ON KLEIN'S COMBINATION THEOREM. II

BY

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It was proven in [2] that every finitely generated Kleinian group with an invariant region of discontinuity can be built up from certain "elementary" groups using two constructions. Both constructions are generalizations of the original combination theorem of Klein [1]. An elaboration of the first construction, the free product with amalgamation, was given in [3]. In this paper we give an elaboration of the second construction.

The basic idea is the following. We are given a Kleinian group G with two non-conjugate cyclic subgroups H_1 and H_2. We wish to adjoin to G a Möbius transformation f, so that $f \circ H_1 \circ f^{-1} = H_2$, and so that G', the group generated by G and f is again a Kleinian group. A set of conditions for this to be so was given in [2]. In this paper we give a more restrictive set of conditions, which enables us to find a fundamental set for G'.

Under these more restrictive conditions, we are also able to assert that there exists a Jordan curve C so that for every limit point z of G', which is not conjugate in G' to a limit point of G, or conjugate to a fixed point of f, there is a sequence $\{j_n\}$ of elements of G', so that $j_n(C)$ is a nested sequence of curves about z (a sequence of Jordan curves $\{C_n\}$ is said to be a *nested sequence of curves about* z, if $z = \bigcap B^n$ where B^n is one of the closed discs bounded by C_n and $B^{n+1} \subset B^n$).

The above assertion is useful for extending to the limit set, invariant mappings from the regular set of one Kleinian group onto the regular set of another. In particular, using this result and the corresponding result in [3], it is shown, in a forthcoming paper by Lipman Bers and myself, that a finitely generated Kleinian group with two invariant regions of discontinuity is a quasiconformal deformation of a Fuchsian group; in particular, the limit set of such a group is a Jordan curve.

1. We begin by introducing some definitions and notations.

We denote the extended complex plane, or Riemann sphere, by Σ. If G is a group of Möbius transformations, and $z \in \Sigma$, then z is called a *regular point for* G ($z \in R(G)$) if there is a neighborhood U of z so that $g(U) \cap U = \varnothing$, for all $g \in G$, $g \neq 1$. If $R(G) \neq \varnothing$ then G is called a *Kleinian group*.

A set D is called a *fundamental set* (FS) for the Kleinian group G, if D satisfies the following:

(i) $D \subset R(G)$,

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(ii) if $x, y \in D$, $g \in G$, $g(x)=y$, then $x=y$, and $g=1$,

(iii) if $x \in R(G)$, then there is a $y \in D$ and a $g \in G$ with $g(x)=y$.

We will use the following notation:

If S is a subset of Σ, and Γ a set of Möbius transformations, then $\pi(S, \Gamma)=\bigcup_{\gamma\in\Gamma} \gamma(S)$. Also, for any set x, we denote the closure of x in Σ by $\bar{x}$.

2. The main result of this paper is the following:

THEOREM. *Let H_1 and H_2 be cyclic subgroups of the Kleinian group G; let D be an FS for G; let $f \notin G$ be a Möbius transformation; and let B_1 and B_2 be disjoint Jordan domains, with disjoint boundary curves C_1 and C_2 respectively. Let $B_3=\Sigma-(\bar{B}_1 \cup \bar{B}_2)$. Assume that*

(1) $\operatorname{int}(B_3 \cap D)\neq\varnothing$;

(2) *for* $i=1, 2$, $\pi(D \cap \bar{B}_i, H_i)=\bar{B}_i \cap R(G)=\bar{B}_i \cap R(H_i)$;

(3) *for* $i=1, 2$, $\overline{D \cap C_i}\subset R(G)$;

(4) $D^*=B_3 \cup C_1$ *is an FS for the group generated by* f;

(5) $f^{-1} \circ H_2 \circ f=H_1$,

(6) $f(C_1)=C_2$.

Then

(1) *G', the group generated by G and f is Kleinian,*

(2) *$D'=D \cap D^*$ is a FS for G',*

(3) *Every relation in G' is a consequence of the relations in G, and the relations of hypothesis* (5).

