THE STRUCTURE AND IDEAL THEORY OF THE PREDUAL OF A BANACH LATTICE

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1. **Introduction.** If V is an ordered Banach space over the real field, then the Banach dual V^* has a natural induced partial ordering. In Theorems 3.1 and 3.2 we present necessary and sufficient conditions for V^* to be a Banach lattice, extending partial results obtained by Choquet [5] and Andô [1]. The theorems include as special cases the characterisations of the predual of an M-space, [8], [14], and the predual of an L-space, [6], called a simplex space in [11]. We show how the theory is a natural generalisation of Choquet simplex theory.

If V^* is a Banach lattice, we study the set of closed ideals of V. For the special case of simplex spaces, the results provide direct proofs of theorems in [11].

2. **Basic theorems on ordered Banach spaces.** An *ordered normed space* V over the real field is defined as a normed vector space V with a closed cone C which is *proper* in the sense that $C \cap (-C) = \{0\}$ and with the partial ordering given by saying that $x \leqq y$ if and only if $y - x \in C$. V is said to be *positively generated* if $C - C = V$ and to have a *monotone norm* if $0 \leqq x \leqq y$ implies $\|x\| \leqq \|y\|$. V is said to be *regular* if it has the properties

(i) if $x, y \in V$ and $-x \leqq y \leqq x$ then $\|y\| \leqq \|x\|$;

(ii) if $x \in V$ and $\varepsilon > 0$ then there is some $y \in V$ with $y \geqq x, -x$ and $\|y\| < \|x\| + \varepsilon$.

A *normed lattice* is a regular ordered normed space which is a lattice under its partial ordering, and a *Banach lattice* is a complete normed lattice. Banach lattices are discussed in many places, for example [7], [12], and we follow the standard terminology. A partially ordered vector space V is said to have the *Riesz separation property* if when $a, b \leqq c, d \in V$ we can find some $x \in V$ with

$$a, b \leqq x \leqq c, d.$$

For alternative formulations of this condition see [1], [9].

LEMMA 2.1. *Let V be an ordered positively generated Banach space with a monotone norm $\| \ \|$, and define*

$$\|x\|_1 = \inf\{\|y\| : x, -x \leqq y \in V\}$$

for all $x \in V$. Then $\| \ \|_1$ is a regular norm on V which is equivalent to $\| \ \|$, and for any $0 \leqq x \in V$ we have $\|x\|_1 = \|x\|$.

For a proof of this see [16].

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Let V be a positively generated ordered Banach space with Banach dual V^*. The set C^* of continuous positive functionals is a weak*-closed proper cone in V^*. By the Hahn-Banach theorem if $x \in V$ then $x \geqq 0$ if and only if $(x, \phi) \geqq 0$ for all $\phi \in C^*$. If V is a positively generated ordered Banach space with a monotone norm then, by [16], every positive functional on V is continuous; also if $0 \leqq x \in V$ then

$$\|x\| = \max \{(x, \phi) : \phi \in C^* \cdot \|\phi\| \leqq 1\}.$$

This follows from the fact that $-x$ is not in the open convex set

$$\{v \in V : \exists c \in C \cdot \|v-c\| < \|x\|\}$$

by the use of the Hahn-Banach theorem. If V is a regular ordered Banach space then for any $0 \leqq \phi \in V^*$ we have

$$\|\phi\| = \sup \{(x, \phi) : x \in C \cdot \|x\| \leqq 1\}.$$

LEMMA 2.2. *If V is a regular ordered normed space then for any ϕ, $\psi \in V^*$ with $-\phi \leqq \psi \leqq \phi$ we have $\|\phi\| \geqq \|\psi\|$.*

For let $x \in V$ and $\|x\| < 1$. If $z \geqq x$, 0 then

$$\begin{aligned}(\psi, x) &= (\psi, z)+(\psi, x-z)\\ &\leqq (\phi, z)+(\phi, z-x)\\ &= (\phi, 2z-x).\end{aligned}$$

If $y \in V$ satisfies $y \geqq x$, $-x$ and $\|y\| < 1$ and we put $z=(x+y)/2$ then we obtain

$$(\psi, x) \leqq \|\phi\| \; \|y\| < \|\phi\|.$$

The result now follows immediately.

Very much more information can be obtained about V^*, but we first need to consider a special case.

