HARMONIC DIFFERENTIAL WITH PRESCRIBED SINGULARITIES

BY
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Introduction

1. Throughout this paper we denote by R an open Riemann surface and by R_0 a relatively compact subdomain of R with the relative boundary ∂R_0 consisting of a finite number of mutually disjoint closed analytic Jordan curves. The open set $R_1 = R - \bar{R}_0$ can be considered to be a neighborhood of the ideal boundary β of R. For the sake of simplicity, we denote by α the common relative boundary $\partial R_0 = \partial R_1$ and we fix the orientation of α positively with respect to the domain R_0.

A harmonic differential σ defined on $\bar{R}_1 = R_1 \cup \alpha$ is called a *harmonic singularity at* β and in case $\int_{R_1} \sigma \wedge {}^*\sigma < \infty$, we say that the singularity σ at β is *removable.* A harmonic differential λ on R is said *to have the harmonic singularity* σ at β if $\lambda - \sigma$ is a removable harmonic singularity at β. The purpose of this paper is to discuss the following

PROBLEM A. *Find a harmonic differential λ on R having a given harmonic singularity σ at β.*

It is known (Ahlfors-Sario [1, p. 300]) that Problem A is solvable if σ and ${}^*\sigma$ are the restrictions to $\bar{R}_1$ of some closed C^1-differentials on R. We shall prove that if $R \notin O_G$, then Problem A is always solvable, and if $R \in O_G$, then Problem A is solvable if and only if $\int_\alpha \sigma = \int_\alpha {}^*\sigma = 0$ (Theorem 2).

2. In Problem A, *we may assume without loss of generality that σ is a C^1-differential on R whose restriction to $\bar{R}_1$ gives a harmonic singularity at β.* In fact, take a subdomain R_σ of R such that $\bar{R}_\sigma \subset R_0$ and σ is harmonic on $R - \bar{R}_\sigma$. We find a C^1-function ϕ on R such that $\phi = 1$ on a neighborhood of $\bar{R}_1$ and $\phi = 0$ on a neighborhood of $\bar{R}_\sigma$. Then $\phi\sigma$ can be considered to be a C^1-differential on R and $\phi\sigma | \bar{R}_1 = \sigma$.

Let $\Gamma = \Gamma(R)$ be the Hilbert space of all square integrable differentials on R which is the completion of square integrable C^∞-differentials on R with respect to the inner product $(\omega_1, \omega_2) = \int \omega_1 \wedge {}^*\omega_2$. We denote by $\Gamma_{e0} = \Gamma_{e0}(R)$ the closure of $\Gamma_{e0}^\infty = \Gamma_{e0}^\infty(R) = \{df\,; f \in C_0^\infty(R)\}$ in Γ, where $C_0^\infty(R)$ is the totality of C^∞-functions on R with compact supports. We also denote by ${}^*\Gamma_{e0} = {}^*\Gamma_{e0}(R) = \{{}^*\omega\,;\, \omega \in \Gamma_{e0}(R)\}$. Then we have the *de Rahm decomposition* of Γ:

$$\Gamma(R) = \Gamma_{e0}(R) \oplus {}^*\Gamma_{e0}(R) \oplus \Gamma_h(R), \tag{1}$$

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where $\Gamma_h = \Gamma_h(R)$ is the totality of harmonic differentials in $\Gamma(R)$. From these remarks, it follows at once that Problem A is equivalent to the following:

PROBLEM B. *Given a C^1-differential σ on R whose restriction to $\bar{R}_1$ is a harmonic singularity at β, find a harmonic differential λ on R such that $\lambda - \sigma \in \Gamma_{e0}(R) \oplus {}^*\Gamma_{e0}(R)$.*

The advantage of this reformulation is that we can now see the precise nature of the solution. That is, we shall show that if the solution λ of Problem B exists, then it is unique and if σ is exact (resp. *exact), then the solution λ is also exact (resp. *exact). A differential is *exact, if, by definition, its *conjugate is exact (Theorem 2).