(4) *If $z \notin R(G')$ and z is conjugate in G' to a point of $R(G)$, but z is not conjugate in G' to a fixed point of f, then there is a sequence j_n of elements of G', so that $j_n(C_1)$ form a nested sequence about z.*

3. It should be remarked that hypothesis 3 is a technicality, and can always be satisfied, if the others are, simply by changing that part of D that lies on C_1.

One can easily construct examples where all the hypotheses of the theorem are satisfied except that H_1 is not cyclic, and one expects that the theorem should be equally true in such cases, but I have been unable to prove this. In all these examples, however, the resulting group G' does not have an invariant region of discontinuity.

4. We list below some elementary consequences of the hypotheses of the theorem. The proofs actually involve only hypotheses 2, 4, 5, and 6, and are left to the reader. Some of the proofs use the well-known fact that if G is Kleinian, then $R(G)$ is dense in Σ.

PROPOSITION 1. *If $\alpha>0$, then $f^\alpha(B_3 \cup B_2 \cup C_2)\subset B_2$, if $\alpha<0$, then*

$$f^\alpha(B_3 \cup B_1 \cup C_1) \subset B_1.$$

PROPOSITION 2. *For $i=1, 2$, $H_i(B_i)=B_i$, $H_i(C_i)=C_i$.*

PROPOSITION 3. $f(C_1 \cap R(G))=C_2 \cap R(G)$.

PROPOSITION 4. *For* $i=1, 2$, *if* $g \in G$, $x, y \in B_i$, $g(x)=y$, *then* $g \in H_i$.

PROPOSITION 5. *If* $z \in B_1$, $g \in G$, *then* $g(z) \notin B_2$.

PROPOSITION 6. *For* $i, j=1, 2$, *if* $z \in B_i$, $g \in G$, *then* $g(z) \notin C_j$.

PROPOSITION 7. *If* $z \in C_1$, $g \in G$, $g(z) \in C_2$, *then* $z \notin R(G)$.

PROPOSITION 8. *For* $i=1, 2$, *if* $z \in C_i$, $g \in G-H_i$, $g(z) \in C_1$, *then* $z \notin R(G)$.

We also recall the following three lemmas from [3].

LEMMA 9 (Lemma 1 in [3]). *Let* $\{g_n\}$ *be a sequence of distinct elements of* G, *and let* $x \in \Sigma$. *Then* $\{z \mid g_n(z) \to x\}$ *is both open and closed in* $R(G)$.

LEMMA 10 (Lemma 2 in [3]). *Let* $z_n \to z \in R(G)$ *and let* $\{g_n\}$ *be a sequence of elements of* G *so that* $g_n(z_n) \to x$. *Then* $g_n(z) \to x$.

LEMMA 11 (Lemma 3 in [3]). *Let* $z_0 \in R(G)$ *and let* $\{g_n\}$ *be a sequence of distinct elements of* G, *with* $g_n(z_0) \to x$. *Then there is a subsequence* $\{g_{n_i}\}$ *so that* $g_{n_i}(z) \to x$, *for all* $z \in \Sigma$ *with at most one exception.*

5. It should be observed that the hypotheses of the theorem imply the hypotheses of Theorem 6 in [2], and so we already know that no two distinct points of $B_3 \cap D$ are equivalent under G'.

Every element $g' \in G'$ can be represented, not uniquely, as a word in the form

$$g' = f^{\alpha_{n+1}} \circ g_n \circ \cdots \circ g_1 \circ f^{\alpha_1},$$

where for $i=1, \ldots, n$, $g_i \in G$, $g_i \neq 1$, and for $i=2, \ldots, n$, $\alpha_i \neq 0$.

Using hypothesis (5), we can assume that, in this presentation, whenever $\alpha_i > 0$ and $g_i \in H_2$, then $\alpha_{i+1} \geqq 0$, and whenever $\alpha_i < 0$ and $g_i \in H_1$, then $\alpha_{i+1} \leqq 0$. We call such a presentation a *standard presentation*. We assume from here on that every element of G' is given by some standard presentation.