LEMMA 2.3. *Let V be a regular ordered normed space with the Riesz separation property. Then V^* is a Banach lattice.*

Let $\phi \in V^*$ and $x \in V$, $x \geqq 0$. Then we define

$$(\phi^+, x) = \sup \{(\phi, y) : 0 \leqq y \leqq x\}.$$

It is easy to see that ϕ^+ can be uniquely extended to a positive linear functional on V, and that ϕ^+ is continuous with $\|\phi^+\| \leqq \|\phi\|$. Clearly, V^* is a lattice with $\phi \vee 0 = \phi^+$. Define $|\phi| = 2\phi^+ - \phi$. For any $y \in V$, $y \geqq 0$ we can find some $a \in V$ with $0 \leqq a \leqq y$ and

$$(\phi, a) \leqq (\phi^+, y) \leqq (\phi, a)+\varepsilon/2,$$

which gives

$$|(|\phi|, y)-(\phi, 2a-y)| < \varepsilon.$$

As $-y \leqq 2a-y \leqq y$ so

$$|(\phi, 2a-y)| \leqq \|\phi\| \; \|y\|$$

from which we conclude that $\| \, |\phi| \, \| \leqq \|\phi\|$. The result now follows immediately.

Let V be an ordered positively generated Banach space and define:

$$\Delta = C^* \cap \{\phi \in V^* : \|\phi\| \leqq 1\}.$$

Δ is a weak*-compact convex set with $0 \in \Delta$ and there is a natural map $\lambda: V \to A_0(\Delta)$, the space of all continuous affine functionals ϕ on Δ such that $\phi(0)=0$. By our previous remarks λ is an order isomorphism of V into $A_0(\Delta)$, and if $\| \quad \|_u$ denotes the supremum norm in $A_0(\Delta)$, we have for all $f \in V$

$$\|\lambda f\|_u \leqq \|f\|.$$

Identifying V with λV we define S as the cone of functions of the form $f_1 \vee \cdots \vee f_n$ where $f_r \in V$. Then if $L=S-S$, L is a vector lattice of continuous functions on Δ. See [5]. For $\phi \in L$ define

$$\|\phi\| = \inf\{\|\psi\| : \psi \in V \cdot \psi \geqq \phi, -\phi\}.$$

Then, provided V is positively generated and has a monotone norm, L becomes a normed lattice and V is canonically embedded as a Banach subspace of L.

LEMMA 2.4. *If V is a regular ordered Banach space then V^* is also regular.*

Let $\phi \in V^*$ and $\|\phi\| \leqq 1$. We can, by the Hahn-Banach theorem, extend ϕ to a functional $\xi \in L^*$ with $\|\xi\| \leqq 1$. As L^* is a Banach lattice so by defining ψ as the restriction to V of $|\xi|$, we see that $\psi \geqq \phi$, $-\phi$ and $\|\psi\| \leqq 1$.

LEMMA 2.5. *If V is an ordered Banach space such that V^* is regular then the map $\lambda: V \to A_0(\Delta)$ is a one-one onto order isomorphism such that for all $f \in V$*

$$\|\lambda f\|_u \leqq \|f\| \leqq 2\|\lambda f\|_u,$$

and for all $0 \leqq f \in V$,

$$\|\lambda f\|_u = \|f\|.$$

That $\|\lambda f\|_u \leqq \|f\|$ is clear. Suppose $f \in V$ and $|(\phi, f)| = \|f\|$ where $\|\phi\|=1$. Then for any $\varepsilon > 0$ we find $\psi \in V^*$ with $\psi \geqq \phi$, $-\phi$, $\|\psi\| < 1+\varepsilon$.

$$|(\phi,f)| = |((\psi+\phi)/2, f) - ((\psi-\phi)/2, f)|$$

so that

$$\max |((\psi \pm \phi)/2, f)| \geqq \tfrac{1}{2}\|f\|.$$

As $(\psi \pm \phi)/2 \geqq 0$ and $\|(\psi \pm \phi)/2\| < 1+\varepsilon$ so

$$\|f\| \leqq 2\|\lambda f\|_u.$$

If $f \geqq 0$ then $((\psi \pm \phi)/2, f) \geqq 0$ so that

$$|(\phi,f)| \leqq \max |((\psi \pm \phi)/2, f)|,$$

from which we see that

$$\|f\| \leqq \|\lambda f\|_u.$$

We see that V maps onto a uniformly closed subspace of $A_0(\Delta)$ which separates the points of Δ, and so must conclude by Lemma 4.3 of [13] that

$$\lambda V = A_0(\Delta).$$

A partial converse to Lemma 2.4 is contained in the following

LEMMA 2.6. *Let V be an ordered Banach space such that V^* has a proper positive dual cone C^* and is regular. Then V is positively generated and for any $x, y \in V$ with $-x \leqq y \leqq x$ we have $\|y\| \leqq \|x\|$.*

For let $\psi \in V^*$ and $\|\psi\| < 1$. Let $\phi \in V^*$ satisfy $\phi \geqq \psi$, $-\psi$ and $\|\phi\| < 1$. Then

$$\begin{aligned}(\psi, y) &= ((\psi+\phi)/2, y)-((\phi-\psi)/2, y)\\ &\leqq ((\psi+\phi)/2, x)+((\phi-\psi)/2, x)\\ &= (\phi, x) < \|x\|.\end{aligned}$$

Thus $\|y\| \leqq \|x\|$. That V is positively generated follows from [1] upon observing that V^* is σ-(0) complete.