3. The key to the solution of our problem is the following: let $\mathscr{D}(\alpha)$ be the totality of continuous differentials ω defined on neighborhoods V_ω of α and $\mathscr{D}_0(\alpha)$ the subclass of $\mathscr{D}(\alpha)$ consisting of differentials ω such that $\int_\alpha \omega = 0$. For each $\omega \in \mathscr{D}(\alpha)$, we consider the quantity

$$K(\alpha, \omega) = \sup\left\{\left|\int_\alpha f\omega\right|^2 \Big/ \int_R df \wedge {}^*df \,;\, f \in C_0^\infty(R), f \not\equiv 0\right\}. \tag{2}$$

We shall show that if $R \notin O_G$, then $K(\alpha, \omega) < \infty$ for any $\omega \in \mathscr{D}(\alpha)$, and if $R \in O_G$, then $K(\alpha, \omega) < \infty$ if and only if $\omega \in \mathscr{D}_0(\alpha)$ (Theorem 1).

Fundamental inequality

4. Fix a point z_0 in R_0 and consider the space $HD_0 = HD_0(R_0)$ of HD-functions on R_0 vanishing at z_0. HD_0 is a Hilbert space with respect to the norm $\int_{R_0} du \wedge {}^*du$. For two functions u and v in HD_0, we set

$$(u, v)_g = \int_{R_0} (1 + g(z, z_0))\, du(z) \wedge {}^*dv(z),$$

where g is the Green's function on R_0. HD_0 is a pre-Hilbert space with this inner product, and we denote it by $HD_{0,g}$. Let $\{u_n\}_{n=1}^\infty$ be a Cauchy sequence in $HD_{0,g}$. Since it is also a Cauchy sequence in HD_0, there exists an element $u \in HD_0$ such that $du_n \to du$ in each parametric neighborhood. Then by Fatou's lemma

$$\int_{R_0} (1 + g(z, z_0))\, d(u - u_n) \wedge {}^*d(u - u_n) \leqq \liminf_{m \to \infty} (u_m - u_n, u_m - u_n)_g.$$

Thus we have that $u \in HD_{0,g}$ and

$$\liminf_{n \to \infty} (u - u_n, u - u_n) \leqq \lim_{n,m \to \infty} (u_m - u_n, u_m - u_n) = 0.$$

This shows that $HD_{0,g}$ is a Hilbert space. Clearly $u \to u$ is a continuous isomorphism of $HD_{0,g}$ onto HD_0. By the closed graph theorem, the norm $(u, u)_g$ is equivalent to $\int_{R_0} du \wedge {}^*du$. Hence in particular, there is a constant K_1 such that

$$\int_{R_0} g\, du \wedge {}^*du \leqq K_1 \int_{R_0} du \wedge {}^*du \tag{3}$$

for any u in HD_0 and hence for any u in $HD(R_0)$.

Let $u \in HD_0 \cap C(\bar{R}_0)$. Since $d^*du^2 = 2\, du \wedge {}^*du$, by Green's formula,

$$\int_\alpha u^2\, {}^*dg = 2 \int_{R_0} g\, du \wedge {}^*du. \tag{4}$$

Hence if $\omega \in \mathscr{D}_0(\alpha)$ and $u \in HD(R) \cap C(\bar{R}_0)$, then by (3), (4), and the fact $\int_\alpha u\omega = \int (u-u(z_0))\omega$, we deduce

$$\left|\int_\alpha u\omega\right|^2 \leqq K_2 \int_{R_0} du \wedge {}^*du, \tag{5}$$

where $K_2 = 2K_1 \int_\alpha \Omega^2\, {}^*dg$ with $\Omega\, {}^*dg = \omega$ on α. Hence by using Dirichlet's principle, we obtain the following:

LEMMA 1 [3]. *Let $\omega \in \mathscr{D}_0(\alpha)$. Then there exists a constant C_ω depending only on ω such that for any $f \in C^1(R_0) \cap C(\bar{R}_0)$,*

$$\left|\int_\alpha f\omega\right|^2 \leqq C_\omega \int_{R_0} df \wedge {}^*df. \tag{6}$$

