# THE WORD PROBLEM AND CONSEQUENCES FOR THE BRAID GROUPS AND MAPPING CLASS GROUPS OF THE 2-SPHERE(1)

BY
RICHARD GILLETTE AND JAMES VAN BUSKIRK

1. **Introduction.** Artin [2] solved the word problem for the group $B_n(E^2)$ of $n$-string braids on the plane with presentation

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} : \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i \ (i+1 < j)\rangle$$

by giving a normal form for such braids together with an effective procedure for putting a braid into this form. Fox and Neuwirth [9] showed that no braid in $B_n(E^2)$ has finite order. Fadell and one of the authors [8] obtained a presentation of the group $B_n(S^2)$ of $n$-string braids on the 2-sphere by adding the relation

$$\sigma_1\sigma_2 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_2\sigma_1 = 1$$

to the presentation of $B_n(E^2)$, and showed that finite order braids exist for each $n>1$. Magnus [10] derived a presentation of the group $M_n$ of mapping classes of the $n$-punctured 2-sphere from that of $B_n(S^2)$ by adding the relation $(\sigma_1\sigma_2 \cdots \sigma_{n-1})^n = 1$. A mapping class of the $n$-punctured 2-sphere can be visualized as an $n$-string braid between concentric 2-spheres, where the inner sphere is free to execute full revolutions.

In this paper we modify Artin's effective procedure for putting a braid on $E^2$ into normal form in order to solve the word problems for $B_n(S^2)$ and $M_n$. The resulting normal forms are then used to prove the following theorems.

THEOREM. *If $n>1$, $B_n(S^2)$ has a unique braid of order 2 which generates its center. If $n>2$, $B_n(S^2)$ has a braid of finite order $m$ if and only if $m$ divides $2n$, $2n-2$ or $2n-4$; and if $n>3$ and $m>2$, there are infinitely many braids of each such order $m$.*

THEOREM. *If $n>2$, $M_n$ is centerless. If $n>2$, $M_n$ contains a mapping class of period $m$ if and only if $m$ divides $n$, $n-1$* or *$n-2$. Further, if $n>3$ and $m>1$, then there are infinitely many mapping classes of each such period $m$.*

Given that $B_n(E^2)$ has no elements of finite order, our results are algebraic consequences of the presentations of $B_n(S^2)$ and $M_n$.

2. **Preliminaries.** The *configuration space* of a manifold $X$ is the space $F_n(X)$ of $n$-tuples of distinct points of $X$. The symmetric group $\Sigma_n$ of degree $n$ acts freely

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on $F_n(X)$ by permuting coordinates, and this action induces a regular covering of the orbit space $F_n(X)/\Sigma_n$ by the configuration space $F_n(X)$ (we will assume that $X$ is a connected 2-manifold). The fundamental group of $F_n(X)/\Sigma_n$ is called the *n-string braid group of X*, and is denoted $B_n(X)$ [7], [9]. The covering map $F_n(X) \to F_n(X)/\Sigma_n$ induces an exact sequence

$$1 \longrightarrow K_n(X) \longrightarrow B_n(X) \xrightarrow{\alpha} \Sigma_n \longrightarrow 1$$

where $K_n(X)$ is the fundamental group of $F_n(X)$. The group $K_n(X)$ will be identified with its image as a normal subgroup of $B_n(X)$. If $b \in B_n(X)$, we will speak of $\alpha(b)$ as the *permutation associated with b*.

Let $\Sigma'_{n-1}$ be the subgroup of $\Sigma_n$ consisting of the permutations which leave the symbol 1 fixed, and let $D_n(X)$ be the fundamental group of $F_n(X)/\Sigma'_{n-1}$. There are associated covering maps

$$F_n(X) \to F_n(X)/\Sigma'_{n-1} \to F_n(X)/\Sigma_n$$

so that $D_n(X)$ may be viewed as a subgroup of $B_n(X)$ containing $K_n(X)$. The projection of $F_n(X)$ onto $F_{n-1}(X)$ which deletes the first coordinate induces a map of $F_n(X)/\Sigma'_{n-1}$ onto $F_{n-1}(X)/\Sigma_{n-1}$ which induces a homomorphism $j\colon D_n(X) \to B_{n-1}(X)$. The kernel of $j$ will be denoted $A_n(X)$, and the action of $j$ will be referred to as *killing the first string*. Algebraically, $D_n(X)$ is the inverse image of $\Sigma'_{n-1}$ under the epimorphism $\alpha$.

The definition of $B_n(X)$ generalizes the concept of an $n$-string braid on the plane $E^2$ which was introduced by Artin in 1925 [1], [2]. Artin showed that if an $n$-string braid on $E^2$ is viewed as a motion of $n$ points $p_1, \dots, p_n$ in $E^2$ (by means of a representative path in $F_n(E^2)$), then this motion extends to an isotopy $\{f_t\colon 0 \leqq t \leqq 1\}$ of $E^2$ having compact support, where $f_0 = 1$ and $f_1$ permutes the points $p_1, \dots, p_n$ among themselves [2, Theorem 6]. The homeomorphism $f_1$ induces an automorphism $\psi$ of the fundamental group of the $n$-punctured plane $E^2 - \{p_1, \dots, p_n\}$; that is, an automorphism of a free group $\mathscr{F}_n = \langle t_1, \dots, t_n : \rangle$. The automorphism $\psi$ has the following properties:

(i) Each $\psi(t_i)$ is a conjugate $T_i^{-1} t_j T_i$ of some $t_j$, where $T_i \in \mathscr{F}_n$.

(ii) $\psi(t_1 t_2 \cdots t_n) = t_1 t_2 \cdots t_n$.

In this way, $B_n(E^2)$ is isomorphic with the group of *all* automorphisms of $\mathscr{F}_n$ having these two properties [2, Theorems 14, 15 and 16]. (The analogous representation for the $n$-string braid group of the 2-sphere is not faithful because it annihilates the center. Compare Theorem I.4.1 [5].)

For $1 \leqq i < n$, let $\sigma_i$ be the braid in $B_n(E^2)$ which corresponds to the automorphism of $\mathscr{F}_n$ taking $t_i$ to $t_{i+1}$ and $t_{i+1}$ to $t_{i+1}^{-1} t_i t_{i+1}$, while leaving all other $t_j$ fixed. For $1 \leqq i < j \leqq n$, define $a_{ij}$ to be the braid

$$\sigma_i^{-1} \sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^2 \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i$$

where $a_{i,i+1}$ is understood to be $\sigma_i^2$, and $a_{ji}$ is defined to be $a_{ij}$. In these terms, the epimorphism $\alpha$ has the explicit expression $\alpha(\sigma_i)=(i\,i+1)$, so that each $a_{ij}$ is in the kernel $K_n(E^2)$ of $\alpha$.

The group $B_n(E^2)$ is generated by $\sigma_1,\ldots,\sigma_{n-1}$ subject to the following defining relations:

(i) $\sigma_i\sigma_j=\sigma_j\sigma_i$ for $1<i+1<j<n$;

(ii) $\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1\leqq i<n-1$

[1], [3], [4], [8], [9]. The group $A_n(E^2)$ is freely generated by $a_{12},\ldots,a_{1n}$, and the exact sequence

$$1\longrightarrow A_n(E^2)\longrightarrow D_n(E^2)\xrightarrow{j} B_{n-1}(E^2)\longrightarrow 1$$

splits, where $D_n(E^2)$ is generated by $a_{12},\ldots,a_{1n},\sigma_2,\ldots,\sigma_{n-1}$, and where $j(\sigma_i)=\sigma_i$ for $i=2,\ldots,n-1$ [4, p. 654]. Notice that here $B_{n-1}(E^2)$ is generated by $\sigma_2,\ldots,\sigma_{n-1}$. As a result, the *fundamental exact sequence* of $B_n(E^2)$,

$$1\longrightarrow A_n(E^2)\longrightarrow K_n(E^2)\xrightarrow{j} K_{n-1}(E^2)\longrightarrow 1,$$

obtained by restricting $j$ to $K_n(E^2)$, is split exact and therefore each braid $k$ in $K_n(E^2)$ is uniquely expressible in the form $k=\mathscr{A}_n\mathscr{A}_{n-1}\cdots\mathscr{A}_2$, where $\mathscr{A}_{n-i+1}$ is a word in the free generators $a_{i,i+1},\ldots,a_{i,n}$ [2]–[4]. We will refer to this expression as the *Artin normal form* for $k$. Beginning with $k$ as a word in $\sigma_1,\ldots,\sigma_{n-1}$ there are effective procedures [2]–[4] for explicitly computing the Artin normal form for $k$. We will refer to such a procedure as *combing the braid* $k$. Observe that the word problem for the presentation of $B_n(E^2)$ given above is solved in two ways. For to determine whether a braid $b$ (given as a word in $\sigma_1,\ldots,\sigma_{n-1}$) is the identity, one can compute explicitly the automorphism of $\mathscr{F}_n$ determined by $b$ [11, p. 174], or, having determined whether $b$ is in $K_n(E^2)$ by computing $\alpha(b)$, one can find the Artin normal form for $b$.

It is known that all of the above results for $B_n(E^2)$ are obtainable by algebraic means from the presentation [3], [4].

Viewing the 2-sphere $S^2$ as the compactified plane, there is an embedding of $F_n(E^2)$ in $F_n(S^2)$ which induces a homomorphism $\beta$ from $B_n(E^2)$ to $B_n(S^2)$, where it may be assumed that $\beta$ preserves permutations; $\alpha\beta=\alpha$. For simplicity we will usually use the same notation for a braid in $B_n(E^2)$ as for its image in $B_n(S^2)$ under $\beta$. We will usually omit the symbol "$S^2$" and write $B_n$ for $B_n(S^2)$, etc.

