THE INTEGRAL COHOMOLOGY RINGS OF GROUPS OF ORDER p^3

BY

GENE LEWIS[1]

0. **Introduction.** Most of the papers in cohomology of groups have been theoretical [1], [3]–[6], [8], [9], [13], [14], rather than computational. The following list describes all the (published and unpublished) computations I am aware of. Atiyah [1, p. 60] calculated the integral (cohomology) ring of the quaternion group Q of order 8. Wall [16] obtained the additive structure of split cyclic-by-cyclic extension. He could not get the ring structure. Cartan-Eilenberg [3, Chapter XII, §7] calculated the cohomology of a cyclic group with any coefficients, and gave an explicit diagonal map. Evens [6] wrote down the cohomology ring of the dihedral group D of order 8. Unfortunately, he did not publish the details of his calculations. Nakaoka [9] calculated the homology ring of symmetric groups with coefficients Z_p and in [10, p. 52] found the cohomology ring of Σ_4, coefficients Z_2. Cardenas [2] calculated the ring of Σ_{p^2}, coefficients Z_p. Tate [15] got the integral homology ring of any finitely generated abelian group.

Finally, Norman Hamilton and Arnold Shapiro got the integral cohomology ring of $Z_p \wr Z_p$, but neglected to write it down.

In this paper, I will find the integral cohomology rings of groups of order p^3, except for Q and D. I merely sketch the abelian cases, as they are trivial.

1. **Review of basic facts.**

1.1. We recall some basic results on change of groups, transfer, and spectral sequences. Throughout this paper, all G-modules are left G-modules, all exact sequences of G-modules are assumed $\boldsymbol{Z}$-split, and all groups are finite (in many places, this last assumption is superfluous). Let G, G' be groups, A, A' be G (resp. G')-modules.

Definition. A *change of groups* $\phi: (G, A) \to (G', A')$ is a pair (φ, α) such that $\varphi: G' \to G$ is a morphism, and $\alpha: A_{(\varphi)} \to A'$ is a G'-morphism.

A change of groups ϕ induces a morphism $\phi^*: H^n(G, A) \to H^n(G', A')$, $n \geqq 0$. ϕ^* is obtained as follows: Let X be a G-projective resolution of $\boldsymbol{Z}$, Y a G'-projective resolution of $\boldsymbol{Z}$. The comparison theorem gives a homotopy class of chain maps $Y \xrightarrow{f} X_{(\varphi)}$. For $\xi \in H^n(G, A)$, with representative cocycle $h: X_n \to A$, we let $\phi^*(h)$ be the image of

$$h \in \operatorname{Hom}_G(X, A) \xrightarrow{\text{incl}} \operatorname{Hom}_{G'}(X_{(\varphi)}, A_{(\varphi)}) \xrightarrow{f^*} \operatorname{Hom}_{G'}(Y, A_{(\varphi)}) \xrightarrow{\alpha_*} \operatorname{Hom}_{G'}(Y, A')$$

Received by the editors October 21, 1966.

[1] This research was done when the author was an NSF Graduate Fellow.

under the composition $\alpha_* \circ f^* \circ \iota$. Let $\phi^*(\xi) \in H^n(G, A')$ be the cohomology class of $\phi^*(h)$. It is easy to see that ϕ^* is independent of the choice of X and Y. If $(G, A) \xrightarrow{\phi} (G', A') \xrightarrow{\psi} (G'', A'')$ are changes of groups, then if we define $(G, A) \xrightarrow{\psi\circ\phi} (G'', A'')$ in the obvious way, it is a change of groups, and $(\psi \circ \phi)^* = \psi^* \circ \phi^*$. In particular, if $\lambda: G \to G$ is an automorphism, we get a change of groups $\lambda: (G, A) \to (G, A_{(\lambda)})$ given by $\lambda: G \to G$ and identity $A_{(\lambda)} \to A_{(\lambda)}$. In terms of standard cocycles, this is given by $[h] \in H^n(G, A) \to [h \circ f] \in H^n(G, A_{(\lambda)})$, where $f: BG \to BG_{(\lambda)}$ is $f([g_1, \ldots, g_n]) = [\lambda(g_1), \ldots, \lambda(g_n)]$, f a G-morphism. For $A = \boldsymbol{Z}$, trivial action, this gives a *right action* of Aut (G) on $H^*(G, \boldsymbol{Z})$, which evidently preserves products. If $A = G$-module, $H \subset G$, subgroup, $\sigma \in N_G(H)$, then σ induces $\lambda_\sigma: H \to H$, $\lambda_\sigma(h) = \sigma^{-1}h\sigma = h^\sigma$. Let $X = G$-projective resolution of $\boldsymbol{Z}$. Then $(H, A) \xrightarrow{\lambda_\sigma} (H, A)$ given by $\lambda_\sigma: H \to H$, $A \xleftarrow{\sigma} A$ is a change of groups. The corresponding $f: X \to X_{(\lambda_\sigma)}$ is multiplication by σ^{-1}. Thus λ_σ^* is got as follows: For $[h] \in H^n(H, A)$, $h: X_n \to A$, cocycle, form σh, where $\sigma h(x) = \sigma \cdot h(\sigma^{-1}x)$. Then $\lambda_\sigma^*[h] = [\sigma h]$. As $\sigma \to \lambda_\sigma$ is an antimorphism of $G \to \text{Aut}\, G$, $[h] \to \lambda_\sigma^*[h]$ gives a *left action* of $N_G(H)$ on $H^*(H, A)$. In terms of standard H-cocycles,

$$\lambda_\sigma^*[h]([h_1, \ldots, h_n]) = \sigma \cdot h([h_1^\sigma, \ldots, h_n^\sigma])$$

[8, p. 117]. For we now use $f: BH \to BH_{(\lambda_\sigma)}$ such that

$$f([h_1, \ldots, h_n]) = [\lambda_\sigma(h_1), \ldots, \lambda_\sigma(h_n)],$$

and then compose with $A \xleftarrow{\sigma} A$.

