THE MONOTONE UNION PROPERTY OF MANIFOLDS

BY
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1. **Introduction.** Let M be a manifold. $U=\bigcup_{i=1}^{\infty} U_i$ is called a *monotone union* of M if (i) each U_i is homeomorphic to M and (ii) $U_i \subset \operatorname{int} U_{i+1}$ (=interior U_{i+1}) for each i. M is said to have the *monotone union property* if each monotone union of M is homeomorphic to the interior of M. Brown [5] has shown that the n-cell has the monotone union property. Kwun [13] has generalized this to all compact combinatorial n-manifolds ($n \neq 4$) with a sphere boundary. Edwards [7, p. 9], [8, p. 421] has given some necessary and sufficient conditions that a monotone union of a compact 3-manifold be homeomorphic to int M. By use of the Regular Neighborhood Annulus Theorem [11, p. 725], Edwards' results may be generalized to compact n-manifolds for all n.

The motivation for this paper was the converse of Kwun's Theorem.

(A) *If M is a compact bounded combinatorial manifold with the monotone union property, is the boundary of M a sphere?*

We shall restrict ourselves to connected compact 3-manifolds whose boundary is either a collection of spheres or contains a torus, i.e., a surface of genus 1. A positive answer is given for (A) for a class of such manifolds in §4 and §5. However, the answer to (A) is no; an example of a compact bounded 3-manifold whose boundary is a torus and which possesses the monotone union property is constructed in §6.

Whitehead [25] has given an example of a monotone union of the solid torus which is not homeomorphic to the interior of any compact bounded manifold. However, there exist manifolds which do not have the monotone union property but they have the property that each monotone union is homeomorphic to the interior of some compact bounded manifold. Manifolds with the latter property are said to have the *bounded union property*. A characterization of a class of manifolds with the bounded union property is given in §4 and §5.

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2. **Whitehead's example.** Let us recall Whitehead's construction in [25]. Let

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T_2 be an unknotted polyhedral solid torus in the three sphere S^3. Let T_1 be a polyhedral solid torus embedded in the interior of T_2 as indicated in Figure 1. Let B be a polyhedral 3-cell in the interior of T_1 as indicated. Clearly we can find a homeomorphism h of S^3 onto itself such that

(i) $h(x)=x$ for $x \in B$;

(ii) $h(T_1)=T_2$.

Let $U=\bigcup_{i=1}^{\infty} h^n(T_1)$. Then U is a monotone union of the solid torus T_1. Whitehead showed that U was contractible, so that T_1 does not have the monotone union property.

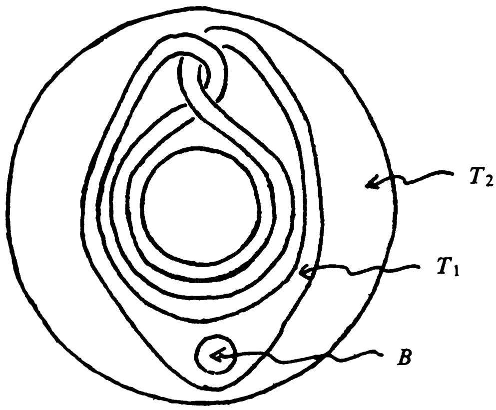

FIGURE 1

PROPOSITION 1. *U is not homeomorphic to the interior of any compact bounded 3-manifold.*

Proof. Suppose to the contrary that U is homeomorphic to int N, where N is a compact bounded 3-manifold. By [3], [16], we may assume that N is a combinatorial manifold. Let $N_1=N-\text{int } T_1$. By Alexander [2, Corollary 14:4], N_1 is a compact combinatorial manifold. Hence, the fundamental group of N_1 is finitely generated [10, Corollary 6.3.10]. However, Newman and Whitehead [18] have shown that the fundamental group of N_1 is infinitely generated. This contradiction establishes the proposition.

3. **The concentricity property for 3-manifolds.** Henceforth we shall assume, without loss of generality, that our manifolds are combinatorial and that all maps are piecewise linear. In [8], Edwards makes the following definition.

DEFINITION. Two compact manifolds M and N with nonempty boundaries and $N \subset \text{int } M$ are said to be *concentric* if Cl $(M-N)$ is homeomorphic to (bdry $M) \times I$, where I is the closed interval [0, 1].

We make the following definition.