LEMMA 12. *Let* $g' = f^{\alpha_{n+1}} \circ \cdots \circ f^{\alpha_1} \in G'$, *and let* $z \in D'$. *If* $\alpha_{n+1} > 0$, *then* $g'(z) \in B_2 \cup (C_2 \cap R(G))$, *and if* $\alpha_{n+1} < 0$, *then* $g'(z) \in B_1$.

Proof. The proof is by induction on n. If $n=0$, then the result follows at once from Propositions 1 and 3.

We first assume that $\alpha_{n+1}, \alpha_n > 0$. Then $z^* = f^{\alpha_n} \circ \cdots \circ f^{\alpha_1}(z) \in B_2 \cup (C_2 \cap R(G))$, and so by Propositions 5, 6, and 7, $g_n(z^*) \in B_2 \cup C_2 \cup B_3$. Now, by Proposition 1, $f^{\alpha_{n+1}} \circ g_n(z^*) = g'(z) \in B_2$.

Similarly, if $\alpha_{n+1}, \alpha_n < 0$, then $z^* = f^{\alpha_n} \circ \cdots \circ f^{\alpha_1}(z) \in B_1$, and so $g_n(z^*) \in B_1 \cup B_3$, and then $g'(z) \in B_1$.

Now if $\alpha_{n+1} < 0$, $\alpha_n > 0$, then $g_n \notin H_2$. Now $z^* = f^{\alpha_n} \circ \cdots \circ f^{\alpha_1}(z) \in B_2 \cup (C_2 \cap R(G))$, then, since $g_n \notin H_2$, $g_n(z^*) \in B_3$, and so $g'(z) \in B_1$.

Similarly, if $\alpha_{n+1} > 0$, $\alpha_n < 0$, then $z^* = f^{\alpha_n} \circ \cdots \circ f^{\alpha_1}(z) \in B_1$ and so, since $g_n \notin H_1$, $g_n(z^*) \in B_2 \cup B_3$, and then $g'(z) \in B_2$.

There remains only the special case that $\alpha_n=0$, so that $n=1$, and $g'=f^{\alpha}\circ g$. Now $z\in D'\subset B_3\cup(C_1\cap R(G))$, and so, by hypothesis (2), and Propositions 5, 6, and 7, $g(z)\in B_3\cup(C_1\cap R(G))$. Now if $\alpha<0$, $g'(z)\in B_1$, and, by Proposition (2), if $\alpha>0$, $g'(z)\in B_2\cup(C_2\cap R(G))$.

LEMMA 13. *If $g'=f^{\alpha_{n+1}}\circ g_n\circ\cdots\circ f^{\alpha_1}\in G'$, where $\alpha_{n+1}=0$, $g_n\neq 1$, and $z\in D'$, then $g'(z)\notin D'$.*

Proof. If $\alpha_n\neq 0$, then by Lemma 9,

$$z^* = f^{\alpha_n}\circ\cdots\circ f^{\alpha_1}(z)\in B_1\cup B_2\cup(C_2\cap R(G)).$$

Now $(B_1\cup B_2\cup C_2)\cap D'=\varnothing$, and by hypothesis (2), if $z^*\in R(G)$, there is a $z_1^*\in(B_1\cup B_2\cup C_2)\cap D$ which is equivalent to z^* under G, and so $g_n(z^*)=g'(z)\notin D'$. If $z^*\notin R(G)$, then $g_n(z^*)=g'(z)\notin R(G)$, and so $g'(z)\notin D'$, since $D'\subset D\subset R(G)$.

If $\alpha_n=0$, then $g'=g_1\in G$ and the result is immediate.

Lemmas 12 and 13 show that if $z\in D'$, $g'\in G'$, $g'\neq 1$, then $g'(z)\notin D'$; i.e., no two distinct points of D' are equivalent under G'. The lemmas also reprove conclusion 3 in the statement of the theorem. In order to prove conclusion 2, we still have to show that $D'\subset R(G')$, and that $\pi(D', G')=R(G')$.