1.2. As Cor commutes with coboundaries, the following commutes

$$\begin{array}{ccc} H^2(H, \boldsymbol{Z}) & \xrightarrow{\text{Cor}} & H^2(G, \boldsymbol{Z}) \\ \delta \uparrow \cong & & \delta \uparrow \cong \\ H'(H, \boldsymbol{Q}/\boldsymbol{Z}) & \xrightarrow{\text{Cor}} & H^1(G, \boldsymbol{Q}/\boldsymbol{Z}) \\ \wr\| & & \wr\| \\ \text{Hom}\,(H, \boldsymbol{Q}/\boldsymbol{Z}) & \xrightarrow{\text{Cor}} & \text{Hom}\,(G, \boldsymbol{Q}/\boldsymbol{Z}) \end{array}$$

Let $G = \bigcup \sigma_i H$, disjoint, $g\sigma_i = \sigma_{\pi(i)} h(i, g)$. Given $f: H \to \boldsymbol{Q}/\boldsymbol{Z}$, morphism, we have: $\text{Cor}\, f: g \to \sum_i f(h(i, g))$.

The proof is an easy calculation, which we leave to the reader.

1.3. If $1 \to H \to G \to K \to 1$ is a group extension, A a G-module, then there is a spectral sequence $E_2 \Rightarrow H^*(G, A)$ [8], where $E_2^{i,j} \cong H^i(K, H^j(H, A))$. In case $A = R$ $=$ a G-ring, the isomorphism $E_2 \cong^\varphi H^*(K, H^*(H, R))$ is an isomorphism of bigraded rings, up to a sign: if $\alpha \in H^i(K, H^j(H, R))$, $\beta \in H^k(K, H^l(H, R))$, then $\varphi(\alpha\beta) = (-1)^{jk}\varphi(\alpha)\varphi(\beta)$. A check on this is provided by noting that in $H^*(K, H^*(H, R))$ we have

(1) $\alpha\beta = (-1)^{ik+jl}\beta\alpha$,

whereas in E_2 we need

(2) $\varphi(\alpha)\varphi(\beta) = (-1)^{(i+j)(k+l)}\varphi(\beta)\varphi(\alpha)$.

With the sign $(-1)^{jk}$, φ transforms (1) into (2). Recall also the commutative diagrams [4, p. 227]

$$\text{I.}\quad \begin{array}{ccc} H^*(K, R^H) & \xrightarrow{\text{inflation}} & H^*(G, R) \\ \cong \downarrow & & \uparrow \text{mono} \\ E_2^{*,0} & \xrightarrow{\text{epi}} & E_\infty^{*,0} \end{array}$$

$$\text{II.}\quad \begin{array}{ccc} H^*(G, R) & \xrightarrow{\text{Res}} & H^*(H, R)^G \\ \text{epi} \downarrow & & \uparrow \cong \\ E_\infty^{0,*} & \xrightarrow{\text{Mono}} & E_2^{0,*} \end{array}$$

COROLLARY (I)$\Rightarrow$: *in a split extension, since* Inf *is then mono, we have* $E_2^{*,0} \cong E_\infty^{*,0}$.

COROLLARY (II)$\Rightarrow$: *universal cycles of* $E_2^{0,*}$ *are in the image of the restriction.*

REMARK. It can be shown that from the E_2 term on, the spectral sequence (including the multiplication and differentials) is an invariant of the extension.

2. **The restriction-corestriction sequences.** Suppose $1 \to H \to G \to K \to 1$ is a group extension, such that K is cyclic of order s, with generator k. The exact sequence

(1) $$0 \longleftarrow Z \xleftarrow{\varepsilon} ZK \xleftarrow{k-1} ZK \xleftarrow{\theta} Z \longleftarrow 0$$

is an exact sequence of G-modules, via inflation. Let $X = \text{Ker}\,\varepsilon = \text{Im}\,(k-1)$. We get:

(2) $0 \longleftarrow Z \xleftarrow{\varepsilon} ZK \xleftarrow{\mu} X \longleftarrow 0$. Apply the long exact sequence of

(2)′ $0 \longleftarrow X \longleftarrow ZK \xleftarrow{\theta} Z \longleftarrow 0$ cohomology. This gives:

(3) $$\cdots \longrightarrow H^{k-1}(G, Z) \xrightarrow{\delta} H^k(G, X) \xrightarrow{\mu_*} H^k(G, ZK) \xrightarrow{\varepsilon_*} H^k(G, Z) \xrightarrow{\delta} \cdots$$

(3)′ $$\cdots \longrightarrow H^{k-1}(G, X) \xrightarrow{\delta} H^k(G, Z) \xrightarrow{\theta_*} H^k(G, ZK) \xrightarrow{\varepsilon_*} H^k(G, X) \xrightarrow{\delta} \cdots.$$

By Shapiro's Lemma, [3, p. 196], $H^k(H, Z) \xrightarrow{\omega} H^k(G, ZK)$ is an isomorphism. If $X = G$-projective resolution of Z, ω arises from the associativity

$$\omega: \text{Hom}_H(ZG \otimes_G X, Z) \cong \text{Hom}_G(X, \text{Hom}_H(ZG, Z)).$$

We check rapidly that $\mathrm{Cor}=\varepsilon_* \circ \omega$, $\mathrm{Res}=\omega^{-1} \circ \theta_*$. Also, the action $k\colon H^*(H, Z) \to H^*(H, Z)$ is carried by ω into the map $k_*\colon H^*(G, ZK) \to H^*(G, ZK)$, induced by $k\colon ZK \to ZK$. Thus:

PROPOSITION 2.1. *Given $1 \to H \to G \to K \to 1 \ni K=\{k\}$, cyclic, we get exact sequences*

$$(4) \qquad \cdots \longrightarrow H^{k-1}(G, Z) \xrightarrow{\delta} H^k(G, X) \xrightarrow{\bar{\mu}_*} H^k(H, Z) \xrightarrow{\mathrm{Cor}} H^k(G, Z) \xrightarrow{\delta} H^{k+1}(G, X) \longrightarrow \cdots$$

$$(4)' \qquad \cdots \longrightarrow H^{k-1}(G, X) \xrightarrow{\delta} H^k(G, Z) \xrightarrow{\mathrm{Res}} H^k(H, Z) \xrightarrow{\bar{\nu}_*} H^*(G, X) \xrightarrow{\delta} H^{k+1}(G, Z).$$

Here $\bar{\mu}_* \circ \bar{\nu}_*=k-1\colon H^k(H, Z) \to H^k(H, Z)$.

REMARK 2.0. We could have used Tate cohomology and obtained the same result.

