

PROPERTIES OF THE SEQUENCE OF CLOSED POWERS OF A MAXIMAL IDEAL IN A SUP-NORM ALGEBRA⁽¹⁾

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1. A sup-norm algebra is a closed point-separating subalgebra A of $C(X)$ containing the constants, where X is a compact Hausdorff space. M_A will denote the (compact Hausdorff) space of multiplicative linear functionals φ on A in the Gelfand topology, and to each such φ we associate its kernel A_φ ; the A_φ are precisely the maximal ideals of A . X can be viewed as a closed subset of M_A , and there is an isomorphism $f \rightarrow \hat{f}$ of A into $C(M_A)$ such that $\hat{f}|X = f$. The isomorphism is given by $\hat{f}(\varphi) = \varphi(f)$. We will frequently write f for \hat{f} . S_A will denote the Šilov boundary of A , that is, the smallest closed subset of X on which each function in A attains its maximum modulus. Each $f \in A$ has the same supremum over S_A , X , and M_A ; this common supremum is the norm of f , written $\|f\|$.

We intend to study the ideals $(A_\varphi^n)^-$. [Note: If I is an ideal in A , I^n is the ideal generated by products $f_1 \cdots f_n$ with $f_i \in I$ if $n \geq 1$, and $I^0 = A$; also \bar{I} denotes the closure of I .] Evidently $(A_\varphi^n)^- \supset (A_\varphi^{n+1})^-$ and if the inclusion is an equality for some n , then it is an equality for all subsequent n . A problem of the sort that will interest us—indeed, in great part the stimulus for this research—is to determine whether one can preassign the cutoff point in this chain of inclusions: given n , is it possible to have $(A_\varphi^n)^- \neq (A_\varphi^{n+1})^-$ but $(A_\varphi^{n+1})^- = (A_\varphi^{n+2})^-$?

More generally, how much control do we have over the sequence of numbers $\{\dim((A_\varphi^n)^- / (A_\varphi^{n+1})^-) : n = 1, 2, \dots\}$? While we will not give an explicit arithmetic answer, we will reduce the problem to a purely algebraic one. In particular, it is possible to obtain any nonincreasing sequence of nonnegative integers, and hence to preassign the cutoff point.

We now will begin to formulate the problem we will actually solve. First, consider a graded algebra (all algebras in this paper are commutative algebras over the complex field C) $Q = \sum_{n=0}^{\infty} Q_n$ and suppose that Q_0 consists of scalar multiples of an identity for Q , and that Q_0 and Q_1 generate Q . If $\{F_\gamma : \gamma \in \Gamma\}$ is a basis for Q_1 and $R = C[\{X_\gamma : \gamma \in \Gamma\}]$ is the polynomial ring in Γ variables with its usual graded algebra structure, the linear extension of the map $X_{\gamma_1} \cdots X_{\gamma_n} \rightarrow F_{\gamma_1} \cdots F_{\gamma_n}$ is a graded algebra homomorphism of R onto Q . Thus Q is isomorphic as a

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graded algebra to R/J where J is some homogeneous ideal in R . (For algebraic terminology and theorems consult [ZS].)

Next, suppose B is a linear algebra and let $\{I_n : n=0, 1, 2, \dots\}$ be a sequence of ideals in B such that $I_0=B$, $I_{n+1} \subset I_n$, and $I_p I_q \subset I_{p+q}$. We associate with $\{I_n\}$ a graded vector space $V = \sum_{n=0}^{\infty} \oplus V_n$ where $I_n : I_n \rightarrow V_n$ is some fixed linear isomorphism of I_n onto a space V_n . Let $U = \sum_{n=0}^{\infty} \oplus U_n$ where $U_n = I_n(I_{n+1}) \subset V_n$, and let Q be the quotient graded vector space $Q = V/U = \sum_{n=0}^{\infty} \oplus Q_n$ where $Q_n = V_n/U_n$. V is a graded algebra if we extend linearly the map given by $b * c = I_{p+q}(I_p^{-1}(b) \cdot I_q^{-1}(c))$ for $b \in V_p$ and $c \in V_q$, and U is a homogeneous ideal of V ; hence the multiplication $(b + U_p) * (c + U_q) = b * c + U_{p+q}$ for $b \in V_p$ and $c \in V_q$ makes Q into a graded algebra.

Let us now apply this chain of ideas to the situation when B is a sup-norm algebra A and $I_n = (A_\phi^n)^-$ for some $\phi \in M_A$. Associate to this situation a graded algebra Q as in the last paragraph, so $Q_n = (A_\phi^n)^- / (A_\phi^{n+1})^-$. Evidently Q_0 consists of scalar multiples of an identity for Q . In general it is false that Q_0 and Q_1 generate Q . (The closure of the polynomials on a countable product of discs yields a counterexample.) However, if Q_1 has finite dimension, say r , we can show that Q_0 and Q_1 generate Q . This amounts to showing that for $n > 1$ the products $b_1 * b_2 * \dots * b_n$ with $b_i \in Q_1$ span Q_n . Let Q'_n denote the span of these products. Q'_n is a finite-dimensional subspace of Q_n , hence is closed in Q_n where the latter has its norm topology as $(A_\phi^n)^- / (A_\phi^{n+1})^-$. It remains to prove density of Q'_n in Q_n .

Let F_1, \dots, F_r be a basis for Q_1 and let F_j have the representative $f_j \in A_\phi$. Let $f \in (A_\phi^n)^-$. Select a sequence $\{g_k\}$ in A_ϕ^n such that $\|f - g_k\| \rightarrow 0$. Write $g_k = \sum_{i=1}^{t_k} g_{ki1} \dots g_{kin}$ where $g_{kij} \in A_\phi$. Write $g_{kij} = \sum_{s=1}^r \alpha_{kij s} f_s + h_{kij}$ where $h_{kij} \in (A_\phi^2)^-$. Then

$$I_n(g_{ki1} \dots g_{kin}) + U_n = I_n\left(\prod_{j=1}^n \left(\sum_{s=1}^r \alpha_{kij s} f_s\right)\right) + U_n = \prod_{j=1}^n \left(\sum_{s=1}^r \alpha_{kij s} F_s\right) \in Q'_n,$$

so $I_n(g_k) + U_n \in Q'_n$. But $\|(I_n(f) + U_n) - (I_n(g_k) + U_n)\| \leq \|f - g_k\| \rightarrow 0$, so $I_n(f) + U_n$ is in the closure of Q'_n . Thus Q'_n is dense in Q_n , hence $Q'_n = Q_n$. Therefore Q_0 and Q_1 generate Q .

Combining with our first remarks about graded algebras, we see that we have proved the following:

THEOREM 1. *Let A be a sup-norm algebra, $\phi \in M_A$. Then the direct sum $\sum_{n=0}^{\infty} \oplus ((A_\phi^n)^- / (A_\phi^{n+1})^-)$ has in a natural way the structure of a graded algebra Q . If $\dim(A_\phi / (A_\phi^2)^-) = r < \infty$ then Q is isomorphic as a graded algebra to $C[X_1, \dots, X_r]/J$ where J is a homogeneous ideal in $C[X_1, \dots, X_r]$ and $J_0 = J_1 = \{0\}$.*

Note that we really only require A to be a Banach algebra with identity. Further, the same arguments work if A is simply a linear algebra with identity, A_ϕ is the kernel of a multiplicative linear functional ϕ , and we take $I_n = (A_\phi^n)^-$ (rather than $(A_\phi^n)^-$); and in this case no finiteness hypothesis is required.

A consequence of Theorem 1 is our first quantitative information: if

$$\dim (A_\varphi / (A_\varphi^2)^-) = r < \infty$$

then $\dim ((A_\varphi^n)^- / (A_\varphi^{n+1})^-) \leq C_{n+r-1, n}$ for all n . In addition, the Hilbert-Serre theorem [ZS, p. 232] says that there is a polynomial G of degree at most $r-1$ such that $\dim ((A_\varphi^n)^- / (A_\varphi^{n+1})^-) = G(n)$ for n large. (By convention, the zero polynomial has degree -1 .)

Theorem 1 can be radically refined if φ has a unique (nonnegative regular Borel) representing measure on X (e.g., if A is Dirichlet or, more generally, logmodular). It is then known [S1] that either Q is 1-dimensional (i.e., $Q_n = 0$ for $n > 0$) or Q is isomorphic to $C[X_1]$, according as φ is or is not a one-point part for A .

The main objective of this paper is to obtain a converse to Theorem 1: if J is a homogeneous ideal in $C[X_1, \dots, X_r]$, we find a sup-norm algebra A and $\varphi \in M_A$ whose corresponding graded algebra is isomorphic to $C[X_1, \dots, X_r]/J$. §§2 and 3 are devoted to this, and it is accomplished in Theorem 3.17. In §4 we show that the algebras constructed to prove Theorem 3.17 are actually local algebras.

The construction we use, which is developed in §3, is useful for producing examples of many pathologies concerned with the $(A_\varphi^n)^-$. For example, one can construct a one-point part φ supporting a nontrivial bounded derivation (Example 5.13 below). §5 is devoted to exploration of the types of algebra to which our construction leads, and to examples.

Let me record here my debt to Professors E. L. Stout and Lynn H. Loomis for valuable consultation during this research; and most especially to Professor Andrew M. Gleason, who gave this work its initial stimulus both by suggesting the study of derivations and by showing the author the algebra (Example 5.14 below) which motivated our construction, and whose ideas have played an important role throughout.

2. We will have to know something about tensor products, and this section is devoted to the relevant material. For our purposes it consists entirely of technical lemmas. The early material is standard, and proofs are omitted; the reader ought to be able to fill them in, or he can consult the literature (see for example [Ge]). The later material on the interplay between tensor products and direct sums is probably new, but is quite easy.

Let Γ be an index set, and for each $\gamma \in \Gamma$, let X_γ be a compact Hausdorff space. If for each γ we are given $f_\gamma \in C(X_\gamma)$ such that $f_\gamma = 1$ for all but finitely many γ , define a function $\bigotimes \{f_\gamma\}$ on $X = \prod X_\gamma$ by

$$(\bigotimes \{f_\gamma\})(\{x_\gamma\}) = \prod f_\gamma(x_\gamma).$$

X is a compact Hausdorff space in the product topology, and $\bigotimes \{f_\gamma\} \in C(X)$.

If W is a subset of a vector space V , $\text{sp}(W)$ will denote the linear span of W ; if V is also a topological vector space, $\text{sp}(W)^-$ is the closure of $\text{sp}(W)$. Suppose for each γ we are given a nonempty subset W_γ of $C(X_\gamma)$, and suppose W_γ contains

1 for all but finitely many γ . $\bigotimes \{W_\gamma\} = \text{sp} \{ \bigotimes \{f_\gamma\} : f_\gamma \in W_\gamma \}$ is the tensor product of the W_γ and $\bigotimes \{W_\gamma\}^-$ is its closure, the closed tensor product of the W_γ . Clearly $\bigotimes \{W_\gamma\} = \bigotimes \{ \text{sp}(W_\gamma) \}$ and $\bigotimes \{W_\gamma\}^- = \bigotimes \{W_\gamma^-\}^-$.

Should Γ be finite we can write $X = X_1 \times \cdots \times X_n$, $f_1 \otimes \cdots \otimes f_n$ and $W_1 \otimes \cdots \otimes W_n$ with the obvious meanings.