DEFINITION. A compact manifold M is said to have the *concentricity property* if for each embedding $h: M \to \text{int } M$, $h(M)$ and M are concentric.

THEOREM 1. *Let M be a compact 3-manifold with connected boundary. M has the monotone union property if and only if M has the concentricity property.*

Proof. If M has the concentricity property, clearly M has the monotone union property. Suppose that M has the monotone union property and let $h: M \to \operatorname{int} M$ be an embedding. For each positive integer j, consider $M_j = M \times j$. Define $h_j: M_j \to M_{j+1}$ by $h_j(x, j) = (h(x), j+1)$. $\{M_j, h_j\}$ induces naturally a direct system [23, p. 18] in the category of topological spaces and continuous maps. The direct limit, N, of $\{M_j, h_j\}$ exists and can be constructed as in [23, p. 3] from $\bigcup M_j$. The natural maps $p_i: M_i \to N$ are continuous by definition and, in our case, are one-one. It is then easily seen that $N = \bigcup p_i M_i$ is a monotone union for M. Hence N is homeomorphic to $\operatorname{int} M$; let $k: N \to \operatorname{int} M$ be the homeomorphism. There exists a homeomorphism $c: (\text{bdry } M) \times I \to M$ such that $c(x, 0) = x$. Hence $k^{-1}c: (\text{bdry } M) \times (0, 1] \to N$ is a homeomorphism. There exists a positive integer n such that $p_n(\text{bdry } M) \subset k^{-1}c\ [(\text{bdry } M) \times (0, 1)]$. Since bdry M is connected, $p_n(\text{bdry } M)$ separates $k^{-1}c(\text{bdry } M \times 1)$ from $k^{-1}c(\text{bdry } M \times \varepsilon)$ for some sufficiently small ε. From Edwards [8, Theorem 1] $p_n(\text{bdry } M_n)$ is concentric with $k^{-1}c(\text{bdry } M \times 1)$. Similarly, $p_{n+1}(\text{bdry } M_{n+1})$ is concentric with $k^{-1}c(\text{bdry } M \times 1)$. Hence, by [8, Theorem 2], $\operatorname{Cl}(p_{n+1}M_{n+1} - p_n M_n)$ is homeomorphic to $(\text{bdry } M) \times I$. Therefore, it follows that M has the concentricity property.

4. **Manifolds whose boundary components are spheres.** We shall prove the following theorem in this section.

THEOREM 2. *Let M be a compact orientable 3-manifold whose boundary components are spheres. Then*

(i) *M has the bounded union property;*

(ii) *M has the monotone union property if and only if the boundary of M is connected.*

PROPOSITION 2. *Let M and N be two compact 3-manifolds such that there exists embeddings $i: M \to \operatorname{int} N$, $j: N \to \operatorname{int} M$. If M has the monotone union property and M is not homeomorphic to N, then N does not have the monotone union property.*

Proof. Let M_i be formed from $M \cup [(\text{bdry } M) \times [1, 2 - 1/i]]$ by identifying $(\text{bdry } M) \times 1$ with bdry M by the map $y \times 1 \to y$ for each positive integer i. Hence $L = \bigcup M_i$ is a monotone union of M and hence L is homeomorphic to $\operatorname{int} M$. By the hypothesis and Theorem 1, we can find for each positive integer k an embedding $j_k: N \to L$ such that $j_k(\text{bdry } N) \subseteq [(\text{bdry } M) \times (2 - 1/(k-1), 2 - 1/k)]$. Clearly $L = \bigcup j_k N$ is then a monotone union for N. Since M is not homeomorphic to N, $\operatorname{int} M$ is not homeomorphic to $\operatorname{int} N$ [8, Theorem 3]. Therefore L is not homeomorphic to $\operatorname{int} N$ and N does not have the monotone union property.

PROPOSITION 3. *Let M satisfy the hypotheses of Theorem 2; if the boundary of M is not connected, then M does not have the monotone union property.*

Proof. Let n be the number of components of bdry M. Suppose bdry $M=\bigcup S_i$, where each S_i is a 2-sphere. Let $\{B_i\}$ be a collection of n distinct 3-cells. Let M' be obtained from $M \cup B_1 \cup \cdots \cup B_n$ by identifying the boundary of each B_i with S_i. Let $A_1, A_2, \ldots, A_n$ be arcs in M such that

(i) $A_i \cap A_j = \varnothing$ for $i \neq j$;

(ii) $A_i \cap B_j$ is a point if $i=j-1$ or j;

(iii) $A_i \cap B_j = \varnothing$ otherwise.