5. Assume that $R \notin O_G$. $R_1 = R - \bar{R}_0$ consists of a finite number of components $R_1^{(1)}, R_1^{(2)}, \ldots, R_1^{(k)}$. Let $\alpha_i = \alpha \cap \partial R_1^{(i)}$. If $R_1^{(i)}$ has positive (resp. null) ideal boundary, then we put $F_i = R_1^{(i)}$ (resp. $F_i = R - (R_1^{(i)})^-$) and orient $\alpha_i = \partial F_i$ positively with respect to F_i. Take the harmonic measure w_i on F_i such that $w_i = 0$ on ∂F_i. Since $R \notin O_G$, at least one of $R_1^{(i)}$ and $R - (R_1^{(i)})^-$ has positive ideal boundary and by our choice of F_i, $w_i > 0$. Take an f in $C_0^\infty(R)$ arbitrary but fixed for the time being. Let F_i' be a subdomain of F_i such that $F_i - (F_i')^-$ is a neighborhood of the ideal boundary of F_i and f vanishes on $F_i - F_i'$ and such that $\partial F_i'$ consists of a finite number of mutually disjoint analytic closed Jordan curves with $\alpha_i \subset \partial F_i'$. We orient $\beta_i = \partial F_i' - \alpha_i$ positively with respect to F_i'. Let u be harmonic in F_i' with boundary values f on $\partial F_i'$. Hence $u = f$ on α_i and $u = 0$ on β_i. By Green's formula

$$\int_{\alpha_i + \beta_i} u^2\, {}^*dw_i - \int_{\alpha_i + \beta_i} w_i\, {}^*du^2 = \int_{F_i'} w_i\, d\, {}^*du^2.$$

Since ${}^*du^2 = 2u{}^*du = 0$ on β_i and $d^*du^2 = 2\, du \wedge {}^*du$, we have

$$\int_{\alpha_i} u^2\, {}^*dw_i = 2 \int_{F_i'} w_i\, du \wedge {}^*du.$$

Hence by noticing $w_i < 1$,

$$\left|\int_{\alpha_i} u\omega\right|^2 \leqq K_\omega^{(i)} \int_{F_i'} du \wedge {}^*du,$$

where $\omega \in \mathscr{D}(\alpha)$ and $K_\omega^{(i)} = 2\int_{\alpha_i} \Omega^2\, {}^*dw_i$ with $\Omega\, {}^*dw_i = \omega$ on α. Thus by $\int_{F_i'} du \wedge {}^*du \leqq \int_{F_i'} df \wedge {}^*df \leqq \int_R df \wedge {}^*df$ and $u = f$ on α_i, we have

$$\left|\int_{\alpha_i} f\omega\right|^2 \leqq K_\omega^{(i)} \int_R df \wedge {}^*df. \tag{7}$$

Let $K_\omega = k \max(K_\omega^{(i)}\ ;\ 1 \leqq i \leqq k)$. Then

$$\left|\int_\alpha f\omega\right| \leqq \sum_{i=1}^{k}\left|\int_{\alpha_i} f\omega\right| \leqq \sqrt{K_\omega \int_R df \wedge {}^*df}.$$

Hence we obtain the following:

LEMMA 2. *Let $R \notin O_G$ and $\omega \in \mathscr{D}(\alpha)$. Then there exists a constant K_ω depending only on ω such that for any $f \in C_0^1(R)$,*

$$\left|\int_\alpha f\omega\right|^2 \leqq K_\omega \int_R df \wedge {}^*df. \tag{8}$$

6. Assume that $R \in O_G$. Let $\omega \in \mathscr{D}(\alpha)$ and assume that there exists a constant C_ω depending only on ω such that

$$\left|\int_\alpha f\omega\right|^2 \leqq C_\omega \int_R df \wedge {}^*df \tag{9}$$

for any $f \in C_0^\infty(R)$. Let Ω be a subdomain of R such that $\Omega \supset \bar{R}_0$ and $\partial\Omega$ consists of a finite number of mutually disjoint analytic closed Jordan curves. Let w_Ω be the continuous function on R such that $w_\Omega = 1$ on $\bar{R}_0$, $w_\Omega = 0$ on $R - \Omega$ and w_Ω is harmonic on $\Omega - \bar{R}_0$. By applying the mollifier, we can find a sequence $\{f_n\}$ in $C_0^\infty(R)$ such that $\int_R d(w_\Omega - f_n) \wedge {}^*d(w_\Omega - f_n) \to 0$ $(n \to \infty)$ and f_n converges uniformly to w_Ω on R. Then from

$$\left|\int_\alpha f_n\omega\right|^2 \leqq C_\omega \int_R df_n \wedge {}^*df_n;$$

$$\left|\int_\alpha (f_n - 1)\omega\right| \leqq \left(\int_\alpha |\omega|\right) \sup |f_n - w_\Omega|$$

it follows that

$$\left|\int_\alpha \omega\right|^2 \leqq C_\omega \int_{\Omega - \bar{R}_0} dw_\Omega \wedge {}^*dw_\Omega.$$