If Γ is a disjoint union $\Gamma = \bigcup \{ \Gamma_\sigma : \sigma \in \Sigma \}$ where Γ_σ is nonempty, there are obvious natural identifications

$$\bigotimes \{W_\gamma : \gamma \in \Gamma\} = \bigotimes \{ \bigotimes \{W_\gamma : \gamma \in \Gamma_\sigma\} : \sigma \in \Sigma \}$$

and

$$\begin{aligned} \bigotimes \{W_\gamma : \gamma \in \Gamma\}^- &= \bigotimes \{ \bigotimes \{W_\gamma : \gamma \in \Gamma_\sigma\} : \sigma \in \Sigma \}^- \\ &= \bigotimes \{ \bigotimes \{W_\gamma : \gamma \in \Gamma_\sigma\}^- : \sigma \in \Sigma \}^-. \end{aligned}$$

Let Γ be a disjoint union $\Gamma = \Gamma^1 \cup \Gamma^2$ with Γ^j nonempty. Write

$$X^j = \prod \{X_\gamma : \gamma \in \Gamma^j\},$$

so $X = X^1 \times X^2$ in a natural fashion. For each $x^2 \in X^2$ we get an imbedding $e_{x^2}: X^1 \rightarrow X$ by $e_{x^2}(x^1) = (x^1, x^2)$, and the $e_{x^2}(X^1)$ for various x^2 provide a disjoint covering of X . The dual map $e_{x^2}^*: C(X) \rightarrow C(X^1)$ is a norm-decreasing algebra homomorphism.

LEMMA 2.1. $e_{x^2}^*(\bigotimes \{W_\gamma : \gamma \in \Gamma\}) \subset \bigotimes \{W_\gamma : \gamma \in \Gamma^1\}$ and $e_{x^2}^*(\bigotimes \{W_\gamma : \gamma \in \Gamma\}^-) \subset \bigotimes \{W_\gamma : \gamma \in \Gamma^1\}^-$. If there exists for each $\gamma \in \Gamma^2$ an $f_\gamma \in W_\gamma$ such that $f_\gamma(x_\gamma^2) \neq 0$, then these inclusions are actually equalities.

In particular, if $\gamma_0 \in \Gamma$ then as a function of x_{γ_0} in X_{γ_0} alone, each $f \in \bigotimes \{W_\gamma\}$ is in $\text{sp}(W_{\gamma_0})$ and each $f \in \bigotimes \{W_\gamma\}^-$ is in $\text{sp}(W_{\gamma_0})^-$.

For each γ let Y_γ be a nonempty closed subset of X_γ and let $Y = \prod Y_\gamma$. The inclusions $l_\gamma: Y_\gamma \rightarrow X_\gamma$ induce an inclusion $l: Y \rightarrow X$ and the dual map $l^*: C(X) \rightarrow C(Y)$ where $l^*(f) = f|_Y$.

LEMMA 2.2. l^* is a norm-decreasing algebra homomorphism, $l^*(\bigotimes \{W_\gamma\}) = \bigotimes \{W_\gamma|Y_\gamma\}$, and $l^*(\bigotimes \{W_\gamma\}^-) \subset \bigotimes \{W_\gamma|Y_\gamma\}^-$. If $\gamma \in \Gamma$ and $f \in \text{sp}(W_\gamma)$ imply $\|f\|_{Y_\gamma} = \|f\|_{X_\gamma}$, then $l^*(\bigotimes \{W_\gamma\}^-)$ is norm-preserving and $l^*(\bigotimes \{W_\gamma\}^-) = \bigotimes \{W_\gamma|Y_\gamma\}^-$. (The norms are the obvious suprema.)

If $W_\gamma = A_\gamma$ a sup-norm algebra on X_γ , then $\bigotimes \{A_\gamma\}$ is a point-separating algebra with 1 on $\prod X_\gamma$ and $\bigotimes \{A_\gamma\}^-$ is a sup-norm algebra on $\prod X_\gamma$. $S_{A_\gamma} \subset X_\gamma \subset M_{A_\gamma}$ and we can think of A_γ as a sup-norm algebra on M_{A_γ} . Let $l_1: \prod S_{A_\gamma} \rightarrow \prod X_\gamma$ and $l_2: \prod X_\gamma \rightarrow \prod M_{A_\gamma}$ be the inclusions, with dual maps $l_1^*: C(\prod X_\gamma) \rightarrow C(\prod S_{A_\gamma})$ and $l_2^*: C(\prod M_{A_\gamma}) \rightarrow C(\prod X_\gamma)$. Lemma 2.2 leads to

THEOREM 2.3. l_1^* maps $\bigotimes \{A_\gamma|X_\gamma\}^-$ isometrically and isomorphically onto $\bigotimes \{A_\gamma|S_{A_\gamma}\}^-$. l_2^* maps $\bigotimes \{A_\gamma\}^-$ isometrically and isomorphically onto $\bigotimes \{A_\gamma|X_\gamma\}^-$. $M_{\bigotimes \{A_\gamma\}^-} = \prod M_{A_\gamma}$ and $S_{\bigotimes \{A_\gamma\}^-} = \prod S_{A_\gamma}$.

Our emphasis will now shift to the behavior of subspaces in tensor products. The set Γ will, until notice to the contrary is given, have cardinality 2.

Let V_1, \dots, V_n be subspaces of a vector space V . In addition to their vector space sum $\text{sp}(V_1 \cup \dots \cup V_n) = V_1 + \dots + V_n \subset V$, we can define their direct sum $V_1 \oplus \dots \oplus V_n$. This, of course, is not a subspace of V ; it is the space of n -tuples (v_1, \dots, v_n) where $v_j \in V_j$ with coordinate-wise addition and scalar multiplication. There is an obvious natural map of $V_1 \oplus \dots \oplus V_n$ into V , and its image is $V_1 + \dots + V_n$. If this map is injective, we write $V_1 \oplus \dots \oplus V_n \equiv V_1 + \dots + V_n$. If V_0 is another subspace of V , the expression $V_1 \oplus \dots \oplus V_n \equiv V_0$ means $V_0 = V_1 + \dots + V_n$ and $V_1 \oplus \dots \oplus V_n \equiv V_1 + \dots + V_n$.

If in addition V is normed then $V_1 \oplus \dots \oplus V_n$ will always be understood with the norm $\|(v_1, \dots, v_n)\| = \|v_1\| + \dots + \|v_n\|$. Then the natural map into V is automatically continuous. If it is actually a homeomorphism (so in particular $V_1 \oplus \dots \oplus V_n \equiv V_1 + \dots + V_n$) we write $V_1 \oplus \dots \oplus V_n \equiv V_1 + \dots + V_n$ topologically. The expression $V_1 \oplus \dots \oplus V_n \equiv V_0$ topologically has the obvious meaning.

LEMMA 2.4. *Suppose $\emptyset \neq W \subset C(X_1)$ and V_1 and V_2 are closed subspaces of $C(X_2)$ such that $V_1 \cap V_2 = \{0\}$. Then $(W \otimes V_1)^- \cap (W \otimes V_2)^- = \{0\}$.*

Proof. Use Lemma 2.1. \square

LEMMA 2.5. *Suppose $\emptyset \neq W \subset C(X_2)$ and V_1, \dots, V_n, V are closed subspaces of $C(X_2)$ such that $V_1 \oplus \dots \oplus V_n \equiv V$. Then*

- (i) $V_{i_1} + \dots + V_{i_q}$ is closed in $C(X_2)$, $1 \leq i_1 < \dots < i_q \leq n$.
- (ii) $V_{i_1} \oplus \dots \oplus V_{i_q} \equiv V_{i_1} + \dots + V_{i_q}$ topologically, $1 \leq i_1 < \dots < i_q \leq n$.
- (iii) $(W \otimes V_1)^- \oplus \dots \oplus (W \otimes V_n)^- \equiv (W \otimes V)^-$ topologically.

Proof. (i) and (ii) are an easy exercise using the open mapping theorem. They permit us (by induction) to reduce (iii) to the case $n=2$, the case $n=1$ being trivial.

Because $V_1 \cap V_2 = \{0\}$, Lemma 2.4 implies $(W \otimes V_1)^- \cap (W \otimes V_2)^- = \{0\}$. Since $(W \otimes V_1)^-, (W \otimes V_2)^-$ and $(W \otimes V)^-$ are closed subspaces of $C(X_1 \times X_2)$, in view of (ii) it remains only to show that $(W \otimes V_1)^- + (W \otimes V_2)^- = (W \otimes V)^-$. Clearly $(W \otimes V_1)^- + (W \otimes V_2)^- \subset (W \otimes V)^-$.

To prove the reverse inclusion, let $f \in (W \otimes V)^-$. We can select a sequence $\{f_k\}$ in $W \otimes V$ such that $\|f - f_k\| \rightarrow 0$. Write $f_k = g_k + h_k$ where $g_k \in W \otimes V_1$ and $h_k \in W \otimes V_2$.

Let K be a positive constant such that $F_1 \in V_1$ and $F_2 \in V_2$ imply $\|F_1\| \leq K\|F_1 + F_2\|$ and $\|F_2\| \leq K\|F_1 + F_2\|$; such a constant exists by (ii). We claim that $\|g_j - g_k\|, \|h_j - h_k\| \leq K\|f_j - f_k\|$ for all j, k . For let $x_1 \in X_1$ be given. Define F_{1jk} and F_{2jk} in $C(X_2)$ by $F_{1jk}(x_2) = g_j(x_1, x_2) - g_k(x_1, x_2)$ and $F_{2jk}(x_2) = h_j(x_1, x_2) - h_k(x_1, x_2)$. Then $F_{1jk} \in V_1$ and $F_{2jk} \in V_2$ by Lemma 2.1, and $F_{1jk}(x_2) + F_{2jk}(x_2) = f_j(x_1, x_2) - f_k(x_1, x_2)$. Thus $\|F_{1jk}\|, \|F_{2jk}\| \leq K\|F_{1jk} + F_{2jk}\| \leq K\|f_j - f_k\|$, or $|(g_j - g_k)(x_1, x_2)|, |(h_j - h_k)(x_1, x_2)| \leq K\|f_j - f_k\|$ for all $x_2 \in X_2$. Letting x_1 vary through X_1 yields $\|g_j - g_k\|, \|h_j - h_k\| \leq K\|f_j - f_k\|$.

Since the sequence $\{f_k\}$ is Cauchy, so are the sequences $\{g_k\}$ and $\{h_k\}$, hence we can find $g \in (W \otimes V_1)^-$ and $h \in (W \otimes V_2)^-$ such that $\|g - g_k\| \rightarrow 0$ and $\|h - h_k\| \rightarrow 0$. Clearly $f = g + h$. \square

We now introduce polydisc algebras. The open unit disc in the complex plane will forevermore be denoted by D , its closure by \bar{D} , its bounding circle by T . Let Σ be a nonempty index set and let $Z = \prod \bar{D}_\sigma$ be a product of Σ copies of \bar{D} . Let \bar{D}_σ have coordinate z_σ , so the typical element of Z is $z = \{z_\sigma\}$. Let D_σ and T_σ be the copies of D and T respectively in \bar{D}_σ . Then $G = \prod T_\sigma \subset Z$. Let λ be normalized Haar measure on G , i.e., the product of the normalized Lebesgue measures on the various T_σ . Two distinct monomials of the form $z_{\sigma_1}^{m_1} \cdots z_{\sigma_n}^{m_n}$ are then orthogonal in $L^2(\lambda)$.