It is shown in [8, p. 245] that a presentation for $B_n$ is obtained by adjoining the single relation $\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_{n-1}\cdots\sigma_2\sigma_1=1$ to the presentation for $B_n(E^2)$. Furthermore, if $n\geqq 4$, $A_n$ is the free group of rank $n-2$ having presentation

$$\langle a_{12},\ldots,a_{1n} : a_{12}\cdots a_{1n} = 1\rangle,$$

and $D_n$ is generated by $a_{12},\ldots,a_{1n},\sigma_2,\ldots,\sigma_{n-1}$ [8] (for algebraic proofs see [13, p. 36]). The sequence

$$1\longrightarrow A_n\longrightarrow D_n\xrightarrow{j} B_{n-1}\longrightarrow 1$$

does not generally split, as we shall see in §4. The restriction of $j$ to $K_n$ induces the *fundamental exact sequence of* $B_n$:

$$1 \longrightarrow A_n \longrightarrow K_n \xrightarrow{j} K_{n-1} \longrightarrow 1.$$

Using topological methods it can be shown that this sequence splits if $n \geqq 4$ [8, p. 256]. Letting $s\colon K_{n-1} \to K_n$ denote a splitting map, there is a semidirect product decomposition $K_n = A_n s(K_{n-1})$. However, the splitting map $s$ cannot be induced by the inclusion map $K_{n-1}(E^2) \to K_n(E^2)$ [8, p. 256]. Therefore the question of *effectively* expressing a braid $k \in K_{n-1}$ as a product $k = \mathscr{A}_n \cdot sj(k)$, $\mathscr{A}_n \in A_n$, depends on explicit formulas for $s$ on the generators of $K_{n-1}$ and ultimately on a procedure, such as Artin's, for expressing a braid in $K_{n-1}$ in terms of the $a_{ij}$'s. Rather than pursue this, we shall use Artin's combing procedure to derive a normal form for braids in $K_n$.

The center of $B_n(E^2)$ is the infinite cyclic group generated by $(\sigma_1\sigma_2\cdots\sigma_{n-1})^n$ [2, p. 121], [4, p. 658] and its image under the homomorphism $\beta\colon B_n(E^2) \to B_n$ is the cyclic group of order 2 generated by the *Dirac braid* $\Delta_n = (\sigma_1\sigma_2\cdots\sigma_{n-1})^n$ [6, p. 239], [8, p. 248]. Thus the center of $B_n$ contains $\Delta_n$. From the fact that the $k$-string braid group on $E^2$ generated by $\sigma_i, \sigma_{i+1}, \ldots, \sigma_{i+k-2}$ has $(\sigma_i\cdots\sigma_{i+k-2})^k$ as the generator of its center and has defining relations which also hold in the subgroup of $B_n$ generated by $\sigma_i, \sigma_{i+1}, \ldots, \sigma_{i+k-2}$ $(i \leqq n-k+1)$, we obtain, by computations found in [8, p. 247], the following

LEMMA 2.1. *In* $B_n$,

(i) $(\sigma_i\cdots\sigma_{i+k-2})^k = a_{i,i+1}\cdots a_{i,i+k-1}(\sigma_{i+1}\cdots\sigma_{i+k-2})^{k-1}$ *and commutes with* $\sigma_j$ $(i \leqq j \leqq i+k-2)$.

(ii) *The Dirac braid* $\Delta_n = (\sigma_1\cdots\sigma_{n-2})^{n-1} = (a_{12}a_{13}\cdots a_{1,n-1})(a_{23}\cdots a_{2,n-1})\cdots(a_{n-2,n-1})$.

LEMMA 2.2. *In* $B_n$,

(iii) $(\sigma_i\cdots\sigma_{k-1})^r\sigma_j(\sigma_i\cdots\sigma_{k-1})^{-r} = \sigma_{j+r}$ $(i \leqq j < k-r)$,

(iv) $(\sigma_2\cdots\sigma_{k-1})^r a_{1,m}(\sigma_2\cdots\sigma_{k-1})^{-r} = a_{1,m+r}$ $(m \leqq k-r)$,

*and with the indices taken* mod $n$,

(v) $a_{i,i+1}a_{i,i+2}\cdots a_{i,i+n-1} = 1$,

(vi) $(\sigma_1\cdots\sigma_{n-1})^r a_{ij}(\sigma_1\cdots\sigma_{n-1})^{-r} = a_{i+r,j+r}$,

(vii) *Any* $n-2$ *of* $a_{1,i}, \ldots, a_{i-1,i}, a_{i,i+1}, \ldots, a_{in}$ *freely generate a subgroup of* $B_n$, *if* $n \geqq 4$.

**Proof.** From the easily derived

$$(\sigma_1\cdots\sigma_{k-1})\sigma_j(\sigma_1\cdots\sigma_{k-1})^{-1} = \sigma_{j+1} \qquad (j < k-1)$$

we obtain (iii) and

$$(*) \qquad (\sigma_1\cdots\sigma_{k-1})a_{ij}(\sigma_1\cdots\sigma_{k-1})^{-1} = a_{i+1,j+1} \qquad (i,j < k).$$

Now (iv) follows from the fact that, by (*),

$$(\sigma_1 \cdots \sigma_{k-1})a_{1,j}(\sigma_1 \cdots \sigma_{k-1})^{-1} = a_{2,j+1} = \sigma_1 a_{1,j+1}\sigma_1^{-1} \qquad (j < k).$$

We next observe that, taking indices mod $n$,

$$a_{i,i+1}a_{i,i+2} \cdots a_{i,i+n-1} = \sigma_i\sigma_{i+1} \cdots \sigma_{n-1}\sigma_{n-1} \cdots \sigma_1\sigma_1 \cdots \sigma_{i-1} = 1,$$

since, by [2, p. 119], if $i<j$, then

$$a_{ij} = a_{ji} = \sigma_i^{-1} \cdots \sigma_{j-2}^{-1}\sigma_{j-1}^2\sigma_{j-2} \cdots \sigma_i = \sigma_{j-1} \cdots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}.$$

We now obtain (vi) from (*) and the fact that $(\sigma_1 \cdots \sigma_{n-1})a_{in}(\sigma_1 \cdots \sigma_{n-1})^{-1}=a_{1,i+1}$ since, by (v), $a_{in}=a_{i,n-1}^{-1} \cdots a_{i,i+1}^{-1}a_{i-1,i}^{-1} \cdots a_{1,i}^{-1}$. Finally, we recall that any $n-2$ of $a_{12}, \ldots, a_{1n}$ freely generate $A_n$, $n \geqq 4$, and note that since, by (vi), the inner automorphism determined by $(\sigma_1 \cdots \sigma_{n-1})^{i-1}$ maps $a_{1,j}$ onto $a_{i,i+j-1}$, (vii) follows.

COROLLARY. *In* $B_4$, $(\sigma_1\sigma_2\sigma_3)a_{12}(\sigma_1\sigma_2\sigma_3)^{-1}=a_{14}\Delta_4$ *and* $(\sigma_1\sigma_2\sigma_3)a_{14}(\sigma_1\sigma_2\sigma_3)^{-1}=a_{12}$.

**Proof.** We only need observe that

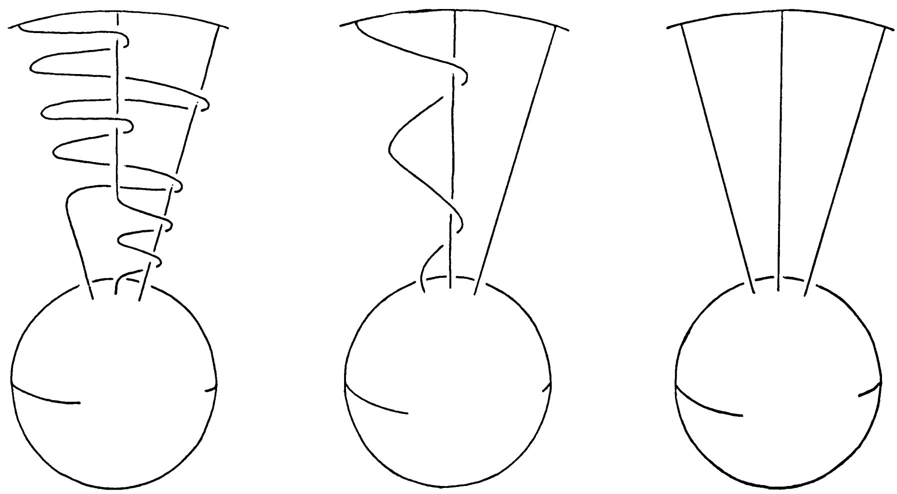
$$(\sigma_1\sigma_2\sigma_3)a_{12}(\sigma_1\sigma_2\sigma_3)^{-1} = a_{23} = a_{13}^{-1}a_{12}^{-1}\Delta_4 = a_{14}\Delta_4$$

by (ii) of Lemma 2.1, and (v) and (vi) of Lemma 2.2.

Let $Q_n=\{p_1, \ldots, p_n\}$ be a set of $n$ distinct points of $S^2$ and let $H(S^2, Q_n)$ be the group of orientation preserving homeomorphisms $h$ of $S^2$ such that $h(Q_n)=Q_n$. Let $D(S^2, Q_n)$ be the normal subgroup of $H(S^2, Q_n)$ consisting of homeomorphisms which are isotopic to the identity, keeping $Q_n$ (pointwise) fixed. The *mapping class group* $M_n$ of the $n$-punctured sphere is defined to be the quotient group $H(S^2, Q_n)/D(S^2, Q_n)$. Obviously $M_0=M_1=1$, while $M_2$ is cyclic of order 2. Magnus has shown [10, p. 624] that a presentation for $M_n$ is obtained by adjoining to the presentation for $B_n(S^2)$ the single relation $(\sigma_1 \cdots \sigma_{n-1})^n=1$. There is thus an epimorphism $\delta_n$: $B_n \to M_n$ with kernel generated by the Dirac braid $\Delta_n$.