REMARK 2.1. If A is any module, then as $0 \leftarrow Z \leftarrow ZK \leftarrow ZK \leftarrow Z \leftarrow 0$ is Z-split, hence so is

$$0 \to \mathrm{Hom}\,(Z, A) \to \mathrm{Hom}\,(ZK, A) \to \mathrm{Hom}\,(ZK, A) \to \mathrm{Hom}\,(Z, A) \to 0.$$

Therefore, this is exact $A \cong \mathrm{Hom}\,(Z, A)$, and if X, $\bar{X}=$image-kernel, we get sequences (4), (4)′ with A in place of Z.

REMARK 2.2. We can get several puny results directly from (4) and (4)′.

(1) If G is a p-group, and $H^{2n}(G, Z)=0$, $H^{2n+1}(G, Z)=0$, then $G=\{1\}$. For let $H \lhd G$, of index p. As $k-1=\bar{\mu}_* \circ \bar{\nu}_*\colon H^k(H, Z) \to H^k(H, Z)$ is an isomorphism, for $k=2n$, $2n+1$, we see that k has no fixed points. But $\{k\}$, $H^k(H, Z)$ are p-groups. Therefore $H^k(H, Z)=0$, $k=2n$, $2n+1$, implies that $H=\{1\}$. By induction this implies that $G=Z_p$, a contradiction. Hence $G=\{1\}$.

(2) If $1 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_r=G$ such that H_i/H_{i-1} cyclic, then $|H^3(G, Z)| \leq \prod_{i=1}^{r-1} |H_i/[H_i, H_i]|$. There is an alternative treatment of these sequences. Let $\bar{X}=$a complete resolution for Z,

$$M = \{0 \leftarrow ZK \leftarrow ZK \leftarrow 0\}, \quad M^* = \mathrm{Hom}\,(M, Z).$$

The bicomplex $\mathrm{Hom}_G\,(X, M^*)$ gives rise to two spectral sequences. If

$$T = H^*(\mathrm{Hom}_G\,(X, M^*)),$$

we find easily the sequences:

$$(5) \qquad \cdots \longrightarrow \hat{H}^{i-1}(G, Z) \xrightarrow{\cup\alpha} \hat{H}^{i+1}(G, Z) \xrightarrow{\rho} T^{i+1} \xrightarrow{\tau} \hat{H}^i(G, Z) \xrightarrow{\cup\alpha} \hat{H}^{i+2}(G, Z)$$

$$(5)' \qquad 0 \longrightarrow \hat{H}^i(H, Z)_K \xrightarrow{\mu} T^{i+1} \xrightarrow{\varepsilon} \hat{H}^{i+1}(G, Z)^K \longrightarrow 0,$$

where $\varepsilon \circ \rho = \text{Res}$, $\tau \circ \mu = \text{Cor}$. Here $\alpha = \inf \chi$, $\chi \in H^2(K, \boldsymbol{Z})$, a maximal generator. Again, this holds for any module in place of $\boldsymbol{Z}$.

3. Multiply periodic resolutions.

3.1. DEFINITION. G is *periodic* if and only if there is a natural isomorphism of functors, $\hat{H}^0(G, \cdot) \approx \hat{H}^n(G, \cdot)$, for some $n > 0$. The least such n is the *period* of G. Call it $n(G)$.

It can be shown that $n(G)$ is always even [3, Chapter XII, §11].

THEOREM 3.0 (ARTIN-TATE). *G is periodic if and only if*

$$G = \text{cyclic}, \quad p \text{ odd},$$
$$= \text{cyclic or generalized quaternion}, \; p = 2.$$

Proof. See [3, Chapter XII, §11].

COROLLARY. *If G is periodic, then so is any subgroup. Thus $Z_p \oplus Z_p \not\subset G$, for any p.*

The least n such that $\hat{H}^0(G, \cdot)_p \approx \hat{H}^n(G, \cdot)_p$ is called the *p-period* of G, denoted by $n_p(G)$. $n_p(G) < \infty$ if and only if G_p is cyclic, p odd, or cyclic or generalized quaternion, $p = 2$, [3, Example 11, p. 265], $n(G) = \text{l.c.m.}\,\{n_p(G) : p \mid |G|\}$.

THEOREM 3.1 (SWAN). *If G is p-periodic, then*

$$n_p(G) = 2|N_G(G_p)/C_G(G_p)|, \quad p \text{ odd},$$
$$= 2, \quad \text{if } G_2 \text{ cyclic},$$
$$= 4, \quad \text{if } G_2 \text{ generalized quaternion}.$$

Proof. [12].

DEFINITION. G is *spheriodic* if and only if G acts freely on S^n, some n.

DEFINITION. G is *uniodic* if and only if G has a fix-point free unitary representation.

It is easy to see that uniodic implies spheriodic implies periodic.

THEOREM 3.2 (MILNOR). *If G is spheriodic, then every involution lies in the center.*

Proof [Amer. J. Math. **79** (1957), 623–630]. As $Z_2 \oplus Z_2 \not\subset G$, there is at most one involution in G.

EXAMPLE. Σ_3 is periodic of period 4, but not spheriodic.

LEMMA 3.3. *Let $1 \to H \to G \xrightarrow{\pi} K \to 1$ be a group extension. If M is an H-projective G-module, N a K-projective module, then $M \otimes N$, with action $g(m \otimes n) = gm \otimes \pi(g)n$, is G-projective.*

Proof. See [4, p. 231]. The proof given there for the split case works in general.

DEFINITION. If C, C' are chain complexes, a morphism $f: C \to C'$ is called a *chain map of degree r* if (1) $f: C_n \to C_{n+r}$ for all n, (2) $f\partial = (-1)^r \partial f$.

LEMMA 3.4. *Let X be a complete resolution of $\mathbf{Z}$, and suppose $f: X \to X$ is a chain map of degree $-n$. Then there exists $\eta \in H^n(G, \mathbf{Z})$ such that, for all G-modules A, the map $f^*: \hat{H}^i(G, A) \to \hat{H}^{i+n}(G, A)$ is just cup-product with η.*

Proof. Let $\varepsilon: X_0 \to \mathbf{Z}$ be the augmentation. Then $\varepsilon \circ f_n: X_n \to \mathbf{Z}$ is an n-cocycle, representing a class $\eta \in H^n(G, \mathbf{Z})$. The verification that $f^* = \eta \cup$ is routine, and we omit details.

DEFINITION. A finite complex C is *n-spherical* if and only if C has the homology of an n-sphere. Thus,

$$C = \{0 \leftarrow \mathbf{Z} \leftarrow C_0 \leftarrow \cdots \leftarrow C_n \leftarrow \mathbf{Z} \leftarrow 0\}$$

is exact.

