INFINITE-PRODUCT MARKOV PROCESSES(1)

BY

T. E. HARRIS

1. **Introduction and summary.** We discuss Markov random functions $x(t) = (x_1(t), x_2(t), \dots)$ where the component processes $x_i(t)$, $i = 1, 2, \dots$ are themselves Markov processes independent of one another. As remarked in the final paragraph, this is one way to study the temporal development of random counting measures or point processes, if we want to identify two sequences $x(t)$ and $x'(t)$ when the coordinates of $x'(t)$ are a permutation of the coordinates of $x(t)$. However the present paper studies some properties of the product process $x(t)$ when we do not make this identification.

If E_i is the state space of $x_i(t)$ and $E = E_1 \times E_2 \times \cdots$ is the state space of $x(t)$, we shall consider stochastically closed subsets of E. If $H \subset E$ is stochastically closed and $x', x'' \in H$, we give conditions under which H is a union of two disjoint stochastically closed sets, one containing x', the other x''. In some cases, every stochastically closed set has two points with this property.

A particular study is made of the case where $x_i(t)$ is a temporally homogeneous additive process in R_1 (similar results hold in R_k). If x', x'', and H are as above and $\sum (x_i' - x_i'')^2 = \infty$, then the above decomposition is possible. If t takes only non-negative integer values, an analogous result holds for a process of sums of independent random variables.

We also obtain an extension (the italicized words in the statement below) of a sufficient condition of Shepp [7] for distinguishability of a random sequence from its translates (but not of Shepp's necessary condition). Let $Z_1, Z_2, \dots$ be independent identically distributed random variables with the common distribution G, and let $y_1, y_2, \dots$ be a sequence of numbers such that $\sum (y_i - b)^2 = \infty$ for each real b; it will be evident later why we assume this rather than merely $\sum y_i^2 = \infty$. Let P_G and P_G^y be the product measures on $R_1 \times R_1 \times \cdots$ corresponding respectively to $Z_1, Z_2, \dots$ and $Z_1 + y_1, Z_2 + y_2, \dots$. Then there is a measurable set Γ in $R_1 \times R_1 \times \cdots$, which depends on the sequence $y_1, y_2, \dots$ *but does not depend on* G, such that $P_G(\Gamma) = 1$, $P_G^y(\Gamma) = 0$.

2. **Definition of the product Markov process.** We use the terminology and notation of E. B. Dynkin [5], [6]. For each $i = 1, 2, \dots$ let

$$X_i = (x_i(t), M_{it}, P_{ix_i})$$

Received by the editors December 1, 1966.

(1) This research was supported in part by the National Science Foundation through Grant NSF GP-6216.

be a nonterminating Markov process with state space $(E_i, \mathscr{B}_i)$ and stationary transition function $P_i(t, x_i, \Gamma) = P_{ix_i}\{x_i(t) \in \Gamma\}$, $\Gamma \in \mathscr{B}_i$, where $P_i(t, x_i, E_i) = 1$, $t \geqq 0$. Here $x_i(t) = x_i(t, \omega_i)$, $\omega_i \in \Omega_i$. It is assumed that $M_{it} \subset M_i^0$, $t \geqq 0$, and that P_{ix_i} is for each $i = 1, 2, \ldots$ and each $x_i \in E_i$ a probability measure on M_i^0. For the rest, each process X_i is assumed to satisfy conditions 3.1 A–G [6, Vol. 1, pp. 77–78].

The *product process* $X = (x(t), M_t, P_x)$ will be defined from the following elements([2]).

$$
\begin{aligned}
x &= (x_1, x_2, \ldots),\ x(t) = (x_1(t), x_2(t), \ldots),\ \omega = (\omega_1, \omega_2, \ldots),\\
E &= \prod E_i,\ \Omega = \prod \Omega_i,\ \mathscr{B} = \prod \mathscr{B}_i,\\
M_t &= \prod M_{it},\ M^0 = \prod M_i^0,\\
P_x &= \prod P_{ix_i},\ P(t, x, \cdot) = \prod P_i(t, x_i, \cdot).
\end{aligned}
\tag{2.1}
$$

It can be verified that the product process X satisfies each of the conditions 3.1 A–G [6, Vol. 1, pp. 77–78]. (The verification is routine and will not be included here.) *Hence, the process $X = (x(t), M_t, P_x)$ is Markov.*

3. **Properties of the product process.** We will make the following additional assumptions in the rest of this paper. In each of A1–A3, i runs through the positive integers.

A1. E_i is a compact metric space (hence separable) with metric D_i, and $\mathscr{B}_i$ is generated by the open sets. (In case E_i is locally compact separable metric, it is compactified as indicated later.)