The algebra B will be $P(Z)$ the closure on Z of the polynomials in the z_σ . Evidently $B = \bigotimes \{P(\bar{D}_\sigma)\}^-$ and because $M_{P(\bar{D}_\sigma)} = \bar{D}_\sigma$ and $S_{P(\bar{D}_\sigma)} = T_\sigma$, it follows from Theorem 2.3 that $M_B = Z$ and $S_B = G$. We next show that analyticity characterizes the functions in B .

LEMMA 2.6. *B consists of those $f \in C(Z)$ which are holomorphic on D_σ as a function of z_σ for each $\sigma \in \Sigma$.*

Proof. Clearly each $f \in B$ is analytic on D_σ as a function of z_σ . The reverse implication is well known if Σ is finite and follows readily from, say, Osgood's lemma [GR, p. 2]. We therefore assume the lemma for finite Σ .

Suppose $f \in C(Z)$ is analytic in each variable separately. Let $\varepsilon > 0$ be given; we approximate f within ε by a polynomial.

Using continuity of f , we can find a finite set $\Sigma_0 \subset \Sigma$ such that $z^0 \in \prod \{\bar{D}_\sigma : \sigma \in \Sigma_0\}$ and $z^1, z^2 \in \prod \{\bar{D}_\sigma : \sigma \in \Sigma - \Sigma_0\}$ imply $|f(z^0, z^1) - f(z^0, z^2)| < \varepsilon/2$. Select

$$z^1 \in \prod \{\bar{D}_\sigma : \sigma \in \Sigma - \Sigma_0\}$$

and define $g \in C(\prod \{\bar{D}_\sigma : \sigma \in \Sigma_0\})$ by $g(z^0) = f(z^0, z^1)$. Using the finite $-\Sigma$ version of the lemma, choose h_1 a polynomial in $\{z_\sigma : \sigma \in \Sigma_0\}$ such that $\|h_1 - g\| < \varepsilon/2$ in $C(\prod \{\bar{D}_\sigma : \sigma \in \Sigma_0\})$. Define h on $\prod \{\bar{D}_\sigma\}$ by $h(z^0, z^2) = h_1(z^0)$. Then h is a polynomial and $\|h - f\| < \varepsilon$. \square

Let q be a nonnegative integer. H_q will denote the homogeneous polynomials of degree q in the z_σ , B_q its closure. If F is a set of nonnegative integers (the symbol F is henceforth reserved for this meaning), set $B_F = \text{sp}(\bigcup \{B_q : q \in F\})^- = \text{sp}(\bigcup \{H_q : q \in F\})^-$. In particular, $B_{\{q\}} = B_q$.

LEMMA 2.7. *Let q be a nonnegative integer. Let $\{g_n\}$ and $\{h_n\}$ be sequences in H_q and $\text{sp}(\bigcup \{H_k : k > q\})$ respectively such that $\|g_n + h_n\| \rightarrow 0$. Then $\|g_n\| \rightarrow 0$ and $\|h_n\| \rightarrow 0$.*

Proof. Otherwise, by passing to a subsequence and normalizing if necessary, we may assume $\|g_n\| \rightarrow 1$ and $\|h_n\| \rightarrow 1$. Select a sequence $\{\rho_n\}$ of positive numbers such that $\rho_n \rightarrow 0$ and $\|g_n + h_n\| \cdot \rho_n^{-q} \rightarrow 0$; for instance, $\rho_n = (\|g_n + h_n\| + n^{-1})^{1/2q}$.

Define $g'_n \in H_q$ and $h'_n \in \text{sp}(\bigcup \{H_k : k > q\})$ by $g'_n(z) = g_n(\rho_n z)$ and $h'_n(z) = h_n(\rho_n z)$. Since h_n is a polynomial of total order at 0 at least $q+1$, Schwarz's lemma [GR, p. 8] yields $\|h'_n\| \leq \rho_n^{q+1} \|h_n\|$, so $\|h'_n\| \cdot \rho_n^{-q} \rightarrow 0$. Since also $\|g'_n + h'_n\| \cdot \rho_n^{-q} \leq \|g_n + h_n\| \cdot \rho_n^{-q} \rightarrow 0$ we obtain $\|g'_n\| \cdot \rho_n^{-q} \rightarrow 0$. But $g'_n \cdot \rho_n^{-q} = g_n$, so $\|g'_n\| \cdot \rho_n^{-q} = \|g_n\| \rightarrow 1$, a contradiction. \square

LEMMA 2.8. *Let q be a nonnegative integer and suppose $k \in F$ implies $k > q$. Then $B_q \oplus B_F \equiv B_{F \cup \{q\}}$ topologically.*

Proof. Clearly $B_q + B_F \subset B_{F \cup \{q\}}$ and the remarks after Lemma 2.5 imply that B_q is orthogonal to B_F in $L^2(\lambda)$, hence $B_q \cap B_F = \{0\}$. Lemma 2.5 then reduces us to showing $B_{F \cup \{q\}} \subset B_q + B_F$.

Let $f \in B_{F \cup \{q\}}$ be given. Select sequences $\{g_n\} \subset H_q$ and $\{h_n\} \subset \text{sp}(\bigcup \{H_k : k \in F\})$ such that $\|f - (g_n + h_n)\| \rightarrow 0$. The sequence $\{g_n + h_n\}$ is Cauchy, hence by Lemma 2.7 so are the sequences $\{g_n\}$ and $\{h_n\}$. Thus we can find $g \in B_q$ and $h \in B_F$ such that $\|g - g_n\| \rightarrow 0$ and $\|h - h_n\| \rightarrow 0$. Clearly $f = g + h$. \square

COROLLARY 2.9. *Let F_0, F_1, \dots, F_n be disjoint and F_1, \dots, F_n finite. Then $B_{F_0} \oplus B_{F_1} \oplus \dots \oplus B_{F_n} \equiv B_{F_0 \cup F_1 \cup \dots \cup F_n}$ topologically.*

Let V be a closed subspace of B , and for each nonnegative integer q let $V_q = V \cap B_q$. If F is any set of nonnegative integers, let $V_F = \text{sp}(\bigcup \{V_q : q \in F\})^-$; in particular, $V_{\{q\}} = V_q$. Finally, call V *homogeneous* if $V = \text{sp}(\bigcup \{V_q : q \geq 0\})^-$.

Similarly, if X is a compact Hausdorff space and U is a closed subspace of $(C(X) \otimes B)^-$, let $U_q = U \cap (C(X) \otimes B_q)^-$ and $U_F = \text{sp}(\bigcup \{U_q : q \in F\})^-$, so $U_{\{q\}} = U_q$. Call U *homogeneous* if $U = \text{sp}(\bigcup \{U_q : q \geq 0\})^-$.

Two remarks are in order here. First, the question of whether U is homogeneous depends on its representation, i.e., on the choice of X and Z . For example, let \bar{D}_j have coordinate z_j for $j = 1, 2$. Let $W_1 \subset P(\bar{D}_1)$ be the linear span of the polynomial $1 + z_1$ and let $W_2 \subset P(\bar{D}_2)$ be the linear span of the polynomial 1, say. Let $U = (W_1 \otimes W_2)^-$. If we let $\bar{D}_1 = X$, $\bar{D}_2 = Z$ then U is a homogeneous subspace of $(C(X) \otimes B)^-$. However, if we let X be a one-point space and $Z = \bar{D}_1 \times \bar{D}_2$, then we may view U as a subspace of $(C(X) \otimes B)^-$, and U is not homogeneous. In using the term, then, we shall always have a fixed representation in mind, and this representation will be clear from the context, so no ambiguity will arise. The second remark is that by letting X be a one-point space, we can make facts about $(C(X) \otimes B)^-$ imply corresponding facts about B .

LEMMA 2.10. *For each nonnegative integer q let N_q be a subspace of $(C(X) \otimes B_q)^-$. Let $U = \text{sp}(\bigcup \{N_q : q \geq 0\})^-$. Then U is homogeneous and $U_q = N_q^-$ for all q .*

Proof. Clearly U is homogeneous and $N_q \subset U_q$. It remains to show that N_q is dense in U_q . Let $f \in U_q$ be given. Choose sequences $\{g_n\} \subset N_q$ and

$$\{h_n\} \subset \text{sp}(\bigcup \{N_k : 0 \leq k \neq q\})$$

such that $\|f - (g_n + h_n)\| \rightarrow 0$. From Corollary 2.9 and Lemma 2.5(iii) it follows that $\|f - g_n\| \rightarrow 0$. \square

COROLLARY 2.11. *Let U and V be homogeneous subspaces of $(C(X) \otimes B)^-$. Then $(U+V)^-$ is a homogeneous subspace of $(C(X) \otimes B)^-$, and $((U+V)^-)_q = (U_q + V_q)^-$ for all q .*

COROLLARY 2.12. *Let $\emptyset \neq W \subset C(X)$ and let V be a homogeneous subspace of B . Then $(W \otimes V)^-$ is a homogeneous subspace of $(C(X) \otimes B)^-$, and $((W \otimes V)^-)_q = (W \otimes V_q)^-$ for all q .*

LEMMA 2.13. *Let F_0, F_1, \dots, F_n be disjoint and F_1, \dots, F_n finite. Suppose U is a homogeneous subspace of $(C(X) \otimes B)^-$. Then*

$$U_{F_0} \oplus U_{F_1} \oplus \cdots \oplus U_{F_n} \equiv U_{F_0 \cup F_1 \cup \cdots \cup F_n}$$

topologically.

Proof. Corollary 2.9 and Lemma 2.5 imply that $(C(X) \otimes B_{F_0})^- \oplus \cdots \oplus (C(X) \otimes B_{F_n})^- \equiv (C(X) \otimes B_{F_0 \cup \cdots \cup F_n})^-$ topologically. U_{F_j} is a closed subspace of $(C(X) \otimes B_{F_j})^-$, so $U_{F_0} \oplus \cdots \oplus U_{F_n}$ is a closed, hence complete, subspace of $(C(X) \otimes B_{F_0})^- \oplus \cdots \oplus (C(X) \otimes B_{F_n})^-$. Consequently $U_{F_0} \oplus \cdots \oplus U_{F_n} \equiv U_{F_0} + \cdots + U_{F_n}$ topologically, and in particular the latter is a complete, hence closed, subspace of $U_{F_0 \cup \cdots \cup F_n}$. We want $U_{F_0} + \cdots + U_{F_n} = U_{F_0 \cup \cdots \cup F_n}$, hence we must show that $U_{F_0} + \cdots + U_{F_n}$ is dense in $U_{F_0 \cup \cdots \cup F_n}$. But this holds because $\text{sp}(\bigcup \{U_q : q \in F_0 \cup \cdots \cup F_n\})$ is contained in $U_{F_0} + \cdots + U_{F_n}$ and is dense in $U_{F_0 \cup \cdots \cup F_n}$. \square

3. This section is devoted to producing algebras and maximal ideals whose associated graded algebras realize preassigned structures. We obtain Theorem 3.17, the main theorem of this paper; in fact, we obtain a slight generalization of it (Theorem 3.16).

To begin with, let A' and B denote sup-norm algebras on $X = M_{A'}$ and $Y = M_B$ respectively. Let A be a sup-norm algebra on X contained in A' and let I be a closed ideal of B . \mathfrak{A} will denote the sup-norm algebra on $X \times Y$ generated by $A' \otimes I$ and $A \otimes B$.

LEMMA 3.1. $\mathfrak{A} = ((A' \otimes I) + (A \otimes B))^- \subset (A' \otimes B)^-$.