We continue to suppose that V is an ordered Banach space and that V^* is regular. Following [5], we now define a *conical measure* μ as a positive functional in L^*. Such a functional defines by restriction to V a point of C^* called the *barycentre* of μ. The set P of conical measures μ with $\|\mu\| \leqq 1$ is compact in the $\sigma(L^*, L)$ topology. We define a closed partial ordering, $\prec$, on P by

$$\mu \prec \nu \quad \text{if} \quad (\mu, f) \leqq (\nu, f) \qquad \text{for all } f \in S.$$

We say $\mu \in P$ is a *representing conical measure* for $x \in \Delta$ if $f(x) \leqq (\mu, f)$ for all $f \in S$, and observe that every $x \in \Delta$ has at least one maximal representing conical measure.

If $x_1, \ldots, x_n \in \Delta$ and $x_1 + \cdots + x_n = x \in \Delta$ then the functional

$$f \rightsquigarrow f(x_1) + \cdots + f(x_n)$$

is called a *discrete* conical measure, and is obviously a representing conical measure for x. We show that such conical measures are dense in P in a very good sense.

LEMMA 2.7. *Suppose $\mu \in P$ is a representing conical measure for $x \in \Delta$ and that $f_r \in S$ for $r = 1, \ldots, n$. Then there is a discrete representing conical measure ν for x such that $\mu(f_r) = \nu(f_r)$ for $r = 1, \ldots, n$.*

We can write the weak*-closed cone $C^* \subseteq V^*$ as a union of closed cones $D_1, \ldots, D_m$ such that $f_r | D_s$ are equal to the restrictions of functions in $A_0(\Delta)$. If $f \in L$ and $\geqq 0$, we define

$$(\mu_{D_s}, f) = \inf\{(\mu, g) : 0 \leqq g \in L \cdot g | D_s > f | D_s\}.$$

Then $\mu_{D_s} \in P$ and $\mu \leqq \mu_{D_1} + \cdots + \mu_{D_m}$. Now we choose $\mu_s \in P$ such that $0 \leqq \mu_s \leqq \mu_{D_s}$ and $\mu = \mu_1 + \cdots + \mu_m$. If $f \in L$ and $f | D_s \geqq 0$ then

$$(f, \mu_s) \geqq (f \wedge 0, \mu_s) \geqq (f \wedge 0, \mu_{D_s}) = 0.$$

Thus μ_s is determined by the value of f on D_s alone. The barycentre x_s of μ_s is in D_s. Finally for $r=1,\ldots,n$

$$(f_r, \mu) = \sum_{s=1}^{m} (f_r, \mu_s) = \sum_{s=1}^{m} (f_r(x_s)).$$

LEMMA 2.8. *Suppose* $f_r \in V$ *and* $f_1 \vee \cdots \vee f_n = f \in S$. *Then*

$$\max\{(f, \mu) : \varepsilon_x < \mu\} = \max\left\{\sum_{r=1}^{n} f(x_r) : x_r \in \Delta. \sum x_r = x\right\}$$

and this quantity is called $\hat{f}(x)$. *The map* $x \rightarrow \hat{f}(x)$ *is upper semicontinuous* (*u.s.c.*).

The set

$$\left\{\{x_r \in \Delta\}_{r=1}^{n} : \sum_{r=1}^{n} x_r = x \in \Delta\right\}$$

is a closed subset of Δ^n and so by the continuity of the f_r we see that the supremum of the right-hand side is always attained. The two sides are equal by Lemma 2.7. That $\hat{f}$ is upper semicontinuous follows from the equation

$$\{x \in \Delta : \hat{f}(x) \geqq \alpha\} = \left\{x = \sum_{r=1}^{n} x_r : \sum_{r=1}^{n} f_r(x_r) \geqq \alpha\right\}.$$

The following extension of the above lemma to semicontinuous functions will be important in our later discussion of closed ideals.