A2. If f is continuous on E_i, then so is $T_{it}f$, where

$$
T_{it}f(x_i) = \mathscr{E}_{ix_i} f(x_i(t)) = \int_{E_i} P_i(t, x_i, dy_i) f(y_i). \tag{3.1}
$$

(the "Feller" property).

A3. P_i is stochastically continuous; i.e., if $U \subset E_i$ is open and $x_i \in U$, then $\lim_{t \downarrow 0} P_i(t, x_i, U) = 1$.

From A1, E is a compact topological space if we take the metric

$$
D(x, y) = \sum_{i=1}^{\infty} \frac{2^{-i} D_i(x_i, y_i)}{1 + D_i(x_i, y_i)}, \tag{3.2}
$$

and the measurable sets $\mathscr{B}$ are generated by the open sets.

THEOREM 3.1. *If X_i satisfies* A1–A3, $i = 1, 2, \ldots$, *then the product process X has a stochastically continuous Feller transition function $P(t, x, \Gamma)$, $\Gamma \in \mathscr{B}$.*

([2]) The fields M_t and M^0 will be enlarged below.

Proof(3). Let f be continuous (hence bounded and uniformly continuous) on E and let y be an arbitrary fixed point of E. For each $x \in E$ and $n=1, 2, \ldots$ put

$$x^{(n)} = (x_1, x_2, \ldots, x_n, y_{n+1}, y_{n+2}, \ldots).$$

Then $D(x, x^{(n)}) \leqq 2^{-n}$, $x \in E$, and hence, given $\varepsilon > 0$, we can choose the positive integer n_0 so that $n \geqq n_0$ implies $|f(x) - f(x^{(n)})| < \varepsilon$, $x \in E$.

Putting $x^{(n)}(t) = (x_1(t), x_2(t), \ldots, x_n(t), y_{n+1}, y_{n+2}, \ldots)$, $g_n(x_1, \ldots, x_n) = f(x^{(n)})$, we obtain

$$T_t f(x) = \mathscr{E}_x[f(x(t)) - f(x^{(n)}(t))] + \mathscr{E}_{x_1 x_2 \cdots x_n} g_n(x_1(t), \ldots, x_n(t)), \tag{3.3}$$

where $\mathscr{E}_{x_1 x_2 \cdots x_n}$ denotes expectation with respect to the finite product process $(x_1(t), x_2(t), \ldots, x_n(t))$. If $n \geqq n_0$, the first term on the right side of (3.3) is bounded by ε. The second term is a continuous function of x, since, using the Stone-Weierstrass theorem, we can approximate g_n uniformly within ε on $E_1 \times E_2 \times \cdots \times E_n$ by a finite sum of products of the form

$$g_{n1}(x_1) g_{n2}(x_2) \cdots g_{nn}(x_n) = h(x_1, \ldots, x_n),$$

where g_{ni} is continuous on E_i. Since each X_i is Feller,

$$\mathscr{E}_x h(x_1(t), \ldots, x_n(t)) = \prod_{i=1}^{n} \mathscr{E}_{x_i} g_{ni}(x_i(t))$$

is continuous in x_i, $i = 1, \ldots, n$, and hence is continuous in x. Hence $T_t f(x)$ is continuous in x.

If $U \subset E$ is open and $x \in U$, we find an open set $U' = U_1 \times U_2 \times \cdots \times U_k \times E_{k+1} \times \cdots$, U_i open in E_i, such that $x \in U' \subset U$. Then $P(t, x, U) \geqq \prod_{i=1}^{k} P_i(t, x_i, U_i)$, and the product approaches 1 as $t \to 0$. ▮

4. **Standard character of the product process.** Since the transition function $P(t, x, \Gamma)$ of X is a stochastically continuous Feller transition function on a compact metric space, P is the transition function of a *standard* Markov process which is equivalent to X. This follows from [6, Theorem 3.14, Vol. 1, p. 104]. We shall not repeat here the complete definition of "standard," but note that a standard process is strongly Markov and has right-continuous sample functions.

Actually, by virtue of assumptions A1–A3, we can and will take each of the processes X_i to be standard. Moreover, following the discussion in [5] and [6] leading to Theorem 3.14 of [6], we see that the process X is standard provided we replace M_t and M^0 by the larger fields $\overline{M}_{t+0}$ and $\overline{M}^0$. We shall suppose that this has been done, and henceforth can and will assume that our process X is standard.

---

(3) Cairoli, in [1] and an earlier note, has announced a number of results about products of two Markov processes; e.g., such properties as being strong, Hunt, standard, Feller, strongly Feller are productive. Since the entire proof for the Feller property of denumerable products is short, we give it here.

5. Decomposition of the state space.

Definition 5.1. A nonempty set $F\subset E$ is *stochastically closed* if $F\in\mathscr{B}$ and

$$P_x\{x(t)\in F,\ t\geqq 0\} = 1,\ x\in F. \tag{5.1}$$

The fact that { } in (5.1) is M^0-measurable follows from [6, §4.5, Vol. 1, pp. 110–111]. A set $F\in\mathscr{B}$ is *indecomposable* if it does not contain two disjoint stochastically closed subsets.