Let N be a regular neighborhood of $(\bigcup B_i) \cup (\bigcup A_i)$. By [11] N is a 3-cell. Let $R = M' - \text{int } N$. Then R is a compact 3-manifold such that the boundary of R is a 2-sphere. By Kwun [13], R has the monotone union property. Let D be a 3-cell contained in the interior of B_1. Let $R' = M' - \text{int } D$. By Newman [17], R and R' are homeomorphic. Therefore we have

$$R \subset \text{int } M \subset M \subset \text{int } R'.$$

Since the boundary of R is connected and the boundary of M is not connected, R and M are not homeomorphic. By Proposition 2, M does not have the monotone union property.

PROPOSITION 4. *Let M satisfy the hypotheses of Theorem* 2; *then M has the bounded union property.*

Proof. We show that M has the bounded union property by induction on the number of components of the boundary of M. If $k=1$, where k is the number of components of the boundary of M, this is Kwun's Theorem. Suppose the theorem is true for all $k<n$. Let $k=n$ and let $U=\bigcup U_i$ be a monotone union of M. It is easy to see that the number of components of $U_{i+1}-U_i$ is n for each i. We have two cases.

Case 1. Suppose there exists a number N such that for every $i>N$, each component of $U_{i+1}-U_i$ meets the boundary of U_{i+1}. Let $A_1, A_2, \ldots, A_n$ be the components of $U_{i+1} - \text{int } U_i$. We want to show that each A_i is homeomorphic to $S^2 \times [0, 1]$; then it would follow easily that U is homeomorphic to int M. Let M' be the closed manifold obtained from U_{i+1} by attaching n disjoint 3-cells along the boundary of U_{i+1}; let M'' be obtained from U_i similarly. Then M' and M'' are homeomorphic. Let M_i be obtained from A_i by attaching two 3-cells along the boundary of A_i. Then

$$M' = M'' \# M_1 \# M_2 \# \cdots \# M_n.$$

($\#$ denotes the "connected sum" operation as defined in [15]. When we later form the connected sum of bounded manifolds, we shall assume as usual that the open 3-cells chosen shall have closures which do not meet their boundaries of the given manifolds.) By Milnor [15], $M_1 \# M_2 \# \cdots \# M_n$ is a 3-sphere and hence each M_i is a 3-sphere. Therefore each A_i is homeomorphic to $S^2 \times [0, 1]$.

Case 2. Suppose for each positive integer N, there exists an integer $i=i(N)>N$

such that some component of $U_{i+1}-U_i$ does not meet the boundary of U_{i+1}. Let $A_1, \ldots, A_n$ be the components of $U_{i+1}-\text{int } U_i$ so ordered that

(i) $A_i \cap \text{bdry } U_{i+1} \neq \varnothing$ for $i=1, 2, \ldots, r$;

(ii) $A_i \cap \text{bdry } U_{i+1} = \varnothing$ for $i=r+1, \ldots, n$.

Each of the A_i, $i=r+1, \ldots, n$, have only one boundary component. Let M', M'', $M_1, M_2, \ldots, M_r$ be defined as in Case 1, except, of course, more than two 3-cells may have to be attached to each A_i. Form M_j, $j=r+1, \ldots, n$, by attaching a 3-cell to A_j. Then

$$M' = M'' \# M_1 \# M_2 \# \cdots \# M_n.$$

As in Case 1, each M_i is a 3-sphere so that the A_i, $i=r+1, \ldots, n$, are each 3-cells. Let R be formed from M by attaching a 3-cell along one of the boundary components of M. From the above remarks, it follows that there exists an embedding $h_i\colon R \to \text{int } U_{i+1}$ such that

$$U_i \subset \text{int } h_i R \subset h_i R \subset \text{int } U_{i+1}.$$

Clearly, $U=\bigcup U_i=\bigcup_N h_{i(N)}R$ is a monotone union of R. By the induction hypotheses, R has the bounded union property; therefore U is homeomorphic to the interior of some compact bounded manifold. Hence M has the bounded union property.