LEMMA 3.5. *If $H \lhd G$, and H possesses a spherical projective complex, then there exists a spherical G-complex which is H-projective.*

Proof. Let $r = |G:H|$, C = an n-spherical projective H-complex. Let $\dot{C}$ be the augmented complex of C, with degrees raised by 1. Form $\dot{C}^r = \dot{C} \otimes \cdots \otimes \dot{C}$ (r times). Let C^r = the deaugmented complex of $\dot{C}^r$, with degrees lowered by 1. Künneth $\Rightarrow C^r$ is $(nr + r - 1)$-spherical.

As shown in [4, p. 231], C^r is a $\Sigma_r \wr H$-complex, where Σ_r acts by permuting factors, and H^r acts pointwise. A choice of left coset representatives for H in G yields a monomorphism $\Phi: G \to \Sigma_r \wr H$. Make C^r a G-module via pullback by Φ. This G-module structure is independent of the choice of coset representatives [4]. $H \lhd G$ implies that $\Phi(H) \subset H^r$. Lemma 3.3, iterated, implies that C^r is H^r-projective, hence H is projective under Φ. Q.E.D.

Suppose now that H and G/H are periodic. By [14], [7] we know that H and G/H possess spherical projective complexes, C and D. By Lemma 3.5, we may suppose that C is a G-complex.

Let

$$C = \{0 \leftarrow \mathbf{Z} \leftarrow C_0 \leftarrow \cdots \leftarrow C_b \leftarrow \mathbf{Z} \leftarrow 0\}.$$

Splicing yields periodic

$$D = \{0 \leftarrow \mathbf{Z} \leftarrow D_0 \leftarrow \cdots \leftarrow D_a \leftarrow \mathbf{Z} \leftarrow 0\}$$

resolutions X, Y of $\mathbf{Z}$, X an H-projective G-resolution, Y a G/H-projective resolution. Let $f: X \to X$, $g: Y \to Y$ be the chain maps of degree $-(b+1)$, $-(a+1)$, respectively, given by

$$f: X_{i+b+1} \xrightarrow{\text{identity}} X_i, \qquad g: Y_{i+a+1} \xrightarrow{\text{identity}} Y_i.$$

(Recall that $a+1$, $b+1$ are necessarily even.) Form $W = Y \otimes X$. This is doubly periodic, and by Lemma 3.3, is a G-projective resolution of $\mathbf{Z}$. If $\varepsilon: W \to \mathbf{Z}$ is the augmentation, then $\varepsilon \circ (g \otimes 1)$, $\varepsilon \circ (1 \otimes f)$ are cocycles.

Set $\eta_1 = [\varepsilon \circ (g \otimes 1)]$, $\eta_2 = [\varepsilon \circ (1 \otimes f)]$, $\eta_1 \in H^{a+1}(G, \mathbf{Z})$ and $\eta_2 \in H^{b+1}(G, \mathbf{Z})$.

By Lemma 3.4, if $\varphi: W \to Z$ is a cocycle, then $\eta_1 \cup [\varphi] = [\varphi \circ (g \times 1)]$. Thus, to multiply $[\varphi]$ by η_1 one translates φ $a+1$ units east. Similarly for η_2.

By iteration of the above procedure, one proves:

THEOREM 3.6. *If $1 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_r = G$ is a chain of subgroups, each normal in G, and such that H_{i+1}/H_i is periodic, then G possesses an $(r+1)$-ply periodic projective resolution of Z.*

REMARK. One might hope that the cohomology classes corresponding to the perpendicular directions in a multiply periodic resolution would be algebraically independent in some sense. This seems incorrect. Thus if $1 \to Z_4 \to Q \to Z_2 \to 1$, Q = quaternion group, let η = inflation of the maximal generator of Z_2. η corresponds to a translation in the obvious doubly periodic resolution. But $\eta^2 = 0$ (see §4). Again, for groups of type II, we will see that the inflations η_1, η_2 of $\chi_A, \chi_B \in H^2(G/Z(G), Z)$ satisfy $\eta_1\eta_2^p = \eta_2\eta_1^p$. But η_1, η_2 correspond to mutually perpendicular directions in a triply periodic resolution of G.

3.2. We now look at the special case $G = (x, y : x^m = y^n = 1, xy = yx)$ such that $n|m$. Set $\Lambda = ZG$. The obvious doubly periodic resolution here is $W = \sum_{i,j \geq 0} \Lambda a_{i,j}$ (direct). Let $a_{i,j} = 0$ if $i < 0$ or $j < 0$. The boundary is

$$\begin{aligned} \partial a_{i,j} &= T_y a_{i-1,j} - T_x a_{i,j-1} && i, j \text{ odd,} \\ &= N_y a_{i-1,j} + N_x a_{i,j-1} && i, j \text{ even,} \\ &= T_y a_{i-1,j} - N_x a_{i,j-1} && i \text{ odd, } j \text{ even,} \\ &= N_y a_{i-1,j} + T_x a_{i,j-1} && i \text{ even, } j \text{ odd,} \end{aligned}$$

with $T_x = x - 1$, $N_x = \sum_{i=0}^{m-1} x^m$, and so on.

The periodicity chain maps $\eta_1, \eta_2: W \to W$ are of degree -2. $\eta_1: a_{i,j} \to a_{i-2,j}$, $\eta_2: a_{i,j} \to a_{i,j-2}$. The associated cocycles are $\bar{\eta}_1: a_{2,0} \to 1$, $a_{1,1} \to 0$, $a_{0,2} \to 0$. Let $\alpha = [\bar{\eta}_1]$, $\beta \in [\bar{\eta}_2]$ in $H^2(G, Z)$. $\bar{\eta}_2: a_{0,2} \to 1$, $a_{1,1} \to 0$, $a_{2,0} \to 0$. By inspection, we see that all 3-cocycles are multiples of f: $a_{03} \to 0$, $a_{12} \to -m/n$, $a_{21} \to 1$, $a_{30} \to 0$. Let $\xi = [f] \in H^3(G, Z)$. We see easily that $H^*(G, Z) = Z[\alpha, \beta, \xi)$ as ring. The only possible relations are of the form $\xi^2 = p(\alpha, \beta)$ = polynomial in α and β. If $|G|$ is odd, then $\xi^2 = 0$. If $|G|$ is even, ξ^2 may or may not be zero.

EXAMPLE. $G = Z_4 \oplus Z_2 = (x, y: x^4 = y^2 = 1, xy = yx)$.