6. In this paragraph, we prove that $D'\subset R(G')$; we already know that int $(D')\subset R(G')$. We denote $D'\cap \operatorname{bd}(D')$ by $\dot{D}$.

LEMMA 14. *Let $z\in\dot{D}\cap B_3$. Then there is a neighborhood U of z so that $g(U)\subset B_3$, for every $g\in G$.*

Proof. Since $g(z)\in B_3$ for every $g\in G$, it suffices to show that z is not an accumulation point of images of $C_1\cup C_2$ under G; i.e., we need to show that there does not exist a sequence of points $\{z_n\}$, $z_n\in C_1\cup C_2$, and a sequence $\{g_n\}$ of elements of G, with $g_n(z_n)\to z$. If we did have such a sequence, then, since $z\in R(G)$, $z_n\in R(G)$, and so by hypothesis (3), we could pick a subsequence, which we again call z_n, so that $z_n\to z^*\in(C_1\cup C_2)\cap R(G)$. Picking a subsequence, we can assume the g_n all distinct, and so by Lemma 10, $g_n(z^*)\to z$, and therefore $z\notin R(G)$.

LEMMA 15. *Let $z\in\dot{D}\cap B_3$, then $z\in R(G')$.*

Proof. Let U be a neighborhood of z, with $g(U)\cap U=\varnothing$, for all $g\in G$, $g\neq 1$, and $g(U)\subset B_3$ for all $g\in G$. Let $g'=f^{\alpha_{n+1}}\circ\cdots\circ f^{\alpha_1}\in G'$. We will show that $g'(U)\cap U=\varnothing$, if $g'\neq 1$.

We first take the case that $\alpha_{n+1}\neq 0$. If $\alpha_1\neq 0$, then $f^{\alpha_1}(U)\subset B_1\cup B_2$. If $\alpha_1=0$, then $g_1(U)\subset B_3$, and so $f^{\alpha_2}\circ f^{\alpha_1}(U)\subset B_1\cup B_2$. The proof now proceeds by induction, exactly as in the proof of Lemma 12. Assume, for simplicity, that $\alpha_n>0$, so that $f^{\alpha_n}\circ\cdots\circ f^{\alpha_1}(U)\subset B_2$. Then $g_n\circ\cdots\circ f^{\alpha_1}(U)\subset B_3$ unless $g_n\subset H_2$, in which case $\alpha_{n+1}>0$ so that $g'(U)\subset B_2$. If $g_n\circ\cdots\circ g^{\alpha_1}(U)\subset B_3$, then, for $\alpha_{n+1}<0$, $g'(U)\subset B_1$

and for $\alpha_{n+1}>0$, $g'(U)\subset B_2$. Therefore, we have shown that $g'(U)\subset B_1$ if $\alpha_{n+1}<0$, and $g'(U)\subset B_2$ if $\alpha_{n+1}>0$.

We now consider the case that $\alpha_{n+1}=0$. If $\alpha_n=0$, then $g'\in G$, and there is nothing to prove, and so we assume for simplicity that $\alpha_n>0$. Then, by the above remark, $f^{\alpha_n}\circ\cdots\circ f^{\alpha_1}(U)\subset B_2$, so that C_2 separates $f^{\alpha_n}\circ\cdots\circ f^{\alpha_1}(U)$ and $g_n^{-1}(U)\subset B_3$. Therefore, $g_n(C_2)$ separates $g'(U)$ and U. This concludes the proof that $\dot{D}\cap B_3\subset R(G)$.

LEMMA 16. *Let $z\in\dot{D}\cap C_1$. Then $z\in R(G')$.*

Proof. It suffices to show that there is a neighborhood U of z satisfying

A. For every pair of distinct points z_1, $z_2\in U$, there are elements g_1', $g_2'\in G'$, with $g_1'(z_1)$, $g_2'(z_2)\in D'$, and $g_1'(z_1)\neq g_2'(z_2)$.

For any neighborhood U of z, we write $U_1=U\cap C_1$, $U_2=U\cap B_3$, $U_3=U-(U_1\cup U_2)$, and we observe that we can always choose U so that $f(U_3)\subset B_3$.