COROLLARY 3.2. *Suppose $y \in \text{hull}(I) \subset Y$. Then for each $f \in \mathfrak{A}$ the function $f_y \in C(X)$ given by $f_y(x) = f(x, y)$ is in A .*

Let $\tau: A \rightarrow \mathfrak{A}$ and $\eta: B \rightarrow \mathfrak{A}$ be the isometric isomorphisms $\tau(a) = a \otimes 1$ and $\eta(b) = 1 \otimes b$. The dual maps are $\tau_*: M_{\mathfrak{A}} \rightarrow M_A$ and $\eta_*: M_{\mathfrak{A}} \rightarrow M_B = Y$. Define $\pi: M_{\mathfrak{A}} \rightarrow M_A \times Y$ by $\pi(\varphi) = (\tau_*(\varphi), \eta_*(\varphi))$.

THEOREM 3.3. π maps $M_{\mathfrak{A}}$ homeomorphically onto $(M_A \times \text{hull}(I)) \cup (X \times Y)$.

Proof. Clearly π is continuous and $\pi(M_{\mathfrak{A}}) \supset X \times Y$.

Let $(\alpha, y) \in M_A \times \text{hull}(I)$ be given. The map $f \rightarrow f_y$ is a homomorphism of \mathfrak{A} onto A (see Corollary 3.2). Define $\varphi \in M_{\mathfrak{A}}$ by $\varphi(f) = \alpha(f_y)$. Then $\pi(\varphi) = (\alpha, y)$.

Therefore $\pi(M_{\mathfrak{A}}) \supset (M_A \times \text{hull}(I)) \cup (X \times Y)$.

Now suppose $\varphi \in M_{\mathfrak{A}}$ and $\pi(\varphi) = (\alpha, y) \in M_A \times Y$. We must show that $(\alpha, y) \in (M_A \times \text{hull}(I)) \cup (X \times Y)$ and that φ is uniquely determined by (α, y) .

Case 1. $y \in \text{hull}(I)$. It suffices to show that $\varphi|(A' \otimes I) = 0$ and $\varphi(a \otimes b) = \alpha(a)b(y)$ for $a \in A, b \in B$.

If $a \in A'$ and $i \in I$ then $(\varphi(a \otimes i))^2 = \varphi(a^2 \otimes i^2) = \varphi(a^2 \otimes i) \cdot \varphi(1 \otimes i) = \varphi(a^2 \otimes i) \cdot i(y) = 0$, so $\varphi(a \otimes i) = 0$. Thus $\varphi|(A' \otimes I) = 0$.

If $a \in A$ and $b \in B$ then

$$\varphi(a \otimes b) = \varphi(a \otimes 1) \cdot \varphi(1 \otimes b) = \alpha(a)b(y).$$

Case 2. $y \notin \text{hull}(I)$. We will show that α is evaluation at some $x \in X$, and that $\varphi(a' \otimes i) = a'(x)i(y)$ and $\varphi(a \otimes b) = a(x)b(y)$ whenever $a \in A, a' \in A', b \in B, i \in I$.

Select $i_0 \in I$ such that $i_0(y) = 1$. Define $L \in A'^*$ by $L(a') = \varphi(a' \otimes i_0)$. Then $L(1) = 1$, so $L \neq 0$. Further, $L(a'_1 \cdot a'_2) = \varphi(a'_1 \cdot a'_2 \otimes i_0) = \varphi(a'_1 \cdot a'_2 \otimes i_0) \cdot \varphi(1 \otimes i_0) = \varphi(a'_1 \otimes i_0) \cdot \varphi(a'_2 \otimes i_0) = L(a'_1) \cdot L(a'_2)$, so L is multiplicative. Thus L is evaluation at some $x \in X$. If $a \in A$ then $\alpha(a) = \varphi(a \otimes 1) \cdot \varphi(1 \otimes i_0) = \varphi(a \otimes i_0) = L(a) = a(x)$, so α is evaluation at x .

If $a \in A, b \in B$ then $\varphi(a \otimes b) = \varphi(a \otimes 1) \cdot \varphi(1 \otimes b) = \alpha(a)b(y) = a(x)b(y)$. If $a' \in A', i \in I$ then $\varphi(a' \otimes i) = \varphi(a' \otimes i) \cdot \varphi(1 \otimes i_0) = \varphi(a' \otimes i_0) \cdot \varphi(1 \otimes i) = L(a')i(y) = a'(x)i(y)$. \square

LEMMA 3.4. *Suppose $y \in S_B - \text{hull}(I)$ and V is an open set in Y containing y . Then we can find $i \in I$ such that $\|i\| = 1$ and $\|i|Y - V\| < 1$.*

PROPOSITION 3.5. $(S_{A'} \times (S_B - \text{hull}(I)))^- \cup (S_A \times S_B) \subset S_{\mathfrak{A}} \subset S_{A'} \times S_B$.

COROLLARY 3.6. *If $\text{int}_{S_B}(\text{hull}(I)) = \emptyset$, then $S_{\mathfrak{A}} = S_{A'} \times S_B$.*

Now let Z and B be as in the discussion preceding Lemma 2.6, i.e., $Z = \prod \{\bar{D}_\sigma : \sigma \in \Sigma\}$ is a product of closed discs and $B = P(Z)$; Z and B will serve as the Y and B of our construction. I will denote a proper homogeneous ideal in B , that is, a proper homogeneous subspace which is an ideal in B ; this is not to be confused with the term "homogeneous ideal" used earlier in connection with polynomial rings. A and A' will be sup-norm algebras on $X = M_{A'}$ with $A \subset A'$ and we will construct \mathfrak{A} . Observe that $0 \in \text{hull}(I)$, so by Theorem 3.3 $M_A \times \{0\} \subset M_{\mathfrak{A}}$.

We begin by describing the permissible I .

LEMMA 3.7. *Let there be given a family $\{J_q : q \geq 0\}$ of subspaces of B such that $J_0 = \{0\}$, $J_q \subset B_q$, and $z_\sigma J_q \subset J_{q+1}$ for each $\sigma \in \Sigma$ and $q \geq 0$. Then $I = \text{sp}(\bigcup \{J_q : q \geq 0\})^-$ is a proper homogeneous ideal in B and $I_q = J_q^-$ for all $q \geq 0$.*

Our next objective is an explicit description of $(\mathfrak{A}_q^B)^-$ for certain $\varphi \in M_{\mathfrak{A}}$.

LEMMA 3.8. \mathfrak{A} is a homogeneous subspace of $(C(X) \otimes B)^-$, and $\mathfrak{A}_q = ((A' \otimes I_q) + (A \otimes B_q))^-$ for all $q \geq 0$.

Proof. Use Corollary 2.12 to obtain such a result for $(A' \otimes I)^-$ and $(A \otimes B)^-$, and then use Lemma 3.1 and Corollary 2.11 to obtain the desired statement. \square

Let $\varphi_0 \in M_A - X$ be given and set $\varphi = (\varphi_0, 0) \in M_{\mathfrak{A}}$.

THEOREM 3.9. $(\mathfrak{A}_\phi^n)^-$ is the closed linear span of $A' \otimes I$, the spaces $A \otimes B_q$ for $q \geq n$, and the spaces $A_{\phi_0}^{n-q} \otimes B_q$ for $0 \leq q < n$.

Proof. Lemma 3.8 proves the theorem for $n=0$. We next prove the theorem for $n=1$. \mathfrak{A}_ϕ obviously contains $A' \otimes I$, $A_{\phi_0} \otimes B_0$ and $A \otimes B_q$ for $q \geq 1$. Conversely, let $f \in \mathfrak{A}_\phi$. Select a sequence $\{f_k\}$ in \mathfrak{A}_ϕ such that $\|f - f_k\| \rightarrow 0$ and $f_k = s_k + t_k + u_k$ where $s_k \in A' \otimes I$, $t_k \in \text{sp}(\bigcup \{A \otimes B_q : q > 0\})$ and $u_k \in A \otimes B_0$; this is possible by Lemma 3.8. Then $\varphi(f_k) = \varphi(s_k) = \varphi(t_k) = 0$, so $\varphi(u_k) = 0$, hence $u_k \in A_{\phi_0} \otimes B_0$. This proves the theorem for $n=1$.

Suppose $N > 1$; we prove the theorem for $n=N$.

Let $a' \in A'$ be given. Since $\text{hull}(A_{\phi_0}^N) = \{\varphi_0\}$ and $\varphi_0 \notin X$, $A_{\phi_0}^N$ generates the unit ideal in A' , so we can find a_1, \dots, a_m in $A_{\phi_0}^N$ and a'_1, \dots, a'_m in A' such that $a' = a_1 a'_1 + \dots + a_m a'_m$. If $i \in I$ then

$$a' \otimes i = (a_1 \otimes 1)(a'_1 \otimes i) + \dots + (a_m \otimes 1)(a'_m \otimes i) \in \mathfrak{A}_\phi^N.$$

Thus $A' \otimes I \subset \mathfrak{A}_\phi^N$.

Next, observe that for $q > 0$, $B_q \subset (B_1^q)^-$; hence for $q \geq 0$, $1 \otimes B_q \subset (\mathfrak{A}_\phi^q)^-$. Also, $A_{\phi_0}^p \otimes 1 \subset \mathfrak{A}_\phi^p$. Therefore $A_{\phi_0}^p \otimes B_q \subset \text{sp}((A_{\phi_0}^p \otimes 1)(1 \otimes B_q)) \subset (\mathfrak{A}_\phi^{p+q})^-$. This leads to $A \otimes B_q \subset (\mathfrak{A}_\phi^N)^-$ if $q \geq N$ and $A_{\phi_0}^{N-q} \otimes B_q \subset (\mathfrak{A}_\phi^N)^-$ if $q < N$. Thus $(\mathfrak{A}_\phi^N)^-$ contains the desired linear span.

For the reverse inclusion, it suffices to show that if f_1, \dots, f_N are in \mathfrak{A}_ϕ then $f_1 \cdots f_N$ is in the desired closed linear span. The case $n=1$ of the theorem implies that we may assume that each f_j is in $(A' \otimes I) + (A_{\phi_0} \otimes B_0) + \text{sp}(\bigcup \{A \otimes B_q : q > 0\})$, and applying Corollary 2.12 to $A' \otimes I$ we see that we may suppose that

$$f_j \in \text{sp}(\bigcup \{A' \otimes I_q : q > 0\}) + (A_{\phi_0} \otimes B_0) + \text{sp}(\bigcup \{A \otimes B_q : q > 0\}).$$

But under these conditions $f_1 \cdots f_N$ is a finite sum of terms of the form

$$u = (\alpha_1 \otimes \beta_1) \cdots (\alpha_{N_1} \otimes \beta_{N_1})(a'_1 \otimes i_1) \cdots (a'_{N_2} \otimes i_{N_2})(a_1 \otimes b_1) \cdots (a_{N_3} \otimes b_{N_3})$$

with $\alpha_r \in A_{\phi_0}$, $\beta_r \in B_0$, $a'_s \in A'$, $i_s \in I_{p_s}$ for some $p_s \geq 1$, $a_t \in A$ and $b_t \in B_{q_t}$ for some $q_t \geq 1$ with $N_1 + N_2 + N_3 = N$. If $N_2 \neq 0$, $u \in A' \otimes I$. If $N_2 = 0$ and $N_1 \neq 0$, set $q = N - N_1 < N$; then $u \in A_{\phi_0}^{N-q} \otimes B_q$. If $N_1 = N_2 = 0$ then $N_3 = N$ and $u \in A \otimes B_q$ for a $q \geq N$. Thus each such u is in the desired closed linear span, hence so is $f_1 \cdots f_N$. \square

COROLLARY 3.10. $(\mathfrak{A}_\phi^n)^-$ is a homogeneous subspace of $(C(X) \otimes B)^-$ and $((\mathfrak{A}_\phi^n)^-)_q = ((A' \otimes I_q) + (A \otimes B_q))^-$ if $q \geq n$, while $((\mathfrak{A}_\phi^n)^-)_q = ((A' \otimes I_q) + (A_{\phi_0}^{n-q} \otimes B_q))^-$ if $0 \leq q < n$.