Since $R \in O_G$, $\int_{\Omega - \bar{R}_0} dw_\Omega \wedge {}^*dw_\Omega \to 0$ as $\Omega \nearrow R$. Thus $\int_\alpha \omega = 0$. Hence this with Lemma 1 gives the following:

LEMMA 3. *Let $R \in O_G$ and $\omega \in \mathscr{D}(\alpha)$. In order that there exists a constant C_ω depending only on ω such that for any $f \in C_0^1(R)$,*

$$\left|\int_\alpha f\omega\right|^2 \leqq C_\omega \int_R df \wedge {}^*df,$$

it is necessary and sufficient that $\omega \in \mathscr{D}_0(\alpha)$.

7. Lemmas 2 and 3 complete the proof of the fact mentioned in 3:

THEOREM 1. *If $R \notin O_G$, then $K(\alpha, \omega) < \infty$ for any $\omega \in \mathscr{D}(\alpha)$, while if $R \in O_G$, then $K(\alpha, \omega) < \infty$ if and only if $\omega \in \mathscr{D}_0(\alpha)$.*

Existence theorem

8. THEOREM 2. *Let σ be a C^1-differential on R such that $\sigma|\bar{R}_1$ gives a harmonic singularity at the ideal boundary β of R. If $R \notin O_G$, then there exists a harmonic differential λ on R such that $\lambda - \sigma \in \Gamma_{e0}(R) \oplus {}^*\Gamma_{e0}(R)$, and if $R \in O_G$, then there exists a harmonic differential λ on R such that $\lambda - \sigma \in \Gamma_{e0}(R) \oplus {}^*\Gamma_{e0}(R)$ if and only if*

$$\int_\alpha \sigma = \int_\alpha {}^*\sigma = 0. \tag{10}$$

*In either case λ is uniquely determined by σ and λ is exact (resp. *exact) if σ is exact (resp. *exact).*

9. First, we prove the existence part. Let σ be arbitrary if $R \notin O_G$ and let σ satisfy (10) if $R \in O_G$. Let $f \in C_0^\infty(R)$. By Green's formula,

$$\int_{R_1} \sigma \wedge {}^*df = \int_{R_1} df \wedge {}^*\sigma = \int_\alpha f{}^*\sigma$$

and

$$\int_{R_1} \sigma \wedge df = -\int_{R_1} df \wedge \sigma = -\int_\alpha f\sigma.$$

Hence for any f_1 and f_2 in $C_0^\infty(R)$, by Theorem 1 and $(df_1, {}^*df_2) = 0$,

$$\left|\int_{R_1} \sigma \wedge {}^*(df_1 + {}^*df_2)\right|^2 \leqq T'_\sigma \int (df_1 + {}^*df_2) \wedge {}^*(df_1 + {}^*df_2),$$

where T'_σ is a constant depending only on σ. Thus the functional

$$T(\theta) = -\int_R \sigma \wedge {}^*\theta$$

defined on $\Gamma_{e0}^\infty(R) \oplus {}^*\Gamma_{e0}^\infty(R)$ satisfies

$$|T(\theta)|^2 \leqq T_\sigma \int_R \theta \wedge {}^*\theta,$$

where $T_\sigma = \int_{R_0} \sigma \wedge {}^*\sigma + T'_\sigma$. Thus T can be extended to a bounded linear functional on

$$\Gamma_{e0}(R) \oplus {}^*\Gamma_{e0}(R) = \overline{\Gamma_{e0}^\infty(R) \oplus^* \Gamma_{e0}^\infty(R)}.$$

