THE CHARACTERS OF THE FINITE SYMPLECTIC GROUP $Sp(4, q)$(1)

BY
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1. **Introduction.** In this paper we calculate all the (complex) irreducible characters of the group $Sp(4, q)$ where q is odd. The conjugacy classes of this group have been determined by Dickson [1a], Springer [4], and Wall [7]. We show that the irreducible characters of the group fall into families in a natural way, just as the conjugacy classes of the group do. Also involved in our work are certain polynomials in q which have properties similar to those of the polynomials Q_ρ^λ defined by Green [2] in his work on the characters of the groups $GL(n, q)$.

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Notation. G is the group of all nonsingular 4×4 matrices X over $F=GF(q)$ (q a power of the odd prime p) satisfying $XAX'=A$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The order of G is $q^4(q^2-1)(q^4-1)$, and the center Z is of order 2. (See e.g. Dickson [1].)

Let κ be a generator of the multiplicative group of $GF(q^4)$, and let $\zeta=\kappa^{q^2-1}$, $\theta=\kappa^{q^2+1}$, $\eta=\theta^{q-1}$, $\gamma=\theta^{q+1}$. Choose a fixed isomorphism from the multiplicative group of $GF(q^4)$ into the multiplicative group of complex numbers, and let $\tilde{\zeta}$, $\tilde{\theta}$, $\tilde{\eta}$, $\tilde{\gamma}$ be the images of ζ, θ, η, γ respectively under this isomorphism.

By a *character* of a finite group we mean a rational integral combination of the complex irreducible characters of the group. If χ, ϕ are class functions on the group, the scalar product (χ, ϕ) is defined as usual.

If ϕ is a character of a subgroup H of G, ϕ^G denotes the character of G induced from ϕ.

Conjugacy classes of G. Each element of G is an element of $GL(4, q)$, and so there correspond to it its characteristic polynomial $f_1^{n_1}f_2^{n_2}\dots$ where $f_1, f_2, \dots$ are distinct irreducible polynomials over F, and certain partitions $\nu_1, \nu_2, \dots$ of the positive integers $n_1, n_2, \dots$ (see [2, p. 406]). Using the results of Wall [7] we see the conjugacy classes of G are given by the table below.

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Class representative	Number of classes	Order of centralizer	Notation
$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$	1, 1	$q^4(q^2-1)(q^4-1)$	A_1, A_1'
$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$	1, 1	$2q^4(q^2-1)$	A_{21}, A_{21}'
$\begin{pmatrix} 1 & \gamma & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} -1 & -\gamma & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$	1, 1		A_{22}, A_{22}'
$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & & \\ & -1 & & \\ & & -1 & 1 \\ & & & -1 \end{pmatrix}$	1, 1	$2q^3(q-1)$	A_{31}, A_{31}'
$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & -\gamma \\ & & & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & & \\ & -1 & & \\ & & -1 & \gamma \\ & & & -1 \end{pmatrix}$	1, 1	$2q^3(q+1)$	A_{32}, A_{32}'
$\begin{pmatrix} 1 & 1 & & \\ & 1 & & 1 \\ -1 & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & & \\ & -1 & & -1 \\ 1 & & -1 & \\ & & & -1 \end{pmatrix}$	1, 1	$2q^2$	A_{41}, A_{41}'
$\begin{pmatrix} 1 & \gamma & & \\ & 1 & & 1 \\ -1 & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} -1 & -\gamma & & \\ & -1 & & -1 \\ 1 & & -1 & \\ & & & -1 \end{pmatrix}$	1, 1		A_{42}, A_{42}'
$\begin{pmatrix} \zeta^i & & & \\ & \zeta^{-i} & & \\ & & \zeta^{qi} & \\ & & & \zeta^{-qi} \end{pmatrix}$	$\frac{1}{4}(q^2-1)$ $i \in R_1$	q^2+1	$B_1(i)$

Class representative	Number of classes	Order of centralizer	Notation
$\begin{pmatrix} \theta^{i} & & & \\ & \theta^{-i} & & \\ & & \theta^{qi} & \\ & & & \theta^{-qi} \end{pmatrix}$	$\frac{1}{4}(q-1)^2$ $i \in R_2$	q^2-1	$B_2(i)$
$\begin{pmatrix} \gamma^{i} & & & \\ & \gamma^{-i} & & \\ & & \gamma^{j} & \\ & & & \gamma^{-j} \end{pmatrix}$	$\frac{1}{8}(q-3)(q-5)$ $i, j \in T_1, i \neq j$	$(q-1)^2$	$B_3(i, j)$
$\begin{pmatrix} \eta^{i} & & & \\ & \eta^{-i} & & \\ & & \eta^{j} & \\ & & & \eta^{-j} \end{pmatrix}$	$\frac{1}{8}(q-1)(q-3)$ $i, j \in T_2, i \neq j$	$(q+1)^2$	$B_4(i, j)$
$\begin{pmatrix} \eta^{i} & & & \\ & \eta^{-i} & & \\ & & \gamma^{j} & \\ & & & \gamma^{-j} \end{pmatrix}$	$\frac{1}{4}(q-1)(q-3)$ $i \in T_2, j \in T_1$	q^2-1	$B_5(i, j)$
$\begin{pmatrix} \eta^{i} & & & \\ & \eta^{-i} & & \\ & & \eta^{i} & \\ & & & \eta^{-i} \end{pmatrix}$	$\frac{1}{2}(q-1)$ $i \in T_2$	$q(q+1)(q^2-1)$	$B_6(i)$
$\begin{pmatrix} \eta^{i} & & 1 & \\ & \eta^{-i} & & 1 \\ & & \eta^{i} & \\ & & & \eta^{-i} \end{pmatrix}$	$\frac{1}{2}(q-1)$ $i \in T_2$	$q(q+1)$	$B_7(i)$
$\begin{pmatrix} \gamma^{i} & & & \\ & \gamma^{-i} & & \\ & & \gamma^{i} & \\ & & & \gamma^{-i} \end{pmatrix}$	$\frac{1}{2}(q-3)$ $i \in T_1$	$q(q-1)(q^2-1)$	$B_8(i)$
$\begin{pmatrix} \gamma^{i} & & 1 & \\ & \gamma^{-i} & & 1 \\ & & \gamma^{i} & \\ & & & \gamma^{-i} \end{pmatrix}$	$\frac{1}{2}(q-3)$ $i \in T_1$	$q(q-1)$	$B_9(i)$
$\begin{pmatrix} \eta^{i} & & & \\ & \eta^{-i} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \eta^{i} & & & \\ & \eta^{-i} & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$	$\frac{1}{2}(q-1)$, $\frac{1}{2}(q-1)$ $i \in T_2$	$q(q+1)(q^2-1)$	$C_1(i)$, $C_1'(i)$

$\begin{pmatrix} \eta^i & & & \\ & \eta^{-i} & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \eta^i & & & \\ & \eta^{-i} & & \\ & & -1 & -1 \\ & & & -1 \end{pmatrix}$ $\begin{pmatrix} \eta^i & & & \\ & \eta^{-i} & & \\ & & 1 & \gamma \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \eta^i & & & \\ & \eta^{-i} & & \\ & & -1 & -\gamma \\ & & & -1 \end{pmatrix}$	$\frac{1}{2}(q-1)$, $\frac{1}{2}(q-1)$ $\frac{1}{2}(q-1)$, $\frac{1}{2}(q-1)$ $i \in T_2$	$2q(q+1)$	$C_{21}(i)$, $C'_{21}(i)$ $C_{22}(i)$, $C'_{22}(i)$
$\begin{pmatrix} \gamma^i & & & \\ & \gamma^{-i} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \gamma^i & & & \\ & \gamma^{-i} & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$	$\frac{1}{2}(q-3)$, $\frac{1}{2}(q-3)$ $i \in T_1$	$q(q-1)(q^2-1)$	$C_3(i)$, $C'_3(i)$
$\begin{pmatrix} \gamma^i & & & \\ & \gamma^{-i} & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} \gamma^i & & & \\ & \gamma^{-i} & & \\ & & -1 & -1 \\ & & & -1 \end{pmatrix}$ $\begin{pmatrix} \gamma^i & & & \\ & \gamma^{-i} & & \\ & & 1 & \gamma \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \gamma^i & & & \\ & \gamma^{-i} & & \\ & & -1 & -\gamma \\ & & & -1 \end{pmatrix}$	$\frac{1}{2}(q-3)$, $\frac{1}{2}(q-3)$ $\frac{1}{2}(q-3)$, $\frac{1}{2}(q-3)$ $i \in T_1$	$2q(q-1)$	$C_{41}(i)$, $C'_{41}(i)$ $C_{42}(i)$, $C'_{42}(i)$
$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$	1	$q^2(q^2-1)^2$	D_1
$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & -1 \\ & & & -1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & -\gamma \\ & & & -1 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \begin{pmatrix} 1 & \gamma & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$	1, 1 1, 1	$2q^2(q^2-1)$	D_{21}, D_{22} D_{23}, D_{24}
$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & -1 & -1 \\ & & & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & -1 & -\gamma \\ & & & -1 \end{pmatrix}$ $\begin{pmatrix} 1 & \gamma & & \\ & 1 & & \\ & & -1 & -1 \\ & & & -1 \end{pmatrix}, \begin{pmatrix} 1 & \gamma & & \\ & 1 & & \\ & & -1 & -\gamma \\ & & & -1 \end{pmatrix}$	1, 1 1, 1	$4q^2$	D_{31}, D_{32} D_{33}, D_{34}

(We remark that in the first column we give as class representatives not necessarily elements of G, but their canonical forms in an extension field of F.)

REMARKS. The sets R_1, R_2, T_1, T_2 of positive integers mentioned in the table are defined as follows.

$R_1=\{1, 2, \ldots, \frac{1}{4}(q^2-1)\}$,

R_2 is a set of $\frac{1}{4}(q-1)^2$ distinct positive integers i such that θ^i, θ^{-i}, θ^{qi}, θ^{-qi} are all distinct,

$T_1=\{1, 2, \ldots, \frac{1}{2}(q-3)\}$, and $T_2=\{1, 2, \ldots, \frac{1}{2}(q-1)\}$.

The elements of the classes $B_1(i), \ldots, B_5(i, j)$, $B_6(i)$, $B_8(i)$ are p-regular. The elements of $B_7(i)$ and $B_9(i)$ have their p-regular factors in $B_6(i)$ and $B_8(i)$ respectively.

LEMMA 1.1. *Let A be the additive group of F, i.e. an elementary abelian group of order q. Then there exist irreducible characters $\alpha \to \varepsilon(\alpha)$ and $\alpha \to \varepsilon'(\alpha)$ of A such that*

$$\sum_{\alpha\in S} \varepsilon(\alpha) = \sum_{\alpha\in S'} \varepsilon'(\alpha) = -\frac{s}{2}(s+(sq)^{1/2}),$$

$$\sum_{\alpha\in S'} \varepsilon(\alpha) = \sum_{\alpha\in S} \varepsilon'(\alpha) = -\frac{s}{2}(s-(sq)^{1/2}),$$

where $s=(-1)^{(q-1)/2}$, S is the set of nonzero elements of F which are squares, and S' is the set of elements of F which are not squares in F.

Proof. We know (see e.g. [5, p. 103]) that there exist characters ϕ, ϕ' of A such that

$$\begin{aligned} \phi(0) &= \phi'(0) = \tfrac{1}{2}(q+s), \\ \phi(\alpha) &= \tfrac{1}{2}(s+(sq)^{1/2})) \qquad \text{if } \alpha \in S, \\ &= \tfrac{1}{2}(s-(sq)^{1/2}) \qquad \text{if } \alpha \in S', \end{aligned}$$

and

$$\begin{aligned} \phi'(\alpha) &= \tfrac{1}{2}(s-(sq)^{1/2}) \qquad \text{if } \alpha \in S\ , \\ &= \tfrac{1}{2}(s+(sq)^{1/2}) \qquad \text{if } \alpha \in S'. \end{aligned}$$

Then $(\phi', \phi')=(\phi, \phi)=\frac{1}{2}(q+s)$, and

$$\begin{aligned} (\phi, \phi') &= 1 \qquad \text{if } q \equiv 1 \pmod 4, \\ &= 0 \qquad \text{if } q \equiv -1 \pmod 4. \end{aligned}$$

ϕ and ϕ' contain the identity character if $q\equiv 1 \pmod 4$ and have no irreducible constituent in common if $q\equiv -1 \pmod 4$. They are each the sum of $\frac{1}{2}(q+s)$ distinct irreducible characters.

Let ε (ε') be a nonidentity irreducible character of A occuring in ϕ' (ϕ) if $q\equiv 1 \pmod 4$, and in ϕ (ϕ') if $q\equiv -1 \pmod 4$. Solving the equations

$$\begin{aligned} (\varepsilon, \phi') &= 1, \qquad (\varepsilon, \phi) = 0 \quad \text{if } q \equiv 1 \pmod 4, \\ (\varepsilon, \phi) &= 1, \qquad (\varepsilon, \phi') = 0 \quad \text{if } q \equiv -1 \pmod 4, \end{aligned}$$

we find that ε, and similarly ε', have the required properties.

(1.2) Let $\tilde{\varepsilon} = -s(s+(sq)^{1/2})/2$, $\tilde{\varepsilon}' = -s(s-(sq)^{1/2})/2$, where $s = (-1)^{(q-1)/2}$

(1.3) We have the identities

$$\tilde{\varepsilon}^2 + \tilde{\varepsilon}'^2 = \tfrac{1}{2}(1+sq),$$
$$2\tilde{\varepsilon}\tilde{\varepsilon}' = \tfrac{1}{2}(1-sq).$$

2. **The Sylow p-subgroup of G.** Consider a Sylow p-subgroup U of G of order q^4 consisting of all matrices of the form

$$\begin{pmatrix} 1 & \lambda & 0 & \lambda\alpha+\beta \\ 0 & 1 & 0 & \alpha \\ -\alpha & \beta & 1 & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\lambda, \alpha, \mu, \beta \in F).$$

This element of U will be denoted by $(\lambda, \alpha, \mu, \beta)$. The elements of the form $(\lambda, 0, \mu, \beta)$ form a subgroup W of U of order q^3. The center is of order q. The commutator subgroup is of order q^2 and consists of elements of the form $(0, 0, \mu, \beta)$.

The conjugacy classes of U are given in the table below. Again we use the results of Wall [7] to determine the class of G in which each class of U lies.

Class representative	No. of classes	Order of Centralizer in U	Class in G
$(0, 0, 0, 0)$	1	q^4	A_1
$(0, 0, \mu, 0)$, $\mu \neq 0$	$q-1$	q^4	A_{21} if $\mu \in S$, A_{22} if $\mu \in S'$
$(0, 0, 0, \beta)$, $\beta \neq 0$	$q-1$	q^3	A_{31}
$(\lambda, 0, 0, 0)$, $\lambda \neq 0$	$q-1$	q^3	A_{21} if $\lambda \in S$, A_{22} if $\lambda \in S'$
$(\lambda, 0, \mu, 0)$, $\lambda \neq 0$, $\mu \neq 0$	$(q-1)^2$	q^3	A_{31} if $-\lambda\mu \in S$, A_{32} if $-\lambda\mu \in S'$
$(0, \alpha, 0, 0)$, $\alpha \neq 0$	$q-1$	q^2	A_{31}
$(\lambda, \alpha, 0, 0)$, $\lambda \neq 0$, $\alpha \neq 0$	$(q-1)^2$	q^2	A_{41} if $\lambda \in S$, A_{42} if $\lambda \in S'$

We now consider certain characters of UZ and induce them to G. Now the "one-parameter subgroups" $\{(\lambda, 0, 0, 0)\}_{\lambda\in F}$, $\{(0, \alpha, 0, 0)\}_{\alpha\in F}$, etc. are all isomorphic to the additive group of F. By making use of these isomorphisms we see that there exist characters $\varepsilon: (\lambda, 0, 0, 0) \to \varepsilon(\lambda)$, $\varepsilon': (\lambda, 0, 0, 0) \to \varepsilon'(\lambda)$ of the subgroup $\{(\lambda, 0, 0, 0)\}$ having the properties stated in Lemma 1.1. Similarly we define characters ε, ε' of each of the three other subgroups.