We first observe that we can find a U so that $U_1\subset R(G)$, and so for every pair z_1, $z_2\in U_1$, we have g_1, $g_2\in H_1$ with $g_i(z_i)\in D'$, $i=1, 2$. If there were pairs z_1, z_2 with $g_1(z_1)=g_2(z_2)$ for arbitrarily small U, then we would have a sequence $z_n\to z$ and elements $g_n\in H_1$ with $g_n(z_n)\to z$; thus by Lemma 10 $g_n(z)\to z$, contradicting the fact that $z\in R(H_1)$.

We next show that there is a U so that U_2 satisfies A. Since $z\in R(G)$, we pick U so that $U_2\subset R(G)$. As in the above if there were no U with U_2 satisfying A, we would have a sequence $z_n\to z$, and a sequence $g_n\in G$ with $g_n(z_n)\to z$, again contradicting the fact that $z\in R(G)$.

Since $f(C_1\cap R(G))=C_2\cap R(G)$, $f(z)\subset R(G)$, and so the same argument as above insures that we can find a U with U_3 satisfying A.

Now we pick a U so that U_1, U_2, and U_3 all satisfy A. Then if U does not satisfy A, there are $z_1\in U_2$, $z_2\in U_3$, and $g\in G$, so that $f^{-1}\circ g(z_1)=z_2$. Therefore, if there were no U satisfying A, then there would be a sequence $z_n\to z$, and a sequence g_n of elements of G so that $g_n(z_n)\to f(z)$. If infinitely many of the g_n were equal, then $z\notin R(G)$, by Proposition 7. If the g_n were all distinct, then $f(z)\notin R(G)$ by Lemma 10. Therefore, there is a neighborhood U of z, satisfying A and so the lemma is proven.

Putting together Lemmas 15 and 16, we have

LEMMA 17. $\dot{D}\subset R(G')$.

7. Thus far we know that $\pi(D', G')\subset R(G')$, and that if $g'\in G'$, $g'\neq 1$, then $g'(D')\cap D'=\varnothing$. In order to complete the proof that D' is a FS for G', we need to prove that $R(G')\subset\pi(D', G')$; we also still have to prove conclusion 4. Let $L_1=\Sigma-R(G)$, and let L_2 be the set of fixed points of f, then $(L_1\cup L_2)\cap R(G')=\varnothing$. Now let T be the set of points about which nothing is known; i.e.

$$T=\Sigma-\pi((L_1\cup L_2\cup D'), G').$$

The proof of the theorem will be complete once we have shown that for every $z \in T$, there is a sequence j'_n of elements of G', so that $j'_n(C_1)$ is a nested sequence of curves about z.

From the way T has been defined, $T \subset R(G) - L_2$, and $T \cap D' = \varnothing$, so that for every $z \in T$, there is a $g \in G$ with $g(z) \in D$, and there is an α with $f^{\alpha}(z) \in D^*$, where either $g \neq 1$, or $\alpha \neq 0$.

We now start with some fixed $z \in T$. Then there is an α_1 so that $f^{\alpha_1}(z) \in D^*$. Since T is invariant under G', $f^{\alpha_1}(z) \in D^* - D$, and so we can find a $g_1 \in G$, $g_1 \neq 1$, with $g_1 \circ f^{\alpha_1}(z) \in D - D^*$, and then we can find $\alpha_2 \neq 0$ so that $f^{\alpha_2} \circ g_1 \circ f^{\alpha_1}(z) \in D^* - D$, and so on. In this way, we get a sequence of elements $g'_n \in G'$, where

$$g'_{2n} = g_n \circ f^{\alpha_n} \circ \cdots \circ f^{\alpha_1}, \quad g'_{2n+1} = f^{\alpha_{n+1}} \circ g_n \circ \cdots \circ f^{\alpha_1},$$

$$g'_{2n}(z) \in D - D^*, \quad g'_{2n+1}(z) \in D^* - D.$$

These elements g'_n are given by certain presentations, we wish to show that these presentations are standard. We first prove

LEMMA 18. $T \cap (C_1 \cup C_2) = \varnothing$.