Proof. Use Theorem 3.9, Lemma 2.10, and Corollaries 2.11 and 2.12. \square

We now associate to \mathfrak{A} and φ the complex of objects in §1, i.e., the graded algebra V , the maps l_n , the ideal U and the quotient graded algebra \mathcal{Q} .

Let $\mathcal{B} = \text{sp}(\bigcup \{B_n : n \geq 0\}) \subset B$ and $\Lambda = \text{sp}(\bigcup \{I_n : n \geq 0\}) \subset \mathcal{B}$. \mathcal{B} has an obvious graded algebra structure, in which Λ is a homogeneous ideal. Define a linear map

$L: \mathcal{B} \rightarrow V$ by extending linearly the map $L(b) = l_n(1 \otimes b)$ for $b \in B_n$. L is a graded algebra homomorphism of \mathcal{B} into V . By composition with the quotient map we obtain a graded algebra homomorphism $L^*: \mathcal{B} \rightarrow Q$.

LEMMA 3.11. $L(I_n) \subset U_n$ for all n . Equivalently, $L^*|_{\Lambda} = 0$.

Proof. $1 \otimes I_n \subset A' \otimes I \subset (\mathfrak{A}_\phi^{n+1})^-$ by Theorem 3.9, hence $L(I_n) = l_n(1 \otimes I_n) \subset l_n((\mathfrak{A}_\phi^{n+1})^-) = U_n$. \square

Thus L^* determines a graded algebra homomorphism $\tilde{L}: \mathcal{B}/\Lambda \rightarrow Q$ where $\mathcal{B}/\Lambda = \sum_{n=0}^{\infty} \oplus (\mathcal{B}/\Lambda)_n$ with $(\mathcal{B}/\Lambda)_n = B_n/I_n$. Our next objective will be to obtain conditions under which \tilde{L} is an isomorphism into Q , that is, under which $\ker(L^*) = \Lambda$. The key lies in the following definition: the homogeneous subspace W of B will be called *complemented* if for each $q \geq 0$ there exists J_q a closed subspace of B_q such that $W_q \oplus J_q \equiv B_q$ topologically. Note that if Σ is finite, that is, if B consists of analytic functions in finitely many variables, then any homogeneous subspace of B is automatically complemented. $\{J_q\}$ is called a complement for W .

LEMMA 3.12. If I is complemented and $\{J_q\}$ is a complement for I , then $(A' \otimes I_q)^- \oplus (A \otimes J_q)^- \equiv ((\mathfrak{A}_\phi^n)^-)_q$ topologically if $q \geq n$, and $(A' \otimes I_q)^- \oplus (A_{\phi_0}^{n-q} \otimes J_q)^- \equiv ((\mathfrak{A}_\phi^n)^-)_q$ topologically if $0 \leq q < n$.

Proof. Lemma 2.5 (iii) leads to $(A' \otimes I_q)^- \oplus (A_{\phi_0}^p \otimes J_q)^- \equiv (A' \otimes I_q)^- + (A_{\phi_0}^p \otimes J_q)^-$ topologically, so the latter is a closed subspace of $(C(X) \otimes B)^-$. In view of Corollary 3.10 and the fact (which also follows from Lemma 2.5 (iii)) that $(A_{\phi_0}^p \otimes B_q)^- = (A_{\phi_0}^p \otimes I_q)^- + (A_{\phi_0}^p \otimes J_q)^-$ we obtain the lemma. \square

PROPOSITION 3.13. If I is complemented then $\Lambda = \ker(L^*)$.

Proof. We know (Lemma 3.11) that $\Lambda \subset \ker(L^*)$. Since $\ker(L^*)$ is a homogeneous ideal in \mathcal{B} , we have only to show that $\ker(L^*) \cap B_n \subset I_n$.

Suppose $b \in \ker(L^*) \cap B_n$. Write $b = b_1 + b_2$ where $b_1 \in I_n$ and $b_2 \in J_n$; here $\{J_q\}$ is a complement for I . We must show that $b_2 = 0$.

$L^*(b) = 0$ implies $1 \otimes b \in ((\mathfrak{A}_\phi^{n+1})^-)_n$, and Lemma 3.12 yields a $g \in (A' \otimes I_n)^-$ and $h \in (A_{\phi_0} \otimes J_n)^-$ such that $1 \otimes b = g + h$. Then $((1 \otimes b_1) - g) \in (C(X) \otimes I_n)^-$ and $(h - (1 \otimes b_2)) \in (C(X) \otimes J_n)^-$ and $(1 \otimes b_1) - g = h - (1 \otimes b_2)$, so Lemma 2.4 implies $h = 1 \otimes b_2$. Since h and $1 \otimes b_2$ are in $A \otimes B$, we may evaluate them at (φ_0, y) for each $y \in Z$, and we obtain $b_2(y) = (1 \otimes b_2)^\wedge(\varphi_0, y) = \hat{h}(\varphi_0, y) = 0$, the last equality because $h \in (A_{\phi_0} \otimes J_n)^-$. Thus $b_2 = 0$. \square

Thus if I is complemented then \tilde{L} is an isomorphism into Q . It remains to determine the size of the image of \tilde{L} ; this will lead to a reasonable condition on A which guarantees that \tilde{L} be onto Q .

LEMMA 3.14. If I is complemented and $\{J_q\}$ is a complement for I , there exist natural isomorphisms v_n of $\sum_{q=0}^n \oplus ((A_{\phi_0}^{n-q} \otimes J_q)^- / (A_{\phi_0}^{n+1-q} \otimes J_q)^-)$ onto Q_n such that $\tilde{L}((\mathcal{B}/\Lambda)_n) = v_n((A \otimes J_n)^- / (A_{\phi_0} \otimes J_n)^-)$.

Proof. Let $F(n)$ consist of all positive integers strictly greater than n . From Corollary 3.10 and Lemma 2.13 we see that $((\mathfrak{A}_\phi^n)^-)_0 \oplus \cdots \oplus ((\mathfrak{A}_\phi^n)^-)_n \oplus ((\mathfrak{A}_\phi^n)^-)^{F(n)} \equiv (\mathfrak{A}_\phi^n)^-$ topologically while

$$((\mathfrak{A}_\phi^{n+1})^-)_0 \oplus \cdots \oplus ((\mathfrak{A}_\phi^{n+1})^-)_n \oplus ((\mathfrak{A}_\phi^{n+1})^-)^{F(n)} \equiv (\mathfrak{A}_\phi^{n+1})^-$$

topologically, where the second direct sum is imbedded in the first in an obvious fashion. Lemma 3.12 yields $((\mathfrak{A}_\phi^n)^-)^{F(n)} = ((\mathfrak{A}_\phi^{n+1})^-)^{F(n)}$ and, for $0 \leq q \leq n$, $(A' \otimes I_q)^- \oplus (A_{\phi_0}^{n-q} \otimes J_q)^- \equiv ((\mathfrak{A}_\phi^n)^-)_q$ topologically while $(A' \otimes I_q)^- \oplus (A_{\phi_0}^{n+1-q} \otimes J_q)^- \equiv ((\mathfrak{A}_\phi^{n+1})^-)_q$ topologically. The map ν_n is then defined by letting

$$\nu_n(f + (A_{\phi_0}^{n+1-q} \otimes J_q)^-) = l_n(f) + U_n$$

for $f \in (A_{\phi_0}^{n-q} \otimes J_q)^-$ and $0 \leq q \leq n$. Clearly ν_n is an isomorphism of the given direct sum onto Q_n . It remains to prove that $L^*(B_n) = \nu_n((A \otimes J_n)^- / (A_{\phi_0} \otimes J_n)^-)$.

If $b \in I_n$ then $L^*(b) = 0$ by Lemma 3.11. If $b \in J_n$ then $L(b) \in l_n((A \otimes J_n)^-)$ so $L^*(b) = \nu_n(l_n^{-1}(L(b)) + (A_{\phi_0} \otimes J_n)^-)$. Thus $L^*(B_n) \subset \nu_n((A \otimes J_n)^- / (A_{\phi_0} \otimes J_n)^-)$.

For the reverse inclusion, let $f \in (A \otimes J_n)^-$ be given. Then $f \in (A \otimes B)^-$, so we can define $b \in J_n$ by $b(y) = \hat{f}(\varphi_0, y)$ for $y \in Z$. Then $1 \otimes b \in (A \otimes J_n)^-$ and $\hat{f}(\varphi_0, y) = (1 \otimes b)^\wedge(\varphi_0, y)$ for all $y \in Z$.

Since $A_{\phi_0} \oplus C \equiv A$ topologically, Lemma 2.5 yields $(A_{\phi_0} \otimes J_n)^- \oplus (C \otimes J_n)^- \equiv (A \otimes J_n)^-$ topologically, so we can write $f - (1 \otimes b) = g + h$ where $g \in (A_{\phi_0} \otimes J_n)^-$ and $h \in (C \otimes J_n)^- = 1 \otimes J_n$. Then $\hat{g}(\varphi_0, y) = 0$ and therefore $\hat{h}(\varphi_0, y) = 0$ for all $y \in Z$. But $\hat{h}(\varphi_0, y) = u(y)$ where $h = 1 \otimes u$, so $h = 0$ and $f - (1 \otimes b) = g \in (A_{\phi_0} \otimes J_n)^-$. Thus $L^*(b) = l_n(1 \otimes b) + U_n = l_n(f - g) + U_n = l_n(f) + U_n = \nu_n(f + (A_{\phi_0} \otimes J_n)^-)$. This completes the proof. \square

COROLLARY 3.15. *If I is complemented, then in order that \tilde{L} be onto Q it is necessary and sufficient that $A_{\phi_0} = (A_{\phi_0}^2)^-$.*

Proof. From Lemma 3.14 we see that $A_{\phi_0} = (A_{\phi_0}^2)^-$ implies that \tilde{L} is onto, and conversely that if \tilde{L} is onto then $(A_{\phi_0} \otimes J_0)^- = (A_{\phi_0}^2 \otimes J_0)^-$. Because I is a proper ideal in B we have $J_0 = B_0 = C$, so $(A_{\phi_0} \otimes J_0)^- = (A_{\phi_0}^2 \otimes J_0)^-$ implies $A_{\phi_0} = (A_{\phi_0}^2)^-$. \square

We shall now select X , A , A' and φ_0 to make $A_{\phi_0} = (A_{\phi_0}^2)^-$. X will denote a torus T^2 and $A' = C(X)$. Let μ denote normalized Lebesgue measure on T^2 and let α be a positive irrational number. Let

$$A = \left\{ f \in C(X) : \int_{T^2} f(x_1, x_2) x_1^{k_1} x_2^{k_2} d\mu(x_1, x_2) = 0 \text{ whenever } k_1 + k_2\alpha > 0 \right\}.$$

In [W1] A is discussed; it is revealed that A is a Dirichlet algebra on X and that $\varphi_0 \in A^*$ defined by $\varphi_0(f) = \int_{T^2} f d\mu$ is a member of $M_A - X$ which is a one-point part for A . Standard Fourier series arguments show that A is the closed linear span of the functions $x_1^{k_1} x_2^{k_2}$ for $k_1 + k_2\alpha \geq 0$, from which it is easy to show directly that $A_{\phi_0} = (A_{\phi_0}^2)^-$; this in fact implies that φ_0 is a one-point part (see [S1]). Alternatively, starting with the assertion that φ_0 is a one-point part, we can conclude from [S1] that $A_{\phi_0} = (A_{\phi_0}^2)^-$.