3. **Normal form and center.** In this section we solve the word problem for the $n$-string braid group of the 2-sphere by producing a normal form for braids in $B_n(S^2)$ and an effective procedure for transforming a word in the generators $\sigma_1, \ldots, \sigma_{n-1}$ into this form. Using the normal form we show that for $n \geqq 3$ the Dirac braid $\Delta_n$ is the only braid of order 2 in $B_n(S^2)$ and that $\Delta_n$ generates the center of $B_n(S^2)$. Similarly we solve the word problem for the mapping class group of the $n$-punctured 2-sphere and show that for $n \geqq 3$, $M_n$ is centerless. Everything proved here depends ultimately on the presentation for $B_n(S^2)$ given in [8] and on the algorithm given by Artin for computing the normal form of a braid in $B_n(E^2)$, where that braid is given in terms of $\sigma_1, \ldots, \sigma_{n-1}$.

We observe that, since each relation in the generators of $B_n(E^2)$ holds for the corresponding generators of $B_n(S^2)$, Artin's "combing" procedure for putting a plane braid into normal form, when applied to a braid in $B_n(S^2)$, yields an



$(a_{12}a_{13})^2 a_{23}^2$ $a_{12}^2$ $1$

equivalent $n$-string braid on $S^2$. However, equivalent braids in $S^2$ may be transformed into distinct Artin forms. For example, the braids shown in the figure are in Artin normal form and distinct when viewed as 3-string braids on $E^2$, yet are equivalent as braids in $B_3(S^2)$.

THEOREM 3.1 (NORMAL FORM). *If $b$ is an $n$-string braid on the* 2*-sphere, $n \geqq 3$, then $b$ is uniquely expressible in the form $b = m\mathscr{A}_n\mathscr{A}_{n-1}\cdots\mathscr{A}_4\Delta_n^{\varepsilon}$, where $m$ depends only on the permutation $\alpha(b)$, $\mathscr{A}_i$ is in the group freely generated by $a_{n-i+1,n-i+2},\ldots,$ $a_{n-i+1,n-1}$, and $\varepsilon$ is* 0 *or* 1. *If $b$ is expressed in terms of $\sigma_1,\ldots,\sigma_{n-1}$, then there is an effective procedure for putting $b$ into this form with $m$ expressed in terms of $\sigma_1,\ldots,$ $\sigma_{n-1}$.*

**Proof.** We will first show that each braid in $B_n$ can be effectively expressed in the prescribed form and then show that this form is, in fact, unique. The normal form we obtain will be a normal form with respect to a "cross section" $\theta$ of $\alpha$, where by this we mean any function $\theta\colon \Sigma_n \to B_n$ such that $\alpha\theta = 1$, $\theta(P)$ is a fixed word in $\sigma_1,\ldots,\sigma_{n-1}$ for each $P \in \Sigma_n$, and $\theta(1) = 1$.

NOTE. There is a cross section $\theta$ of $\alpha\colon B_n \to \Sigma_n$ such that if $b \in B_n$ and $\alpha(b)$ moves no symbol which is less than $r$ or greater than $s$, then $\theta\alpha(b)$ can be expressed in terms of $\sigma_r, \sigma_{r+1},\ldots,\sigma_{s-1}$. To see this, we will decompose $\alpha(b)$ into disjoint cycles of length $>1$, where a particular cycle contains the least symbol which is moved by $\alpha(b)$ and is involved in no preceding cycle. Each of these cycles can be expressed as a product of transpositions involving the smallest symbol in that cycle and finally each such transposition $(ij)$, where $i<j$, can in turn be expressed in the form

$$(i\ i+1)(i+1\ i+2)\cdots(j-2\ j-1)(j-1\ j)(j-2\ j-1)\cdots(i+1\ i+2)(i\ i+1).$$

For example, if $b=\sigma_4\sigma_3^{-1}\sigma_5^3\sigma_4\sigma_7^{-1}\sigma_5^{-1}\sigma_6^2\sigma_8$, then

$$\begin{aligned}\alpha(b) &= (3645)(798) = (36)(34)(35)(79)(78)\\ &= (34)(45)(56)(45)(34)\cdot(34)\cdot(34)(45)(34)\cdot(78)(89)(78)\cdot(78).\end{aligned}$$

This procedure gives $\alpha(b)$ canonically as a product of transpositions involving adjacent symbols, where we choose not to allow cancellations. We define $\theta\alpha(b)$ to be the braid obtained from this expression for $\alpha(b)$ by simply replacing the transposition $(i\,i+1)$ by $\sigma_i$. Thus for our example, $\theta\alpha(b)=\sigma_3\sigma_4\sigma_5\sigma_4\sigma_3^3\sigma_4\sigma_3\sigma_7\sigma_8\sigma_7^2$.

Let $b$ be a braid in $B_n$ which is expressed in the $\sigma_i$'s and define $m$ and $k$ by $m=\theta\alpha(b)$ and $k=m^{-1}b$. Then since $\alpha(m)=\alpha(b)$, $k\in K_n$ and $\theta\alpha(k)=1$. Applying the Artin "combing" procedure to $k$, as written in $\sigma_1,\dots,\sigma_{n-1}$, we obtain $k=\mathscr{A}_n''\mathscr{A}_{n-1}''\cdots\mathscr{A}_2''$ where $\mathscr{A}_i''$ is a power product of $a_{n-i+1,n-i+2},\dots,a_{n-i+1,n}$. Using Lemma 2.2 (v), we can replace each occurence of $a_{i,n}$ in $\mathscr{A}_{n-i+1}''$ by a word in the $a_{ij}$'s, where $1\leqq j<n$, $i\neq j$, and then rewrite the resulting expression in the $\sigma_i$'s, $1\leqq i\leqq n-2$. We speak of this operation as "freeing the last string" of $k$. We next apply Artin's "combing" procedure to $k$, here considered as a plane braid on its first $n-1$ strings so as to insure that its $n$th string remains free, and obtain $k=\mathscr{A}_n'\mathscr{A}_{n-1}'\cdots\mathscr{A}_3'$, where $\mathscr{A}_i'$ is a power product of $a_{n-i+1,n-i+2},\dots,a_{n-i+1,n-1}$ and $\mathscr{A}_3'=a_{n-2,n-1}^{-q}$ for some integer $q$.

Recall, from Lemma 2.1, that $(\sigma_i\cdots\sigma_{n-2})^{n-i}$ commutes with each $\sigma_j$, $i\leqq j\leqq n-2$, and hence with $\mathscr{A}_{n-i+1}'$ and that $(\sigma_i\cdots\sigma_{n-2})^{n-i}=a_{i,i+1}\cdots a_{i,n-1}(\sigma_{i+1}\cdots\sigma_{n-2})^{n-i-1}$, so that $(\sigma_i\cdots\sigma_{n-2})^{(n-i)q}\mathscr{A}_{n-i+1}'=\mathscr{A}_{n-i+1}'(a_{i,i+1}\cdots a_{i,n-1})^q(\sigma_{i+1}\cdots\sigma_{n-2})^{(n-i-1)q}$. Using this and the fact that $\Delta_n=(\sigma_1\cdots\sigma_{n-2})^{n-1}$, we see that

$$\Delta_n^q\cdot k = (\sigma_1\cdots\sigma_{n-2})^{(n-1)q}\mathscr{A}_n'\cdots\mathscr{A}_3' = \mathscr{A}_n\cdots\mathscr{A}_4,$$

where $a_{n-2,n-1}^q\mathscr{A}_3'=1$ and

$$\mathscr{A}_{n-i+1} = \mathscr{A}_{n-i+1}'(a_{i,i+1}\cdots a_{i,n-i})^q.$$

But then $b=mk=m\mathscr{A}_n\cdots\mathscr{A}_4\Delta_n^q=\mathscr{A}_n\cdots\mathscr{A}_4\Delta_n^\varepsilon$, where $\varepsilon=0$ or $1$ according as $q$ is even or odd.

To see that each step in the described process is effectively computable, we consider a braid $k$ in $K_n$ expressed as a word in $\sigma_1,\dots,\sigma_{n-1}$ and enumerate the steps as follows: "Comb" $k$ into Artin form; free the last string; "comb" the resulting braid (the last string remaining free); and finally, in order to eliminate the occurences of $a_{n-2,n-1}$, multiply by a suitable even power of $\Delta_n$, say $\Delta_n^{2r}=1$, by postmultiplying by $\Delta_n^r$ and then premultiplying by $\Delta_n^r$ so as to move the "loop" $(a_{i,i+1}\cdots a_{i,n-1})^r$ of $\Delta_n^r$ into juxtaposition with $\mathscr{A}_{n-i+1}$. As pointed out in §2, "combing" is an effective procedure, while the other two steps are obviously effective.

The proof of uniqueness involves an induction on the number of strings. Of course it suffices to consider braids in $K_n$.

Since $B_3$ has order 12 [8, p. 248] and $B_3/K_3=\Sigma_3$ has order 6, $K_3$ has order 2. Therefore $K_3=\langle\Delta_3 : \Delta_3^2=1\rangle$ because $\Delta_3\in K_3$. This settles the question of uniqueness in case $n=3$.

Before discussing the general case we note some properties of the homomorphism $j: D_n\to B_{n-1}$. For this purpose take $B_{n-1}$ to be generated by $\sigma_2,\dots,\sigma_{n-1}$, so that if $1<i<k\leqq n$, then $\sigma_i$ and $a_{ik}$ ambiguously denote braids in $D_n$ or $B_{n-1}$. As a result of the formulas given in §2, $j(a_{1i})=1$ if $1<i\leqq n$, and $j(a_{ik})=a_{ik}$ if $1<i<k\leqq n$. By Lemma 2.2 (vii), $a_{i,i+1},\dots,a_{i,n-1}$ freely generate a subgroup of $K_n$, and thus, if $i>1$, $j$ carries this group isomorphically onto the subgroup of $K_{n-1}$ freely generated by the same named elements $a_{i,i+1},\dots,a_{i,n-1}$ of $K_{n-1}$. Furthermore, by Lemma 2.1 (ii) $j(\Delta_n)=(a_{23}\cdots a_{2,n-1})\cdots(a_{n-2,n-1})=\Delta_{n-1}$.