Proof. Since $f(C_1) = C_2$, it suffices to show that $T \cap C_1 = \varnothing$. This follows at once from the fact that $D' \cap C_1 = D \cap C_1$, and so $\pi(D', G') \cap C_1 = R(G) \cap C_1$. We now prove

LEMMA 19. *The g'_n are given by standard presentations.*

Proof. Assume that $\alpha_n > 0$, and $g_n \in H_2$. (The proof for $\alpha_n < 0$, $g_n \in H_1$ is essentially the same.) Since $D^* - D \subset B_3 \cup C_1$, $g'_{2n-1}(z) \in B_3$. Then since $g_n \in H_2$, $g'_{2n}(z) \notin B_2$. However, since $D - D^* \subset B_1 \cup \bar{B}_2$, $\alpha_{n+1} < 0$ if and only if $g'_{2n}(z) \in B_2$, and so $\alpha_{n+1} > 0$.

We now set $j_n^{-1} = g'_{2n}$. Let $C^n = C_1$ if $\alpha_{n+1} > 0$ and $C^n = C_2$ if $\alpha_{n+1} < 0$. Correspondingly we set $B^n = \bar{B}_1$ if $\alpha_{n+1} > 0$, and $B^n = \bar{B}_2$ if $\alpha_{n+1} < 0$.

LEMMA 20. $z \in j_n(B^n)$.

Proof. $j_n^{-1}(z) = g'_{2n}(z) \in D - D^* \subset B_1 \cup B_2$. Since $g'_{2n+1}(z) \in D^* - D \subset \bar{B}_3$, $j_n^{-1}(z) \in B_1$ if and only if $\alpha_{n+1} > 0$, and $j_n^{-1}(z) \in B_2$ if and only if $\alpha_{n+1} < 0$.

LEMMA 21. $j_{n+1}(B^{n+1}) \subset j_n(B^n)$.

Proof. It suffices to show that $\operatorname{int}(j_{n+1}(B^{n+1})) \cap \operatorname{int}(j_n(B^n)) \neq \varnothing$, and that $\operatorname{int}(j_{n+1}(B^{n+1})) \cap \operatorname{bd}(j_n(B^n)) = \varnothing$. The first assertion follows at once from Lemmas 18 and 20, and so we have to show that $\operatorname{int}(B^{n+1}) \cap j_{n+1}^{-1} \circ j_n(C^n) = \varnothing$. We assume for simplicity that $C^n = C_1$, so that $\alpha_{n+1} > 0$. Now $j_{n+1}^{-1} \circ j_n = g_{n+1} \circ f^{\alpha_{n+1}}$, and since $a_{n+1} > 0$, $f^{\alpha_{n+1}}(C_1) \subset C_2 \cup B_2$. It follows that $g_{n+1} \circ f^{\alpha_{n+1}}(C_1) \cap B_1 = \varnothing$, and that $g_{n+1} \circ f^{\alpha_{n+1}}(C_1) \cap B_2 \neq \varnothing$ only if $g_{n+1} \in H_2$. By Lemma 19, $g_{n+1} \in H_2$ only if $\alpha_{n+2} > 0$, and then $B^{n+1} = B_1$. This concludes the proof of the lemma.

8. Let $S=\bigcap j_n(B^n)$. We wish to prove that S consists only of the one point z. It is at this point that we need H_1 to be cyclic. We will prove that bd (S) consists of at most a finite number of points. Setting $j'_n=j_n$ if $B^n=B_1$, $j'_n=j_n\circ f$ if $B^n=B_2$, then bd (S) precisely consists of accumulation points of sequences $\{j'_n(z_n)\}$ where $z_n\in C_1$. The proof is essentially the same as that given in [3] (Lemmas 12, 13, and 14), and appears here as Lemmas 23, 24, and 25.