THEOREM 3.16. *If I is a complemented homogeneous ideal in B then there exists a sup-norm algebra \mathfrak{A} and a $\varphi \in M_{\mathfrak{A}}$ such that if Q , \mathcal{B} and Λ have the meanings stated before Lemma 3.11 then \mathcal{B}/Λ and Q are isomorphic as graded algebras.*

Proof. If $I=B$ the theorem is trivial (and can be realized with a one-dimensional algebra). If I is a proper ideal, let X , A , A' and φ_0 be as above and construct \mathfrak{A} on $X \times Z$ in the usual manner. Construct L , L^* and \tilde{L} as usual. By Proposition 3.13 \tilde{L} is an isomorphism of \mathcal{B}/Λ into Q , and by Corollary 3.15 \tilde{L} maps \mathcal{B}/Λ onto Q . \square

THEOREM 3.17. *Let $R=C[X_1, \dots, X_r]$ be the graded polynomial algebra in $r < \infty$ variables. Let J be a homogeneous ideal (algebraic sense) in R . Then there exists a sup-norm algebra \mathfrak{A} and a $\varphi \in M_{\mathfrak{A}}$ such that if Q has the meaning stated before Lemma 3.11 then R/J and Q are isomorphic as graded algebras.*

Proof. Let $\Sigma=\{1, \dots, r\}$ and let $\gamma: R \rightarrow B$ be the obvious imbedding. Let $I= \gamma(J)^-$, so by Lemma 3.9 I is a homogeneous ideal in B and $I_n = \gamma(J_n)$. I is complemented, so Theorem 3.16 enables us to construct \mathfrak{A} and φ such that \mathcal{B}/Λ and Q are isomorphic as graded algebras. Finally, note that \mathcal{B}/Λ and R/J are naturally isomorphic as graded algebras. \square

4. It is a bonus of our construction that the algebra constructed in the proof of Theorem 3.17 is actually a local algebra. This section will be devoted for the most part to proving this fact.

We begin with some terminology; for extensive discussion of these notions see [R]. Let A be a sup-norm algebra on X and let $f \in C(X)$. f is *locally in A on X* [resp., *A -holomorphic on X*] if there is an open covering $\{U_\alpha\}$ of X (which may of course be taken finite) such that $f|U_\alpha \in A|U_\alpha$ [resp., $f|U_\alpha \in (A|U_\alpha)^-$]. A is called *local* [resp., *holomorphically closed*] on X if whenever f is locally in A [resp., A -holomorphic] on X , we have $f \in A$.

A local maximum modulus theorem holds for A -holomorphic functions [R, Theorem 2.2] and leads to the fact that if A is local [resp., holomorphically closed] on X , then A is local [resp., holomorphically closed] on M_A . It will also be useful to know that if X is connected and A is analytic on X (i.e., $f \in A$ and $\text{int}_X f^{-1}(0) \neq \emptyset$ imply $f=0$) then A is local on X ; a trivial argument proves this.

Assume Σ is finite, say $\Sigma=\{1, \dots, r\}$. Let I be a proper homogeneous ideal in B . I is automatically complemented; further, letting J_q be the orthogonal complement of I_q in B_q (where I_q, J_q and B_q are thought of as subspaces of $L^2(\lambda)$, λ being normalized Lebesgue measure on $G=T_1 \times \dots \times T_r \subset Z$), we see that $\{J_q\}$ is a complement for I and $\text{sp}(\bigcup \{J_q : q \geq 0\})^-$ is orthogonal to I in $L^2(\lambda)$.

If $f \in C(X \times Z)$ we can define a number of associated functions.

First, for each $g \in B$ define $f_g \in C(X)$ by $f_g(x) = \int_G g(z) \bar{f}(x, z) d\lambda(z)$. Observe that $\|f_g\| \leq \|g\| \cdot \|f\|$.

Second, for each nonnegative integer q and each $x \in X$, f_q^x will denote the orthogonal projection into B_q of the function $z \rightarrow f(x, z)$. If $e_{q,1}, \dots, e_{q,N(q)}$ is the

usual orthonormal basis of B_q consisting of the monic monomials of degree q —so $N(q) = C_{q+r-1, q} \leq (q+r-1)^{r-1}$ —then $f_q^x = f_{e_{q,1}}(x)e_{q,1} + \cdots + f_{e_{q,N(q)}}(x)e_{q,N(q)}$ and therefore $\|f_q^x\| \leq \|f_{e_{q,1}}\| \|e_{q,1}\| + \cdots + \|f_{e_{q,N(q)}}\| \|e_{q,N(q)}\| \leq N(q)\|f\| \leq (q+r-1)^{r-1}\|f\|$ for all $x \in X$.

Finally, if $0 \leq t < 1$ define $f^t \in C(X \times Z)$ by $f^t(x, z) = f(x, tz)$.

LEMMA 4.1. *If f is analytic on D_σ as a function of z_σ for each $\sigma \in \Sigma$, then*

$$\lim_{K \rightarrow \infty} \left(\sup \left\{ \left| f^t(x, z) - \sum_{q=0}^K f_q^x(z) t^q \right| : (x, z) \in X \times Z \right\} \right) = 0.$$

Proof. $\sum_{q=0}^\infty f_q^x$ is the power series expansion of $z \rightarrow f(x, z)$, so $x \in X$ and $z \in D_1 \times \cdots \times D_r$ imply $f(x, z) = \lim_{K \rightarrow \infty} \sum_{q=0}^K f_q^x(z)$. Since $f_q^x(tz) = f_q^x(z)t^q$ we obtain

$$\begin{aligned} \left| f^t(x, z) - \sum_{q=0}^K f_q^x(z) t^q \right| &= \left| \sum_{q=K+1}^\infty f_q^x(z) t^q \right| \\ &\leq \sum_{q=K+1}^\infty \|f_q^x\| t^q \leq \sum_{q=K+1}^\infty (q+r-1)^{r-1} \|f\| t^q. \end{aligned}$$

The latter expression is independent of (x, z) and goes to zero as K goes to infinity. \square

LEMMA 4.2. *Let $g_{q,1}, \dots, g_{q,N(q)}$ be any orthonormal basis for B_q . Suppose $f \in C(X \times Z)$ and f is analytic on D_σ as a function of z_σ for each $\sigma \in \Sigma$. Then*

$$f \in \text{sp} \left(\bigcup \{f_{g_{q,j}} \otimes g_{q,j} : 0 \leq q < \infty, 1 \leq j \leq N(q)\} \right)^-.$$

Proof. Let $\varepsilon > 0$ be given. Select t , $0 \leq t < 1$, such that $\|f - f^t\| < \varepsilon/2$. By Lemma 4.1, select K so large that

$$\sup \left\{ \left| f^t(x, z) - \sum_{q=0}^K f_q^x(z) t^q \right| : (x, z) \in X \times Z \right\} < \varepsilon/2.$$

Finally, observe that

$$f_q^x(z) = \sum_{j=1}^{N(q)} f_{g_{q,j}}(x) g_{q,j}(z) = \sum_{j=1}^{N(q)} (f_{g_{q,j}} \otimes g_{q,j})(x, z). \quad \square$$

Now let X , A , A' and \mathfrak{A} be as in the discussion preceding Lemma 3.7, and $\Sigma = \{1, \dots, r\}$ as above.

LEMMA 4.3. \mathfrak{A} consists precisely of those $f \in C(X \times Z)$ which satisfy

- (i) f is analytic on D_σ as a function of z_σ for each $\sigma \in \Sigma$.
- (ii) $f_g \in A'$ for each $g \in B$.
- (iii) $f_g \in A$ for each $g \in J = \text{sp} \left(\bigcup \{J_n : n \geq 0\} \right)^-$.

Proof. Suppose $f \in \mathfrak{A}$. (i) and (ii) obviously hold, the latter by Lemma 2.1 and the most elementary facts about integration. In proving (iii), the inclusion $\{f_g : g \in J\} \subset \text{sp} \left(\bigcup \{f_g : g \in J_n \text{ for some } n\} \right)^-$ shows that we need only consider $g \in J_n$. Using Lemmas 3.8 and 3.12, we can select sequences $\{u_k\} \subset A \otimes J_n$ and $\{v_k\} \subset (A' \otimes I_n) + \text{sp} \left(\bigcup \{A' \otimes B_q : 0 \leq q \neq n\} \right)$ such that $\|f - f_k\| \rightarrow 0$ where $f_k =$

$u_k + v_k$. Fix $x \in X$. Then $z \rightarrow v_k(x, z)$ is in $I_n + \text{sp}(\bigcup \{B_q : 0 \leq q \neq n\})$, hence is orthogonal to g . Therefore

$$\begin{aligned}(f_k)_g(x) &= \int_G g(z)^{-1} f_k(x, z) d\lambda(z) \\ &= \int_G g(z)^{-1} u_k(x, z) d\lambda(z) + \int_G g(z)^{-1} v_k(x, z) d\lambda(z) = (u_k)_g(x).\end{aligned}$$

Thus $(f_k)_g = (u_k)_g$. Again, Lemma 2.1 and integration theory yield $(u_k)_g \in A$, hence $(f_k)_g \in A$. Since $\|f_g - (f_k)_g\| \leq \|g\| \cdot \|f - f_k\| \rightarrow 0$, we get $f_g \in A$. Thus (iii) holds.

Conversely, suppose $f \in C(X \times Z)$ satisfies (i)–(iii). Let $g_{q,1}, \dots, g_{q,N(q)}$ be an orthonormal basis for B_q such that each $g_{q,j}$ is in either I_q or J_q . By (ii) and (iii) we obtain $f_{g_{q,j}} \otimes g_{q,j} \in \mathfrak{A}$ for $0 \leq q < \infty$, $1 \leq j \leq N(q)$. By (i) and Lemma 4.2, f is in the closed linear span of the $f_{g_{q,j}} \otimes g_{q,j}$. Thus $f \in \mathfrak{A}$. \square

LEMMA 4.4. *If $0 \leq t < 1$ and if $f \in C(X \times Z)$ is analytic on D_σ as a function of z_σ for each $\sigma \in \Sigma$ and if $g \in B_q$ then $f_g(x)t^{2q} = \int_G g(tz)^{-1} f(x, tz) d\lambda(z)$ for all $x \in X$.*

Proof. Use the observations that $f_q^x(tz) = t^q f_q^x(z)$, $t^q f_q^x = (f_t^x)_q$, $g(tz) = t^q g(z)$ and $\int_G g(z)^{-1} f(x, z) d\lambda(z) = \int_G g(z)^{-1} f_q^x(z) d\lambda(z)$.