Definition 5.2. A stochastically closed set F is *perfectly decomposable* if(4) $F=F_1+F_2$, where F_1 and F_2 are stochastically closed.

Definition 5.3. The points $x'\in F$ and $x''\in F$ are said to *divide* the stochastically closed set F if $F=F_1+F_2$, $x'\in F_1$, $x''\in F_2$, and F_1 and F_2 are stochastically closed.

In many types of Markov processes, the state space is a union of indecomposable stochastically closed sets plus perhaps certain other sets. This is not characteristic of the process X of §4. In fact, in important cases, we will see that every stochastically closed set is perfectly decomposable, and we will give a sufficient condition for recognizing when two points divide a stochastically closed set.

It is convenient to have a sufficient condition for perfect decomposability in terms of transition probabilities. Theorem 6.1 will provide this. The following preliminary lemma is an easy consequence of known results on excessive functions.

Lemma 5.1. *Suppose $E=E_1+E_2$, where*

$$\begin{aligned} P(t,x,E_1) &= 1, \qquad x\in E_1,\quad t\geqq 0,\\ P(t,x,E_2) &= 1, \qquad x\in E_2,\quad t\geqq 0. \end{aligned} \tag{5.2}$$

Then E_1 and E_2 are stochastically closed.

Note that the first equality in (5.2) would not by itself imply the stochastic closure of E_1.

Proof of Lemma 5.1. Define f on E by $f(x)=0$, $x\in E_1$; $f(x)=\infty$, $x\in E_2$. Then f is excessive [6, Vol. 2, p. 1]. That is, $f(x)\geqq\mathscr{E}_xf(x(t))$ and $\lim_{t\downarrow 0}\mathscr{E}_xf(x(t))=f(x)$, $x\in E$. Since X is standard, E_2 cannot be reached from any point of E_1, i.e., E_1 is stochastically closed (ibid., p. 10, Theorem 12.5). A similar argument applies for E_2.

6. A criterion for division.

We define an important subalgebra of $\mathscr{B}=\mathscr{B}_1\times\mathscr{B}_2\times\cdots$.

Definition 6.1. Let $\mathscr{B}_\infty=\bigcap_{n=1}^{\infty}(\mathscr{B}_n\times\mathscr{B}_{n+1}\times\cdots)$. We call the sets in $\mathscr{B}_\infty$ "tail sets."

Theorem 6.1. *Let $H\in\mathscr{B}$ be stochastically closed. Suppose that $x'\in H$, $x''\in H$, and that there is a set $\Gamma\in\mathscr{B}_\infty$ such that $P(t,x',\Gamma)=1$ for almost every (Lebesgue) t while $P(t,x'',\Gamma)=0$ for a.e. t. Then x' and x'' divide H.* (See Definitions 5.1 and 5.3.)

(4) We write $F=F_1+F_2$ to mean that F is the union of the *disjoint* nonempty sets F_1 and F_2.

Note 1. For certain applications, the allowance by the hypotheses of an exceptional null set for t may be useful. It helped in an earlier version (later modified) of the proof of Theorem 7.1.

Note 2. The result is not true for processes X in arbitrary state spaces E where a field analogous to $\mathscr{B}_\infty$ is not defined. (It is the 0-1 property of $\mathscr{B}_\infty$ that counts.) Consider for example a process with states 1, 2, and 3 where 1 and 2 are absorbing and $0<P(t, 3, \{i\})<1$, $t>0$, $i=1, 2$. Take $H=E$, $x'=1$, $x''=2$, $\Gamma=\{1\}$. Then the hypotheses of the theorem (except the meaningless one about $\mathscr{B}_\infty$) are true, but E is not perfectly decomposable.

Proof of Theorem 6.1. First define

$$H_1 = \{x : x \in E, P(t, x, \Gamma) = 1 \text{ for a.e. } t\}. \tag{6.1}$$

Then $x' \in H_1$, which is thus not empty. For fixed $\Gamma_i \in \mathscr{B}_i$, $i=1, 2, \ldots$, the transition function $P_i(t, x_i, \Gamma_i)$ is measurable $\mathscr{B}_{[0,\infty)} \times \mathscr{B}_i$, where $\mathscr{B}_{[0,\infty)}$ is the ordinary Borel algebra of the nonnegative reals (see [6, Vol. 1, §3.17, pp. 98–99]). Since $\Gamma \in \mathscr{B}_n \times \mathscr{B}_{n+1} \times \cdots$, it follows that $P(t, x, \Gamma)$ is measurable $\mathscr{B}_{[0,\infty)} \times \mathscr{B}_n \times \mathscr{B}_{n+1} \times \cdots$ and hence measurable $\mathscr{B}_{[0,\infty)} \times \mathscr{B}_\infty$. Since H_1 is exactly the set of x such that

$$\int_0^\infty (1-P(t, x, \Gamma))\, dt = 0,$$

we see that $H_1 \in \mathscr{B}_\infty$.