Hence there exists a unique element ω in $\Gamma_{e0}(R) \oplus {}^*\Gamma_{e0}(R)$ such that $T(\theta) = \int \omega \wedge {}^*\theta$. Thus in particular,

$$\int_R (\sigma + \omega) \wedge {}^*df = \int_R (\sigma + \omega) \wedge df = 0$$

for any f in $C_0^\infty(R)$. Take a compact subdomain Ω in R; then clearly $\sigma + \omega \in \Gamma(\Omega)$. Taking f in $C_0^\infty(\Omega)$, the above equality shows that $\sigma + \omega \in (\Gamma_{e0}^\infty(\Omega) \oplus {}^*\Gamma_{e0}^\infty(\Omega))^\perp$. By the de Rahm decomposition, $\Gamma_h(\Omega) = \Gamma(\Omega) \ominus (\Gamma_{e0}^\infty(\Omega) \oplus {}^*\Gamma_{e0}(\Omega))$, we conclude that $\sigma+\omega \in \Gamma_h(\Omega)$. Thus $\lambda=\sigma+\omega$ is a harmonic differential on R and $\lambda-\sigma=\omega \in \Gamma_{e0}(R) \oplus {}^*\Gamma_{e0}(R)$.

10. Next assume that $R \in O_G$ and we can find a harmonic differential λ on R such that $\lambda-\sigma=\omega \in \Gamma_{e0}(R) \oplus {}^*\Gamma_{e0}(R)$ for a given σ. We have to show that (10) holds. Clearly, for any $f \in C_0^\infty(R)$,

$$\left|\int_\alpha f\omega\right|^2 = \left|\int_{R_1} df \wedge \omega\right|^2 \leqq \left(\int_R \omega \wedge {}^*\omega\right)\cdot \int_R df \wedge {}^*df$$

and similarly

$$\left|\int_\alpha f{}^*\omega\right|^2 \leqq \left(\int_R \omega \wedge {}^*\omega\right)\cdot \int_R df \wedge {}^*df.$$

Hence by Theorem 1, we must have $\int_\alpha \omega=\int_\alpha {}^*\omega=0$. On the other hand, $\int_\alpha \lambda=\int_{R_0} d\lambda=0$ and $\int_\alpha {}^*\lambda=\int_{R_0} d{}^*\lambda=0$. Thus (10) follows.

11. Finally, we prove the last part of Theorem 2. Let λ_1 and λ_2 be harmonic differential on R such that $\lambda_1-\sigma$ and $\lambda_2-\sigma$ belong to $\Gamma_{e0}(R) \oplus {}^*(\Gamma_{e0}(R))$. Then $\lambda_1-\lambda_2 \in \Gamma_{e0}(R) \oplus {}^*\Gamma_{e0}(R)$. But $\lambda_1-\lambda_2$ is harmonic and thus $\lambda_1-\lambda_2=0$ by the de Rahm decomposition.

Assume that σ is exact (resp. *exact). Then for $\theta \in {}^*\Gamma_{e0}^\infty(R)$ (resp. $\Gamma_{e0}^\infty(R)$), $\int_R \lambda \wedge {}^*\theta=\int_R \sigma \wedge {}^*\theta=0$. This implies $\int_R \omega \wedge {}^*\theta=0$. Thus $\omega \in (\Gamma_{e0}(R) \oplus {}^*\Gamma_{e0}(R)) \ominus {}^*\Gamma_{e0}^\infty(R)=\Gamma_{e0}(R)$ (resp. $\omega \in (\Gamma_{e0}(R) \oplus {}^*\Gamma_{e0}(R)) \ominus \Gamma_{e0}^\infty(R)={}^*\Gamma_{e0}(R)$).

12. *Application to the case of functions.* Let s be a harmonic function on $\bar{R}_1$, arbitrary if $R \notin O_G$, and $\int_\alpha {}^*ds=0$ if $R \in O_G$. Thus by Theorem 2, there exists a harmonic function p on R such that $d(p-s) \in \Gamma_{e0}(R)$. This means that $p-s$ is bounded on R. Thus we constructed a harmonic function p on R which behaves like s at β. If $\int_\alpha {}^*ds=0$, then $\int_\alpha {}^*d(p-s)=0$ and $d(p-s) \in \Gamma_{e0}(R)$. This implies $L_1(p-s)=p-s$ on R_1 (see [3]), where L_1 is Sario's principal operator for R_1. Such an approach to the principal function problem was initiated by Browder [2].

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References

1. L. V. Ahlfors and L. Sario, *Riemann surfaces*, Princeton Math. Series, No. 26, Princeton, N. J., 1960.

2. F. E. Browder, *Principal functions for elliptic systems of differential equations*, Bull. Amer. Math. Soc. **71** (1965), 342–344.

3. M. Nakai and L. Sario, *Construction of principal functions by orthogonal projections*, Canad. J. Math. **18** (1966), 887-896.

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