LEMMA 2.9. *Let* $f_r : \Delta \rightarrow [-\infty, \infty)$ *be upper semicontinuous affine functionals with* $f_r(0)=0$. *Then for* $f=f_1 \vee \cdots \vee f_n$ *and* $x \in \Delta$ *the supremum*

$$\sup\left\{\sum_{r=1}^{n} f(x_r) : \sum_{r=1}^{n} x_r = x. \ x_r \in \Delta\right\}$$

is always attained and is denoted $\hat{f}(x)$. $\hat{f}$ *is an upper semicontinuous function with* $\hat{f}(0)=0$ *and* $\hat{f}(\alpha x)=\alpha\hat{f}(x)$ *for all* $x \in \Delta$.

The proof is as for Lemma 2.8.

3. **The main theorems.**

THEOREM 3.1. *Let V be an ordered Banach space, such that V^* is regular. Then the following statements are equivalent*:

(i) *V is regular and has the Riesz separation property*;

(ii) *V^* is a lattice*;

(iii) *the map* $f \frown\!\!\rightarrow \hat{f}(x)$ *is linear for any* $x \in \Delta$;

(iv) *every* $x \in \Delta$ *has a unique maximal representing conical measure*;

(v) *the map* $\frown\!\!\rightarrow \hat{f}(x)$ *is linear for any* $f \in S$.

As an immediate corollary we have

THEOREM 3.2. *Let V be an ordered Banach space. Then V^* is a Banach lattice if and only if V is regular and has the Riesz separation property.*

We now prove Theorem 3.1. The implication (i) $\to$ (ii) was shown in Lemma 2.3. The implications (ii) $\to$ (iii) $\to$ (iv) $\to$ (v) can be found in [5]. The equivalence (i) $\leftrightarrow$ (ii) is closely related to a Theorem in [1], but is more specific. We now show (v) $\to$ (i).

Suppose $f, g, h \in V$. Then $(f\vee g)^\wedge$ is u.s.c. affine and so we can define $((f\vee g)^\wedge \vee h)^\wedge$ as in Lemma 2.9. It is easy to show that

$$((f \vee g)^\wedge \vee h)^\wedge = (f \vee g \vee h)^\wedge$$

and so is u.s.c. affine.

Next suppose that f, g are u.s.c. affine on Δ and $f, g \leqq h$ where h is affine. Then $(f\vee g)^\wedge \leqq h$ without any continuity assumptions for h.

For $f\in -S$ we define $f^\vee$ by $f^\vee = -(-f)^\wedge$. Now let $f_1, f_2, g_1, g_2 \in V$ and $f_1 \vee f_2 \leqq g_1 \wedge g_2$. Then if $f=(f_1\vee f_2)^\wedge$ and $g=(g_1\wedge g_2)^\vee$ we have $f\leqq g$. We construct a sequence $h_n \in A_0(\Delta)\equiv V$ such that

$$f-1/2^n < h_n < g+1/2^n.$$

Suppose h_n is given. Let $z\in\Delta$ and $z=x+y$ where $x, y\in\Delta$ and

$$(f\vee h_n)^\wedge(z) = f(x)+h_n(y) < (h_n(z)+1/2^n) \wedge (g(z)+1/2^n).$$

Now $(f\vee h_n)^\wedge$ is u.s.c. affine and so by simple convexity arguments to be found in [3], we can find $k_n \in A(\Delta)$ with

$$(f \vee h_n)^\wedge-1/2^{n+1} < k_n < h_n \wedge g+1/2^{n+1},$$

and also satisfying $k_n(0)<0$.

Similarly we can find $1_n \in A(\Delta)$ with

$$f \vee h_n-1/2^{n+1} < 1_n < (h_n \wedge g)^\vee+1/2^{n+1},$$

and also satisfying $h_n(0)>0$.

Putting $h_{n+1}=\lambda k_n+(1-\lambda)1_n$ for a suitable $0<\lambda<1$ we obtain

$$(f \vee h_n)-1/2^{n+1} < h_{n+1} < (h_n \wedge g)+1/2^{n+1},$$

and $h_{n+1}(0)=0$.

Then

$$\|h_n-h_{n+1}\|_u < 1/2^{n+1},$$

so $h_n \to h\in A_0(\Delta)$. It is then clear that

$$f_1 \vee f_2 \leqq f \leqq h \leqq g \leqq g_1 \wedge g_2,$$

so that V has the Riesz separation property.