If $b\in D_n$ then $\theta\alpha(b)$ is expressible in terms of $\sigma_2,\dots,\sigma_{n-1}$ and hence $j\theta\alpha(b)$ $=\theta\alpha(jb)$. Therefore if one applies $j$ to a braid $b\in D_n$ expressed in the normal form $b=\theta\alpha(b)\mathscr{A}_n\mathscr{A}_{n-1}\cdots\mathscr{A}_4\Delta_n^\varepsilon$ one obtains the normal form $j(b)=\theta\alpha(jb)j(\mathscr{A}_{n-1})\cdots$ $j(\mathscr{A}_4)\Delta_{n-1}^\varepsilon$ in $K_{n-1}$.

Suppose now that $n\geqq 4$ and assume the conclusion of the theorem for $K_{n-1}$. Let $\mathscr{A}_n\cdots\mathscr{A}_4\Delta_n^\varepsilon$ and $\mathscr{A}_n'\cdots\mathscr{A}_4'\Delta_n^\delta$ be normal forms for a braid $k$ in $K_n$. Since these normal forms are carried by $j$ onto normal forms for the braid $j(k)$ in $K_{n-1}$, then by the inductive assumption, $\Delta_{n-1}^\varepsilon=\Delta_{n-1}^\delta$ and $j(\mathscr{A}_{n-i+1})=j(\mathscr{A}_{n-i+1}')$ for $2\leqq i\leqq n-3$. Now $\mathscr{A}_{n-i+1}$ and $\mathscr{A}_{n-i+1}'$ are elements of the subgroup of $K_n$ freely generated by $a_{i,i+1},\dots,a_{i,n-1}$ and, as was shown above, the restriction of $j$ to this free group is an isomorphism. Therefore, $\mathscr{A}_{n-i+1}=\mathscr{A}_{n-i+1}'$ for $2\leqq i\leqq n-3$ and $\Delta_n^\varepsilon=\Delta_n^\delta$. But then $\mathscr{A}_n=\mathscr{A}_n'$ and the proof is complete.

LEMMA 3.2. *If $n\geqq 3$, the only braid of finite order $>1$ in the group $K_n$ is the Dirac braid $\Delta_n$.*

**Proof.** This lemma will be proved by induction on the number of strings. If $n=3$, the conclusion follows from the fact that $K_3$ is generated by $\Delta_3$ (see the proof of Theorem 3.1). Suppose now that $n\geqq 4$ and assume the conclusion of the lemma for $K_{n-1}$. Let $b$ be a braid of finite order in $K_n$, where $b$ has normal form $\mathscr{A}_n\mathscr{A}_{n-1}\cdots\mathscr{A}_4\Delta_n^\varepsilon$. The homomorphism $j: K_n\to K_{n-1}$ which kills the first string carries $b$ to a braid $j(b)$ which also has finite order, where $K_{n-1}$ consists of braids on the last $n-1$ strings. As observed in the proof of Theorem 3.1, $j(b)=\mathscr{A}_{n-1}\cdots$ $\mathscr{A}_4\Delta_{n-1}^\varepsilon$ is in normal form in $K_{n-1}$. But then by the inductive assumption, $j(b)$ is 1 or $\Delta_{n-1}$, and hence $b=\mathscr{A}_n\Delta_n^\varepsilon$. Since $\Delta_n$ is central and has order 2, $b\Delta_n^\varepsilon$ must have finite order and, since $b\Delta_n^\varepsilon=\mathscr{A}_n$ is a braid in the free group $A_n$ ($n\geqq 4$), we see that $b\Delta_n^\varepsilon=1$.

We shall see in Lemma 4.8 that $\Delta_n$ is the only finite order $n$-string braid whose associated permutation fixes as many as three symbols.

COROLLARY. *There is an effective procedure for determining if a given braid $b$ in $B_n$ has finite order.*

**Proof.** One simply computes the normal form for $b^r$, where $r$ is the order of the permutation $\alpha(b)$. Notice that this procedure does not depend on the fact (proved topologically in [9]) that $B_n(E^2)$ has no elements of finite order. Since the proof of Lemma 3.2 can be modified to show that $K_n(E^2)$ has no elements of finite order, there is an algebraic procedure for showing that a given braid in $B_n(E^2)$ does not have finite order. This raises the

QUESTION. Can the nonexistence of braids of finite order in $B_n(E^2)$ be derived algebraically from the presentation of $B_n(E^2)$? (It can be shown that it suffices to prove algebraically that if $p$ is an odd prime, then $B_p(E^2)$ has no elements of order $p$.)

The next result follows from the fact that two permutations in $\Sigma_n=\alpha(B_n)$ which have the same number of cycles of each length are conjugate.

LEMMA 3.3. *If $P$ is a permutation on $n$ symbols having the same number of cycles of each length as the permutation $\alpha(b)$ associated with an $n$-string braid $b$ of finite order $m$, then there exists an $n$-string braid of order $m$ with associated permutation $P$. If there are only a finite number $r$ of $n$-string braids having finite order $m$ and associated permutation $P$, then there are precisely $rs$ $n$-string braids of order $m$ having associated permutations with the same number of cycles of each length as $P$, where $s$ is the number of conjugates of $P$ in $\Sigma_n$.*

The $n$-string braids having associated permutations which leave each of the symbols $1, 2, \ldots, k$ fixed form a subgroup, say $D_n^k$, of $D_n$ since the associated permutations form a subgroup of $\alpha(D_n)$, the group of permutations on $n$ symbols which leave the symbol 1 fixed. We will next show that deletion of the first $k$ strings of the braids of $D_n^k$ yields a natural homomorphism of $D_n^k$ onto $B_{n-k}$.

LEMMA 3.4. *The function $\rho_k$ defined by*

$$\rho_k(\sigma_i) = \sigma_i, \qquad \text{if } i > k,$$

*and, for $i<j$,*

$$\begin{aligned} \rho_k(a_{ij}) &= 1 \qquad \text{if } i \leqq k, \\ &= a_{ij} \qquad \text{if } i > k \end{aligned}$$

*is a homomorphism of $D_n^k$ onto $B_{n-k}$, where $B_{n-k}$ is here generated by $\sigma_{k+1}, \ldots, \sigma_{n-1}$.*

**Proof.** If $b \in D_n^k$, then $\alpha(b)$ fixes each of the symbols $1, 2, \ldots, k$ and $b$ can be written in normal form without using $\sigma_1, \sigma_2, \ldots, \sigma_k$, as was noted in the proof of Theorem 3.1. According to Lemma 2.1 (ii), $\Delta_n$ can be expressed in terms of the $a_{j,m}$, $1\leqq j<m<n$. But then, if $k<j<m<n$, the $a_{j,m}$ can in turn be expressed in terms of $\sigma_{k+1}, \ldots, \sigma_{n-2}$. We thus see, by considering the normal form for an arbitrary braid in $D_n^k$, that $D_n^k$ is generated by $\sigma_{k+1}, \ldots, \sigma_{n-1}$ and the $a_{ij}$, where $1<i\leqq k$ and $i<j<n$.

The exact sequence

$$1 \longrightarrow A_n \longrightarrow D_n \xrightarrow{j_n} B_{n-1} \longrightarrow 1,$$

(cf. §2) induces the exact sequence

$$1 \longrightarrow A_n \longrightarrow D_n^k \xrightarrow{j_n} D_{n-1}^{k-1} \longrightarrow 1,$$

where $D_{n-1}^{k-1}$ is the group of all braids on the last $n-1$ strings having permutations which leave each of the symbols $2, 3, \ldots, k$ fixed. Inductively we obtain the exact sequences

$$1 \longrightarrow A_{n-i} \longrightarrow D_{n-i}^{k-i} \xrightarrow{j_{n-i}} D_{n-i-1}^{k-i-1} \longrightarrow 1$$

for $i=0, 1, \ldots, k-2$ and finally the exact sequence

$$1 \longrightarrow A_{n-k+1} \longrightarrow D_{n-k+1}^{1} \xrightarrow{j_{n-k+1}} B_{n-k} \longrightarrow 1.$$

The composition $\rho_k = j_{n-k+1} \cdots j_{n-1} j_n$ is thus seen to be the desired homomorphism of $D_n^k$ onto $B_{n-k}$, and $\rho_k$ is given, as in the statement of the theorem, by specifying its action on a set of generators of $D_n^k$.

Note. In the proof of the following theorem and in the sequel, we utilize the normal form of Theorem 3.1 with respect to various convenient choices of the cross section $\theta$.

Theorem 3.5. *The only braid of order* 2 *in* $B_n$, $n \geqq 3$, *is the Dirac braid* $\Delta_n$.