PROPOSITION 3.7. *$H^*(Z_4 \oplus Z_2, Z) = P[\alpha, \beta] \otimes E(\xi)$. That is, $4\beta = 2\alpha = \xi^2 = 0$ are the only relations.*

Proof. It is enough to show $\xi^2 = 0$. Let $j: Z \xrightarrow{\text{epi}} Z_2$. This gives $j_*: H^*(G, Z) \to H^*(G, Z_2)$, a ring morphism.

$$H^*(Z_4 \oplus Z_2, Z_2) \cong H^*(Z_4, Z_3) \otimes H^*(Z_2, Z_2)$$

by Künneth. [17, p. 68] says that $H^*(Z_2, Z_2) = P[u_1]$, deg $u_1 = 1$. A simple-minded calculation, using [5, Chapter XII, §7], gives $H^*(Z_4, Z_2) = P[v_2] \otimes E(u_2)$, with

$\deg u_2=1$, $\deg v_2=2$. In terms of our resolution, we may take representative cocycles

$$u_1\colon a_{10}\to 1,\quad a_{01}\to 0,\quad u_2\colon a_{01}\to 1,\quad a_{10}\to 0,\quad v_2\colon a_{02}\to 1,\quad a_n\to 0,\quad a_{20}\to 0.$$

Cupping with u_1^2 shifts a cocycle two units east. As ξ is represented by $a_{12}\to -2$, $a_{21}\to 1$, a_{03}, $a_{30}\to 0$, clearly $j_*(\xi)=u_2u_1^2$. Therefore $j_*(\xi^2)=u_2^2u_1^4=0$. By inspection, $j_*\colon H^6(G,Z)\to H^6(G,Z)$ is mono except on powers of β. $\mathrm{Res}_{\langle x\rangle}(\xi)=0$, implies that ξ^2 is no such power. Hence $\xi^2=0$.

REMARK. In §4 we will give a second proof of this fact, using the spectral sequence.

4. Abelian, quaternion and dihedral groups.

4.0. In [1, p. 61], Atiyah showed that if $Q=$quaternions, then $H^*(Q,Z)=Z[\alpha,\beta,\chi]$, $\deg\alpha=\deg\beta=2$, $\deg\chi=4$, with relations $8\chi=2\alpha=2\beta=\alpha^2=\beta^2=0$, $\alpha\beta=4\chi$.

In [6], Evens stated that for $D=$dihedral of order 8, $H^*(D,Z)=Z[\alpha,\beta,\nu,\zeta]$, $\deg\alpha=\deg\beta=2$, $\deg\nu=3$, $\deg\zeta=4$, with relations $4\zeta=2\alpha=2\beta=2\nu=0$, $\nu^2=\alpha\zeta$, $\beta^2=\alpha\beta$.

4.1. R. G. Swan pointed out that using the Bockstein Δ it is easy to get $H^*(Z_p\oplus Z_p,Z)$. Indeed $H^*(Z_p\oplus Z_p,Z_p)\cong H^*(Z_p,Z_p)\otimes H^*(Z_p,Z_p)$ by Künneth. It is well-known that $H^*(Z_p,Z_p)=E(u)\otimes P(v)$, $\deg u=1$, $\deg v=2$, $\Delta u=v$, (p odd), and $H^*(Z_2,Z_2)=P[v]$, $\deg v=1$, $\Delta v=v^2$. The Bockstein arises from the sequence

$$0\longrightarrow Z\xrightarrow{p}Z\xrightarrow{j}Z_p\longrightarrow 0,$$

which yields exact sequences

$$0\longrightarrow H^k(Z_p\oplus Z_p,Z)\longrightarrow H^k(Z_p\oplus Z_p,Z_p)\xrightarrow{\delta}H^{k+1}(Z_p\oplus Z_p,Z)\longrightarrow 0$$

(because $H^*(Z_p\oplus Z_p,Z)$ is of exponent p in positive dimensions: Künneth gives us the additive structure of $H^*(Z_p\oplus Z_p,Z)$). $\Delta=j_*\circ\delta$. As j_* is mono, $\mathrm{Ker}\,\Delta=\mathrm{Ker}\,\delta=H^*(Z_p\oplus Z_p,Z)$. Δ is a derivation, and we can calculate $\mathrm{Ker}\,\Delta$ readily. As j_* is a ring morphism we get:

PROPOSITION 4.1. *$H^*(Z_p\oplus Z_p,Z)=P[\alpha,\beta]\otimes E(\mu)$, $\deg\alpha=\deg\beta=2$, $\deg\mu=3$, p odd; $H^*(Z_2\oplus Z_2,Z)=P[\alpha,\beta]\otimes Z[\mu]$, $\deg\alpha=\deg\beta=2$, $\deg\mu=3$, with relations $\mu^2=\alpha\beta^2+\beta\alpha^2$.*

In a similar fashion we find

PROPOSITION 4.2. *$H^*(Z_p\oplus Z_p\oplus Z_p,Z)=Z[\alpha,\beta,\gamma,\mu,\nu,\chi,\xi]$ (p odd), where $\deg\alpha=\deg\beta=\deg\gamma=2$, $\deg\mu=\deg\nu=\deg\chi=3$, $\deg\xi=4$. The relations are*

$$\mu^2=\nu^2=\chi^2=\xi^2=0,\qquad \alpha\nu+\beta\chi+\gamma\mu=0,$$
$$\gamma\mu=\alpha\xi,\quad \nu\chi=\gamma\xi,\quad \mu\nu=\beta\xi,\quad \mu\xi=\nu\xi=\chi\xi=0.$$

$H^*(Z_2 \oplus Z_2 \oplus Z_2, Z) = \boldsymbol{Z}[\alpha, \beta, \gamma, \mu, \nu, \chi, \xi]$, *where the degrees are the same and*

$$\begin{aligned}&\mu^2 = \alpha\beta^2 + \beta\alpha^2, \quad \nu^2 = \beta\gamma^2 + \gamma\beta^2, \quad \chi^2 = \gamma\alpha^2 + \alpha\gamma^2,\\ &\alpha\nu + \beta\chi + \gamma\mu = 0, \quad \xi^2 = \alpha^2\beta\gamma + \beta^2\gamma\alpha + \gamma^2\alpha\beta,\\ &\chi\mu = \alpha\xi + \alpha\beta\gamma, \quad \nu\chi = \gamma\xi + \alpha\beta\gamma, \quad \mu\nu = \beta\xi + \alpha\beta\gamma,\\ &\mu\xi = \alpha\beta(\nu + \chi), \quad \nu\xi = \beta\gamma(\chi + \mu), \quad \chi\xi = \gamma\alpha(\mu + \nu).\end{aligned}$$

4.2. PROPOSITION 4.3. $H^*(Z_{p^2} \oplus Z_p, Z) = P[\alpha, \beta] \otimes E(\chi)$ *for all* p, *where* $\deg \alpha = \deg \beta = 2$, $\deg \chi = 3$, $p^2\alpha = p\beta = p\chi = 0$.