We now show that V is regular. Let $f \in A_0(\Delta)$ and $z \in \Delta$. Then we can find x, $y \in \Delta$ with $x+y=z$ and satisfying

$$\begin{aligned}\{f \vee (-f)\}^\wedge(z) &= f(x)-f(y)\\ &= f(x-y).\end{aligned}$$

Now $-z \leqq x-y \leqq z$ so by the regularity of V^*

$$\{f \vee (-f)\}^\wedge(z) \leqq \|f\|.$$

Now suppose $\varepsilon > 0$. We define $h_1 \in A_0(\Delta)$ so that

$$\{f \vee (-f)\}^\wedge - \varepsilon/2 < h_1 < \|f\| + \varepsilon/2.$$

Inductively we construct $h_n \in A_0(\Delta)$ with

$$\{f \vee (-f)\}^\wedge - \varepsilon/2^n < h_n,$$

and more exactly with

$$\{f \vee (-f)\}^\wedge \vee h_n - \varepsilon/2^{n+1} < h_{n+1} < h_n + \varepsilon/2^{n+1}.$$

This can be done by the same procedure as above. As before $h_n \to h \in A_0(\Delta)$ and $f \vee (-f) \leqq h$. Thus $0 \leqq h \in V$ and

$$\|h\| = \|h\|_u \leqq \sum_{n=1}^{\infty} \|h_{n+1} - h_n\|_u + \|h_1\|_u < \|f\| + \frac{\varepsilon}{2} + \sum_{n=2}^{\infty} \frac{\varepsilon}{2^n} = \|f\| + \varepsilon.$$

Together with Lemma 2.6 this proves the regularity of V.

4. **Some special cases.** We define an *R-space* V as a regular ordered Banach space with the Riesz separation property. If V is an ordered Banach space we say it is of *type M* if for any $x, y \geqq 0$ we can find $z \geqq x, y$ with

$$\|z\| \leqq \max\{\|x\|, \|y\|\}$$

and of *type L* if for any $x, y \geqq 0$, we have

$$\|x+y\| = \|x\| + \|y\|.$$

LEMMA 4.1. *If V is an R-space of type L then it is a Banach lattice.*

See [2].

LEMMA 4.2. *If V is an R-space of type M then V^* is a Banach lattice of type L.*

The proof of this is trivial.

We can now give an immediate proof of a theorem of Dixmier and Kakutani [8], [14].

THEOREM 4.3. *Let V be an ordered Banach space. Then V is an L-type Banach lattice if and only if V^* is an M-type Banach lattice.*

If V is an L-type Banach lattice, it is an easy corollary of Lemma 2.3 that V^* is an M-type Banach lattice. If V^* is an M-type Banach lattice then V is an R-space and V^{**} is an L-type Banach lattice by Lemma 4.2. As V is canonically embedded in V^{**}, so V is of type L and is an L-type Banach lattice.

THEOREM 4.4. *Let V be an ordered Banach space. Then V is an R-space of type M if and only if V^* is a Banach lattice of type L.*

For the proof see [6]. Such spaces V are referred to as *simplex spaces* in [11].

We show how this theory is related to Choquet boundary theory, as expounded in [3], [9] and [17]. Let V be an ordered Banach space with a distinguished order unit e such that for all $x \in V$ we have

$$\|x\| = \inf\{\alpha : -\alpha e \leqq x \leqq \alpha e\}.$$

Then V is regular and if

$$\Omega = \{\phi \in V^* : \phi \geqq 0 \cdot \phi(e) = 1\},$$

then Ω is a weak*-compact base for the cone C^* in the sense of [10]. V is isometrically and order isomorphic with $A_0(\Delta)$ and with $A(\Omega)$, the space of continuous affine functionals on Ω. The conical measures can be identified with the regular Borel measures on Ω. Ω is called a *simplex* [4] if C^* is lattice-ordered, and it is shown in [9], [15] that this occurs if and only if $A(\Omega)$ has the Riesz separation property.

5. **The ideals in an R-space.** If V is an ordered Banach space, *an order ideal I* in V is a subspace such that if $0 \leqq x \leqq y \in I$, then $x \in I$. An ideal is defined as a positively generated order ideal. We now investigate the properties of the closed ideals of an R-space. These generalise the results on the closed ideals of a simplex space in [11], and provide direct proofs for those theorems.

LEMMA 5.1. *Let I be a closed ideal in an R-space V. Then with the restriction norm and ordering I is an R-space.*

The only part not immediate is the second half of the regularity condition. Let $x \in I$ and let $z \in I$, $z \geqq x, -x$. Let $y \in V$ with $y \geqq x, -x$ and $\|y\| < \|x\| + \varepsilon$. Then by the Riesz separation property there is some $\omega \in V$ with $x, -x \leqq \omega \leqq y, z$. As $0 \leqq \omega \leqq z \in I$ so $\omega \in I$ and as $0 \leqq \omega \leqq y$ so $\|\omega\| \leqq \|y\| < \|x\| + \varepsilon$. Thus I is indeed regular.