The proof of Theorem 2 now follows from Propositions 3 and 4. From the proof of Proposition 4, we also have:

Corollary. *Let M satisfy the hypotheses of Theorem 2 and let $U=\bigcup U_i$ be a monotone union of M. Then U is homeomorphic to the interior of the compact bounded manifold obtained from M by attaching a collection (possibly empty) of 3-cells along some of the boundary components of M.*

5. **Manifolds one of whose boundary components is a torus.**

Theorem 3. *Let M be a compact 3-manifold whose boundary contains a torus T which has the property that the inclusion map $i\colon T \to M$ induces $i_{\#}\colon \pi_1(T) \to \pi_1(M)$ which is either*

(i) *not a monomorphism; or*

(ii) *an epimorphism.*

Then M does not have the bounded union property; thus, in particular, M does not have the monotone union property.

Proof. Suppose $i_{\#}$ is not a monomorphism. By the Loop Theorem [20], there is a simple closed polygonal curve L on T such that L is essential in T but inessential in M. By Dehn's Lemma [21], there is a polyhedral 2-cell D in M whose boundary is L. By using the collar of the boundary of M, we may assume that the interior of D does not meet the boundary of M. Since L is an essential simple closed curve in T, L represents a generator of the fundamental group $\pi_1(T)$. Then we can attach a solid torus R to M by identifying the boundary of R with T so that L and the core of

R bound a nonsingular annulus in R. Let $M_\#$ denote the resulting space and let $D_\#$ be the subset of $M_\#$ corresponding to $D \cup R$. Clearly $D_\#$ is a collapsible subcomplex of $M_\#$. Let N be a regular neighborhood of $D_\#$ in $M_\#$ such that $N \cap (\text{bdry } M - T) = \varnothing$. By [11], N is a 3-cell. Using the notation of §2, let

$$U_i = (M - \text{int } N) \cup \left(\bigcup_{j=0}^{i} h^j(T_1) - \text{int } B \right)$$

where the boundary of N is identified with the boundary of B for $i=0, 1, 2, \ldots$. Clearly each U_i is homeomorphic to M and $U_i \subset \text{int } U_{i+1}$. Hence $U' = \bigcup U_i = M \cup U$ (with appropriate identification) is a monotone union of M. By arguments similar to Proposition 1, M does not have the bounded union property.

Suppose $i_\#$ is an epimorphism. If $i_\#$ is also one-one, consider the double of M, $2M$. By Van Kampen's Theorem [6, p. 71], $\Pi_1(2M)$ is isomorphic to $\Pi_1(\text{bdry } M)$. By [26, p. 305], this is impossible. Hence $i_\#$ is not a monomorphism and this case reduces to the first case.

The proof of the above theorem rests upon the fact that in both cases M can be written in the form $M_1 \# M_2$ where M_2 is a solid torus. In such a situation, M is said *to have a handle*. We generalize this situation as follows. If M is a compact 3-manifold such that $M = M_1 \# M_2$ where M_2 is the closure of the complement of a polyhedral solid torus in the 3-sphere, M is said *to have a pseudohandle*.

THEOREM 4. *Let M be a compact 3-manifold which has a pseudohandle. Then M does not have the monotone union property. If the boundary of M is connected and M has a pseudohandle which is not a handle, then M has the bounded union property.*

We first prove this proposition for a special case.

PROPOSITION 5. *Suppose M is the closure of the complement of a knotted solid torus T in the 3-sphere S^3. If $g\colon M \to \text{int } M$ is an embedding then either*

(i) *$M - g(\text{int } M)$ is homeomorphic to $(\text{bdry } M) \times I$; or*

(ii) *$g(M)$ is contained in a 3-cell in M.*

Proof. Let $T = \text{Cl}(S^3 - M)$ and $T' = \text{Cl}(S^3 - g(M))$. By Alexander [1], T, T' are solid tori. (For the terminology of this proof, see [22]. A summary of results in [22] appears in [9].) Let t be a core (or center line) of T. There exist two possibilities.

Case 1. t misses some meridial disk D of T'. Hence we may assume that T misses some meridial disk of T'; say D. Let N be a regular neighborhood of $D \cup \text{bdry } T'$ in T' such that $N \cap T = \varnothing$. Then $g(M) \cup N$ is a three cell containing $g(M)$ and lying in M.