(2.1) Consider the character of U defined by

$$(\lambda, 0, \mu, \beta) \to 1, \qquad (0, \alpha, 0, 0) \to \varepsilon(\alpha).$$

This character can be extended to a linear character of UZ in two ways. We induce these two characters of UZ to G, and denote the characters obtained by ψ_1 and ψ_1'. Thus the representation associated with ψ_1 (ψ_1') maps $-I$ on I ($-I$). (The same convention will be followed in the rest of this section.)

(2.2) Take the characters

$$(0, \alpha, \mu, \beta) \to 1, \qquad (\lambda, 0, 0, 0) \to \varepsilon(\lambda);$$
$$(0, \alpha, \mu, \beta) \to 1, \qquad (\lambda, 0, 0, 0) \to \varepsilon'(\lambda)$$

and apply the same procedure as in (2.1). The two characters of G obtained from the first character will be denoted by ψ_{21}, ψ_{21}' and those obtained from the second character will be denoted by ψ_{22}, ψ_{22}'. Let $\psi_2=\psi_{21}+\psi_{22}$, $\psi_2'=\psi_{21}'+\psi_{22}'$.

(2.3) Consider the characters

$$(0, 0, \mu, \beta) \to 1, \qquad (0, \alpha, 0, 0) \to \varepsilon(\alpha), \qquad (\lambda, 0, 0, 0) \to \varepsilon(\lambda)$$

and

$$(0, 0, \mu, \beta) \to 1, \qquad (0, \alpha, 0, 0) \to \varepsilon(\alpha), \qquad (\lambda, 0, 0, 0) \to \varepsilon'(\lambda).$$

Using the same procedure and convention as in (2.2), the four characters of G obtained will be denoted by ψ_{31}, ψ_{31}', ψ_{32}, ψ_{32}'.

Let $\psi_3=\psi_{31}+\psi_{32}$, $\psi_3'=\psi_{31}'+\psi_{32}'$.

(2.4) Consider the character

$$(\lambda, 0, \mu, 0) \to 1, \qquad (0, 0, 0, \beta) \to \varepsilon(\beta),$$

of the subgroup W. Extend this in two ways to WZ, and induce to G. We denote the two characters obtained by ψ_4, ψ_4'.

(2.5) Consider the character of W which is the sum of the two characters

$$(0, 0, 0, \beta) \to 1, \qquad (\lambda, 0, 0, 0) \to \varepsilon(\lambda), \qquad (0, 0, \mu, 0) \to \varepsilon(\mu)$$

and

$$(0, 0, 0, \beta) \to 1, \qquad (\lambda, 0, 0, 0) \to \varepsilon(\lambda), \qquad (0, 0, \mu, 0) \to \varepsilon'(\mu).$$

Again adopting the procedure of (2.4), we denote the two characters of G obtained by ψ_5 and ψ_5'.

We now give the values of the characters of G that we have constructed at the classes of G. We have omitted the characters $\psi_1', \ldots$, and also the values of $\psi_1, \ldots$ at $A_1', \ldots, A_{42}'$. At all other classes which are not mentioned the values of the characters are zero.

Each entry in the table is to be multiplied by an integer which depends on the column in which the entry is found. Thus, the value of ψ_1 at A_1 is $\frac{1}{2}(q^2-1)(q^4-1)$, etc.

In computing these characters, we have made use of the identities (1.3), as well as the following facts.

(2.6) The elements of W which lie in A_{21} (A_{22}) are the elements of the form $(0, 0, \mu, 0)$ where $\mu \in S$ (S'), and the elements of the form $(\lambda, 0, \nu^2\lambda, -\nu\lambda)$ where $\lambda \in S$ (S') and ν is any element of F.

Class	A_1	A_{21}	A_{22}	A_{31}	A_{32}	A_{41}	A_{42}
Multiplying factor	$(q^2-1)(q^4-1)$	q^2-1	q^2-1	$q-1$	$q+1$	1	1
ψ_1	$\frac{1}{2}$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-2q-1)$	$\frac{1}{2}(q-1)^2$	$\frac{1}{2}(1-q)$	$\frac{1}{2}(1-q)$
ψ_{21}	$\frac{1}{2}$	$\frac{1}{2}(q-1)+q\tilde{\varepsilon}$	$\frac{1}{2}(q-1)+q\tilde{\varepsilon}'$	$\frac{1}{2}(2q^2-q-1)$	$\frac{1}{2}(1-q)$	$(q-1)\tilde{\varepsilon}$	$(q-1)\tilde{\varepsilon}'$
ψ_{22}	$\frac{1}{2}$	$\frac{1}{2}(q-1)+q\tilde{\varepsilon}'$	$\frac{1}{2}(q-1)+q\tilde{\varepsilon}$	$\frac{1}{2}(2q^2-q-1)$	$\frac{1}{2}(1-q)$	$(q-1)\tilde{\varepsilon}'$	$(q-1)\tilde{\varepsilon}$
$\psi_2\ (=\psi_{21}+\psi_{22})$	1	-1	-1	$2q^2-q-1$	$1-q$	$1-q$	$1-q$
ψ_{31}	$\frac{1}{2}$	$\frac{1}{2}(q-1)+q\tilde{\varepsilon}$	$\frac{1}{2}(q-1)+q\tilde{\varepsilon}'$	$-\frac{1}{2}(1+q)$	$\frac{1}{2}(1-q)$	$-\tilde{\varepsilon}$	$-\tilde{\varepsilon}'$
ψ_{32}	$\frac{1}{2}$	$\frac{1}{2}(q-1)+q\tilde{\varepsilon}'$	$\frac{1}{2}(q-1)+q\tilde{\varepsilon}$	$-\frac{1}{2}(1+q)$	$\frac{1}{2}(1-q)$	$-\tilde{\varepsilon}'$	$-\tilde{\varepsilon}$
$\psi_3\ (=\psi_{31}+\psi_{32})$	1	-1	-1	$-(1+q)$	$1-q$	1	1
ψ_4	$\frac{1}{2}q$	$\frac{1}{2}q(q-1)$	$\frac{1}{2}q(q-1)$	$-q$	0	0	0
ψ_5	q	$-q$	$-q$	$-q$	q	0	0

(2.7) The elements of W which lie in A_{31} are the elements of the form $(0, 0, -2\nu\beta, \beta)$ $(\nu, \beta \in F)$, and the elements of the form $(\lambda, 0, \nu^2\lambda+\mu, -\nu\lambda)$ where ν is any element of F and $-\lambda\mu \in S$.

(2.8) The elements of W which lie in A_{32} are the elements of the form $(\lambda, 0, \nu^2\lambda+\mu, -\nu\lambda)$ where ν is any element of F and $-\lambda\mu \in S'$.

3. **Some subgroups of** G. We first prove a lemma which will be used to show the existence of certain types of subgroups of G.

LEMMA 3.1. *Let $\tilde{G}$ be the algebraic group (over F) of all nonsingular 4×4 matrices X over a universal domain Ω containing F as a subfield, satisfying $XAX'=A$ where A is as in* §1. *If $a \in \tilde{G}$, let $a^{(q)}$ be the element of $\tilde{G}$ obtained by raising every entry in the matrix a to its* qth *power. Let H be a subgroup of $\tilde{G}$. Then there exists an element $y \in \tilde{G}$ such that $y^{-1}Hy\subseteq G$, if and only if there exists an element $z \in \tilde{G}$ such that $z^{-1}az=a^{(q)}$ for all $a \in H$.*

Proof. Clearly G is the subgroup of $\tilde{G}$ consisting of all $a \in \tilde{G}$ such that $a^{(q)}=a$. The lemma then follows from a theorem of Lang [3] which asserts that for any $z \in \tilde{G}$ there exists a $y \in \tilde{G}$ such that $z=yy^{-(q)}$.

We now consider the following subgroups of G.

(3.2) Consider the elements

$$a=\begin{pmatrix}\zeta&&&\\&\zeta^{-1}&&\\&&\zeta^{q}&\\&&&\zeta^{-q}\end{pmatrix},\quad b=\begin{pmatrix}&&&1\\&&-1&\\1&&&\\&1&&\end{pmatrix}$$

in $\tilde{G}$. Then b transforms every element c of $\{a, b\}$ to $c^{(q)}$, and hence, by Lemma 3.1, G contains a subgroup M_1 conjugate to $\{a, b\}$ in $\tilde{G}$. Let $M_1=\{a_1, x_1\}$, where $a_1^{q^2+1}=x_1^8=1$, $x_1^{-1}a_1x_1=a_1^q$. Let $H_1=\{a_1\}$.

We give below the conjugacy classes of M_1 and the classes of G containing them. Sometimes the class of G to which an element belongs depends on q. In these cases we have just indicated the characteristic polynomial of the element. The elements in question are always p-regular.

$1, a_1^{\frac{1}{2}(q^2+1)}$	1, 1	$4(q^2+1)$	A_1, A_1'
a_1^i	$\frac{1}{4}(q^2-1), i \in R_1$	q^2+1	$B_1(i)$
$x_1, x_1^3, a_1^q x_1, a_1^q x_1^3$	1, 1, 1, 1	8	$\{x^4+1\}$
$x_1^2, a_1 x_1^2$	1, 1	8	$\{(x^2+1)^2\}$

In the tables of conjugacy classes of subgroups in this section the columns contain, from left to right, (i) class representative, (ii) number of classes, (iii) order of centralizer in the subgroup, and (iv) class in G. In the following, the use of Lemma 3.1 to prove the existence of subgroups will not be explicitly mentioned unless the details are not straightforward.

(3.3) Let M_2 be a subgroup of G which is conjugate in $\tilde{G}$ to the subgroup of $\tilde{G}$ generated by

$$\begin{pmatrix} \theta & & & \\ & \theta^{-1} & & \\ & & \theta^q & \\ & & & \theta^{-q} \end{pmatrix}, \quad \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}.$$

Then $M_2=\{a_2, x_2, y_2\}$, with

$$a_2^{q^2-1} = x_2^4 = y_2^2 = [x_2, y_2]^{(2)} = 1, \qquad x_2^{-1}a_2x_2 = a_2^q, \qquad y_2^{-1}a_2y_2 = a_2^{-q}.$$

Let $H_2=\{a_2\}$. The conjugacy classes of M_2 are given in the next table.

(3.4) Let $H_3=\{a_3, b_3\}$, $M_3=\{a_3, b_3, x_3, y_3, z_3\}$, where

$$a_3 = \begin{pmatrix} \gamma & & & \\ & \gamma^{-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma & \\ & & & \gamma^{-1} \end{pmatrix}, \quad x_3 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix},$$

$$y_3 = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad z_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & -1 & \end{pmatrix}.$$

(2) If $a, b \in G$, $[a, b]=a^{-1}b^{-1}ab$.

$1, a_2^{\frac{1}{2}(q^2-1)}$	1, 1	$4(q^2-1)$	A_1, A_1'
a_2^i	$\frac{1}{4}(q-1)^2, i \in R_2$	q^2-1	$B_2(i)$
$a_2^{i(q-1)}$	$\frac{1}{2}(q-1), i \in T_2$	$2(q^2-1)$	$B_6(i)$
$a_2^{i(q+1)}$	$\frac{1}{2}(q-3), i \in T_1$	$2(q^2-1)$	$B_8(i)$
x_2	1	$4(q-1)$	$\{(x^2+1)^2\}$
$a_2^{\frac{1}{2}(q-1)}x_2$	1	$4(q-1)$	D_1
$a_2^i x_2$	$\frac{1}{2}(q-3), i \in T_1$	$2(q-1)$	$\{(x^2+\gamma^i)(x^2+\gamma^{-i})\}$
y_2	1	$4(q+1)$	D_1
$a_2^{\frac{1}{2}(q+1)}y_2$	1	$4(q+1)$	$\{(x^2+1)^2\}$
$a_2^i y_2$	$\frac{1}{2}(q-1), i \in T_2$	$2(q+1)$	$\{(x^2-\eta^i)(x^2-\eta^{-i})\}$
$x_2y_2, a_2x_2y_2$	1, 1	8	$\{(x^2+1)^2\}$

Then we have the relations

$$a_3^{q-1} = b_3^{q-1} = [a_3, b_3] = x_3^4 = y_3^4 = z_3^4 = [y_3, z_3] = [a_3, z_3] = [b_3, y_3] = 1,$$

$$x_3^{-1}y_3x_3 = z_3, \quad x_3^{-1}a_3x_3 = b_3, \quad y_3^{-1}a_3y_3 = a_3^{-1}, \quad z_3^{-1}b_3z_3 = b_3^{-1}.$$

The conjugacy classes of M_3 are given below.

$1, (a_3b_3)^{(q-1)/2}$	1, 1	$8(q-1)^2$	A_1, A_1'
$a_3^{(q-1)/2}$	1	$4(q-1)^2$	D_1
$a_3^i, a_3^ib_3^{(q-1)/2}$	$\frac{1}{2}(q-3), \frac{1}{2}(q-3), i \in T_1$	$2(q-1)^2$	$C_3(i), C_3'(i)$
$(a_3b_3)^i$	$\frac{1}{2}(q-3), i \in T_1$	$2(q-1)^2$	$B_8(i)$
$a_3^ib_3^j$	$(q-3)(q-5)/8, i, j \in T_1, i \neq j$	$(q-1)^2$	$B_3(i, j)$
y_3, a_3y_3	1, 1	$8(q-1)$	$\{(x^2+1)(x-1)^2\}$
$(a_3b_3)^{(q-1)/2}y_3, (a_3b_3)^{(q-1)/2}a_3y_3$	1, 1	$8(q-1)$	$\{(x^2+1)(x+1)^2\}$
$a_3^iz_3, a_3^ib_3z_3$	$\frac{1}{2}(q-3), \frac{1}{2}(q-3), i \in T_1$	$4(q-1)$	$\{(x-\gamma^i)(x-\gamma^{-i})(x^2+1)\}$
$y_3z_3, a_3y_3z_3$	1, 1	16	$\{(x^2+1)^2\}$
x_3	1	$4(q-1)$	$\{(x^2+1)^2\}$
$a_3^{(q-1)/2}x_3$	1	$4(q-1)$	D_1
$a_3^ix_3$	$\frac{1}{2}(q-3), i \in T_1$	$2(q-1)$	$\{(x^2+\gamma^i)(x^2+\gamma^{-i})\}$
$x_3y_3, a_3x_3y_3$	1, 1	8	$\{x^4+1\}$

(3.5) Let M_4 be a subgroup of G which is conjugate in $\tilde{G}$ to the subgroup of $\tilde{G}$ generated by

$$\begin{pmatrix} \eta & & & \\ & \eta^{-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \eta & \\ & & & \eta^{-1} \end{pmatrix}, \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix},$$

$$\begin{pmatrix} & 1 & & \\ -1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & -1 & \end{pmatrix}.$$

Then $M_4=\{a_4, b_4, x_4, y_4, z_4\}$, where

$$a_4^{q+1} = b_4^{q+1} = [a_4, b_4] = x_4^2 = y_4^4 = z_4^4 = [y_4, z_4] = [a_4, z_4] = [y_4, b_4] = 1,$$

$$x_4^{-1}y_4x_4 = z_4, \quad x_4^{-1}a_4x_4 = b_4, \quad y_4^{-1}a_4y_4 = a_4^{-1}, \quad z_4^{-1}b_4z_4 = b_4^{-1}.$$

Let $H_4=\{a_4, b_4\}$. The conjugacy classes of M_4 are given below.

