closed left invariant subspace of $m(S)$ which is left introverted and contains the constant functions. Then conditions* (1)–(4) *of Theorem* 1 *are equivalent.*

Proof. We need only to show that (4) $\Rightarrow$ (1). Let $E = X^*$ with the weak* topology and let $Y \subset E$ be the set of all means on X. It suffices to show that $s \to l_s^*$ is an A-representation of S, X on Y.

If $h \in E^*$, then there exists a function $f \in X$ such that $h(\mu) = \mu(f)$ for all $\mu \in E$. Thus, if $\mu \in Y$, then

$$\begin{aligned}(T\mu h)s &= h(l_s^*\mu) = l_s^*\mu(f) = \mu(l_s f)\\ &= \mu_l f(s)\end{aligned}$$

for all $s \in S$. Since X is left introverted, it follows that $T\mu(h|_Y) \in X$ for all $h \in E^*$. We can conclude (just as before) that $T\mu(A(Y)) \subset X$ for all $\mu \in Y$. This completes the proof.

References

1. M. M. Day, *Fixed-point theorems for compact convex sets*, Illinois J. Math. **5** (1961), 585–590. (See Correction, ibid. **8** (1964), p. 713.)

2. T. Mitchell, *Function algebras, means, and fixed points*, Trans. Amer. Math. Soc. **130** (1968), 117–126.

3. R. R. Phelps, *Lectures on Choquet's Theorem*, Van Nostrand Mathematical Studies, No. 7, Van Nostrand, Princeton, N. J., 1966.

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