Next note that

$$P(t, x'', H_1) = 0, \quad \textit{all } t \geqq 0. \tag{6.2}$$

For, from the assumption of the theorem, letting $t \geqq 0$ be fixed,

$$0 = P(s+t, x'', \Gamma) = \int P(t, x'', dy)P(s, y, \Gamma) \quad \text{for a.e. } s, \tag{6.3}$$

i.e., for a.e. s we have: $P(s, y, \Gamma)=0$ for a.e. $y(P(t, x'', \cdot)$ measure). Hence for a.e. $y(P(t, x'', \cdot)$ measure) we have: $P(s, y, \Gamma)=0$ for a.e. s. That is, $P(t, x'', H_1)=0$. This proves (6.2).

Next, letting $s \geqq 0$ be fixed, note that if $x \in H_1$, then $P(s+t, x, \Gamma)=1$ for a.e. t, implying by an argument like that above that for a.e. $y(P(s, x, \cdot)$-measure) we have $P(t, y, \Gamma)=1$ for a.e. t, i.e., $y \in H_1$.

Hence,

$$P(s, x, H_1) = 1, \quad \textit{all } s \geqq 0, x \in H_1. \tag{6.4}$$

Suppose that for some $t_0 \geqq 0$ and $x \in E$ we have $P(t_0, x, H_1)>0$. Since H_1 is a tail event and $P(t_0, x, \cdot)$ is a product measure, $P(t_0, x, H_1)$ is 1 or 0, and hence is 1. Using this fact, the relation

$$P(t_0+h, x, H_1) = \int_{H_1} P(t_0, x, dy)P(h, y, H_1), \qquad h \geqq 0, \tag{6.5}$$

and (6.4), we see that

$$P(t_0, x, H_1) > 0 \Rightarrow P(t, x, H_1) = 1, \qquad all\ t \geqq t_0. \tag{6.6}$$

Now define

$$\begin{aligned} H'' &= \{x : x \in E, P(t, x, H_1) = 0, all\ t \geqq 0\}, \\ H' &= E - H''. \end{aligned} \tag{6.7}$$

Note that $x'' \in H''$, from (6.2), and $x' \in H_1 \subset H'$, from (6.1) and (6.4). Moreover, arguing as we did for H_1 above, we see that $H'' \in \mathscr{B}_\infty$ and hence $H' \in \mathscr{B}_\infty$.

Let f be the indicator function of H_1. From (6.6) and the definition of H' we see that

$$\mathscr{E}_x \int_0^\infty f(x(s))\, ds = \int_0^\infty P(s, x, H_1)\, ds = \infty, \qquad x \in H', \tag{6.8}$$

while $\mathscr{E}_x \int_t^\infty f(x(s))\, ds = 0$, $x \in H''$, $t \geqq 0$. Hence

$$0 = \mathscr{E}_x \int_t^\infty f(x(s))\, ds = \mathscr{E}_x \mathscr{E}_{x(t)} \int_0^\infty f(x(s))\, ds, \qquad x \in H'', \quad t \geqq 0. \tag{6.9}$$

Because of (6.8), (6.9) implies $P_x\{x(t) \in H'\} = 0$, $x \in H''$, $t \geqq 0$, or

$$P(t, x, H'') = 1, \qquad x \in H'', \quad t \geqq 0. \tag{6.10}$$

Now suppose $x \in H'$. From (6.6) and the definition of H', we can find a positive number t such that

$$\begin{aligned} 1 = P(t+s, x, H_1) &= \int_{H'} P(s, x, dy) P(t, y, H_1) + \int_{H''} P(s, x, dy) P(t, y, H_1) \\ &= \int_{H'} P(s, x, dy) P(t, y, H_1), \qquad s \geqq 0, \end{aligned} \tag{6.11}$$

and the last integral in (6.11) can be 1 only if $P(s, x, H') = 1$, $s \geqq 0$. Hence

$$P(t, x, H') = 1, \quad x \in H', \quad t \geqq 0. \tag{6.12}$$

Lemma 5.1, (6.10), and (6.12) imply the stochastic closure of H' and H''. Since H is stochastically closed by assumption, we have $H = (H \cap H') + (H \cap H'')$, where $H \cap H'$ and $H \cap H''$ are stochastically closed and $x' \in H \cap H'$, $x'' \in H \cap H''$. ∎