**Proof.** If $n=3$, then $B_3$, as a $ZS$-metacyclic group of order 12 [8, p. 248], contains a single element of order 2. We will proceed by induction on $n$. Suppose that there exists a braid $b$, other than $\Delta_4$, of order 2 in $B_4$. Then by Lemma 3.2, $\alpha(b)$ has order 2 and hence we may assume that $\alpha(b)=(34)$ or $(13)(24)$ according to Lemma 3.3. The former possibility will be ruled out when we verify the inductive step so we assume $\alpha(b)=(13)(24)$. Now if $b \neq \Delta_4$ has order 2, then so does $b\Delta_4$, hence we may assume that $b=(\sigma_1\sigma_2\sigma_3)^2\mathscr{A}_4$, where $\mathscr{A}_4 = w(a_{12}, a_{14})$, a word in the free generators $a_{12}$ and $a_{14}$ of $A_4$. By the corollary to Lemma 2.2, $(\sigma_1\sigma_2\sigma_3)^2 a_{1,i}(\sigma_1\sigma_2\sigma_3)^{-2} = a_{1,i}\Delta_4$ when $i=2$ or 4, and therefore $b^2=(\sigma_1\sigma_2\sigma_3)^4[w(a_{12}, a_{14})]^2\Delta_4^r$, where $r$ is the length of the word $w$. But then, since $(\sigma_1\sigma_2\sigma_3)^4=\Delta_4$, it follows that $[w(a_{12}, a_{14})]^2 = \Delta_4^{r+1}$ and that $w(a_{12}, a_{14})$, as a finite order braid in the free group $A_4$, is the identity braid. Thus $b^2=(\sigma_1\sigma_2\sigma_3)^4=\Delta_4 \neq 1$, a contradiction.

Suppose that $n \geqq 5$, that the result is known for braids on one or two less strings, and that there exists a braid $b$, other than $\Delta_n$, of order 2 in $B_n$. Then $\alpha(b)$ has order 2 in $\Sigma_n$.

*Case* 1. If $n$ is odd, or if $n$ is even but $\alpha(b)$ is not the product of $n/2$ disjoint transpositions, then we may assume that $\alpha(b)$ leaves the symbol 1 fixed. Consequently $b=m\mathscr{A}_n \cdots \mathscr{A}_4\Delta_n^{\varepsilon}$ where $m$ is a word in $\sigma_2, \ldots, \sigma_{n-1}$ such that $\alpha(b)=\alpha(m)$. But then, on killing the first string of $b$, we obtain $j(b)=m\mathscr{A}_{n-1}\cdots\mathscr{A}_4\Delta_{n-1}^{\varepsilon}$ according to Lemma 2.1, where $j(b)$ is a braid in $B_{n-1}$ which is here generated by $\sigma_2, \ldots, \sigma_{n-1}$. Hence by the inductive assumption $m=\mathscr{A}_{n-1}=\cdots=\mathscr{A}_4=1$ and consequently $b=\mathscr{A}_n\Delta_n^{\varepsilon}$. Thus $\mathscr{A}_n^2=\Delta_n^{2\varepsilon}$, and $\mathscr{A}_n$, as a braid of finite order in $A_n$, is the identity braid. But then $b=\Delta_n^{\varepsilon}$, a contradiction.

*Case* 2. If $n$ is even and $\alpha(b)$ is the product of $n/2$ disjoint transpositions, then we may assume that $\alpha(b)=(12)(34)\cdots(n-2\ \ n-1)$ and hence that $b=\sigma_1^{-1}\sigma_3\cdots\sigma_{n-2}\mathscr{A}_n\cdots\mathscr{A}_4\Delta_n^{\varepsilon}$. Since $\sigma_1$ commutes with $\mathscr{A}_{n-2}\cdots\mathscr{A}_4$, we see that

$$b^2 = \sigma_1^{-2}\sigma_3\sigma_5\cdots\sigma_{n-1}\mathscr{A}'_{n-1}\mathscr{A}'_n\mathscr{A}_{n-2}\cdots\mathscr{A}_4\sigma_3\sigma_5\cdots\sigma_{n-1}\mathscr{A}_n\cdots\mathscr{A}_4,$$

where $\mathscr{A}'_{n-1}=\sigma_1\mathscr{A}_n\sigma_1^{-1}$, $\mathscr{A}'_n=\sigma_1\mathscr{A}_{n-1}\sigma_1^{-1}$, and hence, by Lemma 3.4, that

$$\rho_2(b^2) = (\sigma_3\sigma_5\cdots\sigma_{n-1}\mathscr{A}_{n-2}\cdots\mathscr{A}_4)^2 = 1.$$

But then, by the inductive assumption,

$$\sigma_3\sigma_5\cdots\sigma_{n-1}\mathscr{A}_{n-2}\cdots\mathscr{A}_4 = \Delta_{n-2} \quad \text{or} \quad 1,$$

contrary to $\alpha\rho_2(b)\neq 1$.

LEMMA 3.6. *If an n-string braid b has finite order m, then its associated permutation $\alpha(b)$ has order m or $m/2$ according as m is odd or even. Further, for each odd k, if b has order 2k, then $b\Delta_n$ has order k; while if b has odd order k, then $b\Delta_n$ has order 2k.*

**Proof.** If $b$ has finite order $m$ and $\alpha(b)$ has order $k$, then $b^k$, as a finite order braid in $K_n$, is $\Delta_n$ or 1 according to Lemma 3.2. Thus $m=2k$ or $k$ and hence, if $m$ is odd, $m=k$. However if $m$ is even, then $m=2k$, since the assumption that $m=k$ implies, by Theorem 3.5, that $b^{k/2}=\Delta_n$, contrary to $\alpha(b)$ having order $k$. The second conclusion follows from the fact that $\Delta_n$ is central and is the unique order 2 braid in $B_n$.

We recall, from the discussion preceding Lemma 2.1, that if $n\geqq 3$, then the cyclic group of order 2 generated by the Dirac braid $\Delta_n$ is contained in the center $Z(B_n)$ of $B_n$.

THEOREM 3.7. *If $n\geqq 3$, then the center of $B_n$ is the cyclic subgroup of order 2 generated by the Dirac braid. If $n\geqq 2$, then the center of $B_n$ is the unique subgroup of order 2 in $B_n$.*

**Proof.** We first note that, for $n\geqq 3$, the symmetric group $\Sigma_n$ of degree $n$ is centerless and hence that $Z(B_n)$ is contained in the kernel $K_n$ of the homomorphism $\alpha\colon B_n\to\Sigma_n$.

In particular, since $K_3$ was determined in proving Theorem 3.1, we have $\langle\Delta_3\colon \Delta_3^2=1\rangle\subseteq Z(B_3)\subseteq K_3=\langle\Delta_3\colon \Delta_3^2=1\rangle$.

We now assume the result holds for $B_{n-1}$, $n\geqq 4$, and consider an arbitrary braid $z\neq 1$ in $Z(B_n)$, where $z$ has normal form $\mathscr{A}_n\mathscr{A}_{n-1}\cdots\mathscr{A}_4\Delta_n^{\varepsilon}$, since $z\in K_n$. The homomorphism $j\colon K_n\to K_{n-1}$ with kernel $A_n$ maps $z\Delta_n^{\varepsilon}$ onto $\mathscr{A}_{n-1}\cdots\mathscr{A}_4\in Z(B_{n-1})$, where $B_{n-1}$ is here generated by $\sigma_2,\ldots,\sigma_{n-1}$. By the inductive assumption, $\mathscr{A}_{n-1}\cdots\mathscr{A}_4=\Delta_{n-1}^{\eta}$, where $\eta=0$ or 1. But then $\mathscr{A}_{n-1}\cdots\mathscr{A}_4\Delta_{n-1}^{\eta}=1$ and is in normal form (in $B_{n-1}$), so that $\mathscr{A}_{n-1}=\cdots=\mathscr{A}_4=1$ and $\eta=0$. Since the homomorphism $j$ restricted to $A_i$ is, for $i=4,\ldots,n-1$, an isomorphism between free groups of rank $i-2$, $z\Delta_n^\varepsilon=\mathscr{A}_n$ and is in the center of $B_n$, which is contrary to $A_n$ being free of rank $n-2\geqq 2$ unless $\mathscr{A}_n=1$. Thus $z=\Delta_n$. Finally, since $B_2$ is a cyclic group of order 2 in which $\Delta_2=\sigma_1^2$ is the identity and $B_1=\pi_1(S^2)=1$, we completely determine the center of $B_n$ and obtain, in view of Theorem 3.5, the second assertion.

Just as Artin's combing procedure can be applied to a braid of $B_n$, our procedure for putting a braid of $B_n$ in normal form, when applied to a mapping class $c$ in $M_n$, yields an equivalent mapping class, since each relation in the generators of $B_n$ holds for the corresponding generators of $M_n$.

THEOREM 3.8 (NORMAL FORM). *If $c$ is a mapping class of the $n$-punctured 2-sphere, $n\geqq 3$, then $c$ is uniquely expressible in the form $c=m\mathscr{A}_n\cdots\mathscr{A}_4$, where $m$ depends only on the permutation $\alpha(c)$ and $\mathscr{A}_i$ is in the group freely generated by $a_{n-i+1,n-i+2}$, $\ldots, a_{n-i+1,n-1}$. If $c$ is expressed in terms of $\sigma_1,\ldots,\sigma_{n-1}$, then there is an effective procedure for putting $b$ into this form with $m$ expressed in terms of $\sigma_1,\ldots,\sigma_{n-1}$.*

**Proof.** Let $c$ be a mapping class in $M_n$ which is expressed in the $\sigma_i$'s. Then putting $c$ into the normal form of a braid in $B_n$, we obtain $c=m\mathscr{A}_n\cdots\mathscr{A}_4\Delta_n^\varepsilon$ and, since $\Delta_n$ is trivial in $M_n$, $c=m\mathscr{A}_n\cdots\mathscr{A}_4$. This form is effectively computable, hence we have only to establish uniqueness. Assume $m\mathscr{A}_n\cdots\mathscr{A}_4=m'\mathscr{A}'_n\cdots\mathscr{A}'_4$ in $M_n$ and hence that $\mathscr{A}_n\cdots\mathscr{A}_4=\mathscr{A}'_n\cdots\mathscr{A}'_4$. But then, since the $n$-string braids $\mathscr{A}_n\cdots\mathscr{A}_4$ and $\mathscr{A}'_n\cdots\mathscr{A}'_4$ have the same image under $\delta_n$, $\mathscr{A}_n\cdots\mathscr{A}_4=\mathscr{A}'_n\cdots\mathscr{A}'_4\Delta_n^\eta$ in $B_n$ and thus, by Theorem 3.1, $\mathscr{A}_i=\mathscr{A}_i'$ in $B_n$ and hence in $M_n$, for $i=4,\ldots,n$.