LEMMA 22. *The j'_n are all distinct.*

Since $j_{n+1}(B^{n+1})\subset j_n(B^n)$, it suffices to show that $j'_{n+1}\neq j'_n$. Now $(j'_{n+1})^{-1}\circ j'_n$ has one of the forms $g_{n+1}\circ f^{\alpha_{n+1}}$, $f^{-1}\circ g_{n+1}\circ f^{\alpha_{n+1}}\circ f$, $g_{n+1}\circ f^{\alpha_{n+1}}\circ f$, $f^{-1}\circ g_{n+1}\circ f^{\alpha_{n+1}}$. Using Lemmas 12 and 13, since $g_{n+1}\neq 1$, these are all different from the identity.

LEMMA 23. *If H_1 is finite cyclic, then* bd (S) *contains at most one point.*

Proof. Let $x\in$ bd (S), then there is a sequence of points $z_n\in C_1$ with $j'_n(z_n)\to x$. We now assume that $z_n\to y\in C_1$, and since $R(G')\cap C_1=R(H)\cap C_1=C_1$, $j'_n(y)\to x$, by Lemma 10. Now by Lemma 9, $j'_n(C_1)\to x$.

LEMMA 24. *If H_1 is a parabolic cyclic group, then* bd (S) *contains at most one point.*

Proof. The proof consists of picking appropriate subsequences, and to avoid cumbersome notation we will call the subsequence by the same name as the original sequence.

Let $y\in C_1$ be the fixed point of H_1, and choose a subsequence so that $j'_n(y)\to x$. Assume that there is a sequence $\{z_n\}$ of points of C_1 with $j'_n(z_n)\to x'\neq x$. Then choose a subsequence so that $z_n\neq y$ for all n. Now pick $h_n\in H_1$, so that $h_n(z_n)=w_n\in D\cap C_1$, and again pick a subsequence so that $w_n\to w$; by hypothesis 3, $w\in R(G')$. Now by Lemma 10, $j^*_n(w)=j'_n\circ h_n^{-1}(w)\to x'$, and $j^*_n(y)\to x$. Also, by Lemma 9, $j^*_n\circ h(w)\to x'$, for every $h\in H$.

Let $h\neq 1$ be some element of H_1, and set $h^n=j^*_n\circ h\circ (j^*_n)^{-1}$. Let $y_n=j^*_n(y)$, and assume without loss of generality, that all the $j^*_n(C_1)$ lie in some bounded part of the plane. Then h^n can be uniquely represented by a matrix

$$h^n \sim \begin{pmatrix} 1+p_ny_n & -p_ny_n^2 \\ p_n & 1-p_ny_n \end{pmatrix}.$$

We observe that $p_n=(u_n-u'_n)(u_n-y_n)^{-1}(u'_n-y_n)^{-1}$, where $u_n=j^*_n(w)$, and $u'_n=j^*_n\circ h(w)$, and so since $x\neq x'$, h^n converges to the identity, contradicting the fact that G' is a Kleinian group.

LEMMA 25. *If H_1 is a loxodromic (or hyperbolic) cyclic group, then* bd (S) *contains at most two points.*

Proof. As in the preceding lemma, the proof consists of picking appropriate subsequences, which we will again call by the same name as the original sequence.

Since H_1 is loxodromic, H_1 has two fixed points, y_1 and y_2, on C_1. We first pick a subsequence so that $j'_n(y_1) \to x_1$, $j'_n(y_2) \to x_2$. Now assume that there is an $x \in \text{bd}(S)$, $x \neq x_1, x_2$. Then there is a sequence $\{z_n\}$ of points of C_1 with $j'_n(z_n) \to x$. We can assume that $z_n \neq y_1, y_2$, for all n, and find $h_n \in H_1$, so that $h_n(z_n) = w_n \in D \cap C_1$. Pick a subsequence so that $w_n \to w \in R(G')$. Then by Lemma 10, $j^*_n(w) = j'_n \circ h_n^{-1}(w) \to x$. Now by Lemma 11, we can find a subsequence so that $j^*_n(t) \to x$ for all $t \in \Sigma$ with at most one exception. Therefore either $j^*_n(y_1) \to x$ or $j^*_n(y_2) \to x$, contradicting the assumption that $x \neq x_1, x_2$.

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