COROLLARY TO THEOREM 6.1. *If E contains two disjoint stochastically closed $\mathscr{B}_\infty$-measurable sets K' and K'', then E is perfectly decomposable.* (*Take any $x' \in K'$, $x'' \in K''$, and put $\Gamma = K'$.*)

7. The case of additive processes. To get more specific results we now consider the case $E_i = R_1 \cup \{\infty\}$, $i = 1, 2, \ldots$, where R_1 is the real line and ∞ is a compactifying point. Let $\mathscr{B}_i$ be the σ-algebra generated by adding $\{\infty\}$ to the linear Borel sets. Assume that the part of X_i on R_1 is a temporally homogeneous additive process (same law for each i), while $P_1(t, \infty, \{\infty\})=1$ and $P_1(t, x_1, \{\infty\})=0$, $t \geqq 0$, $x_1 \in R_1$. To avoid trivialities we assume that $x_i(t)$, for fixed $t>0$, is not constant with probability 1.

It can be verified, and is essentially known, that each process X_i satisfies A1–A3 of §3, and thus may be assumed standard. Hence we may apply the results of §§3–6.

DEFINITION 7.1. Let $F=\{x : x \in E, x_i \in R_1, i=1, 2, \ldots\}$. Put $\theta=(0, 0, \ldots) \in F$. Note that F is stochastically closed.

THEOREM 7.1. *Suppose $H \subset F$ is stochastically closed, and x', $x'' \in H$ with $\sum_{i=1}^{\infty} (x_i'-x_i'')^2=\infty$. Then x' and x'' divide H (Definition 5.3). (Here ∞ means "infinitely large," and is not the compactifying point.)*

Proof. We will show how to construct a set Γ satisfying the conditions of Theorem 6.1. Note that if $x_i(t)$ has a finite second moment for $x_i(0) \neq \infty$, and if $\mathscr{E}_\theta x_i(t)=0$, we can immediately define(5)

$$\Gamma = \left\{ x : \lim \frac{\sum_{j=1}^{n_k} (x_j''-x_j')(x_j-x_j')}{\sum_{j=1}^{n_k} (x_j''-x_j')^2} = 0 \right\},$$

where the sequence $\{n_k\}$ will be defined in (7.4). Similar criteria are familiar and are suggested by considering the likelihood ratio in the Gaussian case.

The additivity of the processes X_i enables us to take $x'=(0, 0, \ldots)=\theta$ without loss of generality. We then put $y_i=x_i''$, $i=1, 2, \ldots$, and thus $\sum y_i^2=\infty$.

We shall use the function ϕ defined on $R_1 \cup \{\infty\}$:

$$\begin{aligned} \phi(u) &= 1 \quad \text{if } u > 0 \text{ or } u = \infty, \\ &= 0 \quad \text{if } u = 0, \\ &= -1 \quad \text{if } u < 0. \end{aligned} \tag{7.1}$$

Note that if the sequence $\{|y_i|\}$ is unbounded, the solution is particularly easy. Suppose $y_{n_1} < y_{n_2} < \cdots$.

Put

$$\Gamma_1 = \bigcup_{n=1}^{\infty} \left\{ x : \limsup_{i \to \infty} \frac{h_n(x_{n_1}) + \cdots + h_n(x_{n_i})}{i} > 0 \right\}, \tag{7.2}$$

where h_n is the indicator function of the interval $(-n, n)$. From the strong law of large numbers, $P(t, \theta, \Gamma_1)=1$ and $P(t, y, \Gamma_1)=0$, $t \geqq 0$. Hence Γ_1 can be the set Γ of Theorem 6.1. Note that the definition of Γ_1, while depending on y, does not depend on the distribution of the $x_i(t)$.

Thus, for the remainder of the proof we can and will assume

$$|y_i| \leqq K, \qquad i = 1, 2, \ldots, \tag{7.3}$$

y being a fixed sequence such that $\sum y_i^2=\infty$.

(5) In the fraction put $x_j-x_j'=0$, say, if $x_j=\infty$.

(a) Assume first that if $x_i(0)\neq\infty$, then $x_i(t)-x_i(0)$ has for each $t\geqq 0$ a symmetric distribution. Let $n_1<n_2<\cdots$ be positive integers such that

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{n_k} y_j^2\right)^{-1} < \infty. \tag{7.4}$$

Define $T_{k,n}(x)$ and $\psi_n(x)$, $x\in E$, $k, n=1, 2, \ldots$, by

$$T_{k,n}(x) = \frac{\sum_{i=1}^{n_k} y_i \int_{-n}^{n} \phi(x_i+s)\,ds}{\sum_{i=1}^{n_k} y_i^2}, \tag{7.5}$$

$$\psi_n(x) = \limsup_{k\to\infty} |T_{k,n}(x)|,$$

agreeing that $x_i+s=\infty$ if $x_i=\infty$ in (7.5). Put

$$\Gamma^{(n)} = \{x : x\in E, \psi_n(x) = 0\}, \quad \Gamma_2 = \bigcap_{n=1}^{\infty} \Gamma^{(n)}. \tag{7.6}$$

Obviously $\Gamma^{(n)}$ and Γ_2 are $\mathscr{B}_\infty$-measurable. We shall see that $P(t, \theta, \Gamma^{(n)})=1$, $t>0$, while $P(t, y, \Gamma^{(n)})=0$ for all sufficiently large n, so that $P(t, \theta, \Gamma_2)=1$ and $P(t, y, \Gamma_2)=0$.