By combining Theorem 4.6 of [12] with the representation of $M_n$ as a factor group of $B_n(E^2)$ [10], we could obtain a different normal form for $M_n$, but we will not pursue this here.

THEOREM 3.9. *If $n\geqq 3$, then $M_n$ is centerless.*

**Proof.** As in proving Theorem 3.7, we note that the center of $M_n$ is contained in the kernel of $\alpha\colon M_n\to\Sigma_n$, and proceed by induction on $n$. The basis for the induction follows from the fact that

$$\begin{aligned} M_3 &= \langle \sigma_1,\sigma_2\colon \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ \sigma_1\sigma_2^2\sigma_1 = 1,\ (\sigma_1\sigma_2)^3 = 1\rangle \\ &= \langle \sigma_1,\sigma_2\colon (\sigma_1\sigma_2)^3 = \sigma_1^2 = \sigma_2^2 = 1\rangle, \end{aligned}$$

is the centerless group $\Sigma_3$.

4. **Finite orders.** In this section we determine all possible orders of elements in $B_n$ and $M_n$ and indicate when there are only finitely many elements of a given order. The fact that there are no finite order braids in the plane will be used to show that there cannot exist braids of certain finite orders in $B_n$ [9, p. 126]. To exhibit braids of a given order we will use a suitable factorization of the Dirac

braid. We will then use the normal form to determine when there are only finitely many braids of a given order.

LEMMA 4.1. *If* $n \geqq 5$ *and* $d$ *is a divisor of* $2n$ *which is odd and at least* $n/(n-4)$ *or even and at least* $2n/(n-4)$, *then there are infinitely many* $n$-*string braids of order* $d$.

**Proof.** Suppose $q < n-3$ divides $n$. Then the $n$-string braids $a_{12}^{-m}a_{q+1,q+2}^{m}$ are in normal form and hence the braids

$$b(m,q) = (\sigma_1 \cdots \sigma_{n-1})^q a_{12}^{-m} a_{q+1,q+2}^{m} = a_{q+1,q+2}^{-m}(\sigma_1 \cdots \sigma_{n-1})^q a_{q+1,q+2}^{m}$$

are, for given $q$, distinct for distinct $m$. We next note that the order of $b(m,q)$ is at least $n/q$, the order of its associated permutation $\alpha(b(m,q)) = (1\ n \cdots 2)^q$, and that $[b(m,q)]^{n/q} = a_{q+1,q+2}^{-m}(\sigma_1 \cdots \sigma_{n-1})^n a_{q+1,q+2}^{m} = \Delta_n$. We thus see that if $d$ is an odd divisor of $2n$ which is at least $n/(n-4)$, then, by Lemma 3.6, each of the infinitely many distinct $b(m, n/d)\Delta_n$ has order $d$; while, if $d$ is an even divisor of $2n$ which is at least $2n/(n-4)$, then each of the infinitely many distinct $b(m, 2n/d)$ has order $d$.

LEMMA 4.2. *If* $d$ *divides* $2n-2$, *where* $n \geqq 4$ *and* $d \geqq 3$, *then there are infinitely many* $n$-*string braids of order* $d$.

**Proof.** Suppose $q < n-2$ divides $n$. Then, for given $q$, the $n$-string braids $b(m,q) = (\sigma_2 \cdots \sigma_{n-1})^q a_{12}^{m} a_{1,q+2}^{-m} = a_{1,q+2}^{m}(\sigma_2 \cdots \sigma_{n-1})^q a_{1,q+2}^{-m}$ are distinct for distinct $m$, each has order $\geqq (n-1)/q$ and $[b(m,q)]^{n/q} = \Delta_n$. Thus if $d$ is an odd divisor of $2n-2$ which is at least $(n-1)/(n-3)$, then each of the infinitely many distinct $b(m, (n-1)/d)\Delta_n$ has order $d$; while, if $d \geqq (2n-2)/(n-3)$ is an even divisor of $2n-2$, then each of the infinitely many distinct $b(m, (2n-2)/d)$ has order $d$. We next note that $(n-1)/(n-3) \leqq 3$ for each $n \geqq 4$, and that $(2n-2)/(n-3) \leqq 4$ for each $n \geqq 5$. Finally, if $n=4$, then the only divisor of $2n-2$ which is even and at least 4 is $6 = (2n-2)/(n-3)$.

LEMMA 4.3. *There are infinitely many* 4-*string braids of order* 8.

**Proof.** Since the 4-string braids $a_{12}^{-m}(a_{13}^{-1}a_{12}^{-1})^m$ are words in the free generators $a_{12}$ and $a_{13}$ of $A_4$, the braids $b(m) = \sigma_1\sigma_2\sigma_3 a_{12}^{-m}(a_{13}^{-1}a_{12}^{-1})^m = a_{23}^{-m}\sigma_1\sigma_2\sigma_3(a_{23}\Delta_4)^m$ are distinct for distinct $m$ and each has order at least 4, the order of its associated permutation (4321). But then, since $[b(m)]^4 = a_{23}^{-m}(\sigma_1\sigma_2\sigma_3)^4 a_{23}^{m}\ \Delta_4^{4m} = \Delta_4^{4m+1}$, each of the infinitely many distinct $b(m)$ has order 8.

We next note that there are but two distinct braids of order 4 among the squares of the infinitely many 4-string braids of order 8 found above, since

$$[b(m)]^2 = a_{23}^{-m}(\sigma_1\sigma_2\sigma_3)^2 a_{23}^{m} = a_{23}^{-m}a_{14}^{m}(\sigma_1\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2\Delta_4^{m}.$$

In proving the next result we will show that there are, in fact, no other 4-string braids of order 4 with associated permutation (13)(24).

THEOREM 4.4. *There are only six* 4-*string braids of order* 4 *having associated permutations which consist of two disjoint transpositions. However, if* $n$ *is even and at* *least* 6, *then there are infinitely many n-string braids of order* 4 *with $n/2$ disjoint transpositions as associated permutation.*

**Proof.** We first note that there are only three permutations on four symbols consisting of two disjoint transpositions and that if $b$ is a 4-string braid of order 4 having associated permutation $\alpha(b)=(13)(24)$, then $b=(\sigma_1\sigma_2\sigma_3)^2w(a_{12}, a_{14})\Delta_4^{\varepsilon}$, where $w(a_{12}, a_{14})$ is a word in the free generators $a_{12}$ and $a_{14}$ of $A_4$ and $\varepsilon=0$ or 1. Since $(\sigma_1\sigma_2\sigma_3)^2a_{1,i}(\sigma_1\sigma_2\sigma_3)^{-2}=a_{1,i}\Delta_4$, for $i=2,4$ (Corollary to Lemma 2.2), $\Delta_4=b^2=w(a_{12}\Delta_4, a_{14}\Delta_4)\Delta_4w(a_{12}, a_{14})=[w(a_{12}, a_{14})]^2\Delta_4^{\eta+1}$, where $\eta$ is the length of $w(a_{12}, a_{14})$. But then $w(a_{12}, a_{14})$, as a finite order braid in the free group $A_4$, is the identity braid and hence $b=(\sigma_1\sigma_2\sigma_3)^2\Delta_4^{\varepsilon}$, where $\varepsilon = 0$ or 1. The first conclusion now follows from Lemma 3.3.

In proving Lemma 4.1 we exhibited, for each even integer $n\geqq 8$, infinitely many $n$-string braids with associated permutation $\alpha((\sigma_1\sigma_2\cdots\sigma_{n-1})^{n/2})=(n\cdots 2\ 1)^{n/2}$, the product of $n/2$ disjoint cycles. The second conclusion now follows from the fact that the 6-string braids $b(m)=(\sigma_1\sigma_2\cdots\sigma_5)^3(a_{12}a_{13}a_{14}a_{15})^m a_{34}^m$ have associated permutation (14)(25)(36), are distinct for distinct $m$, and each has order 4 since, by (vi) of Lemma 2.2, $[b(m)]^2=(\sigma_1\cdots\sigma_5)^3a_{16}^{-m}a_{34}^m a_{34}^{-m}a_{16}^m(\sigma_1\cdots\sigma_5)^3=\Delta_6$.

The existence of infinitely many 4-string braids of order 4 depends, in view of the preceeding theorem and Lemma 3.6, on our finding infinitely many such braids with a single transposition as associated permutation. This is established as a special case of the following

Lemma 4.5. *If $d$ divides $2n-4$, where $n\geqq 4$ and $d\geqq 3$, then there are infinitely many $n$-braids of order $d$.*

**Proof.** If $1\leqq q<n-2$, then the $n$-string braids $a_{12}^{m+1}a_{13}a_{14}\cdots a_{1,q+1}a_{1,q+2}^{-m}$ are reduced words in the free generators $a_{12},\ldots, a_{1,n-1}$ of $A_n$ and hence the braids

$$\begin{aligned} b(m, q) &= a_{12}^{m+1}a_{13}\cdots a_{1,q+1}a_{1,q+2}^{-m}(\sigma_2\cdots\sigma_{n-2})^q \\ &= a_{12}^m a_{12}a_{13}\cdots a_{1,q+1}(\sigma_2\cdots\sigma_{n-2})^q a_{12}^{-m} \\ &= a_{12}^m\left[\prod_{i=0}^{q-1}(\sigma_2\cdots\sigma_{n-2})^{-i}a_{12}(\sigma_2\cdots\sigma_{n-2})^i\right](\sigma_2\cdots\sigma_{n-2})^q a_{12}^{-m} \\ &= a_{12}^m(a_{12}\sigma_2\cdots\sigma_{n-2})^q a_{12}^{-m} \end{aligned}$$

are, for given $q$, distinct for distinct $m$. We next note that the order of $b(m, q)$ is at least $(n-2)/q$, the order of its associated permutation $((n-1)\cdots 3\ 2)^q$, and that $[b(m, q)]^{(n-2)/q}=a_{12}^m(a_{12}\sigma_2\cdots\sigma_{n-2})^{n-2}a_{12}^{-m}=\Delta_n$, since $(a_{12}\sigma_2\cdots\sigma_{n-2})^{n-2}=a_{12}\cdots a_{1,n-1}(\sigma_2\cdots\sigma_{n-2})^{n-2}=(a_{12}\cdots a_{1,n-1})(a_{23}\cdots a_{2,n-1})\cdots(a_{n-2,n-1})=\Delta_n$. But then, if $d$ is odd, each of the infinitely many distinct $b(m, (n-2)/d)\Delta_n$ has order $d$; while, if $d$ is even, each of the infinitely many distinct $b(m, (2n-4)/d)$ has order $d$.