LEMMA 5.2. *Let I, J be closed ideals in an R-space V. Then $I \cap J$ is a closed ideal.*

We need only verify that $I \cap J$ is positively generated. Let $x \in I \cap J$ and let $y \in I$, $z \in J$ be such that $x, -x \leqq y$ and $x, -x \leqq z$. Then we can find $\omega \in V$ such that $x, -x \leqq \omega \leqq y, z$. As $0 \leqq \omega \leqq y$ so $\omega \in I$ and as $0 \leqq \omega \leqq z$ so $\omega \in J$. This proves the lemma.

THEOREM 5.3. *Let I, J be closed ideals in an R-space V. Then $I+J$ is a closed ideal.*

We first show that $I+J$ is positively generated. Let $x=i+j$ where $i \in I$ and $j \in J$. Then let $i_1 \in I$ and $j_1 \in J$ be such that $i, -i \leqq i_1$ and $j, -j \leqq j_1$. If $x_1 = i_1 + j_1$ then $x_1 \geqq 0$ and $x, -x \leqq x_1$.

If in particular in the above $x \geqq 0$ then by the Riesz separation property as $0 \leqq x \leqq i_1 + j_1$ we can find $0 \leqq i_2 \leqq i_1 \in I$ and $0 \leqq j_2 \leqq j_1 \in I$ such that $x = i_2 + j_2$. Thus the positive cone of $I+J$ is the sum of the positive cones of I and J. It is now immediate that $I+J$ is an order ideal. As in Theorem 5.1 we see that it is regular.

We now have only to show that $I+J$ is a closed subspace of V. Suppose $\varepsilon > 0$ is given. Let $z \in I+J$ and let $z, -z \leqq i+j$ where $0 \leqq i \in I$, $0 \leqq j \in J$ and $\|i+j\| < \|z\| + \varepsilon$. Now observe that $z-j, -i \leqq i, z+j$ so by the Riesz separation property if we let $x \in V$ satisfy

$$z-j,\ -i \leqq x \leqq i,\ z+j$$

and put $y = z - x$ we have

$$-i \leqq x \leqq i \quad \text{and} \quad -j \leqq y \leqq j$$

so that $x \in I$ and $y \in J$. We also have

$$\|x\| \leqq \|i\| \leqq \|i+j\| < \|z\| + \varepsilon \quad \text{and} \quad \|y\| \leqq \|j\| \leqq \|j+i\| < \|z\| + \varepsilon.$$

Now let $z_n \in I+J$ and $\sum \|z_n\| < \infty$. Let $z_n = x_n + y_n$ where $x_n \in I$, $\|x_n\| < 2\|z_n\|$ and $y_n \in J$, $\|y_n\| < 2\|z_n\|$. Then we see that $\sum x_n \to x \in I$ and $\sum y_n \to y \in J$, so that $\sum z_n \to x + y \in I+J$. This proves that $I+J$ is closed.

THEOREM 5.4. *Let I be a closed ideal in an R-space V and let V/I be given the quotient norm and the positive cone which is the image of the positive cone of V. Then V/I is an R-space.*

Let $\tilde{C}$ be the positive cone in V/I so that $\tilde{C} = \pi C$ where $\pi\colon V \to V/I$ is the quotient map. Then $\tilde{C} \cap (-\tilde{C}) = \{0\}$. For let $x, y \in C$ and $\pi x = -\pi y$. Then $\pi(x+y) = 0$ so that $x + y \in I$. Then $x, y \in I$ so that $\pi x = \pi y = 0$.

It is clear that V/I is a positively generated partially ordered space, but we do not yet show that its positive cone is closed.

Suppose $0 \leqq \tilde{y} \in V/I$. For some $x \in V$ we have $\pi x = \tilde{y}$ and $\|x\| < \|\tilde{y}\| + \varepsilon/2$. There is also some $z \in V$ with $z \geqq 0$ and $\pi z = \tilde{y}$. As I is positively generated there is some $v \in V$ such that $v \geqq x, z$ and $\pi v = \tilde{y}$. Also as V is regular there is some $w \in V$ such that $w \geqq x, 0$ and $\|w\| < \|x\| + \varepsilon/2$. If now $y \in V$ is chosen so that $0, x \leqq y \leqq w, v$ then $0 \leqq y \in V$, $\pi y = \tilde{y}$ and $\|y\| < \|\tilde{y}\| + \varepsilon$.