Since ϕ is an odd function on R_1 and $x_i(t)-x_i(0)$ has a symmetric distribution, $\mathscr{E}_\theta\phi(x_i(t)+s)+\mathscr{E}_\theta\phi(x_i(t)-s)=0$ and hence $\mathscr{E}_\theta T_{k,n}(x(t))=0$. Letting $\sigma_{k,n}^2(x)$ be the variance of $T_{k,n}(x(t))$ calculated for $x(0)=x$, we have

$$\sigma_{k,n}^2(x) \leqq \frac{\sum_{i=1}^{n_k} y_i^2(2n)^2}{\left(\sum_{i=1}^{n_k} y_i^2\right)^2} \leqq 4n^2\left(\sum_{i=1}^{n_k} y_i^2\right)^{-1}, \quad x\in E, \tag{7.7}$$

and $\sum_{k=1}^{\infty}\sigma_{k,n}^2(x)<\infty$ because of (7.4). Hence $P_\theta\{\lim_{k\to\infty} T_{k,n}(x(t))=0\}=1$. That is,

$$P(t, \theta, \Gamma^{(n)}) = 1, \qquad n = 1, 2, \ldots, \quad t\geqq 0. \tag{7.8}$$

Because of the symmetry of the distribution of $x_i(t)$, we have for $b>0$

$$\begin{aligned} b\mathscr{E}_\theta\phi(x_i(t)+b) = b\{&P_\theta(x_i(t) > b)+P_\theta(0 \leqq x_i(t) \leqq b) \\ &+P_\theta(-b < x_i(t) < 0)-P_\theta(x_i(t) < -b)\}, \end{aligned} \tag{7.9}$$

whence

$$b\mathscr{E}_\theta\phi(x_i(t)+b) \geqq |b|P_\theta(0 \leqq x_i(t) \leqq |b|). \tag{7.10}$$

A similar argument shows that (7.10) is still true for $b<0$.

Now recall (7.3) and consider, for $n \geqq K \geqq y_i \geqq 0$ (we consider $y_i < 0$ below),

$$(7.11)\qquad \begin{aligned} y_i\mathscr{E}_y \int_{-n}^{n} \phi(x_i(t)+s)\,ds &= y_i\mathscr{E}_\theta \int_{-n}^{n} \phi(x_i(t)+y_i+s)\,ds \\ &= y_i\mathscr{E}_\theta \int_{-(n-y_i)}^{n-y_i} \phi(x_i(t)+s)\,ds + y_i\mathscr{E}_\theta \int_{n-y_i}^{n+y_i} \phi(x_i(t)+s)\,ds. \end{aligned}$$

Of the last two expectations in (7.11), the first vanishes for the same reason that $\mathscr{E}_\theta T_{k,n}=0$ above. As for the second, using (7.10), we find

$$(7.12)\qquad y_i\mathscr{E}_\theta \int_{n-y_i}^{n+y_i} \phi(x_i(t)+s)\,ds \geqq y_i \int_{n-y_i}^{n+y_i} P_\theta(0 \leqq x_i(t) \leqq s)\,ds,$$

and hence, from (7.11) and (7.12),

$$(7.13)\qquad y_i\mathscr{E}_y \int_{-n}^{n} \phi(x_i(t)+s)\,ds \geqq 2y_i^2 P_\theta(0 \leqq x_1(t) \leqq n-K).$$

Almost the same argument shows that (7.13) is also true if $n \geqq K$ and $y_i < 0$, and hence it holds whenever $n \geqq K$. From (7.5) and (7.13),

$$(7.14)\qquad \mathscr{E}_y T_{k,n}(x(t)) \geqq 2P_\theta(0 \leqq x_1(t) \leqq n-K).$$

Using (7.14) and (7.7) and noting from (7.4) that $\lim_{k\to\infty} \sigma_{k,n}^2(y)=0$, we see that when n is large enough so that the right side of (7.14) is positive, $P(t, y, \Gamma^{(n)})=0$, $t>0$. From this and (7.8) we see that Γ_2 as defined by (7.6) can be the set Γ of Theorem 6.1. Note that Γ_2 was defined without reference to the distribution of the $x_i(t)$.