We summarize our results in the following

Lemma 4.6. *If $n\geqq 3$, then there exist $n$-string braids of each order $d$ dividing $2n$, $2n-2$ or $2n-4$. Further, if $n\geqq 4$ and $d>2$, then there are infinitely many such braids.*

**Proof.** We first note that the Dirac braid $\Delta_n$ is, for $n \geqq 3$, the unique $n$-string braid of order 2 and that, since $B_3$ is a $ZS$-metacyclic group of order 12 [8, p. 248], there are two 3-string braids of order 3, six of order 4 and two of order 6, in addition to $\Delta_3$ and the identity. There exist, by Lemma 4.2, infinitely many 4-string braids of each of the orders 3 and 6 and, by Lemmas 4.5 and 4.1, infinitely many of each of the orders 4 and 8. It only remains for us to consider the second conclusion for $n \geqq 5$.

If $n \geqq 5$, then the second conclusion follows for divisors of $2n-2$ and $2n-4$ by Lemmas 4.2 and 4.5, and for divisors of $2n$ by Lemma 4.1 and the existence, by Lemma 4.5, of infinitely many 6-string braids of order 4.

We will use the next four lemmas to show that if $m$ is any positive integer which does not divide $2n$, $2n-2$ or $2n-4$ then no $n$-string braid has order $m$.

Lemma 4.7. *The subgroup of $B_n$ generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-3}$ is isomorphic to the corresponding subgroup of $B_n(E^2)$. Therefore no braid on $S^2$ having its last two strings free can have finite order $>1$.*

**Proof.** Primes will be used to distinguish braids on the plane from corresponding braids on the 2-sphere. Let $B_{n-3}(E^2)$ be the subgroup of $B_n(E^2)$ generated by $\sigma_1', \sigma_2', \ldots, \sigma_{n-3}'$ and $h$ the restriction of the epimorphism $\beta\colon B_n(E^2) \to B_n$ given by $h(\sigma_i') = \sigma_i$ for $1 \leqq i \leqq n-3$. The kernel of $h$ is contained in $K_n(E^2)$, the kernel of $\alpha'\colon B_n(E^2) \to \Sigma_n$, since $h$ preserves permutations. Because $\sigma_1', \ldots, \sigma_{n-3}'$ generate $B_{n-3}(E^2)$, any braid $b'$ in the kernel of $h$ has Artin normal form

$$b' = w_n(a_{12}', \ldots, a_{1,n-2}') \cdots w_4(a_{n-3,n-2}'),$$

a word in the $a_{ij}'$ where $i<j\leqq n-2$. Now

$$h(b') = w_n(a_{12}, \ldots, a_{1,n-2}) \cdots w_4(a_{n-3,n-2}),$$

which we recognize as the normal form for a braid in $B_n$. But then, since $h(b')$ is the identity braid in $B_n$, it follows that each of the words $w_n, \ldots, w_4$ is the identity. Thus $b'=1$ and hence $h$ is an isomorphism.

There is however, for each $n \geqq 3$, an $n$-string braid of finite order $>1$ having its last string free. For example, each of the finite order $n$-string braids, $n \geqq 4$, exhibited in Lemma 4.5 has its last string free, as does the order four 3-string braid $\sigma_1$. We thus see that, although $S^2$ is the one point compactification of $E^2$, the $n$-string braid group of $E^2$ is not just the subgroup of $B_{n+1}$ consisting of those $(n+1)$-string braids which have a free "string at infinity".

Lemma 4.8. *If the permutation associated with an n-string braid leaves at least three, but not all n, symbols fixed, then the braid does not have finite order.*

**Proof.** According to Lemma 3.3 it suffices to consider the subgroup $G_n$ of $B_n$ consisting of all braids having permutations which fix each of the symbols $n-2$, $n-1$ and $n$. If $b \in G_n$ then $b$ has normal form $m\mathscr{A}_n \cdots \mathscr{A}_4 \Delta_n^{\varepsilon}$ where $m$ is (as was noted in the proof of Theorem 3.1) expressible in terms of $\sigma_1, \ldots, \sigma_{n-4}$. Let $H_n$ consist of all braids in $G_n$ which have normal form $m\mathscr{A}_n \cdots \mathscr{A}_4$. We will first show that $H_n$ is a subgroup of $G_n$, and hence that $G_n$ is the direct product of $H_n$ and the center $\langle \Delta_n : \Delta_n^2 = 1 \rangle$ of $B_n$. We will next show that $H_n$ is isomorphic to a subgroup of $B_n(E^2)$, and hence has no elements of finite order. It will then follow that a braid of finite order in $G_n$ is a power of $\Delta_n$ and is thus associated with the identity permutation.

Each element in $H_n$ is expressible in terms of the following braids:

(*) $\qquad \sigma_1, \sigma_2, \ldots, \sigma_{n-4}; \quad a_{ij}$ for $1 \leqq i \leqq n-3$ and $i < j \leqq n-1$.

In fact, $H_n$ is the group generated by the braids in (*) because the transformation formulas of Artin [2, Theorem 8, p. 120 and (29), p. 119] permit one to rearrange a word in the generators (*) until it is brought into the normal form of an element of $H_n$. Of course, these computations are valid in both $B_n$ and $B_n(E^2)$ so that there is a subgroup $H_n(E^2)$ of $B_n(E^2)$ which corresponds isomorphically to $H_n$ by associating a normal form $m\mathscr{A}_n \cdots \mathscr{A}_4$ in $H_n$ with the same normal form considered as a braid in $B_n(E^2)$. In view of our choice of normal form, this isomorphism between $H_n$ and $H_n(E^2)$ carries each of the generators in (*) onto the same named generators of $H_n(E^2)$.

By letting $G_n = K_n$ (i.e. $m = 1$) in the above proof, we obtain the following

COROLLARY. *Let $A'_{n-i+1}$ be the subgroup of $K_n$ freely generated by $a_{i,i+1}, \ldots, a_{i,n-1}$, for $2 \leqq i \leqq n-3$. Then if $n \geqq 3$, $A_n A'_{n-1} \cdots A'_4$ is a subgroup of $K_n$ and $K_n = A_n A'_{n-1} \cdots A'_4 Z(B_n)$, where $Z(B_n) = \langle \Delta_n : \Delta_n^2 = 1 \rangle$ is a direct factor of $K_n$.*

LEMMA 4.9. *If the order $m$ of a permutation $P$ on $n$ symbols does not divide $2n$, $2n-2$ or $2n-4$, then some power of $P$ fixes each of at least three, but not all $n$, symbols.*

**Proof.** We first note that the hypothesis is not satisfied unless $n > 7$ and that if the length of some cycle of $P$ is a multiple of each of the other cycle lengths, then $m \leqq n$. Thus if $m > n$, then $P$ has cycles $Q$ and $R$ such that the length $q$ of $Q$ is less than but does not divide the length $r$ of $R$. But then, since $R^r$ is the identity permutation, $P^r$ fixes each of the $r$ symbols moved by $R$, where $m > n \geqq q + r > r > q > 1$.

We may thus assume that $m < n-2$ and that $P$ fixes fewer than three symbols, then, since $m > 4$, $P$ contains a cycle of length $>2$ and hence a cycle of shortest length $s > 2$. If $P$ has two transpositions or a transposition and a fixed symbol, then $P^2$ is not the identity permutation but fixes at least three symbols and we are finished. Thus we assume $P$, which fixes at most two symbols, either has no transposition and at most two fixed symbols, or has a single transposition and no fixed symbols. Now if some cycle of $P$ has length $>s$, then $P^s$ is not the identity permutation but fixes more than 3 symbols. If not, then $P$ has, say, $k$ cycles of length $s$ and $ks = n$, $n-1$, or $n-2$. In any case, $m = s$ or $2s$ and is thus a divisor of $2n$, $2n-2$, or $2n-4$, contrary to assumption.

SCHOLIUM. If $P$ is a permutation of order $m$ on $n$ symbols such that $P^r$ leaves at most two symbols fixed for each $r<m$, then all cycles of $P$ have the same length $s$, except for the possible occurence of at most two fixed symbols or, in case $s=2^k>2$, a single transposition.

**Proof.** It only remains for us to note that if all cycles of $P$, except for a single transposition, have length $s$ which is not a positive power of 2, then either $s=1$ and there is nothing to prove, since then $m=2$, or an odd prime $p$ divides $s$. But in that case, $P^p$ fixes at least $p$ symbols, where $2<p\leqq s<2s=m$, contrary to assumption.

LEMMA 4.10. *If an n-string braid has finite order m, then m divides* $2n$, $2n-2$ *or* $2n-4$.