$1,\ (a_4b_4)^{(q-1)/2}$	$1, 1$	$8(q+1)^2$	$A_1,\ A_1'$
$a_4^{(q+1)/2}$	1	$4(q+1)^2$	D_1
$a_4^i,\ a_4^ib_4^{(q+1)/2}$	$\frac{1}{2}(q-1), \frac{1}{2}(q-1),\ i \in T_2$	$2(q+1)^2$	$C_1(i),\ C_1'(i)$
$(a_4b_4)^i$	$\frac{1}{2}(q-1),\ i \in T_2$	$2(q+1)^2$	$B_6(i)$
$a_4^ib_4^j$	$(q-1)(q-3)/8,$ $i, j \in T_2,\ i \neq j$	$(q+1)^2$	$B_4(i, j)$
$(a_4b_4)^{(q+1)/2}y_4,\ (a_4b_4)^{(q+1)/2}a_4y_4$	$1, 1$	$8(q+1)$	$\{(x^2+1)(x+1)^2\}$
$y_4,\ a_4y_4$	$1, 1$	$8(q+1)$	$\{(x^2+1)(x-1)^2\}$
$a_4^iz_4,\ a_4^ib_4z_4$	$\frac{1}{2}(q-1), \frac{1}{2}(q-1),\ i \in T_2$	$4(q+1)$	$\{(x-\eta^i)(x-\eta^{-i})$ $(x^2+1)\}$
$y_4z_4,\ a_4y_4z_4$	$1, 1$	16	$\{(x^2+1)^2\}$
x_4	1	$4(q+1)$	D_1
$a_4^{(q+1)/2}x_4$	1	$4(q+1)$	$\{(x^2+1)^2\}$
$a_4^ix_4$	$\frac{1}{2}(q-1),\ i \in T_2$	$2(q+1)$	$\{(x^2-\eta^i)(x^2-\eta^{-i})\}$
$x_4y_4,\ a_4x_4y_4$	$1, 1$	8	$\{x^4+1\}$

(3.6) Let M_5 be a subgroup of G which is conjugate in $\tilde{G}$ to the subgroup generated by

$$\begin{pmatrix} \eta & & & \\ & \eta^{-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma & \\ & & & \gamma^{-1} \end{pmatrix}, \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & -1 & \end{pmatrix}.$$

Then $M_5 = \{a_5, b_5, y_5, z_5\}$, where $a_5^{q+1} = b_5^{q-1} = y_5^4 = z_5^4 = [a_5, b_5] = [y_5, z_5] = [a_5, z_5] = [b_5, y_5] = 1$, $y_5^{-1} a_5 y_5 = a_5^{-1}$, $z_5^{-1} b_5 z_5 = b_5^{-1}$. Let $H_5 = \{a_5, b_5\}$.

The conjugacy classes of M_5 are given below.

$1, a_5^{(q+1)/2} b_5^{(q-1)/2}$	1, 1	$4(q^2-1)$	A_1, A_1'
$a_5^{(q+1)/2}, b_5^{(q-1)/2}$	1, 1	$4(q^2-1)$	D_1
$a_5^i, a_5^i b_5^{(q-1)/2}$	$\frac{1}{2}(q-1), \frac{1}{2}(q-1), i \in T_2$	$2(q^2-1)$	$C_1(i), C_1'(i)$
$b_5^i, b_5^i a_5^{(q+1)/2}$	$\frac{1}{2}(q-3), \frac{1}{2}(q-3), i \in T_1$	$2(q^2-1)$	$C_3(i), C_3'(i)$
$a_5^i b_5^j$	$\frac{1}{4}(q-1)(q-3), i \in T_2, j \in T_1$	q^2-1	$B_5(i, j)$
$y_5, a_5 y_5$	1, 1	$8(q-1)$	$\{(x^2+1)(x-1)^2\}$
$y_5 b_5^{(q-1)/2}, a_5 y_5 b_5^{(q-1)/2}$	1, 1	$8(q-1)$	$\{(x^2+1)(x+1)^2\}$
$z_5, b_5 z_5$	1, 1	$8(q+1)$	$\{(x^2+1)(x-1)^2\}$
$z_5 a_5^{(q+1)/2}, b_5 z_5 a_5^{(q-1)/2}$	1, 1	$8(q+1)$	$\{(x^2+1)(x+1)^2\}$
$a_5^i z_5, a_5^i b_5 z_5$	$\frac{1}{2}(q-1), \frac{1}{2}(q-1), i \in T_2$	$4(q+1)$	$\{(x-\eta^i)(x-\eta^{-i})(x^2+1)\}$
$b_5^i y_5, b_5^i a_5 y_5$	$\frac{1}{2}(q-3), \frac{1}{2}(q-3), i \in T_1$	$4(q-1)$	$\{(x-\gamma^i)(x-\gamma^{-i})(x^2+1)\}$
$y_5 z_5, y_5 b_5 z_5, a_5 y_5 z_5, a_5 b_5 y_5 z_5$	1, 1, 1, 1	16	$\{(x^2+1)^2\}$

(3.7) Let K be the subgroup of G consisting of all elements of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A, B are 2×2 matrices. Then K is of order $q^2(q^2-1)^2$, and $K \cong S_p(2, q) \times S_p(2, q) \cong SL(2, q) \times SL(2, q)$.

(3.8) Let K_1 be the subgroup of G consisting of all elements of the form

$$\begin{pmatrix} \gamma^i & \cdot & \cdot & \beta\gamma^i \\ \cdot & \gamma^{-i} & \cdot & \cdot \\ \cdot & \beta\gamma^{-i} & \gamma^{-i} & \cdot \\ \cdot & \cdot & \cdot & \gamma^i \end{pmatrix} \qquad (\beta \in F).$$

Then $|K_1| = q(q-1)$, and K_1 is the direct product of a cyclic group of order $q-1$ and an elementary abelian group of order q. Let

$$u = \begin{pmatrix} \gamma & & & \\ & \gamma^{-1} & & \\ & & \gamma^{-1} & \\ & & & \gamma \end{pmatrix}, \quad b_\beta = \begin{pmatrix} 1 & \cdot & \cdot & \beta \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \beta & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad v = \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}.$$

Let $K_1' = \{K_1, v\}$. Then v centralizes each b_β and transforms u into u^{-1}. We now give the classes of K_1'.

$1,\ u^{\frac{1}{2}(q-1)}$	$1,\ 1$	$2q(q-1)$	$A_1,\ A_1'$
$b_\beta,\ u^{\frac{1}{2}(q-1)}b_\beta$	$q-1, q-1, 0\neq\beta\in F$	$2q(q-1)$	$A_{31},\ A_{31}'$
u^i	$\frac{1}{2}(q-3),\ i\in T_1$	$q(q-1)$	$B_8(i)$
$u^i b_\beta$	$\frac{1}{2}(q-3)(q-1),\ i\in T_1,\ \beta\neq 0$	$q(q-1)$	$B_9(i)$
$v,\ uv$	$1,\ 1$	$4q,\ 4q$	D_1
$b_\beta v,\ b_\beta uv$	$q-1, q-1, \beta\neq 0$	$4q,\ 4q$	See remark below

REMARK. $b_\beta uv$, $b_\beta v$ are conjugate in G to

$$\begin{pmatrix} 1 & 2\beta\gamma^{-1} & & \\ & 1 & & \\ & & -1 & 2\beta\gamma^{-1} \\ & & & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2\beta & & \\ & 1 & & \\ & & -1 & 2\beta \\ & & & -1 \end{pmatrix} \text{ respectively.}$$

Hence, of the $q-1$ elements $b_\beta u$, $\frac{1}{2}(q-1)$ elements belong to D_{31} and $\frac{1}{2}(q-1)$ to D_{34} if $q\equiv 1 \pmod 4$. Similarly, $\frac{1}{2}(q-1)$ of the elements $b_\beta uv$ belong to D_{31} and $\frac{1}{2}(q-1)$ to D_{34} if $q\equiv 1 \pmod 4$. If $q\equiv -1 \pmod 4$, D_{31} and D_{34} are to be replaced by D_{32} and D_{33} in the above statements.

(3.9) Let K_2 be the subgroup of $\tilde{G}$ of all elements of the form

$$\begin{pmatrix} \eta^i & & & \delta\beta\eta^i \\ & \eta^{-i} & & \\ & \eta^{-i}\delta\beta & \eta^{-i} & \\ & & & \eta^i \end{pmatrix} \quad (\beta\in F,\ \delta \text{ an element of } \Omega \text{ such that } \delta^2=\gamma).$$

Now $\delta^q = -\delta$. Let $\tilde{K}_2' = \{\tilde{K}_2, v\}$ where

$$v = \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}.$$

Then the element

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \end{pmatrix}$$

transforms every element c of $\tilde{K}_2'$ into $c^{(q)}$. Hence there exist $z \in \tilde{G}$ such that $z^{-1}\tilde{K}_2'z = K_2' \subseteq G$. Let $z^{-1}\tilde{K}_2 z = K_2$,

$$z^{-1}\begin{pmatrix} \eta & & & \\ & \eta^{-1} & & \\ & & \eta^{-1} & \\ & & & \eta \end{pmatrix} z = w,$$

$$z^{-1}\begin{pmatrix} 1 & & & \delta\beta \\ & 1 & & \\ & \delta\beta & 1 & \\ & & & 1 \end{pmatrix} z = d_\beta, \qquad z^{-1}vz = y.$$

The conjugacy classes of K_2' are given below.

$1,\ w^{(q+1)/2}$	$1,\ 1$	$2q(q+1)$	$A_1,\ A_1'$
$d_\beta,\ w^{(q+1)/2}d_\beta$	$q-1, q-1,\ \beta \neq 0$	$2q(q+1)$	$A_{32},\ A_{32}'$
w^i	$\frac{1}{2}(q-1),\ i \in T_2$	$q(q+1)$	$B_6(i)$
$w^i d_\beta$	$\frac{1}{2}(q-1)^2,\ i \in T_2,\ \beta \neq 0$	$q(q+1)$	$B_7(i)$
$w,\ yw$	$1,\ 1$	$4q$	D_1
$d_\beta w,\ d_\beta yw$	$q-1, q-1,\ \beta \neq 0$	$4q$	See remark below

REMARK. Here $\frac{1}{2}(q-1)$ of the elements $d_\beta w$ belong to D_{32} and $\frac{1}{2}(q-1)$ to D_{33} if $q \equiv 1 \pmod 4$. Similarly $\frac{1}{2}(q-1)$ of the elements $d_\beta yw$ belong to D_{32} and $\frac{1}{2}(q-1)$ to D_{33} if $q \equiv 1 \pmod 4$. If $q \equiv -1 \pmod 4$, replace D_{32} and D_{33} by D_{31} and D_{34} in the above statements.

(3.10) Let L_1 be the subgroup of G consisting of all elements of the form

$$\begin{pmatrix} \gamma^i & \cdot & \cdot & \cdot \\ \cdot & \gamma^{-i} & \cdot & \cdot \\ \cdot & \cdot & \pm 1 & \beta \\ \cdot & \cdot & \cdot & \pm 1 \end{pmatrix} \qquad (\beta \in F).$$

Then $|L_1| = 2q(q-1)$. Let

$$u_1 = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad \text{and} \quad L_1' = \{L_1, u_1\}.$$

We put

$$g_1 = \begin{pmatrix} \gamma & \cdot & \cdot & \cdot \\ \cdot & \gamma^{-1} & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad h_\beta = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \beta \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

The conjugacy classes of L_1' are easily written down and they will be omitted here.

(3.11) Let $\tilde{L}_2$ be the subgroup of $\tilde{G}$ generated by all elements of the form

$$\begin{pmatrix} \eta & & & \\ & \eta^{-1} & & \\ & & \pm 1 & \beta \\ & & & \pm 1 \end{pmatrix} \quad (\beta \in F) \quad \text{and} \quad \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Then $z^{-1}\tilde{L}_2 z = L_2' \subseteq G$, for some $z \in \tilde{G}$. Let

$$g_2 = z^{-1}\begin{pmatrix} \eta & & & \\ & \eta^{-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} z, \qquad k_\beta = z^{-1}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \beta \\ & & & 1 \end{pmatrix} z,$$

$$c_2 = z^{-1}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} z, \quad \text{and} \quad u_2 = z^{-1}\begin{pmatrix} & 1 & & \\ -1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix} z.$$

Let $L_2 = \{g_2, c_2, k_\beta(\beta \in F)\}$. Again, the conjugacy classes of L_2' will be omitted. We note that, in fact, z can be chosen such that L_2' is a subgroup of the subgroup K of (3.7).

CERTAIN CHARACTERS OF SUBGROUPS OF G.

(3.12) Consider the subgroups K_1, K_1' of (3.8), and the character $u \to \tilde{\gamma}^j$, $b_\beta \to \varepsilon(\beta)$ of K_1, where j is any integer.

[Here we make use of the isomorphism $\beta \to b_\beta$ between the additive group of F and the subgroup of K_1 consisting of all the b_β.]

Induce this character to G and let $\rho(j)$ be the character of G obtained in this way.

Now consider the characters

$$\alpha_{11}: u \to 1, \qquad b_\beta \to \varepsilon(\beta), \qquad v \to -1,$$
$$\alpha_{12}: u \to 1, \qquad b_\beta \to \varepsilon(\beta), \qquad v \to 1,$$

of K_1', and induce the characters α_{11}, α_{12} to G.

(3.13) Similarly we construct a character $\sigma(j)$ of G which is the induced character of the following character of K_2: $w \to \tilde{\eta}^j$, $d_\beta \to \varepsilon(\beta)$ (j any integer). We also consider the following characters of K_2', and induce them to G.

$$\alpha_{21}: w \to 1, \quad d_\beta \to \varepsilon(\beta), \quad y \to -1,$$
$$\alpha_{22}: w \to 1, \quad d_\beta \to \varepsilon(\beta), \quad y \to 1.$$

The values of $\rho(j)$, $\sigma(j)$, α_{11}^G, α_{12}^G, α_{21}^G, α_{22}^G at the classes of G are given below. At classes not mentioned the values are zero. Also, where there is no entry in the table the value is zero. (We will stick to this convention throughout this paper.)

	A_1	A_1'	A_{31}	A_{31}'	A_{32}	A_{32}'
$\rho(j)$	$q^3(q+1)(q^4-1)$	$(-1)^j q^3(q+1)(q^4-1)$	$-2q^2$	$(-1)^{j+1}2q^2$		
$\sigma(j)$	$q^3(q-1)(q^4-1)$	$(-1)^j q^3(q-1)(q^4-1)$			$-2q^2$	$(-1)^{j+1}2q^2$
α_{11}^G	$\frac{1}{2}q^3(q+1)(q^4-1)$	$\frac{1}{2}q^3(q+1)(q^4-1)$	$-q^2$	$-q^2$		
α_{12}^G	$\frac{1}{2}q^3(q+1)(q^4-1)$	$\frac{1}{2}q^3(q+1)(q^4-1)$	$-q^2$	$-q^2$		
α_{21}^G	$\frac{1}{2}q^3(q-1)(q^4-1)$	$\frac{1}{2}q^3(q-1)(q^4-1)$			$-q^2$	$-q^2$
α_{22}^G	$\frac{1}{2}q^3(q-1)(q^4-1)$	$\frac{1}{2}q^3(q-1)(q^4-1)$			$-q^2$	$-q^2$

	$B_6(i)$	$B_7(i)$	$B_8(i)$	$B_9(i)$	D_1	D_{31}	D_{32}	D_{33}	D_{34}
$\rho(j)$			$(q^2-1)(\tilde{\gamma}^{ij}+\tilde{\gamma}^{-ij})$	$-(\tilde{\gamma}^{ij}+\tilde{\gamma}^{ij})$					
$\sigma(j)$	$(q^2-1)(\tilde{\eta}^{ij}+\tilde{\eta}^{-ij})$	$-(\tilde{\eta}^{ij}+\tilde{\eta}^{-ij})$							
α_{11}^G						q			q
			q^2-1	-1	$-\frac{1}{2}q(q^2-1)^2$		q	q	
α_{12}^G						$-q$			$-q$
			q^2-1	-1	$\frac{1}{2}q(q^2-1)^2$		$-q$	$-q$	
α_{21}^G							q	q	
	q^2-1	-1			$-\frac{1}{2}q(q^2-1)^2$	q			q
α_{22}^G							$-q$	$-q$	
	q^2-1	-1			$\frac{1}{2}q(q^2-1)^2$	$-q$			$-q$

REMARK. In reading the values at D_{31} to D_{34}, the upper entry is to be taken if $q \equiv 1 \pmod 4$ and the lower entry if $q \equiv -1 \pmod 4$.