(b) Drop the assumption of symmetry. Suppose, however, that

$$(7.15)\qquad \sum_{i=1}^{\infty} (y_i-b)^2 = \infty \qquad \textit{for each } b, \quad -\infty < b < \infty.$$

(The contrary case is simple, but is essentially different and is treated below.) It follows from (7.15) by routine arguments that will not be given here, that there exist distinct positive integers $m_1, m_2, \ldots$ (the sequence $\{m_i\}$ is not necessarily increasing) such that

$$(7.16)\qquad (y_{m_1}-y_{m_2})^2+(y_{m_3}-y_{m_4})^2+\cdots = \infty.$$

Define the mapping ξ of E onto E by

$$\xi(x) = (x_{m_1}-x_{m_2}, x_{m_3}-x_{m_4}, \ldots),$$

where it is agreed that if either x_{m_i} or $x_{m_{i+1}}=\infty$, then so is their difference. Let $w(t)=(w_1(t), w_2(t), \ldots)=\xi(x(t))$. If $x(0)=\theta$, then the $w_i(t)$ are symmetric independent random variables.

Put $y'=\xi(y)$, and let $n_1'<n_2'\cdots$ be a sequence such that (7.4) holds with each component of y replaced by the corresponding one of y' and $\{n_k\}$ replaced by $\{n_k'\}$. Let Γ_2' be constructed just as was Γ_2, except that y is replaced by y' and $\{n_k\}$

by $\{n_k'\}$. Then $P_\theta(w(t) \in \Gamma_2')=1$ and $P_y(w(t) \in \Gamma_2')=0$, from the argument of (a) above. Putting $\Gamma_3=\xi^{-1}(\Gamma_2')$, we see that $P(t, \theta, \Gamma_3)=1$, $P(t, y, \Gamma_3)=0$, $t>0$, and hence Γ_3 can serve as Γ of Theorem 6.1. Also, Γ_3 was defined without reference to the distribution of the $x_i(t)$.

(c) Finally, suppose that for some $b \neq 0$,

$$\sum (y_i - b)^2 < \infty. \tag{7.17}$$

The difference of this case from the others is evident if we suppose each $x_j(t)$ has a finite variance and $\mathscr{E}_\theta(x_j(t))=\mathscr{E}_\theta(x_j(t)-t)^2=t$. If $x(0)=\theta$, then the components of $x(2)$ have means 2 and variances 2, while if $x(\theta)=(1, 1, \ldots)$, the components of $x(1)$ have means 2 and variances 1. That is, distinguishing the starting point θ from the starting point $(1, 1, \ldots)$ requires distinguishing not only translations but also changes of scale.

The canonical form of the logarithm of the characteristic function of $x_j(t)$, if $x_j(0)=0$, is

$$tR(s)+itI(s), \quad -\infty < s < \infty, \tag{7.18}$$

where R and I are finite functions whose precise form is immaterial. Pick α so that

$$0 < b\alpha < 2\pi \quad \text{and} \quad R(\alpha) \neq 0. \tag{7.19}$$

This is possible since the nondegeneracy of $x_i(t)$ implies that R cannot vanish identically in a neighborhood of 0.

Letting Re ( ) and Im ( ) denote real and imaginary parts and agreeing that $e^{i\alpha\infty}=0$, we put

$$\Gamma_4 = \Bigg\{x : x \in E, \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{i\alpha x_j} \text{ exists } (= U(x), \text{ say}) \text{ and is } \neq 0, \text{ and} \\ \exp\left[\frac{iI(\alpha)}{R(\alpha)} \operatorname{Re}(\log U(x)) - i \operatorname{Im}(\log U(x))\right] = 1\Bigg\}. \tag{7.20}$$

It is easily verified that $P_\theta\{x(t) \in \Gamma_4\}=1$, $t>0$. On the other hand, we have, using (7.15),

$$P_\theta\left\{\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{i\alpha x_j(t)+i\alpha y_j} = \exp\left[tR(\alpha)+itI(\alpha)+ib\alpha\right]\right\} = 1,$$

which shows that $P_\theta\{y+x(t) \in \Gamma_4\}=0$, $t>0$. Hence (7.20) furnishes an appropriate Γ. This completes the proof of Theorem 7.1. ∎

An easy corollary of Theorem 7.1 is that *every stochastically closed subset H of F is perfectly decomposable.* For if $x' \in H$, then

$$P_{x'}\left\{\sum_{i=1}^{\infty} (x_i(t)-x_i')^2 < \infty\right\} = 0, \qquad t > 0.$$

Hence there must be an $x'' \in H$ such that $\sum (x_i''-x_i')^2=\infty$.