Case 2. t meets each meridial disk of T'. If t is also a center line of T', then clearly $T' - \text{int } T$ is homeomorphic to $(\text{bdry } M) \times I$. Suppose t is not a center line of T'. Let t' be a center line of T'. Then t' is a companion knot of t. Hence the genus of t is strictly greater than the genus of t' [22, p. 192]. Therefore $g(M)$ and M are not homeomorphic. This contradiction establishes the proposition.

PROPOSITION 6. *Suppose M is the closure of the complement of a knotted solid torus T in S^3; then M has the bounded union property. If $U=\bigcup U_i$ is a monotone union for M, U is either homeomorphic to an open* 3*-cell or* int M.

Proof. There are two cases to consider.

Case 1. For every positive integer N, suppose that there exists an integer $i=i(N)>N$ such that U_{i-1} is contained in a 3-cell C_i in U_i. Then

$$U = \bigcup U_i = \bigcup_N C_{i(N)}$$

is an open 3-cell.

Case 2. Suppose that there exists an integer N such that for every $i>N$, U_{i-1} is not contained in a 3-cell in U_i. By Proposition 5, it follows that $U-U_N$ is homeomorphic to bdry $M\times(0, 1)$ and hence U is homeomorphic to int M.

PROPOSITION 7. *Let M be a compact* 3*-manifold which has a pseudohandle; say $M=M_1 \# M_2$, where M_2 is the closure of the complement of a polyhedral solid torus in the* 3*-sphere. Let $U=\bigcup U_i$ be a monotone union of M. Then U is homeomorphic to $M_1 \# (\bigcup C_i)$ where $\bigcup C_i$ is a monotone union of M_2.*

Proof. We can write $U_1=V_0\cup V_1$ such that $V_0\cap V_1=S$ is a 2-sphere missing the boundary of U_1 and such that V_0 is homeomorphic to the complement of an open 3-cell in M_1 and V_1 is homeomorphic to the complement of an open 3-cell in M_2. Consider any U_i. Then $U_i=V_0\cup V_i$ where $V_0\cap V_i=S$. From [15], V_i is homeomorphic to the complement of an open 3-cell in M_2. Let B be a 3-cell and let C_i be formed from $V_i\cup B$ by identifying S with the boundary of B. The proposition then follows.

Theorem 4 now follows from Propositions 5, 6, and 7. From Lemma 5 and Theorem 1 of [8], we can prove a converse of Theorem 4.

THEOREM 5. *Let M be a compact* 3*-manifold whose boundary is a torus. If M has the bounded union property but not the monotone union property, then M has a pseudohandle which is not a handle.*

6. The counterexample.

THEOREM 6. *There exists a compact* 3*-manifold M such that the boundary of M is a torus and M has the monotone union property.*

Construction of M. Let P_1, P_2 be solid tori. Let c, a be longitudinal curves of P_1, P_2, respectively, and let d, b be meridian curves of P_1, P_2, respectively, such that $c\cap d$ and $a\cap b$ are single points [22], [7, p. 2]; for example, see Figure 2. $S^2\times S^1$, the product of the 2-sphere with the 1-sphere, can be formed from $P_1\cup P_2$ by identifying their boundaries by a homeomorphism h: bdry $P_2\to$ bdry P_1 such that the induced map $H: \pi_1(\text{bdry } P_2)\to\pi_1(\text{bdry } P_1)$ has the property that $H(a)=c$ and $H(b)=d$ [19]. (We shall denote a homotopy class by one of its representatives if no ambiguity results.)

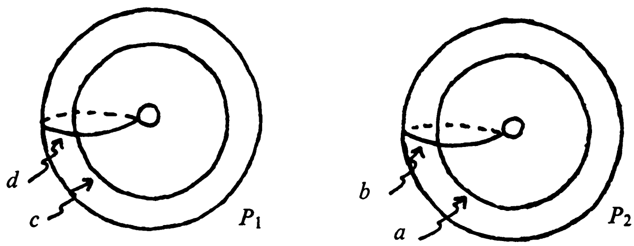

FIGURE 2

Let Γ be the simple closed curve in int P_1 as indicated in Figure 3. Let N be a regular neighborhood of Γ in int P_1 and let $M=(S^2 \times S^1)-\text{int } N$.

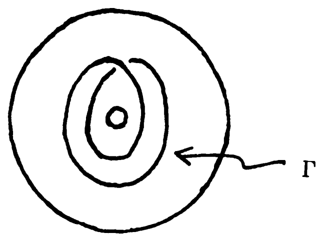

FIGURE 3

LEMMA 1. $\pi_1(M)=\{f, h\colon h\bar{f}\bar{h}\bar{f}, (fh)^2(\bar{f}\bar{h})^2\}$.