PROPOSITION 4.5. *If A and A' are local [resp., holomorphically closed] on X then \mathfrak{A} is local [resp., holomorphically closed] on $X \times Z$ and $M_{\mathfrak{A}}$.*

Proof. In view of our remarks at the beginning of this section, it is enough to prove the result for $X \times Z$. Suppose $f \in C(X \times Z)$ is locally in \mathfrak{A} [resp., \mathfrak{A} -holomorphic] on $X \times Z$. We check that f satisfies the three conditions of Lemma 4.3. (i) is obvious.

(ii) and (iii) reduce to showing that f_g is locally in A' [resp., A' -holomorphic] on X for all $g \in B_q$, and that f_g is locally in A [resp., A -holomorphic] on X for all $g \in J_q$.

Let $x_0 \in X$ be given. Let U be an open neighborhood of x_0 in X and t a positive constant such that $f|V \in \mathfrak{A}|V$ [resp., $f|V \in (\mathfrak{A}|V)^{-}$] if $V = U \times tD_1 \times \dots \times tD_r$. Let $f' \in \mathfrak{A}$ [resp., $\{f_k\} \subset \mathfrak{A}$] be such that $f|V = f'|V$ [resp., $\|(f - f_k)|V\| \rightarrow 0$]. Lemma 4.4 then yields, for $g \in B_q$, $f_g|U = f'_g|U$ [resp., $\|(f_g - (f_k)_g)|U\| \leq \|g\| \cdot \|(f - f_k)|V\|/t^{2q} \rightarrow 0$], from which the desired conclusions can be read. \square

Now let X , A , and A' be as in the discussion preceding Theorem 3.16. Trivially, A' is holomorphically closed on X . In view of Proposition 4.5, to see that the algebra \mathfrak{A} constructed in Theorem 3.17 is local on $X \times Z$ and on $M_{\mathfrak{A}}$ if J is a proper ideal (if $J = B$ then \mathfrak{A} is trivially local, being one-dimensional) it suffices to show that A is local on X . Since X is connected, we need only show that A is analytic on X . A variety of routes lead to this conclusion; we choose to quote Wermer's expository text [W1] to point out that A is antisymmetric and is a maximal subalgebra of $C(X)$ [p. 69], hence is analytic on X [p. 62]. Thus we obtain

THEOREM 4.6. *The algebra \mathfrak{A} constructed in the proof of Theorem 3.17 is local on $X \times Z$ and on $M_{\mathfrak{A}}$.*

Theorem 4.6 has a partial converse, whose proof we leave to the reader.

PROPOSITION 4.7. *With notations as in the beginning of §3, assume that \mathfrak{A} is local [resp., holomorphically closed] on $X \times Y$. Then*

- (i) *A is local [resp., holomorphically closed] on X .*
- (ii) *B is local [resp., holomorphically closed] on Y .*
- (iii) *Either $I = \{0\}$ or A' is local [resp., holomorphically closed] on X .*

5. This section will be devoted to miscellany of two types. First, we shall see to what sorts of algebras our construction may lead us. In §4 we found conditions guaranteeing that \mathfrak{A} be local; our work on this section will be far more transparent, and will deal for the most part with notions such as essential, pervasive, etc.

Second, we shall be interested in bounded derivations. We shall give examples of such phenomena as nontrivial derivations at one-point parts, and nonlocal derivations in algebras satisfying reasonably strong local belonging conditions. The thread which unites these examples to the main body of our work is the construction, which is used to produce many of the examples.

We begin by defining a number of properties which are frequently studied together. For proofs of some of the unproved assertions below, we refer the reader to [HS].

Let A be a sup-norm algebra on X . A is *maximal* on X if $A \neq C(X)$, but if whenever A' is a sup-norm algebra on X containing A , either $A' = A$ or $A' = C(X)$; automatically $S_A = X$. A is *essential* if there is no proper closed subset F of X such that A contains every function in $C(X)$ vanishing identically on F ; this property is independent of the choice of boundary X for A . A is *pervasive* on X if whenever F is a proper closed subset of X , $A|F$ is dense in $C(F)$; again, this implies $S_A = X$. A is *analytic* on X if no function of A other than the zero function vanishes identically on a nonvoid open subset of X . A is *antisymmetric* if it contains no nonconstant real-valued functions; this property is independent of the choice of boundary X for A . The following chain of implications is well known: A is maximal and essential on $X \Rightarrow A$ is a proper pervasive subalgebra of $C(X) \Rightarrow A$ is analytic on $X \Rightarrow A$ is an integral domain $\Rightarrow A$ is antisymmetric $\Rightarrow A$ is essential.

We shall present criteria for \mathfrak{A} to be maximal, essential, etc. Observe that some of the properties, e.g., being essential, occur virtually all the time while others, e.g., being maximal, occur quite rarely. Proofs are easy, and are left to the reader.

Criterion 5.1. With notations as in the discussion preceding Lemma 3.7, except that I need not be proper, \mathfrak{A} will be maximal (on $S_{\mathfrak{A}}$, of course) precisely when X consists of a single point and Σ consists of just one element.

Criterion 5.2. With notations as in the beginning of §3, \mathfrak{A} will be essential if at least one of the following three conditions holds:

- (i) A' is essential.
- (ii) A is essential and $I = \{0\}$.
- (iii) B is essential.

Criterion 5.3. With notations as in the beginning of §3, \mathfrak{A} will be pervasive (on $S_{\mathfrak{A}}$, of course) precisely when at least one of the following five conditions holds:

- (i) X consists of a single point and B is pervasive.
- (ii) Y consists of a single point and A is pervasive.
- (iii) Y consists of a single point, $I=B$, and A' is pervasive.
- (iv) $A=C(X)$ and $B=C(Y)$.
- (v) $A'=C(X)$ and $I=B=C(Y)$.

Criterion 5.4. With notations as in the beginning of §3, \mathfrak{A} will be analytic on $S_{A'} \times S_B$ precisely when either

- (i) A is analytic on $S_{A'}$, B is analytic on S_B , and $I=\{0\}$.
- (ii) A' is analytic on $S_{A'}$ and B is analytic on S_B .

If any of these conditions holds then \mathfrak{A} is analytic on $S_{\mathfrak{A}}$.

Similarly, if \mathfrak{A} is analytic on $M_{\mathfrak{A}}$ then at least one of the following three conditions holds:

- (a) A is analytic on M_A , B is analytic on Y , and $I=\{0\}$.
- (b) A is analytic on M_A , B is analytic on Y , A' is analytic on X , and $\{0\} \neq I \neq B$.
- (c) A' is analytic on X , B is analytic on Y , and $I=B$.

Conversely, (a) or (c) implies that \mathfrak{A} is analytic on $M_{\mathfrak{A}}$.

Criterion 5.5. With notations as in the beginning of §3, if \mathfrak{A} is an integral domain then either A' and B are integral domains, or A and B are integral domains and $I=\{0\}$.

Criterion 5.6. With notations as in the beginning of §3, \mathfrak{A} is antisymmetric precisely when either A and B are antisymmetric and $I \neq B$, or A' and B are antisymmetric.

Combining these criteria, we obtain the

PROPOSITION 5.7. *The algebra \mathfrak{A} constructed in the proof of Theorem 3.16 [resp., Theorem 3.17] for the case $I \neq B$ [resp., $J \neq R$] is always essential and antisymmetric, and is never maximal or pervasive. It is analytic on $X \times G$ precisely when it is analytic on $M_{\mathfrak{A}}$; this happens precisely when $I=\{0\}$ [resp., $J=\{0\}$], which is precisely when \mathfrak{A} is an integral domain.*

The parts in $M_{\mathfrak{A}}$ may be of some interest, especially in view of recent work by Garnett [Ga]. We can obtain a wide variety of parts (including disconnected ones) via our construction. Recall that $\varphi \sim \varphi' \Leftrightarrow \|\varphi - \varphi'\|_{A^*} < 2$ is an equivalence relation on $M_A[\text{Gl}]$, and that the equivalence classes are the (Gleason) parts for A .

PROPOSITION 5.8. *With notations as in the beginning of §3, suppose the A and A' parts relations coincide on X . Then the parts for \mathfrak{A} are precisely the sets $P = (P_1 \times P_2) \cap M_{\mathfrak{A}}$ where P_1 is a part for A in M_A and P_2 is a part for B in Y .*

COROLLARY 5.9. *The set $\{\varphi_0\} \times (\text{hull}(J) \cap \prod_{i=1}^r D_{\sigma_i})$ is a single part for the algebra constructed in the proof of Theorem 3.17 for the case $J \neq R$.*

PROPOSITION 5.10. *With notations as in the beginning of §3, let N denote any cardinal number and assume \mathfrak{A} has a system of N generators. Then B has a system of N or fewer generators. If $I \neq B$ then A has a system of N or fewer generators, and if $I \neq \{0\}$ then A' has a system of N or fewer generators.*

Conversely, let N_1, N_2, N_3, N_4 denote cardinal numbers. Assume A has a system of N_1 generators, B has a system of N_2 generators, A' is the closed A -module spanned by 1 and N_3 other functions, and I is the closed ideal of B generated by N_4 elements of I . Then \mathfrak{A} has a system of $N_1 + N_2 + N_3 N_4$ or fewer generators.

Proof. The first half is trivial. For the second, let $\{a_{n_1} : n_1 \in N_1\}$, $\{b_{n_2} : n_2 \in N_2\}$, $\{a'_{n_3} : n_3 \in N_3\}$ and $\{i_{n_4} : n_4 \in N_4\}$ be the four systems of generators described in the hypotheses. It is easy to see that the union of the systems $\{a_{n_1} \otimes 1 : n_1 \in N_1\}$, $\{1 \otimes b_{n_2} : n_2 \in N_2\}$, and $\{a'_{n_3} \otimes i_{n_4} : n_3 \in N_3, n_4 \in N_4\}$ generates \mathfrak{A} . \square

COROLLARY 5.11. *If A and B are finitely generated, if A' is a finitely generated closed A -module, and if I is a finitely generated closed ideal in B , then \mathfrak{A} is finitely generated.*

We are now almost ready to begin the examples. We need one more fact which will make Proposition 5.10 useful.

LEMMA 5.12. *Let r be a positive integer, let \bar{D}^r have coordinates $z = (z_1, \dots, z_r)$, let $A' = C(T^r)$, and let A be the restriction to T^r of the functions continuous on \bar{D}^r and analytic on D^r . Then there exists $g \in A'$ such that A' is the closed A -module generated by 1 and g .*

Proof. First, suppose $r = 1$ and $z = z_1$. Define h on T by $h(z) = \sum_{n=1}^{\infty} \bar{z}^{2N} / n!$ where the bar denotes complex conjugation. Consideration of the functions $m! z^{2^m - 2^N} h(z)$ for $m > N$ shows that \bar{z}^{2^N} is in the closed A -module generated by 1 and h , for every positive integer N .

Next let r be arbitrary and define g by $g(z_1, \dots, z_r) = h(z_1, \dots, z_r)$ where h is as above. Then our discussion of h shows that $(z_1, \dots, z_r)^{-2^N}$ is in the closed A -module generated by g , for every positive integer N . It is then easy to complete the proof with the aid of the Stone-Weierstrass theorem. \square

Let A be a sup-norm algebra, L a linear functional on A , $\varphi \in M_A$. L is called a *derivation* of A at φ if L satisfies the usual product rule for derivatives: $L(fg) = L(f) \cdot \varphi(g) + \varphi(f) \cdot L(g)$. This is equivalent to annihilating both the constants and the ideal A_φ^2 . This suggests considering also functionals which annihilate A_φ^n for some n ; if L is also continuous, we write $L \in (A_\varphi^n)^\perp$ where $(A_\varphi^n)^\perp \subset A^*$.