Proof. Let $G = (x, y\colon x^{p^2} = y^p = 1, xy = yx)$. Consider the spectral sequence of the extension $1 \to Z_{p^2} \to G \to Z_p \to 1$. $E_2^{i,j} \cong H^i(Z_p, H^j(Z_{p^2}, Z))$, $E_2^{2,0} \cong Z_{p^2}\beta$, $E_2^{1,2} \cong Z_p\chi$. Using the multiplication rule in [3, Chapter XII, §7], we find that α, β, χ generate E_2. Thus,

$$E_2^{*,0} = \sum_{i=0}^{\infty} Z\alpha^i, \quad E_2^{0,*} = \sum_{i=0}^{\infty} Z\beta^i, \quad E_2^{*,2} = \sum_{i=0}^{\infty} Z\alpha^i\beta \oplus Z\alpha^i\chi.$$

Also,

$$\alpha\colon E_2^{i,j} \xrightarrow{\cong} E_2^{2+2,j} \quad (i, j \geqq 0), \qquad \beta\colon E_2^{i,j} \xrightarrow{\cong} E_2^{i,j+2} \quad (i \geqq 0, j > 0).$$

As the extension is split, the terms on the base are never hit. (See §1.3, Corollary I.) This means that α, β, χ are universal cycles, whence $E_2 = E_\infty$. If p odd, then $\deg \chi = 3$ implies that $\chi^2 = 0$ in $H^*(G, \boldsymbol{Z})$. Thus for p odd we have our result. If $p = 2$, we must prove $\chi^2 = 0$. In both cases we will be very explicit about α and β. We choose $\alpha = \inf \chi_y \in H^2(Z_2, Z)$, $\beta = \inf \chi_x \in H^2(Z_4, Z)$, χ_x, χ_y being maximal generators. As $H^2(G, \boldsymbol{Z}) \cong^{\delta} \operatorname{Hom}(G, \boldsymbol{Q}/\boldsymbol{Z})$, and as Inf and δ commute, we can take $\alpha\colon x \to 0, y \to 1/p$, $\beta\colon x \to 1/p^2, y \to 0$. Now let $p = 2$. Define $\lambda \in \operatorname{Aut}(Z_4 \oplus Z_2)$ by $\lambda\colon y \to yx^2, x \to xy$. We find that $\alpha^\lambda = \alpha + 2\beta$, $\beta^\lambda = \beta + \alpha$, where we let λ also designate the induced ring automorphism of $H^*(Z_4 \oplus Z_2, Z)$. As $H^3(G, \boldsymbol{Z}) = \boldsymbol{Z}_2\chi$, we must have $\chi^\lambda = \chi$. Since $\chi^2 = 0$ in $E_2 = E_\infty$, any relation will be of the form $\chi^2 = a\alpha\beta^2 + b\alpha^2\beta + c\alpha^3$. Apply λ to this. We find

$$\begin{aligned}\chi^2 = a\alpha\beta^2 + b\alpha^2\beta + c\alpha^3 &= a(\alpha+2\beta)(\beta^2+\alpha^2) + b(\alpha+2\beta)^2(\beta+\alpha) + c(\alpha+2\beta)^3\\ &= 2a\beta^3 + a\alpha\beta^2 + b\alpha^2\beta + (a+b+c)\alpha^3.\end{aligned}$$

As β^3, $\alpha^2\beta$, $\alpha\beta^2$, α^3 are linearly independent in $E_2 = E_\infty$, hence(2) in $H^*(G, \boldsymbol{Z})$, we conclude that $a + b \equiv 0$ (2), $2a \equiv 0$ (4). Hence $a \equiv b \equiv 0$ (2). So our relation boils down to $\chi^2 = c\alpha^3$. But a glance at the spectral sequence of $0 \to Z_2 \to G \to Z_4 \to 0$ shows that $\operatorname{Res}_{Z_2}(\chi) = 0$, $\operatorname{Res}_{Z_2}(\alpha) = \chi_y$. Applying Res_{Z_2} to our relation thus gives: $0 = c\chi_y^3$, implying $c = 0$. Thus $\chi^2 = 0$ as desired. Q.E.D.

REMARK 4.4. In the above discussion we have used α, β, χ to denote elements of $E_2 = E_\infty$ *and* elements of $H^*(G, Z)$ which map onto the former under the projection maps π_r in the exact sequences

$$0 \to H^i(G, \boldsymbol{Z})_{r+1} \to H^i(G, \boldsymbol{Z})_r \to E_\infty^{r,i-r} \to 0.$$

We will continue this convenient abuse of notation in future sections.

(2) As $\beta^i = \inf \chi_x^i$, we know that $p^2\beta^i = 0$ in $H^*(G, \boldsymbol{Z})$.

5. **Groups of type** I. C. T. C. Wall has calculated the additive structure of $H^*(G, Z)$, where G is a split extension of cyclic groups [16], as follows:

Proposition 5.1 (Wall). *Let* $G=(x, y : x^r=y^s=1, y^{-1}xy=x^t)$. *Then* $H^{2i}(G, Z)=Z_s \oplus \sum_{k<i} Z_{q_k} \oplus Z_{h_i}$, *where* $h_i=(t^i-1, r)$, $k_i=(\sum_{j=0}^{s-1} t^{ij}, r)$, $H^{2i+1}(G, Z)=\sum_{k<i} Z_{q_k}$ *and* $q_i=h_ik_i/r$.