Proof. Consider the link α given in Figure 4.
Then

$$\begin{aligned}\pi_1(S^3-\alpha) &= \pi_1(P_1-\Gamma)\\ &= \{e, f, g, h, i\colon h\bar{e}\bar{i}e, i\bar{f}\bar{h}f, g\bar{h}\bar{f}h, f\bar{h}\bar{e}h, fe\bar{f}\bar{g}\}\\ &= \{e, f, h, i\colon h\bar{e}\bar{i}e, i\bar{f}\bar{h}f, f\bar{h}\bar{e}h, fe\bar{f}\bar{h}\bar{f}h\}\\ &= \{e, f, h\colon eh\bar{e}\bar{f}\bar{h}f, f\bar{h}\bar{e}h, fe\bar{f}\bar{h}\bar{f}h\}\\ &= \{f, h\colon (fh)^2(\bar{f}\bar{h})^2\}.\end{aligned}$$

Note that c is homotopic to h and d is homotopic to $\bar{e}\bar{f}=h\bar{f}\bar{h}\bar{f}$. By Van Kampen's Theorem [6, p. 71], we have that

$$\begin{aligned}\pi_1(M) &= \{a, b, f, h\colon b.\,\bar{a}h, \bar{b}h\bar{f}\bar{h}\bar{f}, ab\bar{a}\bar{b}, (fh)^2(\bar{f}\bar{h})^2\}\\ &= \{f, h\colon h\bar{f}\bar{h}\bar{f}, (fh)^2(\bar{f}\bar{h})^2\}.\end{aligned}$$

LEMMA 2. $H_1(M)=Z \oplus Z_2$; i.e., *the first homology group of M is isomorphic to the direct sum of the integers and the integers* mod 2.

Proof. Since $H_1(M)$ is obtained by abelianizing $\pi_1(M)$, we have that

$$H_1(M) = \{f, h\colon f^2, fh\bar{f}\bar{h}\}.$$

LEMMA 3. *Let* i: bdry $M \to M$ *be the inclusion map. Then the induced map* $i_{\#}\colon \pi_1(\text{bdry } M) \to \pi_1(M)$ *is a monomorphism.*

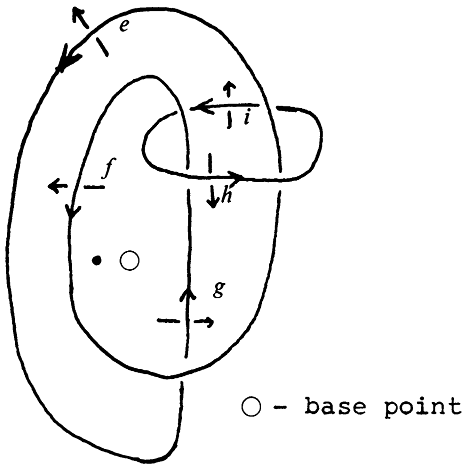

FIGURE 4

Proof. Suppose $i_{\#}$ is not one-one. From the remarks following Theorem 3, $M=M_1 \# M_2$ where M_2 is a solid torus. Therefore $S^2\times S^1=M_1 \# (M_2\cup N)$. By Milnor [15], $M_2\cup N$ is a 3-sphere. For, it follows from Van Kampen's Theorem that M_1 cannot be a 3-sphere. Then $C=(M_2\cup N)\cap(S^2\times S^1)$ is a closed 3-cell in $S^2\times S^1$ containing Γ. But Γ is essential in $S^2\times S^1$, and therefore cannot lie in a 3-cell in $S^2\times S^1$. This contradiction establishes Lemma 3.

LEMMA 4. $i_{\#}\pi_1(\text{bdry } M)=\{f, h^2: h^2\bar{f}\bar{h}^2 f\}$.

Proof. From Figure 4, we obtain the presentation

$$i_{\#}\pi_1(\text{bdry } M) = \{f, hfh: f(hfh)\bar{f}(\bar{h}\bar{f}\bar{h})\}.$$

Note that $hfh=(\bar{f}f)hfh=\bar{f}(fhf)h=\bar{f}(h)h=\bar{f}h^2$ and the conclusion follows.