(3.14) We now introduce certain characters of K. Since $K \cong SL(2, q) \times SL(2, q)$, any two characters ρ and σ of $SL(2, q)$ give rise to a character $\rho \times \sigma$ of K which has as its value at

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

the value of ρ at A multiplied by the value of σ at B. Now there exist families of characters $\lambda(i)$, $\lambda'(j)$, (i, j are any integers) of degree $1+q$, $1-q$ respectively of $SL(2,q)$. (See e.g. [2].) Hence we have characters $\lambda(i)\times\lambda'(j)$, $\lambda(i)\times\lambda(j)$, $\lambda'(i)\times\lambda'(j)$ of K.

We also give below four characters μ_i ($i=1,2,3,4$) of $SL(2,q)$ which will be used to construct characters of K (see [5, p. 103]). Let $t=\frac{1}{2}(q-1)$.

	$\begin{pmatrix}1 & \\ & 1\end{pmatrix}$	$\begin{pmatrix}-1 & \\ & -1\end{pmatrix}$	$\begin{pmatrix}\gamma^i & \\ & \gamma^{-i}\end{pmatrix}, i\in T_1$	$\begin{pmatrix}\eta^i & \\ & \eta^{-i}\end{pmatrix}, i\in T_2$
$\lambda(j)$	$1+q$	$(-1)^j(1+q)$	$\tilde\gamma^{ij}+\tilde\gamma^{-ij}$	
$\lambda'(j)$	$1-q$	$(-1)^j(1-q)$		$\tilde\eta^{ij}+\tilde\eta^{-ij}$
μ_1	$\frac{1}{2}(1+q)$	$(-1)^t\frac{1}{2}(1+q)$	$(-1)^i$	
μ_2	$\frac{1}{2}(1+q)$	$(-1)^t\frac{1}{2}(1+q)$	$(-1)^i$	
μ_3	$\frac{1}{2}(1-q)$	$(-1)^{t+1}\frac{1}{2}(1-q)$		$(-1)^i$
μ_4	$\frac{1}{2}(1-q)$	$(-1)^{t+1}\frac{1}{2}(1-q)$		$(-1)^i$

	$\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}$	$\begin{pmatrix}1 & \gamma\\ & 1\end{pmatrix}$	$\begin{pmatrix}-1 & -1\\ & -1\end{pmatrix}$	$\begin{pmatrix}-1 & -\gamma\\ & -1\end{pmatrix}$
$\lambda(j)$	1	1	$(-1)^j$	$(-1)^j$
$\lambda'(j)$	1	1	$(-1)^j$	$(-1)^j$
μ_1	$-\tilde\varepsilon$	$-\tilde\varepsilon'$	$(-1)^{t+1}\tilde\varepsilon$	$(-1)^{t+1}\tilde\varepsilon'$
μ_2	$-\tilde\varepsilon'$	$-\tilde\varepsilon$	$(-1)^{t+1}\tilde\varepsilon'$	$(-1)^{t+1}\tilde\varepsilon$
μ_3	$-\tilde\varepsilon$	$-\tilde\varepsilon'$	$(-1)^t\tilde\varepsilon$	$(-1)^t\tilde\varepsilon'$
μ_4	$-\tilde\varepsilon'$	$-\tilde\varepsilon$	$(-1)^t\tilde\varepsilon'$	$(-1)^t\tilde\varepsilon$

(3.15) We now consider the subgroups L_1', L_2' of (3.10) and (3.11). Let δ_i ($i=1,2,3,4$) be the characters of L_1' given by

$$\delta_1: g_1\to -1,\quad c_1\to(-1)^t,\quad u_1\to 1,\quad h_\beta\to\varepsilon(\beta),$$
$$\delta_2: g_1\to -1,\quad c_1\to(-1)^t,\quad u_1\to 1,\quad h_\beta\to\varepsilon'(\beta),$$
$$\delta_3: g_1\to -1,\quad c_1\to(-1)^{t+1},\quad u_1\to 1,\quad h_\beta\to\varepsilon(\beta),$$
$$\delta_4: g_1\to -1,\quad c_1\to(-1)^{t+1},\quad u_1\to 1,\quad h_\beta\to\varepsilon'(\beta).$$

Let δ_i' ($i=1,2,3,4$) be the characters of L_2' obtained by replacing g_1 by g_2, c_1 by c_2, u_1 by u_2, and h_β by k_β in the definitions of δ_i ($i=1,2,3,4$).

We give below the induced character δ_1^G. The others can be obtained in a similar way.

A_1, A_1'	A_{21}, A_{21}'	A_{22}, A_{22}'	$C_3(i)$	$C_3'(i)$	$C_{41}(i)$
$\frac{1}{4}q^3(q+1)(q^4-1)$	$\frac{1}{2}q^3(q+1)\tilde\varepsilon$	$\frac{1}{2}q^3(q+1)\tilde\varepsilon'$	$(-1)^i\frac{1}{2}(q^2-1)$	$(-1)^{i+t}\frac{1}{2}(q^2-1)$	$(-1)^i\tilde\varepsilon$

$C_{42}(i)$	$C'_{41}(i)$	$C'_{42}(i)$	D_1	D_{21}, D_{23}	D_{22}, D_{24}
$(-1)^i\bar{\varepsilon}'$	$(-1)^{i+t}\bar{\varepsilon}$	$(-1)^{i+t}\bar{\varepsilon}'$	$(-1)^t\frac{1}{2}q(q+1)(q^2-1)$	$(-1)^t\frac{1}{2}q(q+1)\bar{\varepsilon}$	$(-1)^t\frac{1}{2}q(q+1)\bar{\varepsilon}'$

We will also need characters δ_5, δ'_5 of degree 2 of L'_1, L_2 respectively defined as follows.

δ_5 is the sum of the characters

$$g_1 \to 1, \quad u_1 \to 1, \quad c_1 \to 1, \quad h_\beta \to \varepsilon(\beta);$$
$$g_1 \to 1, \quad u_1 \to 1, \quad c_1 \to 1, \quad h_\beta \to \varepsilon'(\beta) \text{ of } L'_1.$$

δ'_5 is the character of L'_2 obtained by replacing g_1 by g_2, u_1 by u_2, c_1 by c_2 and h_β by k_β in the above definition.

4. **Families of characters of G corresponding to the families of classes $B_1(i)$ to $B_9(i)$.**

(4.1) Consider the subgroups H_1, H_2 defined in (3.2) and (3.3), and the linear characters $\beta_1(j)$, $\beta_2(j)$ of H_1, H_2 respectively given by

$$\beta_1(j): a_1 \to \tilde{\zeta}^j, \qquad \beta_2(j): a_2 \to \tilde{\theta}^j \qquad (j \text{ any integer}).$$

We define characters $\chi_1(j)$, $\chi_2(j)$ (for any integer j) of G by

$$\begin{aligned}\chi_1(j) &= \beta_1^G(j)-\psi_1-\psi_2-\tfrac{1}{2}(3q-5)\psi_3-\psi_4-(q-2)\psi_5, \quad \text{if } j \text{ is even,}\\ &= \beta_1^G(j)-\psi'_1-\psi'_2-\tfrac{1}{2}(3q-5)\psi'_3-\psi'_4-(q-2)\psi'_5, \quad \text{if } j \text{ is odd.}\end{aligned}$$

$$\begin{aligned}\chi_2(j) &= \beta_2^G(j)-\rho(j)-\sigma(j)+\psi_1+\psi_2+\tfrac{1}{2}(3q-1)\psi_3+\psi_4+(q-2)\psi_5, \quad \text{if } j \text{ is even,}\\ &= \beta_2^G(j)-\rho(j)-\sigma(j)+\psi'_1+\psi'_2+\tfrac{1}{2}(3q-1)\psi'_3+\psi'_4+(q-2)\psi'_5, \quad \text{if } j \text{ is odd.}\end{aligned}$$

(For the definitions of $\psi_1, \psi_2, \ldots$, see §2.)

We thus get two families $\{\chi_1(j)\}$ and $\{\chi_2(j)\}$ of characters of G of degrees $(1-q^2)^2$ and $1-q^4$ respectively. We note that the values of all the characters of a family at $A_1, \ldots, A_{42}$ are the same, and these values are polynomials in q.

(4.2) Consider the characters $\lambda(k)\times\lambda(l)$, $\lambda'(k)\times\lambda'(l)$, $\lambda(k)\times\lambda'(l)$ of the subgroup K, where k, l are integers. Let

$$\begin{aligned}\chi_3(k,l) &= [\lambda(k)\times\lambda(l)]^G-\rho(k+l)-\rho(k-l)+2\psi_1+2\psi_2-(2-3q)\psi_3+2\psi_4-2(1-q)\psi_5, \\ &\qquad \text{if } k+l \text{ is even,}\\ &= [\lambda(k)\times\lambda(l)]^G-\rho(k+l)-\rho(k-l)+2\psi'_1+2\psi'_2-(2-3q)\psi'_3+2\psi'_4-2(1-q)\psi'_5, \\ &\qquad \text{if } k+l \text{ is odd.}\end{aligned}$$

$$\begin{aligned}\chi_4(k,l) &= [\lambda'(k)\times\lambda'(l)]^G-\sigma(k+l)-\sigma(k-l)+2\psi_1+2\psi_2-(2-3q)\psi_3+2\psi_4-2(3-q)\psi_5, \\ &\qquad \text{if } k+l \text{ is even,}\\ &= [\lambda'(k)\times\lambda'(l)]^G-\sigma(k+l)-\sigma(k-l)+2\psi'_1+2\psi'_2-(2-3q)\psi'_3+2\psi'_4-2(3-q)\psi'_5, \\ &\qquad \text{if } k+l \text{ is odd.}\end{aligned}$$

$$\begin{aligned}\chi_5(k,l) &= [\lambda'(k)\times\lambda(l)]^G+\psi_3, \quad \text{if } k+l \text{ is even,}\\ &= [\lambda'(k)\times\lambda(l)]^G+\psi'_3, \quad \text{if } k+l \text{ is odd.}\end{aligned}$$

These three families of characters are of degrees $(1+q)^2(q^2+1)$, $(1-q)^2(1+q^2)$ and $1-q^4$ respectively. The values of these characters at the classes of G are given in §8. We note that $\chi_1(j)=\chi_1(k)$ if $j\equiv k \pmod{q^2+1}$, etc.

In order to consider the irreducibility of these characters, we prove the following two lemmas.

LEMMA 4.3. *Consider the* 5×5 *matrix*

$$(R_{\alpha\beta}) = \begin{pmatrix} (1-q^2)^2 & 1-q^4 & (1+q)^2(1+q^2) & (1-q)^2(1+q^2) & 1-q^4 \\ 1-q^2 & 1-q^2 & (1+q)^2 & (1-q)^2 & 1+q^2 \\ 1+q & 1-q & 1+q & 1-3q & 1+q \\ 1-q & 1+q & 1+3q & 1-q & 1-q \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Between the columns of this matrix we have the relations

$$\sum_\alpha \frac{1}{s_\alpha} R_{\alpha\beta}R_{\alpha\gamma} = \delta_{\beta\gamma}\frac{z_\beta}{t_\beta},$$

where $s_1=q^4(q^2-1)(q^4-1)$, $s_2=q^4(q^2-1)$, $s_3=2q^3(q-1)$, $s_4=2q^3(q+1)$ $s_5=q^2$, $z_1=z_2=z_5=4$, $z_3=z_4=8$, $t_1=q^2+1$, $t_2=q^2-1$, $t_3=(q-1)^2$, $t_4=(q+1)^2$, $t_5=q^2-1$.

LEMMA 4.4. *Consider the* 2×2 *matrix*

$$(K_{\alpha\beta}) = \begin{pmatrix} 1+q & 1-q \\ 1 & 1 \end{pmatrix}$$

Between the columns of this matrix we have the relations

$$\sum_\alpha \frac{1}{a_\alpha} K_{\alpha\beta}K_{\beta\gamma} = \delta_{\beta\gamma}\frac{2}{b_\beta},$$

where $a_1=q(q^2-1)$, $a_2=q$, $b_1=q-1$, $b_2=q+1$. (*Compare* [2, p. 431].)

Since the proofs of these lemmas are a routine verification of their statements, they will be omitted. The meaning of these lemmas will become clear when we prove the next lemma.

Consider the linear characters

$$a_1 \to \tilde{\zeta}^j;; \qquad a_2 \to \tilde{\theta}^j; \qquad a_3 \to \tilde{\gamma}^k, \qquad b_3 \to \tilde{\gamma}^l;$$
$$a_4 \to \tilde{\eta}^k, \qquad b_4 \to \tilde{\eta}^l; \qquad a_5 \to \tilde{\eta}^k, \qquad b_5 \to \tilde{\gamma}^l;$$

of H_1, H_2, H_3, H_4, H_5 respectively (j, k, l are any integers). Let $\phi_1(j)$, $\phi_2(j)$, $\phi_3(k, l)$, $\phi_4(k, l)$, $\phi_5(k, l)$ be the characters of M_1, M_2, M_3, M_4, M_5 respectively induced from these characters.

LEMMA 4.5. *The scalar product of two characters belonging to distinct families*

$\{\chi_1(j)\}, \ldots$ *is zero. For characters belonging to the same family, we have the following relations.*

$$(4.6)\qquad \begin{aligned}(\chi_i(j), \chi_i(k)) &= (\phi_i(j), \phi_i(k)) \qquad (i = 1, 2)\\ (\chi_i(k, l), \chi_i(m, n)) &= (\phi_i(k, l), \phi_i(m, n)) \qquad (i = 3, 4, 5).\end{aligned}$$

(*Compare* [2, p. 431].)

Proof. We first remark that the entries in the five columns of the matrix $(R_{\alpha\beta})$ of (4.3) are just the values of the five families of characters $\{\chi_1(j)\}$, $\{\chi_2(j)\}$, $\{\chi_3(k, l)\}$, $\{\chi_4(k, l)\}$, $\{\chi_5(k, l)\}$ at the classes A_1, A_{21} (or A_{22}), A_{31}, A_{32}, and A_{41} (or A_{42}) respectively. Further,

$$s_1 = \frac{|G|}{|A_1|}, \quad s_2 = \frac{|G|}{|A_{21}|+|A_{22}|}, \quad s_3 = \frac{|G|}{|A_{31}|}, \quad s_4 = \frac{|G|}{|A_{32}|}, \quad s_5 = \frac{|G|}{|A_{41}|+|A_{42}|},$$

$|t_i| = |H_i|$, $z_i | M_i : H_i|$. Next, we see that since each H_i is normal in M_i, $\phi_1(j), \ldots, \phi_5(k, l)$ vanish outside $H_1, \ldots, H_5$ respectively. Also $\chi_1(j), \ldots, \chi_5(k, l)$ vanish on all elements of G, whose p-regular factors do not lie in $H_1, \ldots, H_5$ respectively.

To prove the first assertion of the lemma consider a scalar product of two characters from different families $\{\chi_i(j)\}$ $(i=1, 2)$, $\{\chi_i(k, l)\}$ $(i=3, 4, 5)$. We consider the contribution to the scalar product from all elements of G whose p-regular factors are conjugate to a fixed element of G. In other words, we consider separately the contributions from $\{A_1, \ldots, A_{42}\}$, $\{A_1', \ldots, A_{42}'\}$, $\{B_1(i)\}, \ldots, \{B_5(i, j)\}$, $\{B_6(i), B_7(i)\}$, $\{B_8(i), B_9(i)\}$, $\{C_1(i), C_{21}(i), C_{22}(i)\}$, $\{C_1'(i), C_{21}'(i), C_{22}'(i)\}$, $\{C_3(i), C_{41}(i), C_{42}(i)\}$, $\{C_3'(i), C_{41}'(i), C_{42}'(i)\}$, $\{D_1, \ldots, D_{34}\}$ (where $i, j, \ldots$ run over suitable index sets). Using Lemmas 4.3 and 4.4 we see that each such contribution must be zero.