Wall also noted that in the spectral sequence of

$$1 \to Z_r \to G \to Z_s \to 1,$$

$E_2=E_\infty$. I will consider the special case $r=p^2$, $s=p$, $t=1+p$. As $p=2$ implies $G=D$, I will suppose p odd. As

$$\begin{aligned} h_i &= p \quad \text{if } p\nmid i, \\ &= p^2 \quad \text{if } p\mid i, \end{aligned}$$

$k_i=p$, so

$$\begin{aligned} q_i &= 1 \quad p\nmid i, \\ &= p \quad p\mid i. \end{aligned}$$

Proposition 5.1 now gives us the additive structure. In particular, $H^{2i}(G, Z)=Z_p \oplus Z_p$ if $i<p$, $=Z_p \oplus Z_{p^2}$ if $i=p$, and $H^{2i+1}(G, Z)=0$ for $0<i<p$. Let $H=\langle x\rangle$, $K=\langle y\rangle$. We have $E_2^{i,j} \simeq H^i(K, H^j(H, Z))$. $H^*(H, Z)=P[\beta]$, $\deg \beta=2$, $p^2\beta=0$. As $H \lhd G$, y acts on $H^*(H, Z)$. $H^2(H, Z) \simeq^{\delta} \operatorname{Hom}(H, Q/Z)$, and the action of y commutes with δ. $y^{-1}xy=x^{1+p}$.

Let $\beta: H \to Q/Z$. $\beta(x)=1/p^2$. Then

$$y\beta(x) = y\cdot\beta(y^{-1}xy) = (1+p)/p^2.$$

Thus $y\beta=(1+p)\beta$, and $y\beta^i=(1+p)^i\beta^i=(1+pi)\beta^i$, since $p^2\beta=0$. Thus, we find:

$$\begin{aligned} H^i(H, Z)^K &= \beta^i \quad \text{if } p\mid i, \\ &= \beta_i \quad \text{if } p\nmid i, \end{aligned}$$

where $\beta_i=p\beta^i$.

By periodicity,

$$\begin{aligned} H^i(K, H^j(H, Z)) &\overset{\phi}{\cong} \hat{H}^0(K, H^j(H, Z)) \quad i \text{ even}, \\ &\overset{\phi}{\cong} \hat{H}^1(K, H^j(H, Z)) \quad i \text{ odd}, \end{aligned}$$

where $\phi=\alpha\cup$, $\alpha \in H^2(K, Z)$, a maximal generator. As $N_yH^j(H, Z)=N_y(\beta^i)=p\beta^i$, so $\operatorname{Ker} N_y=p\beta^i$. Hence

$$\begin{aligned} \hat{H}^0(K, H^j(H, Z)) &= 0, \quad \text{if } p\nmid i, \\ &= Z\bar{\beta}^i, \quad \text{if } p\mid i, \end{aligned}$$

where $p\bar{\beta}=0$.

$$\begin{aligned} \hat{H}^1(K, H^j(H, Z)) &= 0, \quad \text{if } p\nmid i, \\ &= pZ\beta^i, \quad \text{if } p\mid i. \end{aligned}$$

Applying ϕ, we get: (setting $\zeta=\beta^p$) (and letting $\chi \in E_2^{1,2p}$ be a generator) $E_2^{0,2j}=Z\beta_{j_0}\zeta^a$, where $j=j_0+ap$, $0\leqq j_0<p$, $E_2^{i,j}=0$ if $2p\nmid j$ *or* j odd, $i\geqq 0$. $E_2^{*,0}=\sum_{i=0}^{\infty} Z\alpha^i$, $E_2^{i,2pj}=Z\alpha^{i/2}\zeta^{j-1}$, i even, $=Z\chi\alpha^{(i-1)/2}\zeta^{j-1}$, i odd, $j>0$.

We can now see that $E_2=E_\infty$. For as extension (*) is split, the terms on the base are never hit. Thus, $\alpha, \beta_1, \ldots, \beta_{p-1}, \zeta, \chi$ are all universal cycles, and generate E_2. Therefore $E_2=E_\infty$. χ is of odd degree, so $\chi^2=0$. In $E_2=E_\infty$, $\beta_2\chi=\beta_i\beta_j=0$, all i, j, and $p^2\zeta=0$. I claim $p^2\zeta=0$ in $H^*(G, Z)$. For if not then ζ would be a maximal generator, implying G is periodic [3, Chapter XII, §11]. Contradiction. Done.

Conceivably, $\beta_i\beta_j=d_{i,j}\alpha^{i+j}$, $d_{i,j}\neq 0$, or $\beta_i\chi=d_i\alpha^i\chi$, $d_i\neq 0$. To investigate, we use the restriction-corestriction sequences. From our E_∞ term we find

$$|H^i(G, \boldsymbol{Z})| = 0 \quad i \text{ odd}, \qquad 0<i<2p.$$
$$= p^2 \quad i \text{ even},$$

Proposition 2.1 yields

$$0 \longrightarrow H^{2i-1}(G, Z) \longrightarrow H^{2i}(G, X) \longrightarrow H^{2i}(H, Z)$$
$$\xrightarrow{\text{Cor}} H^{2i}(G, Z) \longrightarrow H^{2i+1}(G, X) \longrightarrow 0$$

and

$$0 \longrightarrow H^{2i-1}(G, X) \longrightarrow H^{2i}(G, Z) \xrightarrow{\text{Res}} H^{2i}(H, Z)$$
$$\longrightarrow H^{2i}(G, X) \longrightarrow H^{2i+1}(G, Z) \longrightarrow 0.$$

As Im $(\text{Res}_{2i})=\boldsymbol{Z}\beta_i$, of order p (cf. Corollary II, §1), we get

$$|H^{2i-1}(G, X)| = p, \quad |H^{2i}(G, X)| = p, \qquad 1 \leqq i < p.$$

Hence $|\text{Im}\,(\text{Cor}_{2i})|=p$ $(1\leqq i<p)$. Thus $\text{Cor}\,\beta^i\neq 0$, $p\,\text{Cor}\,\beta^i=0$ $(1\leqq i<p)$. The formula $\text{Cor}\,(\text{Res}\,(a)\cdot b)=a\,\text{Cor}\,b$ [3, Chapter XII, §8], and $\text{Res}\,\alpha=0$, implies $\alpha\,\text{Cor}\,\beta^i=0$. And $\text{Res Cor}\,\beta^i=N_y\beta^i=p\beta^i=\beta_i$. Thus, in $H^*(G, Z)$ we can choose $\beta_i=\text{Cor}\,\beta^i$ (see Remark 4.4), so that $p\beta_i=0$, $\alpha\beta_i=0$ in H^*. Now suppose $\beta_i\beta_j=d\alpha^{i+j}$. Thus implying $0=\alpha\beta_i\beta_j=d\alpha^{i+j+1}$ which implies $d=0$. And if $\beta_i\chi=d\alpha^i\chi$, then $0=\alpha\beta_i\chi=d\alpha^{i+1}\chi$. As $\alpha^{i+1}\chi\neq 0$ in H^*, so $d=0$. We summarize our results in the following