LEMMA 5. *$i_{\#}\pi_1(\text{bdry } M)$ is a maximal subgroup of $\pi_1(M)$.*

Proof. Note that the subgroup $\{f:\}$ generated by f in $\pi_1(M)$ is normal in $\pi_1(M)$ for

$$hf\bar{h} = (\bar{f}f)hf\bar{h} = \bar{f}(fhf)\bar{h} = \bar{f}h\bar{h} = \bar{f}.$$

Define a homomorphism $\phi: \pi_1(M)\to\{h_1:\}$ by $\phi(h)=h_1$ and $\phi(f)=1$. Then $\phi \mid i_{\#}\pi_1(\text{bdry } M) : i_{\#}\pi_1(\text{bdry } M)\to\{h_1^2:\}$ is an epimorphism.

Note $\{h_1^2:\}$ is maximal in $\{h_1:\}$. Clearly $\{f:\}\subseteq i_{\#}\pi_1(\text{bdry } M)$. It follows from [24, pp. 140–141] that $i_{\#}\pi_1(\text{bdry } M)$ is maximal in $\pi_1(M)$.

Suppose $g: M\to \text{int } M$ is an embedding. Let $R=M-\text{int } g(M)$, $T=g(\text{bdry } M)$.

LEMMA 6. *T bounds a solid torus in $S^2\times S^1$.*

Proof. By Kinoshita [12, p. 791], there is a simple closed curve S on T such that S does not bound a disk on T but bounds a disk D in $S^2\times S^1$ in such a way that $D\cap T=\text{bdry } D=S$. Let N be a regular neighborhood of D in $S^2\times S^1$ which meets

T in a regular neighborhood of S in T. Then $T' = \mathrm{Cl}\,(T-N) \cup \mathrm{Cl}\,(\mathrm{bdry}\, N-T)$ is a 2-sphere which, by Milnor [15], bounds a 3-cell N' in $S^2 \times S^1$. Then $N \cup N'$ is a solid torus which is bounded by T.

LEMMA 7. *The inclusion map $j_T\colon T \to R$ induces a monomorphism*

$$j_{T\#}\colon \pi_1(T) \to \pi_1(R).$$

Proof. Suppose to the contrary that $j_{T\#}$ is not a monomorphism. Then, as in the proof of Theorem 3, there is a 2-cell $D \subseteq R$ such that bdry $D = D \cap T$ is essential in T. Let N' be a regular neighborhood of D in R which meets T in a regular neighborhood of bdry D in T [11, p. 735]. Then

$$\mathrm{Cl}\,(T-N') \cup \mathrm{Cl}\,(\mathrm{bdry}\, N'-T)$$

is a 2-sphere which bounds a 3-cell in $S^2 \times S^1$ containing Γ. This contradiction establishes the lemma.

Proof of Theorem 6. By Van Kampen's Theorem, $\pi_1(M)$ is the free product of $\pi_1(gM) = \pi_1(M)$ and $\pi_1(R)$ with amalgamated subgroup $i_\# \pi_1(\mathrm{bdry}\, M)$. By Lemmas 3 and 7, the natural maps $\pi_1(T) \to \pi_1(gM)$ and $\pi_1(T) \to \pi_1(R)$ are monomorphisms. Hence by [14, p. 199], the inclusion map $j\colon R \to M$ induces a monomorphism $j_\#\colon \Pi_1(R) \to \Pi_1(M)$. Hence we have

$$i_\# \pi_1(\mathrm{bdry}\, M) \subseteq j_\# \pi_1(R) \subseteq \pi_1(M).$$

By Lemma 6, R is embeddable in the 3-sphere and hence $H_1(R)$ has no element of finite order. Hence by Lemma 2, $j_\# \pi_1(R) \neq \pi_1(M)$. Therefore, by Lemma 5, $i_\# \pi_1(\mathrm{bdry}\, M) = j_\# \pi_1(R)$; therefore the inclusion map $i'\colon \mathrm{bdry}\, M \to R$ induces an isomorphism $i'_\#\colon \pi_1(\mathrm{bdry}\, M) \to \pi_1(R)$. Since bdry M and R are aspherical [21], bdry M is a deformation retract of R [4, p. 446]. Hence as argued on [4, p. 446], R is homeomorphic to $(\mathrm{bdry}\, M) \times I$. Hence M has the concentricity property. By Theorem 1, M has the monotone union property.

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