8. **Extensions and relation to other results.** Essentially the same methods apply if each $x_i(t)$ is an additive process in a p-dimensional Euclidean space, $x_i(t)=(x_{i1}(t),\dots,x_{ip}(t))$, with $\|x_i(t)\|^2=\sum_j (x_{ij}(t))^2$, assuming that each component $x_{ij}(t)$ is nondegenerate. Theorem 7.1 is true if we take $F=\{x$: all components of each x_i are finite$\}$ and assume that $\sum \|x_i'-x_i''\|^2=\infty$. The definition of $T_{k,n}(x)$ in (7.5) is changed to

$$\frac{\sum_{i=1}^{n_k}\sum_{j=1}^{p} y_{ij}\int_{-n}^{n}\phi(x_{ij}+s)\,ds}{\sum_{i=1}^{n_k}\|y_i\|^2}.$$

If t is discrete, $t=0, 1, 2,\dots$, then Theorem 6.1 still holds, if we replace "almost every t" in the hypotheses by "every $t=0, 1,\dots$." In Theorem 7.1, we can then assume that $x_i(t+1)-x_i(t)$, $t=0, 1,\dots$, $i=1, 2,\dots$, are arbitrary identically distributed independent random variables. The proof of Theorem 7.1 is unchanged, with one exception. In case (c), where $\sum (y_i-b)^2<\infty$ for some $b\neq 0$, we take α close enough to 0 so that $\mathscr{E}_\theta e^{i\alpha x_1(1)}\neq 0$, besides requiring (7.19).

Shepp [7] considered the following problem. Let $Z_1, Z_2,\dots$ be independent random variables with a common distribution G, let $y=(y_1, y_2,\dots)$ be a nonrandom sequence, and let P_G and P_G^y be the probability measures on $R_1\times R_1\times\cdots$ generated respectively by $Z_1, Z_2,\dots$ and $Z_1+y_1, Z_2+y_2,\dots$. Shepp showed by another method(6) that if $\sum y_i^2=\infty$, there is a measurable Γ_0 in $R_1\times R_1\times\cdots$ such that $P_G(\Gamma_0)=1$ and $P_G^y(\Gamma_0)=0$; this is part of his necessary and sufficient conditions. Since we have found a set Γ such that $P(t, \theta, \Gamma)=1$ and $P(t, y, \Gamma)=0$, $t=1, 2,\dots$ in the discrete case, we obtain Shepp's sufficient condition by putting $t=1$ and $\Gamma_0=\Gamma\cap F$ (see Definition 7.1). However, we can now state the following extension of Shepp's result, noting that except for the case (c) above where $\sum (y_i-b)^2<\infty$ for some b, our set Γ_0 is constructed without reference to the distribution of the $x_i(t)$.

THEOREM 8.1. *Let $Z_1, Z_2,\dots$ be independent identically distributed random variables and let $y_1, y_2,\dots$ be a nonrandom sequence such that $\sum_{i=1}^{\infty}(y_i-b)^2=\infty$ for each real b. Then there is a measurable set Γ_0 in $R_1\times R_1\times\cdots$, whose construction depends on $y_1, y_2,\dots$ but not on the distribution of the Z_i, such that $P_G(\Gamma_0)=1$ and $P_G^y(\Gamma_0)=0$.*

A similar result holds if Z_i are p-dimensional vectors.

9. **Point processes or counting measures.** For some purposes it is appropriate to identify the two sequences $(x_1, x_2,\dots)$ and $(x_1', x_2',\dots)$ if one is obtained by permuting the coordinates of the other. The equivalence classes so obtained are point

(6) Use was made of a well-known theorem of Kakutani on product measures. The sufficient condition of Shepp also follows from recent results of Dudley [4], but without the distribution-free property of Theorem 8.1 that is crucial for our purposes.

processes or counting measures. Thus we have a way of treating Markov processes whose states are infinite counting measures. In particular, for the case where the $x_i(t)$ are additive processes, we have the kind of situation considered by Doob [3, p. 404], Dobrushin [2], and others. The author hopes to discuss such processes in a later paper.

References

1. R. Cairoli, *Sur les fonctions séparément excessives*, C. R. Acad. Sci. Paris **261** (1965), 4323–4326.

2. R. L. Dobrushin, *On the law of Poisson for the distribution of particles in space*, Ukrain. Mat. Ž. **8** (1956), 127–134. (Russian)

3. J. L. Doob, *Stochastic processes*, Wiley, New York, 1953.

4. R. M. Dudley, *Singularity of measures on linear spaces*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **6** (1966), 129–132.

5. E. B. Dynkin, *Theory of Markov processes*, Prentice-Hall, Englewood Cliffs, N. J., 1961.

6. ———, *Markov processes*, Vols. 1 and 2, Springer, Berlin, 1965.

7. L. A. Shepp, *Distinguishing a sequence of random variables from a translate of itself*, Ann. Math. Statist. **36** (1965), 1107–1112.

University of Southern California,
Los Angeles, California