Suppose $0 \leqq \tilde{y} \leqq \tilde{z} \in V/I$ and

$$0 \leqq a \in V \text{ has } \pi a = \tilde{y},$$

$$0 \leqq b \in V \text{ has } \pi b = \tilde{z} - \tilde{y},$$

$$0 \leqq c \in V \text{ has } \pi c = \tilde{z} \quad \text{and} \quad \|c\| < \|\tilde{z}\| + \varepsilon.$$

Then $\pi(a+b)=\pi c=\tilde{z}$. As I is an ideal we can find $z \in V$ such that $0 \leqq z \leqq a+b$, c and $\pi z=\tilde{z}$. By the Riesz decomposition property we can find $y, x \in V$ such that $0 \leqq y \leqq a$ and $0 \leqq x \leqq b$ and $x+y=z$. Then $0 \leqq \pi y \leqq \tilde{y}$ and $0 \leqq \pi x \leqq \tilde{z}-\tilde{y}$ and $\pi y+\pi x=\tilde{z}$. Thus $\pi y=\tilde{y}$. We now have $0 \leqq y \leqq z$, $\pi y=\tilde{y}$, $\pi z=\tilde{z}$ and $\|z\| \leqq \|c\| < \|\tilde{z}\|+\varepsilon$. We immediately obtain

$$\|\tilde{y}\| \leqq \|y\| \leqq \|z\| < \|\tilde{z}\|+\varepsilon$$

and, as $\varepsilon>0$ is arbitrary, so $\|\tilde{y}\| \leqq \|\tilde{z}\|$.

Suppose $-\tilde{z} \leqq \tilde{y} \leqq \tilde{z} \in V/I$. Then $0 \leqq \tilde{y}+\tilde{z} \leqq 2\tilde{z}$. We can from the previous paragraph find $0 \leqq u \leqq v \in V$ with $\pi u=\tilde{y}+\tilde{z}$, $\pi v=2\tilde{z}$ and $\|v\|<2\|\tilde{z}\|+\varepsilon$. Then $-v/2 \leqq u-v/2 \leqq v/2$ and $\pi(u-v/2)=\tilde{y}$. Thus $\|\tilde{y}\| \leqq \|u-v/2\| \leqq \|v/2\| < \|\tilde{z}\|+\varepsilon/2$. Thus $\|\tilde{y}\| \leqq \|\tilde{z}\|$, and we have shown that V/I is a regular partially ordered Banach space.

We can now show that the positive cone in V/I is closed. Let $\tilde{x}_n \in V/I$ where $\sum \|\tilde{x}_n\|<\infty$ and $\sum_{r=1}^n \tilde{x}_r \geqq 0$ for all n. Let $\tilde{x}_n=\tilde{a}_n-\tilde{b}_n$ where $\tilde{a}_n \geqq 0$, $\tilde{b}_n \geqq 0$ and $\sum \|\tilde{a}_r\|<\infty$, $\sum \|\tilde{b}_r\|<\infty$. We have $\sum_{r=1}^n \tilde{a}_r \geqq \sum_{r=1}^n \tilde{b}_r$. Now let $0 \leqq a_n \in V$ where $\pi a_n=\tilde{a}_n$ and $\sum \|a_n\|<\infty$. $\sum a_n$ converges in V to a limit $a \geqq 0$. We now construct $0 \leqq b_n \in V$ such that $\sum_{r=1}^n b_r \leqq \sum_{r=1}^n a_n$ and $\|b_n\|<2\|\tilde{b}_n\|$. Suppose $b_1, \ldots, b_{n-1}$ have been so constructed. As $\tilde{b}_n \geqq 0$ so we can find $0 \leqq c_n \in V$ with $\pi c_n=\tilde{b}_n$ and $\|c_n\| < 2\|b_n\|$. As

$$\tilde{b}_{n+1} \leqq \sum_{r=1}^{n} \tilde{a}_r-\sum_{r=1}^{n-1} \tilde{b}_r,$$

so we can find $d_n \in V$ with $\pi d_n=\tilde{b}_n$ and

$$d_n \leqq \sum_{r=1}^{n} a_r-\sum_{r=1}^{n-1} b_r = k_n.$$

We can now find $e_n \in V$ with $e_n \leqq c_n$, d_n and $\pi e_n=\tilde{b}_n$. As 0, $e_n \leqq c_n$, k_n, so we can find $b_n \in V$ with 0, $e_n \leqq b_n \leqq c_n$, k_n. Then $0 \leqq b_n$, $\|b_n\| \leqq \|c_n\|<2\|\tilde{b}_n\|$, $\pi b_n=\tilde{b}_n$ and

$$\sum_{r=1}^{n} b_r \leqq \sum_{r=1}^{n} a_r \leqq a.$$

It is now clear that $\sum_{r=1}^n b_r \rightarrow b \leqq a$. Then $\sum_{r=1}^n (a_r-b_r) \rightarrow a-b \geqq 0$. Thus $\sum_{r=1}^n \tilde{x}_r \rightarrow x=\pi(a-b) \geqq 0$, and we have shown V/I has a closed cone.