Similarly we compare the contribution from each of the sets to the left-hand side of (4.6), with the contribution from the intersection with M_i of the p-regular class contained in the set to the right hand side of (4.6). We see then, again using Lemmas 4.3 and 4.4, that (4.6) holds. The details are omitted.

Using Lemma 4.5 we can now construct *irreducible* characters of G. It is easy to see that $\phi_1(j)$ $(\phi_2(j))$ is an irreducible character of M_1 (M_2) if and only if $j \in R_1$ $(j \in R_2)$. Hence, corresponding to these values of j, we get $\frac{1}{4}(q^2-1)$ irreducible characters in the family $\{\chi_1(j)\}$ and $\frac{1}{4}(q-1)^2$ irreducible characters in the family $\{-\chi_2(j)\}$. Similarly $\chi_3(k, l)$, $\chi_4(k, l)$, $\chi_5(k, l)$ are irreducible if and only if $k, l \in T_1$, $k, l \in T_2$, $k \in T_2, l \in T_1$ respectively. We have thus constructed families of irreducible characters which correspond to the families of classes $\{B_1(i)\}, \ldots, \{B_5(i, j)\}$, and in each family there are as many irreducible characters as there are conjugacy classes in the corresponding family.

We now construct four families of characters which correspond to the families of classes $\{B_k(i)\}$ $(k=6, 7, 8, 9)$.

LEMMA 4.7. *For each integer k, the function $f(k)$ which takes values on G as given in the table below is a character of G.*

A_1, A_1'	$A_{21}, A_{22}, A_{21}', A_{21}'$	A_{31}, A_{31}'	A_{32}, A_{32}'	$A_{41}, A_{41}', A_{42}, A_{42}'$	$B_2(i)$
$2(1+q^2)$	2	$2(1+q)$	$2(1-q)$	2	$\tilde{\gamma}^{ik}+\tilde{\gamma}^{-ik}+\tilde{\eta}^{ik}+\tilde{\eta}^{-ik}$

$B_3(i,j)$	$B_4(i,j)$	$B_6(i), B_7(i)$	$B_8(i), B_9(i)$	$C_1(i), C_1'(i)$
$(\tilde{\gamma}^{ik}+\tilde{\gamma}^{-ik})(\tilde{\gamma}^{jk}+\tilde{\gamma}^{-jk})$	$(\tilde{\eta}^{ik}+\tilde{\eta}^{-ik})$ $(\tilde{\eta}^{jk}+\tilde{\eta}^{-jk})$	$\tilde{\eta}^{2ik}+\tilde{\eta}^{-2ik}+2$	$\tilde{\gamma}^{2ik}+\tilde{\gamma}^{-2ik}+2$	$(1-q)(\tilde{\eta}^{ik}+\tilde{\eta}^{-ik})$

$C_{21}(i), C_{21}'(i)$ $C_{22}(i), C_{22}'(i)$	$C_3(i), C_3'(i)$	$C_{41}(i), C_{41}'(i)$, $C_{42}(i), C_{42}'(i)$	D_1	D_{21}, D_{22}, D_{23}, D_{24}	D_{31}, D_{32}, D_{33}, D_{34}
$\tilde{\eta}^{ik}+\tilde{\eta}^{-ik}$	$(1+q)(\tilde{\gamma}^{ik}+\tilde{\gamma}^{-ik})$	$\tilde{\gamma}^{ik}+\tilde{\gamma}^{-ik}$	$2(1+q^2)(-1)^k$	$2(-1)^k$	$2(-1)^k$

Proof. Let $\tau(k)$ be the character of degree 6 of G which has as its value at an element a of G, the image under the isomorphism (from the multiplicative group of F into the multiplicative group of complex numbers) described in §1, of the second elementary symmetric function in the kth powers of the characteristic roots of a. (To see that such a character exists, see [2, p. 415]. This is σ_2^k in Green's notation.)

Then we see that $f(k)=\tau(k)-\chi_5(k,k)-\chi_1(k(q+1)-2\theta_0)$ where θ_0 is the identity character of G. Formally,

$$f(k) = \tfrac{1}{2}\chi_2(k(q+1))+\tfrac{1}{2}\chi_2(k(q-1))+\tfrac{1}{2}\chi_3(k,k)+\tfrac{1}{2}\chi_4(k,k). \tag{4.8}$$

Suppose k is not a multiple of $\frac{1}{2}(q+1)$ or $\frac{1}{2}(q-1)$. Then, using Lemma 4.5 we see that $(f(k), f(k))=2$. Hence $f(k)=\zeta_1+\zeta_2$ where either ζ_i or $-\zeta_i$ is irreducible $(i=1, 2)$. Now let, for a fixed k,

$$g_1 = \chi_4(k,k), \qquad g_2 = \chi_2(k(q-1)),$$
$$g_3 = \chi_3(k,k), \qquad g_4 = \chi_2(k(q+1)).$$

Then $(g_i, g_j)=2\delta_{ij}$ and $(g_i, f(k))=1$, each i. Hence the g_i must be of the form $\zeta_1+\alpha$, $\zeta_1-\alpha$, $\zeta_2+\beta$, $\zeta_2-\beta$ where α and β are characters distinct from the ζ_i and from each other, and either $\alpha(\beta)$ or $-\alpha(-\beta)$ is irreducible. We can assume that $g_1=\zeta_1+\alpha$.

Case 1. Suppose $g_4=\zeta_1-\alpha$; then $\frac{1}{2}(g_1+g_4)$ is a character. Consider the restriction of g_1+g_4 to the cyclic subgroup H_2.

	$1, a_2^{(q^2-1)/2}$	a_2^i, $i\not\equiv 0 \pmod{q+1}$, $i\not\equiv 0 \pmod{q-1}$	$a_2^{i(q+1)}$, $i=1,\dots,q-2$, $i\neq\frac{1}{2}(q-1)$	$a_2^{i(q-1)}$, $i=1,\dots,q$, $i\neq\frac{1}{2}(q+1)$
g_1+g_4	$2(1-q)(1+q^2)$	$2(\tilde{\gamma}^{ik}+\tilde{\gamma}^{-ik})$	$(1-q)(\tilde{\gamma}^{2ik}+\tilde{\gamma}^{-2ik})$	$(1-q)(\tilde{\eta}^{ik}+\tilde{\eta}^{-ik})^2$ $+2(1+q)$
	2	$\tilde{\gamma}^{ik}+\tilde{\gamma}^{-ik}$	$\tilde{\gamma}^{2ik}+\tilde{\gamma}^{-2ik}$	2
h_1	$4+2(q-1)(1+q^2)$		$(1+q)(\tilde{\gamma}^{2ik}+\tilde{\gamma}^{-2ik})$	$(q-1)(\tilde{\eta}^{2ik}+\tilde{\eta}^{-2ik})$

Since the function in the second row of the table is a character, the function h_1 in the third row is a character of H_2. If $\frac{1}{2}(g_1+g_4)$ is a character of G, then $\frac{1}{2}h_1$ is a character of H_2. We shall prove that this is impossible. Now $h_1=4(q-1)h_2+h_3+h_4$, where h_2, h_3, h_4 are the characters of H_2 given below.

h_2	$\frac{1}{2}(q^2-1)$			
h_3	$2(q+1)$		$(q+1)(\tilde{\gamma}^{2ik}+\tilde{\gamma}^{-2ik})$	
h_4	$2(q-1)$			$(q-1)(\tilde{\eta}^{2ik}+\tilde{\eta}^{-2ik})$

Suppose $\tilde{\gamma}^{2k}\neq 1, -1$ and $\tilde{\eta}^{2k}\neq 1, -1$. Then there is an irreducible character of H_2 occuring in h_3 with multiplicity 1 which does not occur in h_4 (e.g. the character defined by $a_2 \to \tilde{\theta}^{2k+q-1}$). This shows that $\frac{1}{2}h_1$ cannot be a character of H_2.

If $\tilde{\gamma}^{2k}=1$ then k is a multiple of $\frac{1}{2}(q-1)$ and if $\tilde{\eta}^{2k}=1$ then k is a multiple of $\frac{1}{2}(q+1)$. But, since $\tilde{\gamma}^k=\tilde{\theta}^{k(q+1)}$, $\tilde{\eta}^k=\tilde{\theta}^{k(q-1)}$, the case $\tilde{\gamma}^{2k}=\tilde{\eta}^{2k}=-1$ is impossible.

Case 2. Suppose $\frac{1}{2}(g_1+g_3)$ is a character. We restrict g_1+g_3 to the subgroup H_2.

g_1+g_3	$2(1+q^2)^2$		$(1+q)(\tilde{\gamma}^{ik}+\tilde{\gamma}^{-ik})^2$	$(1-q)(\tilde{\eta}^{ik}+\tilde{\eta}^{-ik})^2$
h_5	$2(1+q^2)^2-4$		$(1+q)(\tilde{\gamma}^{2ik}+\tilde{\gamma}^{-2ik})$	$(1-q)(\tilde{\eta}^{2ik}+\tilde{\eta}^{-2ik})$

The function h_5 is a character of H_2. If $\frac{1}{2}(g_1+g_3)$ is a character of G, then $\frac{1}{2}h_5$ is a character of H_2. An argument similar to that of Case 1 shows that this is impossible.

Hence we have shown

LEMMA 4.9. *Let k be an integer which is not a multiple of $\frac{1}{2}(q+1)$ or $\frac{1}{2}(q-1)$. Then the class functions $\frac{1}{2}\chi_4(k,k)+\frac{1}{2}\chi_2(k(q-1))$, $\frac{1}{2}\chi_4(k,k)-\frac{1}{2}\chi_2(k(q-1))$, $\frac{1}{2}\chi_3(k,k)+\frac{1}{2}\chi_2(k(q+1))$, $\frac{1}{2}\chi_3(k,k)-\frac{1}{2}\chi_2(k(q+1))$ are characters of G. We denote these families by $\{\chi_6(k)\}$, $\{\chi_7(k)\}$, $\{\chi_8(k)\}$ and $\{\chi_9(k)\}$. They are of degrees $(1-q)(1+q^2)$, $q(q-1)(1+q^2)$, $(1+q)(1+q^2)$ and $q(1+q)(1+q^2)$ respectively. We get $\frac{1}{2}(q-3)$ irreducible characters in each of the families $\{-\chi_6(k)\}$, $\{\chi_7(k)\}$, $\{\chi_8(k)\}$ and $\{\chi_9(k)\}$.*

The remaining values of k will be considered in (7.4). We shall then prove that $\frac{1}{2}\chi_4(\frac{1}{2}(q-1), \frac{1}{2}(q-1))\pm\frac{1}{2}\chi_2(\frac{1}{2}(q-1)^2)$ are also irreducible characters.

5. **Characters corresponding to the families of classes $\{C_1(i)\},\dots,\{C_{42}(i)\}$, $\{C_1'(i)\},\dots,\{C_{42}'(i)\}$.**

(5.1) Consider the subgroup K, and, for any integer k, the characters $\lambda(k)\times\lambda_0$, $\lambda'(k)\times\lambda_0$ of K, where λ_0 is the identity character of $SL(2,q)$. Define characters $\xi_1(k)$, $\xi_3(k)$ of G by

$$\xi_1(k) = [\lambda'(k)\times\lambda_0]^G-\sigma(k)+\psi_1+\psi_2-\tfrac{1}{2}(1-3q)\psi_3+\psi_4+(q-3)\psi_5, \qquad \text{if } k \text{ is even,}$$

$$\xi_3(k) = [\lambda(k)\times\lambda_0]^G-\rho(k)+\psi_1+\psi_2-\tfrac{1}{2}(1-3q)\psi_3+\psi_4+(q-1)\psi_5, \qquad \text{if } k \text{ is even.}$$

If k is odd, replace ψ_i in the above definitions by ψ_i'. The same convention will be followed in the rest of the paper.

Then we see that

$$\xi_1(k) = \tfrac{1}{2}\chi_4(k, q+1)+\tfrac{1}{2}\chi_5(k, q-1),$$
$$\xi_3(k) = \tfrac{1}{2}\chi_3(k, q-1)+\tfrac{1}{2}\chi_5(q+1, k). \tag{5.2}$$

Define $\xi_1'(k)$, $\xi_3'(k)$ by

$$\xi_1'(k) = -\chi_4(k, q+1)+\xi_1(k),$$
$$\xi_3'(k) = \chi_3(k, q-1)-\xi_3(k).$$

Again by using Lemma 4.5 we see that for $k \in T_2$, $\{-\xi_1(k)\}$ and $\{-\xi_1'(k)\}$ are irreducible characters of degrees $(q-1)(q^2+1)$, $q(q-1)(q^2+1)$ respectively. For $k \in T_1$, $\{\xi_3(k)\}$, $\{\xi_3'(k)\}$ are irreducible of degrees $(q+1)(q^2+1)$, $q(q+1)(q^2+1)$ respectively.

(5.3) Consider the characters $\mu_i \times \lambda'(k)$, $\mu_i \times \lambda(k)$ $(i=1, 2, 3, 4)$ of K, for any integer k. Let

$$\xi_{21}(k) = [\mu_1 \times \lambda'(k)]^G+\psi_{32}, \qquad \text{if } k+t \text{ is even},$$
$$\xi_{22}(k) = [\mu_2 \times \lambda'(k)]^G+\psi_{31}, \qquad \text{if } k+t \text{ is even},$$
$$\xi_{21}'(k) = [\mu_3 \times \lambda'(k)]^G-\sigma(k+t+1)+\psi_1+\psi_2-\tfrac{1}{2}(1-3q)\psi_3+\psi_4+(q-3)\psi_5-\psi_{31}, \quad \text{if } k+t+1 \text{ is even},$$
$$\xi_{22}'(k) = [\mu_4 \times \lambda'(k)]^G-\sigma(k+t+1)+\psi_1+\psi_2-\tfrac{1}{2}(1-3q)\psi_3+\psi_4+(q-3)\psi_5-\psi_{32}, \quad \text{if } k+t+1 \text{ is even},$$
$$\xi_{41}(k) = [\mu_1 \times \lambda(k)]^G-\rho(k+t)+\psi_1+\psi_2-\tfrac{1}{2}(1-3q)\psi_3+\psi_4+(q-1)\psi_5-\psi_{32}, \quad \text{if } k+t \text{ is even},$$
$$\xi_{42}(k) = [\mu_2 \times \lambda(k)]^G-\rho(k+t)+\psi_1+\psi_2-\tfrac{1}{2}(1-3q)\psi_3+\psi_4+(q-1)\psi_5-\psi_{31}, \quad \text{if } k+t \text{ is even},$$
$$\xi_{41}'(k) = [\mu_3 \times \lambda(k)]^G+\psi_{31}, \qquad \text{if } k+t+1 \text{ is even},$$
$$\xi_{42}'(k) = [\mu_4 \times \lambda(k)]^G+\psi_{32}, \qquad \text{if } k+t+1 \text{ is even}.$$

Then we have equations

$$\begin{aligned} \xi_{21}(k)+\xi_{22}(k) &= \chi_5(k, \tfrac{1}{2}(q-1)),\\ \xi_{21}'(k)+\xi_{22}'(k) &= \chi_4(k, \tfrac{1}{2}(q+1)),\\ \xi_{41}(k)+\xi_{42}(k) &= \chi_3(\tfrac{1}{2}(q-1), k),\\ \xi_{41}'(k)+\xi_{42}'(k) &= \chi_5(\tfrac{1}{2}(q+1), k). \end{aligned} \tag{5.4}$$

We can verify directly that

$$\begin{aligned} (\xi_{21}(k), \xi_{22}(k)) &= 0, \qquad (\xi_{21}'(k), \xi_{22}'(k)) = 0,\\ (\xi_{41}(k), \xi_{42}(k)) &= 0, \qquad (\xi_{41}'(k), \xi_{42}'(k)) = 0. \end{aligned} \tag{5.5}$$

Hence, by a further application of Lemma 4.5, we see that for $k \in T_1$, we get four families $\{-\xi_{41}'(k)\}$, $\{-\xi_{42}'(k)\}$, $\{\xi_{41}(k)\}$, $\{\xi_{42}(k)\}$ of irreducible characters of degrees $\tfrac{1}{2}(q^4-1)$, $\tfrac{1}{2}(q^4-1)$, $\tfrac{1}{2}(q+1)^2(q^2+1)$, $\tfrac{1}{2}(q+1)^2(q^2+1)$ respectively. For $k \in T_2$ we get four families $\{-\xi_{21}(k)\}$, $\{-\xi_{22}(k)\}$, $\{\xi'_{21}(k)\}$, $\{\xi'_{22}(k)\}$ of irreducible characters of degree $\frac{1}{2}(q^4-1)$, $\frac{1}{2}(q^4-1)$, $\frac{1}{2}(q-1)^2(q^2+1)$, $\frac{1}{2}(q-1)^2(q^2+1)$ respectively.