**Proof.** As noted in the discussion following Theorem 3.7, $B_1$ is trivial and $B_2$ is cyclic of order 2. We also note that the orders of 3-string braids, as listed in the proof of Lemma 4.6, satisfy the conclusion. Let $b$ be an $n$-string braid of finite order $m$, where $n>4$. Since each power of $b$ has finite order, $(\alpha(b))^r=\alpha(b^r)$ fixes at most two symbols for each $r<m$, by Lemma 4.8, and, by the scholium to Lemma 4.9, all cycles of the permutation $\alpha(b)$ have the same length, say $s$, except possibly for at most two cycles of length 1 or, if $s=2^k>2$, a single transposition. In any case, $s$ divides $n$, $n-1$ or $n-2$. But then, since by Lemma 3.6 $m=s$ or $2s$, it follows that $m$ divides $2n$, $2n-2$ or $2n-4$.

Notice that we have not excluded the possibility that the permutation associated with some finite order braid might have cycles of equal length $2^k>2$ except for a single transposition.

Summarizing the results of Lemmas 4.6 and 4.10 we obtain the following.

THEOREM 4.11. *If* $n\geqq 3$, *then there exist n-string braids of order m if and only if m is a divisor of* $2n$, $2n-2$ *or* $2n-4$. *Further, there are infinitely many such braids if and only if* $n\geqq 4$ *and* $m>2$.

Recall that when $X$ is $E^2$ the splitting of the exact sequence

$$1\to A_n(X)\to D_n(X)\to B_{n-1}(X)\to 1 \tag{1}$$

induces a splitting of the fundamental exact sequence

$$1\to A_n(X)\to K_n(X)\to K_{n-1}(X)\to 1. \tag{2}$$

When $X$ is $S^2$, the following result implies that (1) does not generally split although, by a geometric argument [8, p. 256], or by arguments used in proving Lemma 4.8, (2) does.

THEOREM 4.12. *If* $n\geqq 6$ *then* $B_n$ *is not isomorphically embeddable in* $B_{n+1}$.

**Proof.** It suffices for us to note that if $n\geqq 4$, then, by Theorem 4.11, $B_n$ has a braid of order $2n-4$ yet if $n\geqq 6$, then $B_{n+1}$ does not. For if there were an

$(n+1)$-string braid of order $2n-4$, then, by Theorem 4.11, $2n-4$ would divide $2n+2$, $2n$ or $2n-2$ with quotient $\geqq 2$. But then $2(2n-4)\leqq 2n+2$ and hence $n\leqq 5$.

Assume that $n\geqq 3$ and note that the exact sequence

$$1 \longrightarrow Z(B_n) \longrightarrow B_n \xrightarrow{\delta} M_n \longrightarrow 1$$

induces the exact sequence

$$1 \to Z(B_n) \to K_n \to L_n \to 1 \tag{3}$$

where $L_n$ is the subgroup of all mapping classes in $M_n$ having identity permutation. Using Theorem 3.8 and the corollary to Lemma 4.8, $K_n = L_n \times Z(B_n)$. Using this and the fact that $M_2 = \langle \sigma_1 : \sigma_1^2 = 1\rangle$ we obtain the following

LEMMA 4.13. *If $n\geqq 3$, then the exact sequence* (3) *splits and, if $n\geqq 4$, $L_n = A_n A'_{n-1}\cdots A'_4$. Furthermore, if $c$ is a mapping class in $M_n$ of finite period $m$, then its associated permutation $\alpha(c)$ has order $m$.*

Suppose $c$ is a mapping class of finite period $k$ in $M_n$ which is expressed as a power product of $\sigma_1,\ldots,\sigma_{n-1}$. Putting $c$ into the normal form of a braid in $B_n$, we obtain $c = m\mathscr{A}_n\cdots\mathscr{A}_4\Delta_n^{\varepsilon}$. Thus $c = m\mathscr{A}_n\cdots\mathscr{A}_4$ is in normal form and hence, by the previous lemma $\alpha(m)$ is a permutation of order $k$. We consider the $n$-string braid $b = m\mathscr{A}_n\cdots\mathscr{A}_4\Delta_n^{\varepsilon}$ and observe that if $k$ is even, then $b^k = (m\mathscr{A}_n\cdots\mathscr{A}_4)^k\Delta_n^{k\varepsilon} = \Delta_n^{\eta}$ and, in view of Lemma 3.6, $b$ has order $2k$; whereas if $k$ is odd, then $(b\Delta_n)^k = \Delta_n^{\mu}$ and, in view of Lemma 3.6, $b\Delta_n$ has order $2k$. Thus a mapping class in $M_n$, $n\geqq 3$, cannot have finite period $k$ unless there is a braid in $B_n$ of order $2k$, in which case $k$ divides $n$, $n-1$, or $n-2$, by Lemma 4.10.

Conversely, suppose $k$ is a divisor of $n$, $n-1$ or $n-2$. There are, by Lemma 4.6, $n$-string braids in $B_n$ of order $k$ or $2k$, according as $k$ is odd or even, having permutations of order $k$. And, if $k>1$ and $n\geqq 4$, there are infinitely many such braids. Now $\delta_n$ maps these braids onto mapping classes in $M_n$ having finite periods and hence period $k$, the order of their permutations. But $\delta_n$ is one or two to one according as $k$ is odd or even. We have thus establishing the following

THEOREM 4.14. *If $n\geqq 3$, then there exists a mapping class of the $n$-punctured 2-sphere having period $m$ if and only if $m$ is a divisor of $n$, $n-1$ or $n-2$. Further, there are infinitely many such mapping classes if and only if $n\geqq 4$ and $m>1$.*

The next result follows from the fact that, for $n\geqq 4$, $B_n$ has infinitely many elements of each of the orders 3 and 4, and a unique element of order 2, while $M_n$ has infinitely many elements of each of the orders 2 and 3.

THEOREM 4.15. *If $n\geqq 4$, then $M_n$ is not isomorphically embeddable in $B_n$ and hence the exact sequence*

$$1 \to \langle \Delta_n : \Delta_n^2 = 1\rangle \to B_n \to M_n \to 1$$

*does not split.*

In view of Lemma 3.2, Theorem 4.4 and the following two results we see, by Lemma 3.3, that the braids of orders 2, 4 and 8 in $B_4$ can be completely classified by their normal forms.

THEOREM 4.16. *Each of the infinitely many* 4-*string braids of order* 8 *is a conjugate of* $(\sigma_1\sigma_2\sigma_3)\Delta_4^\varepsilon$, *where* $\varepsilon=0$ *or* 1. *In particular, if b has order* 8 *in* $B_4$ *and* $\alpha(b)=(4321)$, *then in normal form*

$$b = \sigma_1\sigma_2\sigma_3 a_{12}^{p_1}a_{14}^{q_1}a_{12}^{p_2}a_{14}^{q_2}\cdots a_{12}^{p_m}a_{14}^{q_m}\Delta_4^\varepsilon$$

*where* $p_i=-q_{m-i+1}$ $(i=1,2,\ldots,m)$ *and* $p_i\neq 0$ $(i=2,\ldots,m)$.

**Proof.** By Lemma 3.3 the first statement in the theorem follows from the second. Since $b^2$ has order 4 and permutation (13)(24), $b^2=(\sigma_1\sigma_2\sigma_3)^2\Delta_4^\eta$ according to Theorem 4.4. In normal form, $b=\sigma_1\sigma_2\sigma_3 w(a_{12},a_{14})\Delta_4^\varepsilon$ and, by the corollary to Lemma 2.2,

$$b^2 = \sigma_1\sigma_2\sigma_3 w(a_{12},a_{14})w(a_{14},a_{12})\sigma_1\sigma_2\sigma_3\Delta_4^\mu$$

where $\mu$ is 0 or 1 according as the exponent sum of the $a_{12}$'s in $w(a_{12},a_{14})$ is even or odd. Thus $w(a_{12},a_{14})w(a_{14},a_{12})=\Delta_4^{\eta-\mu}=1$, since $A_4$ is free. We next note that if

$$w(a_{12},a_{14}) = a_{12}^{p_1}a_{14}^{q_1}a_{12}^{p_2}a_{14}^{q_2}\cdots a_{12}^{p_m}a_{14}^{q_m}$$

then $p_i+q_{m-i+1}=0$, since $w(a_{12},a_{14})$ and $w(a_{14},a_{12})$ are inverses, and set

$$\begin{aligned} v(a_{12},a_{14}) &= a_{12}^{p_1}a_{14}^{q_1}\cdots a_{14}^{q_r} && \text{if } m=2r,\\ &= a_{12}^{p_1}a_{14}^{q_1}\cdots a_{12}^{p_r} && \text{if } m=2r-1. \end{aligned}$$

But then, since $w(a_{12},a_{14})=v(a_{12},a_{14})[v(a_{14},a_{12})]^{-1}$ and

$$\sigma_1\sigma_2\sigma_3 v(a_{12},a_{14}) = v(a_{14},a_{12})\sigma_1\sigma_2\sigma_3\Delta_4^m,$$

we see that

$$b = v(a_{14},a_{12})\sigma_1\sigma_2\sigma_3[v(a_{14},a_{12})]^{-1}\Delta_4^{m+\varepsilon}.$$

Considerations similar to those in the preceding proof give us the following

THEOREM 4.17. *Each of the infinitely many* 4-*string braids of order* 4 *with a single transposition as associated permutation is a conjugate of* $\sigma_2 a_{12}\Delta_4^\varepsilon$, *where* $\varepsilon=0$ *or* 1. *In particular, if b has order* 4 *in* $B_4$ *and* $\alpha(b)=(23)$, *then in normal form*

$$b = \sigma_2 a_{12} a_{13}^{p_1}a_{12}^{q_1}a_{13}^{p_2}a_{12}^{q_2}\cdots a_{13}^{p_m}a_{12}^{q_m}\Delta_4^\varepsilon$$

*where* $p_i=-q_{m-i+1}$ $(i=1,2,\ldots,m)$ *and* $p_i\neq 0$ $(i=2,\ldots,m)$.

QUESTION. Can the braids of finite order in $B_n$ be classified by their normal forms?

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University of Oregon,
Eugene, Oregon