Finally we show that V/I has the Riesz separation property. Suppose $\tilde{a}$, $\tilde{b} \leqq \tilde{c}$, $\tilde{d}$ and let $\pi a=\tilde{a}$, $\pi b=\tilde{b}$. We can find c, $d \geqq a$, b with $\pi c=\tilde{c}$ and $\pi d=\tilde{d}$. By the Riesz separation property for V we can find $e \in V$ with a, $b \leqq e \leqq c$, d and if $\tilde{e}=\pi e$ we have $\tilde{a}$, $\tilde{b} \leqq \tilde{e} \leqq \tilde{c}$, $\tilde{d}$. This concludes the proof of the theorem.

Another way of investigating closed ideals, pursued for simplex spaces in [11], is to characterise their annihilators in the dual space.

Returning to the notation of §2, we define a conical face in Δ as a closed convex set F with $0 \in F$ such that if x, $y \in \Delta$ and $\alpha x+\beta y \in \Delta$ for any $\alpha>0$, $\beta>0$ then $x, y \in F$.

THEOREM 5.5. *Let V be an R-space, V^* its dual and $\Delta=\{\phi \in V^* : \phi \geqq 0$ and $\|\phi\| \leqq 1\}$. Then the maps $I \to I^0 \to I^0 \cap \Delta$ determine a one-one correspondence between the set of all closed ideals I in V, the set of all weak*-closed lattice ideals in V^*, and the set of all conical faces in Δ. I may naturally be identified with the space of all continuous affine functionals f on Δ with $f|I^0 \cap \Delta=0$.*

It is elementary to show that if I is a closed ideal then I^0 is a weak*-closed lattice ideal and $F=I^0 \cap \Delta$ is a conical face. Now let F be a conical face. Let J be the subspace of V^* generated by Δ. Then J is obviously an ideal and it meets the unit ball B in a weak*-closed set. Specifically,

$$J \cap B = (1+\varepsilon)(\Delta-\Delta) \cap B.$$

By a well-known theorem on Banach spaces [18] it follows that J is a weak*-closed subspace. J is a lattice ideal with $J \cap \Delta=F$. Let I_1 be defined as the subspace of V such that $f \in I$ and $\phi \in F$ implies $(f, \phi)=0$. Then $I_1={}^0J$ is a closed subspace of $V=A^0(\Delta)$. I_1 determines and is determined by Δ, and if Δ is derived from a closed ideal I then $I_1=I$.

All we need to show is that I_1 is always an ideal. It is clearly an order ideal. Suppose $f \in I_1$. Let g be the function g: $\Delta \to (-\infty, \infty]$ defined by

$$\begin{aligned} g(x) &= 0 \qquad \text{if } x \in F, \\ &= \infty \qquad \text{if } x \notin F. \end{aligned}$$

Then g is l.s.c. affine and f, $-f \leqq g$. If we can find $h \in A_0(\Delta)$ with f, $-f \leqq h \leqq g$, then $h|F=0$, so we see that I, is positively generated and so is an ideal. The theorem is completed by the following separation theorem.

THEOREM 5.6. *Let V be an R-space and*

$$\Delta = \{\phi \in V^*: \phi \geqq 0 \cdot \|\phi\| \leqq 1\}.$$

If $-f_1, \ldots, -f_n, g_1, \ldots, g_m$ are l.s.c. affine functionals on Δ vanishing at the origin such that $f_i \leqq g_j$ for all i, j, then we can find $h \in A_0(\Delta)$ with $f_i \leqq h \leqq g_j$ for all i, j.

First observe that if $k_1, \ldots, k_p$ are u.s.c. affine functionals vanishing at the origin then $(k_1 \vee \cdots \vee k_p)^\wedge$ defined as in Lemma 2.7 is easily shown to be u.s.c. affine. The result now follows by a simple application of the technique of forcing convergence developed in the first part of the proof of Theorem 3.1.

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