We remark that there does not appear to be a clearly defined correspondence between irreducible characters and conjugacy classes in this and subsequent sections. However, we can say that the set of characters $\{\xi_1(k)\}$, $\{\xi'_1(k)\}, \ldots,$ corresponds to the set of classes $\{C_1(i)\}$, $\{C'_1(i)\}, \ldots.$

6. **Characters corresponding to the classes** $D_1, \ldots, D_{34}$. Consider the following characters of G:

$$f_1 = \xi'_{41}(q-1), \qquad f_2 = \xi'_{21}(q+1),$$
$$f_3 = \xi'_{42}(q-1), \qquad f_4 = \xi'_{22}(q+1),$$
$$g = \xi_1(\tfrac{1}{2}(q+1)), \qquad h = \xi'_1(\tfrac{1}{2}(q+1)).$$

Using (5.2), (5.4) and (5.5) we can show that

$$(f_i, f_j) = 2\delta_{ij}, \qquad (g, f_i) = 1 \quad (\text{all } i),$$
$$(h, f_1) = (h, f_3) = 1, \qquad (h, f_2) = (h, f_4) = -1.$$

It then follows that the f_i must be of the form $\zeta_1+\zeta_2$, $\zeta_1-\zeta_2$, $\zeta_3+\zeta_4$, $\zeta_3-\zeta_4$, where either ζ_i or $-\zeta_i$ is irreducible ($i=1, 2, 3, 4$). We consider the values of $\frac{1}{2}(f_1+f_i)$ ($i=2, 3, 4$) at the class A_{41} of G. We find that only $\frac{1}{2}(f_1+f_4)$ is integral at A_{41}. Hence the ζ_i must be the characters $\frac{1}{2}(f_1 \pm f_4)$ and $\frac{1}{2}(f_2 \pm f_3)$. Thus, we have characters

$$\tfrac{1}{2}[\xi'_{41}(q-1)+\xi'_{22}(q+1)], \qquad \tfrac{1}{2}[\xi'_{42}(q-1)+\xi'_{21}(q+1)],$$
$$\tfrac{1}{2}[\xi'_{41}(q-1)-\xi'_{22}(q+1)], \qquad \tfrac{1}{2}[\xi'_{42}(q-1)-\xi'_{21}(q+1)],$$

which will be denoted by Φ_1, Φ_2, Φ_3, Φ_4, and are of degrees $\frac{1}{2}(1-q)(1+q^2)$, $\frac{1}{2}(1-q)(1+q^2)$, $\frac{1}{2}q(1-q)(1+q^2)$, $\frac{1}{2}q(1-q)(1+q^2)$ respectively. The $(-\Phi_i)$ are irreducible, for each i.

Similarly we can show that the functions

$$\tfrac{1}{2}[\xi_{41}(q-1)+\xi_{22}(q+1)], \qquad \tfrac{1}{2}[\xi_{42}(q-1)+\xi_{21}(q+1)],$$
$$\tfrac{1}{2}[\xi_{41}(q-1)-\xi_{22}(q+1)], \qquad \tfrac{1}{2}[\xi_{42}(q-1)-\xi_{21}(q+1)],$$

are irreducible characters of degrees $\frac{1}{2}(1+q)(1+q^2)$, $\frac{1}{2}(1+q)(1+q^2)$, $\frac{1}{2}q(1+q)(1+q^2)$, $\frac{1}{2}q(1+q)(1+q^2)$ respectively. They will be denoted by Φ_5, Φ_6, Φ_7, Φ_8 respectively. We now construct a further irreducible character of degree $q(q^2+1)$. Consider the characters $\mu_1 \times \mu_2$, $\mu_3 \times \mu_4$ of K. Let τ_1 be the character α_{11} (see (3.12)) of K'_1 if $q \equiv -1 \pmod 4$, and α_{12} if $q \equiv 1 \pmod 4$. Let τ_2 be the character α_{21} of K'_2 if $q \equiv -1 \pmod 4$ and α_{22} if $q \equiv 1 \pmod 4$. Then let

$$\Phi_9 = [\mu_1 \times \mu_2]^G - [\mu_3 \times \mu_4]^G + \tau_2^G - \tau_1^G + \psi_5.$$

Then

$$\Phi_9 = \tfrac{1}{4}\chi_3(\tfrac{1}{2}(q-1), \tfrac{1}{2}(q-1)) - \tfrac{1}{4}\chi_4(\tfrac{1}{2}(q+1), \tfrac{1}{2}(q+1)),$$

showing that Φ_9 is an irreducible character of G.

7. **Characters corresponding to the classes** $A_1, \ldots, A_{42}, A'_1, \ldots, A'_{42}$. There are fourteen irreducible characters of this type, of which two are the identity character and the Steinberg character [6].

(7.1) Consider the subgroups M_2, M_3, M_4, and characters ζ_2, ζ_3, ζ_4 of M_2, M_3, M_4 respectively defined by

$$\begin{array}{lllll} \zeta_2: a_2 \to -1, & x_2 \to 1, & y_2 \to 1, & & \\ \zeta_3: a_3 \to -1, & b_3 \to -1, & x_3 \to 1, & y_3 \to 1, & z_3 \to 1, \\ \zeta_4: a_4 \to -1, & b_4 \to -1, & x_4 \to 1, & y_4 \to 1, & z_4 \to 1. \end{array}$$

We compute $-\zeta_2^G+\zeta_3^G+\zeta_4^G$. By the choice of the subgroups and the characters, the contribution to this character from the elements of M_2, M_3, M_4 outside H_2, H_3, H_4 is zero. (This situation will be encountered several times in this section.)

Consider the characters δ_1, δ_2, δ'_3, δ'_4 of (3.15). Let

$$\begin{aligned} \theta_1 &= -\zeta_2^G+\zeta_3^G+\zeta_4^G-\delta_1^G-\delta_3'^G+\psi_{21}-(1-q)\psi_{31}-\tfrac{1}{2}(1-q)\psi_5, \\ \theta_2 &= -\zeta_2^G+\zeta_3^G+\zeta_4^G-\delta_2^G-\delta_4'^G+\psi_{22}-(1-q)\psi_{32}-\tfrac{1}{2}(1-q)\psi_5. \end{aligned}$$

Then, θ_1, θ_2 are of degree $\frac{1}{2}q^2(q^2+1)$ and

$$\theta_1+\theta_2 = \tfrac{1}{4}\chi_3(\tfrac{1}{2}(q-1), \tfrac{1}{2}(q-1))+\tfrac{1}{4}\chi_4(\tfrac{1}{2}(q+1), \tfrac{1}{2}(q+1))-\tfrac{1}{2}\chi_2(\tfrac{1}{2}(q^2-1)).$$

Thus, $(\theta_1+\theta_2, \theta_1+\theta_2)=2$. We can verify directly that $(\theta_1, \theta_2)=0$. Hence θ_1 and θ_2 are irreducible.

Now define characters θ_3, θ_4 of degree $\frac{1}{2}(1+q^2)$ by

$$\begin{aligned} \theta_3+\theta_1 &= [\mu_3\times\mu_3]^G+[\mu_1\times\mu_2]^G-\tau_1^G-\tau_2^G+2\psi_1+\psi_2+(2q-1)\psi_3+(q-2)\psi_5-\psi_{31}, \\ \theta_4+\theta_2 &= [\mu_4\times\mu_4]^G+[\mu_1\times\mu_2]^G-\tau_1^G-\tau_2^G+2\psi_1+\psi_2+(2q-1)\psi_3+(q-2)\psi_5-\psi_{32}, \end{aligned}$$

where

$$\begin{array}{lll} \tau_1 = \alpha_{12}, & \tau_2 = \alpha_{21} & \text{if } q \equiv 1 \pmod 4, \text{ and} \\ \tau_1 = \alpha_{11}, & \tau_2 = \alpha_{22} & \text{if } q \equiv -1 \pmod 4. \end{array}$$

Then

$$\theta_3+\theta_4 = \tfrac{1}{4}\chi_3(\tfrac{1}{2}(q-1), \tfrac{1}{2}(q-1))+\tfrac{1}{4}\chi_4(\tfrac{1}{2}(q+1), \tfrac{1}{2}(q+1))+\tfrac{1}{2}\chi_2(\tfrac{1}{2}(q^2-1)),$$

and again we can show that θ_3, θ_4 are irreducible.

(7.2) Consider the subgroups M_1, M_5 and characters ζ_1, ζ_5 of M_1, M_5 respectively given by

$$\begin{array}{llll} \zeta_1: a_1 \to -1, & x_1 \to 1, & & \\ \zeta_5: a_5 \to -1, & b_5 \to -1, & y_5 \to 1, & z_5 \to 1. \end{array}$$

Define characters θ_5, θ_6 of G by

$$\begin{aligned} \theta_5 &= -\zeta_1^G+\zeta_5^G-\delta_3^G-\delta_1'^G+\psi'_{21}+(q-1)\psi'_{31}+\tfrac{1}{2}(q-1)\psi'_5, \\ \theta_6 &= -\zeta_1^G+\zeta_5^G-\delta_4^G-\delta_2'^G+\psi'_{22}+(q-1)\psi'_{32}+\tfrac{1}{2}(q-1)\psi'_5, \end{aligned}$$

and characters θ_7, θ_8 by

$$\theta_5+\theta_7 = [\mu_3\times(\mu_1+\mu_2)]^G+\psi'_{21},$$
$$\theta_6+\theta_8 = [\mu_4\times(\mu_1+\mu_2)]^G+\psi'_{22}.$$

Then

$$\theta_5+\theta_6 = -\chi_1(\tfrac{1}{2}(q^2+1))+\chi_5(\tfrac{1}{2}(q+1),\tfrac{1}{2}(q-1)),$$
$$\theta_7+\theta_8 = \chi_1(\tfrac{1}{2}(q^2+1))+\chi_5(\tfrac{1}{2}(q+1),\tfrac{1}{2}(q-1)),$$

and again it can be shown that $-\theta_5$, $-\theta_6$, $-\theta_7$, $-\theta_8$ are irreducible characters of degrees $\frac{1}{2}q^2(q^2-1)$, $\frac{1}{2}q^2(q^2-1)$, $\frac{1}{2}(q^2-1)$, $\frac{1}{2}(q^2-1)$ respectively.

(7.3) We first construct four characters K_1, K_2, K_3, K_4 of G of degrees $q(q^2+1)$, $q(q^2+1)$, q^2, q^2 respectively, which will be used to construct four irreducible characters of G.

Take the subgroups M_3, M_4 and characters ρ_{31}, ρ_{32} of M_3 and ρ_{41}, ρ_{42} of M_4 given by

$$\begin{array}{lllll}
\rho_{31}\colon a_3\to 1, & b_3\to 1, & x_3\to 1, & y_3\to 1, & z_3\to 1,\\
\rho_{32}\colon a_3\to 1, & b_3\to 1, & x_3\to -1, & y_3\to -1, & z_3\to -1,\\
\rho_{41}\colon a_4\to 1, & b_4\to 1, & x_4\to 1, & y_4\to 1, & z_4\to 1,\\
\rho_{42}\colon a_4\to 1, & b_4\to 1, & x_4\to -1, & y_4\to -1, & z_4\to -1.
\end{array}$$

Consider the characters of degree $q(q^2+1)$

$$\rho_{31}^G+\rho_{32}^G-\rho_{41}^G-\rho_{42}^G+\alpha_{21}^G-\alpha_{11}^G+\delta_5'^G-\delta_5^G+\psi_5,$$
$$\rho_{31}^G+\rho_{32}^G-\rho_{41}^G-\rho_{42}^G+\alpha_{22}^G-\alpha_{12}^G+\delta_5'^G-\delta_5^G+\psi_5.$$

(For the definitions of δ_5, δ_5' see (3.15).) These two characters are identical on all classes of G except D_{31}, D_{32}, D_{33}, D_{34}. On these classes one of them, which we denote by K_1, takes values $(q, -q, -q, q)$; the other, which we denote by K_2, takes values $(-q, q, q, -q)$. Let Γ_1, Γ_2 be class functions on G which take values $(q, -q, -q, q)$ and $(-q, q, q, -q)$ respectively on D_{31}, D_{32}, D_{33}, D_{34}, and vanish on all other classes of G. Then

$$K_1 = \tfrac{1}{4}\chi_3(q-1,q-1)-\tfrac{1}{4}\chi_4(q+1,q+1)+\Gamma_1,$$
$$K_2 = \tfrac{1}{4}\chi_3(q-1,q-1)-\tfrac{1}{4}\chi_4(q+1,q+1)+\Gamma_2.$$

Thus $(K_1, K_2)=0$ and $(K_1, K_1)=(K_2, K_2)=2$.

Now consider the following characters of M_1, M_2, M_3, M_4, M_5 respectively:

$$\begin{array}{lllll}
\sigma_1\colon a_1\to 1, & x_1\to 1, & & & \\
\sigma_2\colon a_2\to 1, & x_2\to -1, & y_2\to -1, & & \\
\sigma_3\colon a_3\to 1, & b_3\to 1, & x_3\to 1, & y_3\to 1, & z_3\to 1,\\
\sigma_4\colon a_4\to 1, & b_4\to 1, & x_4\to 1, & y_4\to 1, & z_4\to 1,\\
\sigma_5\colon a_5\to 1, & b_5\to 1, & y_5\to 1, & z_5\to 1. &
\end{array}$$

Consider the characters of degree q^2

$$-\sigma_1^G+\sigma_2^G+\sigma_3^G+\sigma_4^G-\sigma_5^G+\alpha_{11}^G+\alpha_{22}^G+\psi_1+\psi_2+\tfrac{3}{2}(q-1)\psi_3+\psi_4+(q-2)\psi_5,$$
$$-\sigma_1^G+\sigma_2^G+\sigma_3^G+\sigma_4^G-\sigma_5^G+\alpha_{12}^G+\alpha_{21}^G+\psi_1+\psi_2+\tfrac{3}{2}(q-1)\psi_3+\psi_4+(q-2)\psi_5.$$

(Note that, as in (7.1), the contribution to $-\sigma_1^G+\sigma_2^G+\sigma_3^G+\sigma_4^G-\sigma_5^G$ from the classes of the M_i outside H_i is zero.)

Again, we get two characters K_3, K_4 which differ only on D_{31}, D_{32}, D_{33}, D_{34}, and

$$\begin{aligned}K_3 &= -\tfrac{1}{4}\chi_1(q^2+1)+\tfrac{1}{4}\chi_2(q^2-1)+\tfrac{1}{8}\chi_3(q-1,q-1)\\ &\quad+\tfrac{1}{8}\chi_4(q+1,q+1)-\tfrac{1}{4}\chi_5(q+1,q-1)+\Gamma_1,\\ K_4 &= -\tfrac{1}{4}\chi_1(q^2+1)+\tfrac{1}{4}\chi_2(q^2-1)+\tfrac{1}{8}\chi_3(q-1,q-1)\\ &\quad+\tfrac{1}{8}\chi_4(q+1,q+1)-\tfrac{1}{4}\chi_5(q+1,q-1)+\Gamma_2,\end{aligned}$$

where Γ_1, Γ_2 are the functions defined above.