The most familiar examples come from differentiation of analytic functions. These derivations are trivially local in the following sense: a linear functional L on A is *local* at $\varphi \in M_A$ if whenever $f \in A$ vanishes on a neighborhood of φ in M_A , it follows that $L(f) = 0$. Since analytic functions form one of the models for sup-norm algebra theory, the next obvious question is, when can we guarantee that a derivation be local?

It has been known at least since the appearance of Eva Kallin's famous example [K] that continuous nonlocal derivations may exist. On the other hand, the results of [S1] show that the hypothesis of unique representing measures precludes the possibility of continuous nonlocal derivations.

It is suggestive that the first nonlocal derivation occurs in the first nonlocal algebra, and an obvious conjecture arises. The conjecture is rendered plausible by the fact that both uses of the term "local" stipulate that all the information one wants can be obtained by looking at small open sets. Nonetheless, as we shall see in Examples 5.13 through 5.15 below, the conjecture is false even under very strong hypotheses.

EXAMPLE 5.13. Let n be an integer greater than 1. A and A' will be as in the discussion preceding Theorem 3.16, i.e., $A' = C(T^2)$ and A is a Dirichlet algebra on $X = T^2$ having a one-point part φ_0 off $X = S_A$. B will be the disc algebra on $Z = \bar{D}$ (with coordinate z) and I will be the closed ideal in B generated by z^n (that is, I consists of those functions in B which vanish at 0 along with their first $n-1$ derivatives). \mathfrak{A} is just the algebra constructed in proving Theorem 3.17 for the case $r=1$, J the ideal of $C[X_1]$ generated by X_1^n . Therefore $(\mathfrak{A}_\varphi^q)^- / (\mathfrak{A}_\varphi^{q+1})^-$ is one-dimensional if $0 \leq q < n$, and $(\mathfrak{A}_\varphi^q)^- = (\mathfrak{A}_\varphi^{q+1})^-$ if $q \geq n$. By Theorem 4.6, \mathfrak{A} is local (on $M_\mathfrak{A}$). By Criterion 5.6 \mathfrak{A} is antisymmetric, hence essential.

Since $\text{hull}(J) = \{0\}$, Corollary 5.9 implies that φ is a one-point part for \mathfrak{A} . Since there is a nonzero bounded derivation of \mathfrak{A} at φ because $\mathfrak{A}_\varphi \neq (\mathfrak{A}_\varphi^2)^-$, we see that just about the weakest imaginable converse to a recent result of Andrew Browder is false. Browder shows [B] that if φ is not isolated in M_A in the A^* norm topology, then φ supports a (not necessarily continuous) nontrivial derivation of A .

The function $1 \otimes z$ is in \mathfrak{A} and vanishes on the neighborhood $M_A \times \{0\}$ of φ in $M_\mathfrak{A}$, but Lemma 3.12 implies it is not in $(\mathfrak{A}_\varphi^2)^-$; hence there is a nonlocal bounded derivation of \mathfrak{A} at φ .

The function $1 \otimes z^k$ provides a generator for $(\mathfrak{A}_\varphi^k)^- / (\mathfrak{A}_\varphi^{k+1})^-$ for $1 \leq k \leq n-1$.

EXAMPLE 5.14. Let $X = T$, $A' = C(X)$, A the restriction to X of the disc algebra. Let B be the disc algebra on $Z = \bar{D}$, let I be the closed ideal of B generated by z^2 , let $\varphi_0 = 0 \in \bar{D} = M_A$, and $\varphi = (\varphi_0, 0) \in M_\mathfrak{A}$ where \mathfrak{A} is constructed in the usual way. By Proposition 4.5, \mathfrak{A} is local (on $M_\mathfrak{A}$). Criterion 5.6 says \mathfrak{A} is antisymmetric. Proposition 5.10 and Lemma 5.12 show that \mathfrak{A} is generated by only three functions. By Lemma 3.12 $(\mathfrak{A}_\varphi^n)^- \neq (\mathfrak{A}_\varphi^{n+1})^-$ for all n .

The function $1 \otimes z$ is in \mathfrak{A} and vanishes on the neighborhood $M_A \times \{0\}$ of φ in $M_\mathfrak{A}$, but Lemma 3.12 implies $1 \otimes z \notin (\mathfrak{A}_\varphi^2)^-$. Thus φ supports a nonlocal continuous derivation of \mathfrak{A} .

This example was called to the author's attention by Professor A. M. Gleason, and it led the author to the construction in §3.

Note that if we let I be generated by z rather than z^2 , the resulting algebra \mathfrak{A} is a familiar Dirichlet algebra on T^2 [W1, p. 69], and the same argument shows that it is triply generated, a fact which may be of independent interest.

EXAMPLE 5.15. Let $m > 1$ be an integer, let $X = \text{bdry}_{C^m}(D^m)$, let $A' = C(X)$ and let A be the polydisc algebra on \bar{D}^m restricted to X , so $S_A = T^m$ and $M_A = \bar{D}^m$. It is not difficult to show that A is holomorphically closed on X .

Let B be the disc algebra on $Z = \bar{D}$, let I be the closed ideal of B generated by z^2 ,

let $\varphi_0 = 0 \in \bar{D}^m = M_A$, and let $\varphi = (\varphi_0, 0) \in M_{\mathfrak{A}}$ where \mathfrak{A} is constructed in the usual way. By Proposition 4.5, \mathfrak{A} is holomorphically closed (on $M_{\mathfrak{A}}$). Criterion 5.6 says \mathfrak{A} is antisymmetric. From Lemma 3.12 we see that $(\mathfrak{A}_\varphi^n)^- \neq (\mathfrak{A}_\varphi^{n+1})^-$ for all n .

As usual, the function $1 \otimes z$ is in \mathfrak{A} and vanishes on the neighborhood $M_A \times \{0\}$ of φ in $M_{\mathfrak{A}}$, but $1 \otimes z \notin (\mathfrak{A}_\varphi^2)^-$, so a nonlocal continuous derivation of \mathfrak{A} at φ exists. Thus even holomorphic closure of \mathfrak{A} —as strong a local belonging property as one could wish—need not guarantee localness of derivations.

EXAMPLE 5.16. Let X, A, A' and φ_0 be as in Example 5.13. Let $Z = \prod_{n=1}^{\infty} \bar{D}_n$ where \bar{D}_n is a closed disc with coordinate z_n , let $B = P(Z)$, and let I be the closed ideal of B generated by the functions $\{z_n^n : n = 1, 2, \dots\}$. Then $\text{hull}(I) = \{0\}$ and I is a homogeneous ideal in B (in our nonalgebraic sense of the word “homogeneous”). Construct \mathfrak{A} as usual, so (Criterion 5.6) \mathfrak{A} is antisymmetric, and let $\varphi = (\varphi_0, 0) \in M_{\mathfrak{A}}$, so (Proposition 5.8) φ is a one-point part for \mathfrak{A} .

Corollary 3.10 can be used to see that $1 \otimes z_{n+1}^n \notin (\mathfrak{A}_\varphi^{n+1})^-$ while of course $1 \otimes z_{n+1}^n \in \mathfrak{A}_\varphi^n$. Thus $(\mathfrak{A}_\varphi^n)^- \neq (\mathfrak{A}_\varphi^{n+1})^-$ for all n , although φ is a one-point part. Note also that $1 \otimes z_2$ vanishes on the neighborhood $M_A \times \{0\}$ of φ in $M_{\mathfrak{A}}$ but $1 \otimes z_2 \notin (\mathfrak{A}_\varphi^2)^-$, hence there is a nonlocal continuous derivation of \mathfrak{A} at φ .

Close inspection will reveal that the nonlocal derivations we have been producing are local somewhere on $M_{\mathfrak{A}}$, though not at the point φ under consideration. For example, the nonlocal derivation indicated in Example 5.13 is actually local at every point of $X \times Z$. We now indicate how to construct nowhere local derivations.

EXAMPLE 5.17. For $j = 1, 2$ let A_j be a sup-norm algebra on $X_j = M_{A_j}$ and let L_j be a continuous nonlocal derivation of $\varphi_j \in X_j$. Let X denote the union of X_1 and X_2 , disjoint except for the identification of φ_1 with φ_2 ; let φ denote this common point of X . Let $A = \{f \in C(X) : f|X_j \in A_j \text{ for } j = 1, 2\}$. Then $M_A = X$.

Define L by $L(f) = L_1(f|X_1) + L_2(f|X_2)$. L is then a bounded derivation of A at φ which is local at no point of X .

Many properties of interest which A_1 and A_2 may have carry over to A , e.g., localness, holomorphic closure, antisymmetry, essentiality. If φ_1 and φ_2 are one-point parts, so is φ . Thus combining our previous examples, one can construct nowhere local continuous point derivations in algebras satisfying all sorts of decent conditions.

All of our examples so far have utilized analytic structures in the maximal ideal space, even if the analytic structure has not been at the point in question. We close by using a “Swiss cheese” algebra to point out that even with no analytic structure anywhere in M_A , it is possible to have $(A_\varphi^n)^- \neq (A_\varphi^{n+1})^-$ for all n .

EXAMPLE 5.18. Let $\{D_k : k = 1, 2, \dots\}$ be a sequence of mutually disjoint discs (with centers b_k and radii r_k) contained in D_0 , the open unit disc in \mathbb{C} . Assume $\sum_{k=1}^{\infty} r_k < \infty$ and let $X = \bar{D}_0 - \bigcup_{k=1}^{\infty} D_k$, a compact set in the plane. Let A denote the uniform closure on X of the rational functions whose poles lie off X . A is a sup-norm algebra on X , and $M_A = X$ (see [W1, p. 77]).

It is possible to choose the D_k so that $\bigcup_{k=1}^{\infty} D_k$ is dense in D_0 and does not contain 0, and so that $\sum_{k=1}^{\infty} r_k/(|b_k| - r_k)^m < \infty$ for $m=0, 1, 2, \dots$. Then X has no planar interior, so M_A has no analytic structure of any kind. Let $\varphi \in M_A$ be evaluation at 0.

Let T_k be the circle bounding D_k . μ will be the complex Borel measure on X supported by $\bigcup_{k=0}^{\infty} T_k$ and given by $d\mu(z) = dz/2\pi iz$ on T_0 and $d\mu(z) = -dz/2\pi iz$ on T_k , $k > 0$. For $m=0, 1, 2, \dots$ let $d\mu_m(z) = z^{-m} d\mu(z)$. μ_m is a complex measure; in fact,

$$\|\mu_m\| \leq 1 + \sum_{k=1}^{\infty} r_k/(|b_k| - r_k)^{m+1} < \infty.$$

It is easy to show that μ_m annihilates A_{φ}^{m+1} . However, clearly $z^m \in A_{\varphi}^m$ but $\int z^m d\mu_m(z) = \int d\mu(z) = 1$. Thus $(A_{\varphi}^m)^{\perp} \neq (A_{\varphi}^{m+1})^{\perp}$ for all m .

Wermer [W2] has apparently proved two interesting facts along these lines. First, if $\sum_{k=1}^{\infty} r_k \cdot |\log r_k| < \infty$ then there exists a nontrivial bounded derivation at almost every point of X . Second, and of great interest, is the fact that the D_k may be chosen in such a manner that there exists no bounded derivation of A at any point of X .

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