We have the following table for the scalar products (K_i, K_j).

	K_1	K_2	K_3	K_4
K_1	2	0	1	-1
K_2		2	-1	1
K_3			2	0
K_4				2

Since $(K_1, K_3)=1$ we can assume that $K_1=\nu_1+\nu_2$, $K_3=\nu_1-\nu_4$ where ν_i or $-\nu_i$ is irreducible $(i=1, 2, 4)$. Then K_2 must be either $-\nu_1+\nu_2$ or $\nu_3+\nu_4$ for some ν_3 which is distinct from ν_1, ν_2, ν_4 and is such that $-\nu_3$ or ν_3 is irreducible. But if $K_2=-\nu_1+\nu_2$ then either $\tfrac{1}{2}(K_1-K_2)$ or its negative is an irreducible character, which is impossible since it is of degree 0. Hence we have $K_1=\nu_1+\nu_2$, $K_2=\nu_3+\nu_4$, $K_3=\nu_1-\nu_4$, $K_4=\nu_3-\nu_2$. These equations are not sufficient to compute the ν_i. For this purpose consider

$$K_5 = \xi_3(q-1)-\theta_0.$$

Now

$$\begin{aligned}\theta_0 &= \tfrac{1}{4}\chi_1(q^2+1)+\tfrac{1}{4}\chi_2(q^2-1)+\tfrac{1}{8}\chi_3(q-1,q-1)\\ &\quad+\tfrac{1}{8}\chi_4(q+1,q+1)+\tfrac{1}{4}\chi_5(q+1,q-1),\end{aligned}$$

and

$$\xi_3(q-1) = \tfrac{1}{2}\chi_3(q-1,q-1)+\tfrac{1}{2}\chi_5(q+1,q-1),$$

and so $(K_5, K_i)=0$ $(i=1, 2, 3, 4)$. Thus $K_5=\nu_1+\nu_4$ or $K_5=\nu_2+\nu_3$.

Suppose $K_5=\nu_2+\nu_3$. We consider the restriction of the class function $\tfrac{1}{2}(K_4+K_5)$

to the abelian subgroup V of order $4q^2$ consisting of all elements of the form

$$\begin{pmatrix} \pm 1 & \lambda & & \\ & \pm 1 & & \\ & & \pm 1 & \mu \\ & & & \pm 1 \end{pmatrix} \qquad (\lambda, \mu \in F).$$

This function has the following values at the classes of G which meet V.

A_1, A_1'	$A_{21}, A_{21}', A_{22}, A_{22}'$	A_{31}, A_{31}'	A_{32}, A_{32}'	$A_{41}, A_{41}', A_{42}, A_{42}'$	D_1	$D_{21}, D_{22}, D_{23}, D_{24}$	D_{31}, D_{34}	D_{32}, D_{33}
$\frac{1}{2}q(1+q^2)$	$\frac{1}{2}q(1+q)$	q	0	0	$\frac{1}{2}(1+q)^2$	$\frac{1}{2}(1+q)$	$\frac{1}{2}(1-q)$	$\frac{1}{2}(1+q)$

[We remark that $\frac{1}{2}(K_3+K_5)$ has the same values as $\frac{1}{2}(K_4+K_5)$ on all the classes of G except D_{31}, D_{32}, D_{33}, D_{34}. At these classes it has values $\frac{1}{2}(1+q)$, $\frac{1}{2}(1-q)$, $\frac{1}{2}(1-q)$, $\frac{1}{2}(1+q)$ respectively.]

We consider the scalar product of the restriction of $\frac{1}{2}(K_4+K_5)$ to V with the character of V defined by

$$\begin{pmatrix} 1 & \lambda & & \\ & 1 & & \\ & & 1 & \mu \\ & & & 1 \end{pmatrix} \to \varepsilon(\lambda)\varepsilon(\mu), \qquad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \to 1,$$

$$\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \to 1.$$

This scalar product is seen to be

$$\tfrac{1}{4}(q+1), \qquad \text{if } q \equiv 1 \pmod 4,$$
$$\tfrac{1}{4}(q+3), \qquad \text{if } q \equiv -1 \pmod 4.$$

This shows that $\frac{1}{2}(K_4+K_5)$ cannot be a character. Hence $K_5=\nu_1+\nu_4$, and this enables us to compute the characters ν_1, ν_2, ν_3, ν_4. We thus get four irreducible characters of degrees $q(1+q)^2/2$, $q(1-q)^2/2$, $q(1+q^2)/2$, $q(1+q^2)/2$, and we denote them by θ_9, θ_{10}, θ_{11}, θ_{12}.

(7.4) We now show that the functions $\frac{1}{2}\chi_4(k, k)\pm\frac{1}{2}\chi_2(k(q-1))$, $\frac{1}{2}\chi_3(k, k)$ $\pm\frac{1}{2}\chi_2(k(q+1))$ are characters of G also for the values of k which were omitted in §4, i.e. for k a multiple of $\frac{1}{2}(q-1)$ or $\frac{1}{2}(q+1)$. We note that, by (4.8), for a fixed k it is enough to show this for one of these four functions.

Case 1. k is an odd multiple of $\frac{1}{2}(q-1)$. Then $\chi_3(k, k)=\chi_3(\frac{1}{2}(q-1), \frac{1}{2}(q-1))$, $\chi_2(k(q+1))=\chi_2(\frac{1}{2}(q^2-1))$, and $\frac{1}{2}\chi_3(k, k)+\frac{1}{2}\chi_2(k(q+1))=\Phi_9+\theta_3+\theta_4$.

Case 2. k is an odd multiple of $\frac{1}{2}(q+1)$. In this case

$$\tfrac{1}{2}\chi_4(k,k)+\tfrac{1}{2}\chi_2(k(q-1)) = -\Phi_9+\theta_3+\theta_4.$$

Case 3. k is an even multiple of $\frac{1}{2}(q-1)$. In this case

$$\begin{aligned}\tfrac{1}{2}\chi_3(k,k)+\tfrac{1}{2}\chi_2(k(q+1)) &= \tfrac{1}{2}\chi_3(q-1,q-1)+\tfrac{1}{2}\chi_2(q^2-1)\\ &= \theta_9+\theta_{11}+\theta_0.\end{aligned}$$

Case 4. k is an even multiple of $\frac{1}{2}(q+1)$. Then

$$\tfrac{1}{2}\chi_4(k,k)+\tfrac{1}{2}\chi_2(k(q-1)) = -\theta_{10}-\theta_{12}+\theta_0.$$

We can now define characters $\chi_6(k)$, $\chi_7(k)$, $\chi_8(k)$, $\chi_9(k)$ as in Lemma 4.9 for any integer k. We also get one more irreducible character in each of the families $\{-\chi_6(k)\}$ and $\{\chi_7(k)\}$.

(7.5) We now give a simple construction for the Steinberg character [6] of G, which will be denoted by θ_{13}.

Let Δ_2, Δ_3, Δ_4, Δ_5 be the identity characters of M_2, M_3, M_4, M_5 respectively, and let Δ_1 be the character of M_1 defined by $a_1 \to 1$, $x_1 \to -1$. Then the character $\Delta_1^G-\Delta_2^G+\Delta_3^G+\Delta_4^G-\Delta_5^G$ is of degree q^4, and is irreducible since it is equal to

$$\tfrac{1}{4}\chi_1(q^2+1)-\tfrac{1}{4}\chi_2(q^2-1)+\tfrac{1}{8}\chi_3(q-1,q-1)+\tfrac{1}{8}\chi_4(q+1,q+1)-\tfrac{1}{4}\chi_5(q+1,q-1).$$

We have now exhausted all the irreducible characters of G, since we have as many as there are conjugacy classes of G.

8. **Table of characters.** The table that follows contains the characters of G that were obtained in the previous sections. In the case of families of characters indexed by parameters, the parameters take rational integral values. The values of the parameters for which the characters are irreducible are indicated in the table. In some cases it is the negative of a character that is irreducible, and this will not be mentioned explicitly in the table.

The values of the characters at the classes $A'_{21},\ldots,A'_{42}$ are omitted, since they can be obtained from the values at $A_{21},\ldots,A_{42}$, A'_1. Similarly the values of the characters at the classes $C'_1(i),\ldots,C'_{42}(i)$, D_{23}, D_{24} are omitted.

It is sufficient to give the values of one character from the pair $\{\xi_{21}(k),\xi_{22}(k)\}$, for the values of the other are then got by replacing $\tilde{\varepsilon}$ by $\tilde{\varepsilon}'$ and $\tilde{\varepsilon}'$ by $\tilde{\varepsilon}$. A similar statement holds for the pairs $\{\xi'_{21}(k),\xi'_{22}(k)\}$, $\{\xi_{41}(k),\xi_{42}(k)\}$, $\{\xi'_{41}(k),\xi'_{42}(k)\}$, $\{\Phi_1,\Phi_2\}$, $\{\Phi_3,\Phi_4\}$, $\{\Phi_5,\Phi_6\}$, $\{\Phi_7,\Phi_8\}$, $\{\theta_1,\theta_2\}$, $\{\theta_3,\theta_4\}$, $\{\theta_5,\theta_6\}$, and $\{\theta_7,\theta_8\}$.

Finally we omit the values of the characters at the classes A_{22}, A_{42}, $C_{22}(i)$, $C_{42}(i)$, D_{22}, D_{33}, D_{34}, as these can be obtained from the classes A_{21}, A_{41}, $C_{21}(i)$, $C_{22}(i)$, D_{21}, D_{32}, D_{31} by replacing $\tilde{\varepsilon}$ by $\tilde{\varepsilon}'$ and $\tilde{\varepsilon}'$ by $\tilde{\varepsilon}$. Again, the absence of an entry in the table indicates that the corresponding value is zero. We also use the abbreviations $t=\frac{1}{2}(q-1)$, $\alpha_j=\tilde{\gamma}^j+\tilde{\gamma}^{-j}$, $\beta_j=\tilde{\eta}^j+\tilde{\eta}^{-j}$, and $s(k,l)=(-1)^k+(-1)^l$.

Character	$\chi_1(j)$	$\chi_2(j)$	$\chi_3(k, l)$	$\chi_4(k, l)$	$\chi_5(k, l)$	$\chi_6(k)$	$\chi_7(k)$	$\chi_8(k)$
When irreducible	$j \in R_1$	$j \in R_2$	$k, l \in T_1, k \neq l$	$k, l \in T_2, k \neq l$	$k \in T_2, l \in T_1$	$k \in T_2$	$k \in T_2$	$k \in T_1$
A_1	$(1-q^2)^2$	$1-q^4$	$(1+q)^2(1+q^2)$	$(1-q)^2(1+q^2)$	$1-q^4$	$(1-q)(1+q^2)$	$q(q-1)(1+q^2)$	$(1+q)(1+q^2)$
A_1'	$(-1)^j(1-q^2)^2$	$(-1)^j(1-q^4)$	$(-1)^{k+l}(1+q)^2 (1+q^2)$	$(-1)^{k+l}(1-q)^2 (1+q^2)$	$(-1)^{k+l}(1-q^4)$	$(1-q)(1+q^2)$	$q(q-1)(1+q^2)$	$(1+q)(1+q^2)$
A_{21}	$1-q^2$	$1-q^2$	$(1+q)^2$	$(1-q)^2$	$1+q^2$	$1-q$	$q(q-1)$	$1+q$
A_{31}	$1-q$	$1+q$	$1+3q$	$1-q$	$1-q$	1	$-q$	$1+2q$
A_{32}	$1+q$	$1-q$	$1+q$	$1-3q$	$1+q$	$1-2q$	$-q$	1
A_{41}	1	1	1	1	1	1		1
$B_1(i)$	$\tilde{\zeta}^{ij}+\tilde{\zeta}^{-ij} + \tilde{\zeta}^{qij}+\tilde{\zeta}^{-qij}$							
$B_2(i)$		$\tilde{\theta}^{ij}+\tilde{\theta}^{-ij} + \tilde{\theta}^{qij}+\tilde{\theta}^{-qij}$				β_{ik}	$-\beta_{ik}$	α_{ik}
$B_3(i, j)$			$\alpha_{ik}\alpha_{jl}+\alpha_{jk}\alpha_{il}$					$\alpha_{ik}\alpha_{jk}$
$B_4(i, j)$				$\beta_{ik}\beta_{jl}+\beta_{jk}\beta_{il}$		$\beta_{ik}\beta_{jk}$	$\beta_{ik}\beta_{jk}$	

Character	$\chi_1(j)$	$\chi_2(j)$	$\chi_3(k,l)$	$\chi_4(k,l)$	$\chi_5(k,l)$	$\chi_6(k)$	$\chi_7(k)$	$\chi_8(k)$
When irreducible	$j \in R_1$	$j \in R_2$	$k, l \in T_1, k \neq l$	$k, l \in T_2, k \neq l$	$k \in T_2, l \in T_1$	$k \in T_2$	$k \in T_2$	$k \in T_1$
$B_5(i,j)$					$\beta_{ik}\alpha_{jl}$			
$B_6(i)$		$(1+q)\beta_{ij}$		$(1-q)\beta_{ik}\beta_{il}$		$\beta_{2ik}+1-q$	$-q\beta_{2ik}+1-q$	$1+q$
$B_7(i)$		β_{ij}		$\beta_{ik}\beta_{il}$		$\beta_{2ik}+1$	1	1
$B_8(i)$		$(1-q)\alpha_{ij}$	$(1+q)\alpha_{ik}\alpha_{il}$			$1-q$	$-(1-q)$	$\alpha_{2ik}+1+q$
$B_9(i)$		α_{ij}	$\alpha_{ik}\alpha_{il}$			1	-1	$\alpha_{2ik}+1$
$C_1(i)$				$(1-q)(\beta_{ik}+\beta_{il})$	$(1+q)\beta_{ik}$	$(1-q)\beta_{ik}$	$(1-q)\beta_{ik}$	
$C_{21}(i)$				$\beta_{ik}+\beta_{il}$	β_{ik}	β_{ik}	β_{ik}	
$C_3(i)$			$(1+q)(\alpha_{ik}+\alpha_{il})$		$(1-q)\alpha_{il}$			$(1+q)\alpha_{ik}$
$C_{41}(i)$			$\alpha_{ik}+\alpha_{il}$		α_{il}			α_{ik}
D_1			$(1+q)^2 s(k,l)$	$(1-q)^2 s(k,l)$	$(1-q^2)s(k,l)$	$(-1)^k(1-q)^2$	$(-1)^k(1-q)^2$	$(-1)^k(1+q)^2$
D_{21}			$(1+q)s(k,l)$	$(1-q)s(k,l)$	$(1-q)(-1)^l$ $+(1+q)(-1)^k$	$(-1)^k(1-q)$	$(-1)^k(1-q)$	$(-1)^k(1+q)$
D_{31}			$s(k,l)$	$s(k,l)$	$s(k,l)$	$(-1)^k$	$(-1)^k$	$(-1)^k$
D_{32}			$s(k,l)$	$s(k,l)$	$s(k,l)$	$(-1)^k$	$(-1)^k$	$(-1)^k$

Character	$\chi_9(k)$	$\xi_1(k)$	$\xi_1'(k)$	$\xi_3(k)$	$\xi_3'(k)$	$\xi_{21}(k)$	$\xi_{21}'(k)$
When irreducible	$k \in T_1$	$k \in T_2$	$k \in T_2$	$k \in T_1$	$k \in T_1$	$k \in T_2$	$k \in T_2$
A_1	$q(1+q)(1+q^2)$	$(1-q)(1+q^2)$	$q(1-q)(1+q^2)$	$(1+q)(1+q^2)$	$q(1+q)(1+q^2)$	$\frac{1}{2}(1-q^4)$	$\frac{1}{2}(1-q)^2(1+q^2)$
A_1'	$q(1+q)(1+q^2)$	$(-1)^k(1-q)(1+q^2)$	$(-1)^k q(1-q)(1+q^2)$	$(-1)^k(1+q)(1+q^2)$	$(-1)^k q(1+q)(1+q^2)$	$(-1)^{k+t}\frac{1}{2}(1-q^4)$	$(-1)^{k+t+1}\frac{1}{2}(1-q)^2(1+q^2)$
A_{21}	$q(1+q)$	$1+q^2-q$	q	$1+q+q^2$	q	$\frac{1}{2}(1+q)+q(1-q)\tilde{\varepsilon}$	$\frac{1}{2}(1-q)-q(q-1)\tilde{\varepsilon}$
A_{31}	q	$1-q$		$1+q$	$2q$	$\frac{1}{2}(1-q)$	$\frac{1}{2}(1-q)$
A_{32}	q	$1-q$	$2q$	$1+q$		$\frac{1}{2}(1+q)$	$\frac{1}{2}(1-3q)$
A_{41}		1		1		$-\tilde{\varepsilon}'$	$-\tilde{\varepsilon}'$
$B_1(i)$							
$B_2(i)$	$-\alpha_{ik}$						
$B_3(i,j)$	$\alpha_{ik}\alpha_{jk}$			$\alpha_{ik}+\alpha_{jk}$	$\alpha_{ik}+\alpha_{jk}$		
$B_4(i,j)$		$\beta_{ik}+\beta_{jk}$	$-(\beta_{ik}+\beta_{jk})$				$-\beta_{ik}-\beta_{jk}$
$B_5(i,j)$		β_{ik}	β_{ik}	α_{jk}	$-\alpha_{jk}$	$-\beta_{ik}$	
$B_6(i)$	$-(1+q)$	$(1-q)\beta_{ik}$	$(q-1)\beta_{ik}$				$(q-1)\beta_{ik}$

Character	$\chi_9(k)$	$\xi_1(k)$	$\xi_1'(k)$	$\xi_3(k)$	$\xi_3'(k)$	$\xi_{21}(k)$	$\xi_{21}'(k)$
When irreducible	$k \in T_1$	$k \in T_2$	$k \in T_2$	$k \in T_1$	$k \in T_1$	$k \in T_2$	$k \in T_2$
$B_7(i)$	-1	β_{ik}	$-\beta_{ik}$				$-\beta_{ik}$
$B_8(i)$	$q\alpha_{2ik}+1+q$			$(1+q)\alpha_{ik}$	$(1+q)\alpha_{ik}$		
$B_9(i)$	1			α_{ik}	α_{ik}		
$C_1(i)$		$1-q+\beta_{ik}$	$q-1+q\beta_{ik}$	$1+q$	$-(1+q)$	$\frac{1}{2}(1+q)\beta_{ik}$	$q-1+\frac{1}{2}(1-q)\beta_{ik}$
$C_{21}(i)$		$1+\beta_{ik}$	-1	1	-1	$-\beta_{ik}\bar{\varepsilon}$	$-1-\bar{\varepsilon}\beta_{ik}$
$C_3(i)$	$(1+q)\alpha_{ik}$	$1-q$	$1-q$	$1+q+\alpha_{ik}$	$1+q+q\alpha_{ik}$	$-(1-q)$	
$C_{41}(i)$	α_{ik}	1	1	$1+\alpha_{ik}$	1	-1	
D_1	$(-1)^k(1+q)^2$	$(1-q)s(2, k)$	$q(1-q)s(2, k)$	$(1+q)s(2, k)$	$q(1+q)s(2, k)$	$\frac{1}{2}(1-q^2)s(k, t)$	$\frac{1}{2}(1-q)^2 s(k, t+1)$
D_{21}	$(-1)^k(1+q)$	$s(2, k)-q$	$q(-1)^k$	$s(2, k)+q$	$q(-1)^k$	$(-1)^k\frac{1}{2}(1+q)$ $+(-1)^{t+1}(1-q)\bar{\varepsilon}$	$(-1)^k\frac{1}{2}(1-q)$ $+(-1)^t(1-q)\bar{\varepsilon}$
D_{31}	$(-1)^k$	$s(2, k)$		$s(2, k)$		$s(k+1, t+1)\bar{\varepsilon}$	$s(k+1, t)\bar{\varepsilon}$
D_{32}	$(-1)^k$	$s(2, k)$		$s(2, k)$		$(-1)^{k+1}\bar{\varepsilon}$ $+(-1)^{t+1}\bar{\varepsilon}'$	$(-1)^{k+1}\bar{\varepsilon}$ $+(-1)^t\bar{\varepsilon}'$

Character	$\xi_{41}(k)$	$\xi'_{41}(k)$	Φ_1	Φ_3	Φ_5	Φ_7	Φ_9
When irreducible	$k \in T_1$	$k \in T_1$					
A_1	$\frac{1}{2}(1+q)^2(1+q^2)$	$\frac{1}{2}(1-q^4)$	$\frac{1}{2}(1-q)(1+q^2)$	$\frac{1}{2}q(1-q)(1+q^2)$	$\frac{1}{2}(1+q)(1+q^2)$	$\frac{1}{2}q(1+q)(1+q^2)$	$q(1+q^2)$
A'_1	$(-1)^{k+t}\frac{1}{2}(1+q)^2(1+q^2)$	$(-1)^{k+t+1}\frac{1}{2}(1-q^4)$	$(-1)^{t+1}\frac{1}{2}(1-q)(1+q^2)$	$(-1)^{t+1}\frac{1}{2}q(1-q)(1+q^2)$	$(-1)^t\frac{1}{2}(1+q)(1+q^2)$	$(-1)^t\frac{1}{2}q(1+q)(1+q^2)$	$q(1+q^2)$
A_{21}	$\frac{1}{2}(1+q)-q(1+q)\tilde{\varepsilon}$	$\frac{1}{2}(1-q)-q(1+q)\tilde{\varepsilon}$	$\frac{1}{2}(1-q)^2-q\tilde{\varepsilon}$	$\frac{1}{2}q(1-q)-q^2\tilde{\varepsilon}$	$\frac{1}{2}(1+q)^2+q\tilde{\varepsilon}'$	$\frac{1}{2}q(1+q)+q^2\tilde{\varepsilon}'$	q
A_{31}	$\frac{1}{2}(1+3q)$	$\frac{1}{2}(1-q)$	$\frac{1}{2}(1-q)$		$\frac{1}{2}(1+q)$	q	q
A_{32}	$\frac{1}{2}(1+q)$	$\frac{1}{2}(1+q)$	$\frac{1}{2}(1-q)$	q	$\frac{1}{2}(1+q)$		q
A_{41}	$-\tilde{\varepsilon}$	$-\tilde{\varepsilon}$	$-\tilde{\varepsilon}$		$-\tilde{\varepsilon}$		
$B_1(i)$							
$B_2(i)$							
$B_3(i,j)$	$-\alpha_{ik}-\alpha_{jk}$				-2	-2	$2(-1)^{i+j}$
$B_4(i,j)$			-2	2			$2(-1)^{i+j+1}$
$B_5(i,j)$		$-\alpha_{jk}$	-1	-1	-1	1	
$B_6(i)$			$q-1$	$1-q$			$q-1$

Character	$\xi_{41}(k)$	$\xi'_{41}(k)$	Φ_1	Φ_3	Φ_5	Φ_7	Φ_9
When irreducible	$k \in T_1$	$k \in T_1$					
$B_7(i)$			-1	1			-1
$B_8(i)$	$-(1+q)\alpha_{ik}$				$-(1+q)$	$-(1+q)$	$q+1$
$B_9(i)$	$-\alpha_{ik}$				-1	-1	1
$C_1(i)$		$-(1+q)$	$-\frac{1}{2}(1+q)$	$-\frac{1}{2}(1+q)$	$\frac{1}{2}(1+q)$	$-\frac{1}{2}(1+q)$	$(q-1)(-1)^t$
$C_{21}(i)$		-1	$\tilde{\varepsilon}$	$\tilde{\varepsilon}'$	$-\tilde{\varepsilon}'$	$\tilde{\varepsilon}'$	$(-1)^{t+1}$
$C_3(i)$	$-(1+q)$ $+\frac{1}{2}(1+q)\alpha_{ik}$	$\frac{1}{2}(1-q)\alpha_{ik}$	$\frac{1}{2}(1-q)$	$\frac{1}{2}(1-q)$	$-\frac{1}{2}(1-q)$	$\frac{1}{2}(1-q)$	$(q+1)(-1)^t$
$C_{41}(i)$	$-1-\tilde{\varepsilon}\alpha_{ik}$	$-\alpha_{ik}\tilde{\varepsilon}$	$-\tilde{\varepsilon}$	$-\tilde{\varepsilon}$	$\tilde{\varepsilon}'$	$-\tilde{\varepsilon}$	$(-1)^t$
D_1	$\frac{1}{2}(1+q)^2 s(k,t)$	$\frac{1}{2}(1-q)^2 s(k,t+1)$	$\frac{1}{2}(1-q)s(2,t+1)$	$\frac{1}{2}q(1-q)s(2,t+1)$	$\frac{1}{2}(1+q)s(2,t)$	$\frac{1}{2}q(1+q)s(2,t)$	$(-1)^t(1+q^2)$
D_{21}	$(-1)^k\frac{1}{2}(1+q)$ $+(-1)^{t+1}(1+q)\tilde{\varepsilon}$	$(-1)^k\frac{1}{2}(1-q)$ $+(-1)^t(1+q)\tilde{\varepsilon}$	$\frac{1}{2}(1-q)s(2,t+1)$ $+q(-1)^t\tilde{\varepsilon}$	$\frac{1}{2}(1-q)(-1)^t$ $+(-1)^t\tilde{\varepsilon}$	$\frac{1}{2}(1+q)s(2,t)$ $+q(-1)^t\tilde{\varepsilon}'$	$\frac{1}{2}(1+q)(-1)^t$ $+(-1)^t\tilde{\varepsilon}'$	$(-1)^t$
D_{31}	$s(k+1,t+1)\tilde{\varepsilon}$	$s(k+1,t)\tilde{\varepsilon}$	$\frac{1}{2}s(2,t+1)$	$\frac{1}{2}s(1,t)(\tilde{\varepsilon}-\tilde{\varepsilon}')$	$\frac{1}{2}s(2,t)$	$\frac{1}{2}s(2,t)(\tilde{\varepsilon}'-\tilde{\varepsilon})$	$(-1)^t$
D_{32}	$(-1)^{k+1}\tilde{\varepsilon}$ $+(-1)^{t+1}\tilde{\varepsilon}'$	$(-1)^{k+1}\tilde{\varepsilon}+(-1)^t\tilde{\varepsilon}'$	$\frac{1}{2}s(2,t+1)$	$\frac{1}{2}s(2,t)(\tilde{\varepsilon}'-\tilde{\varepsilon})$	$\frac{1}{2}s(2,t)$	$\frac{1}{2}s(2,t+1)(\tilde{\varepsilon}'-\tilde{\varepsilon})$	$(-1)^t$

Character	θ_1	θ_3	θ_5	θ_7	θ_9	θ_{10}	θ_{11}	θ_{12}	θ_{13}
When irreducible									
A_1	$\frac{1}{2}q^2(1+q)^2$	$\frac{1}{2}(1+q^2)$	$\frac{1}{2}q^2(1-q^2)$	$\frac{1}{2}(1-q^2)$	$\frac{1}{2}q(1+q)^2$	$\frac{1}{2}q(1-q)^2$	$\frac{1}{2}q(1+q^2)$	$\frac{1}{2}q(1+q^2)$	q^4
A_1'	$\frac{1}{2}q^2(1+q^2)$	$\frac{1}{2}(1+q^2)$	$-\frac{1}{2}q^2(1-q^2)$	$-\frac{1}{2}(1-q^2)$	$\frac{1}{2}q(1+q)^2$	$\frac{1}{2}q(1-q)^2$	$\frac{1}{2}q(1+q^2)$	$\frac{1}{2}q(1+q^2)$	q^4
A_{21}	$-q^2\bar{\varepsilon}$	$\frac{1}{2}(1+q)+q\bar{\varepsilon}$	$-q^2\bar{\varepsilon}$	$\frac{1}{2}(1-q)-q\bar{\varepsilon}$	$\frac{1}{2}q(1+q)$	$\frac{1}{2}q(1-q)$	$\frac{1}{2}q(1-q)$	$\frac{1}{2}q(1+q)$	
A_{31}		$\frac{1}{2}(1+q)$		$\frac{1}{2}(1-q)$	q		q		
A_{32}		$\frac{1}{2}(1-q)$		$\frac{1}{2}(1+q)$		q		q	
A_{41}		$-\bar{\varepsilon}'$		$-\bar{\varepsilon}$					
$B_1(i)$			$(-1)^{i+1}$	$(-1)^i$	-1	1			1
$B_2(i)$	$(-1)^{i+1}$	$(-1)^i$					1	-1	-1
$B_3(i,j)$	$(-1)^{i+j}$	$(-1)^{i+j}$			2		1	1	1
$B_4(i,j)$	$(-1)^{i+j}$	$(-1)^{i+j}$				-2	-1	-1	1
$B_5(i,j)$			$(-1)^{i+j}$	$(-1)^{i+j}$			-1	1	-1
$B_6(i)$	$-q$	1				$q-1$	q	-1	$-q$

Character	θ_1	θ_3	θ_5	θ_7	θ_9	θ_{10}	θ_{11}	θ_{12}	θ_{13}
When irreducible									
$B_7(i)$		1				-1		-1	
$B_8(i)$	q	1			$1+q$		1	q	q
$B_9(i)$		1			1		1		
$C_1(i)$	$\frac{1}{2}(1-q)(-1)^t$	$\frac{1}{2}(1-q)(-1)^t$	$\frac{1}{2}(1+q)(-1)^t$	$\frac{1}{2}(1+q)(-1)^t$		$q-1$	-1	q	$-q$
$C_{21}(i)$	$(-1)^{t+1}\tilde{\varepsilon}$	$(-1)^{t+1}\tilde{\varepsilon}$	$(-1)^{t+1}\tilde{\varepsilon}$	$(-1)^{t+1}\tilde{\varepsilon}'$		-1	-1		
$C_3(i)$	$\frac{1}{2}(1+q)(-1)^t$	$\frac{1}{2}(1+q)(-1)^t$	$\frac{1}{2}(1-q)(-1)^t$	$\frac{1}{2}(1-q)(-1)^t$	$1+q$		q	1	q
$C_{41}(i)$	$(-1)^{t+1}\tilde{\varepsilon}$	$(-1)^{t+1}\tilde{\varepsilon}'$	$(-1)^{t+1}\tilde{\varepsilon}$	$(-1)^{t+1}\tilde{\varepsilon}$	1			1	
D_1	$(-1)^t q$	$(-1)^t q$			$\frac{1}{2}(1+q)^2$	$-\frac{1}{2}(1-q)^2$	$\frac{1}{2}(q^2-1)+q$	$\frac{1}{2}(1+2q-q^2)$	q^2
D_{21}	$(-1)^{t+1}q\tilde{\varepsilon}$	$(-1)^t[\frac{1}{2}(1+q)+\tilde{\varepsilon}]$	$(-1)^t q\tilde{\varepsilon}$	$(-1)^t$ $[\frac{1}{2}(1-q)+\tilde{\varepsilon}]$	$\frac{1}{2}(1+q)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(1+q)$	
D_{31}		$(-1)^{t+1}(\tilde{\varepsilon}'-\tilde{\varepsilon})$			$\frac{1}{2}(1+q)$	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(1+q)$	$\frac{1}{2}(1-q)$	
D_{32}				$(-1)^t(\tilde{\varepsilon}'-\tilde{\varepsilon})$	$\frac{1}{2}(1-q)$	$-\frac{1}{2}(1+q)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(1+q)$	

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