THEOREM 5.2. *If* $G=(x, y : x^{p^2}=y^p=1, y^{-1}xy=x^{1+p})$, *then the ring* $H^*(G, \boldsymbol{Z}) =\boldsymbol{Z}[\alpha, \chi, \zeta, \beta_1, \ldots, \beta_{p-1}]$, $\deg\alpha=2$, $\deg\beta_i=2i$, $\deg\zeta=2p$, $\deg\chi=2p+1$, *with relations* $p^2\zeta=p\alpha=p\chi=p\beta_i=0$, $\chi^2=0$, $\beta_i\alpha=\beta_i\chi=\beta_i\beta_j=0$, *all* i, j. *If* $\beta\in H^2(\langle x\rangle, \boldsymbol{Z})$ *is a maximal generator, we may take* $\beta_i=\text{Cor}\,\beta^i$.

6. **Groups of type** II.

6.1. Let $G=(A, B : A^p=B^p=[A, B]^p=[A, [A, B]]=[B, [A, B]]=1)$. Let $C=[A, B]=B^{-1}A^{-1}BA$. Thus $A^{-1}BA=BC$. Set $\alpha_j=|\hat{H}^j(G, \boldsymbol{Z})|$. Our calculation of $H^*(G, \boldsymbol{Z})$ will proceed in leisurely steps.

Step 1. A proof of the inequality $\alpha_{j+2p}\leqq p^{p+1}\alpha_j$, all j. Also, the observation that for $i\leqq p$, $\alpha^i, \alpha^{i-1}\beta, \ldots, \beta^i$ are linearly independent, and that for $i<p$, $\alpha^i\mu, \alpha^{i-1}\beta\mu, \ldots, \alpha\beta^{i-1}\mu$ are likewise linearly independent. (Here, $\alpha, \beta\in H^2(G, \boldsymbol{Z})$, $\mu\in H^3(G, \boldsymbol{Z})$ are certain cohomology classes.)

Step 2. The calculation of the E_2 terms of the spectral sequences of the extensions

$$1 \to \langle c \rangle \to G \to \langle \bar{A}, \bar{B} \rangle \to 1$$

and

$$1 \to H \to G \to \langle A \rangle \to 1 \quad \text{(split)},$$

where $H = \langle B, C \rangle$.

Step 3. Using the restriction-corestriction sequences, we get enough information to calculate the E_∞ terms.

Step 4. By observing the effect of automorphisms on the generators of H^*, we find the relations.

6.2. Let $H = \langle B, C \rangle$, $H \lhd G$. This has a 1-dimensional unitary representation as follows: B acts as identity, C as multiplication by ζ, where $\zeta = e^{2\pi i/p}$. Form $\mathbf{C}G \otimes_H \mathbf{C}$, the induced representation. $1 \otimes 1, A \otimes 1, \ldots, A^{p-1} \otimes 1$ form a basis. For this basis, A, B, C have matrices

$$A \sim \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad B \sim \begin{pmatrix} 1 & & & & \\ & \zeta & & & \\ & & \zeta^2 & & \\ & & & \ddots & \\ & & & & \zeta^{p-1} \end{pmatrix},$$

$$C \sim \begin{pmatrix} \zeta & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta \end{pmatrix}.$$

The induced representation is again unitary. Thus H acts on $S^1 \subset \mathbf{C}$, and G acts on $S^{2p-1} \subset \mathbf{C}^p$, with C acting freely. Using the fact $S^{2p-1} = S^1 * \cdots * S^1$ (p-fold join), we will construct a G-invariant decomposition for S^{2p-1}. In S^1, take $e = 1$, $f = \{z \mid 0 \leqq \arg z \leqq 2\pi/p\}$ as generating cells for an H-invariant decomposition. Thus the 0-cells are $e, \zeta e, \ldots, \zeta^{p-1}e$, the 1-cells $f, \zeta f, \ldots, \zeta^{p-1}f$, and set $\partial f = \zeta e - e$. Let $C(S^1)$ be the corresponding complex, i.e. $C(S^1) = \{C_0 \xleftarrow{\partial} C_1\}$. If $\dot{C}(S^1) =$ augmented complex with dimensions raised by 1, form $\dot{C}(S^{2p-1}) = \dot{C}(S^1) \otimes \cdots \otimes \dot{C}(S^1)$ $= p$-fold tensor product. Finally, set $C(S^{2p-1}) =$ deaugmented complex of $\dot{C}(S^{2p-1})$, with dimensions lowered by 1. As in §3, $C(S^{2p-1})$ is a G-complex and is C-projective (actually C-free). Take $G = \bigcup_{i=0}^{p-1} A^i H$, yielding the monomorphism $\Phi: G \to \Sigma_p \wr H$. $C(S^{2p-1})$ is made a G-complex via Φ. If $x_1 \otimes \cdots \otimes x_p$ is a typical decomposable element, then $C(x_1 \otimes \cdots \otimes x_p) = \zeta x_1 \otimes \cdots \otimes \zeta x_p$, $B(x_1 \otimes \cdots \otimes x_p) = x_1 \otimes \zeta x_2 \otimes \cdots \otimes \zeta^{p-1} x_p$, $A(x_1 \otimes \cdots \otimes x_p) = (-1)^\nu x_p \otimes x_1 \otimes \cdots \otimes x_{p-1}$ where $\nu = \deg x_p (\sum_{i=1}^{p-1} \deg x_i)$. Write

$$C(S^{2p-1}) = \{C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_{p-1} \leftarrow \cdots \leftarrow C_{2p-1}\}.$$

We will examine the modules C_i, and will show that they are G-free except for C_0, C_1, C_{p-1} and C_{2p-1}. Let $x_1 \otimes \cdots \otimes x_p \in C_n$ be a decomposable element. Thus, each x_i is 1, or e, or f. The elements

$$B^iC^jx_1 \otimes \cdots \otimes x_p = \zeta^j x_i \otimes \zeta^{i+j}x_2 \otimes \zeta^{2i+j}x_3 \otimes \cdots \otimes \zeta^{(p-1)i+j}x_p$$

will be distinct ($1 \leqq i, j < p$), hence Z-independent, unless $\zeta^{ki+j}x_{k+1} = x_{k+1}$ ($0 \leqq k < p$). As the $\{ki+j\}$ are incongruent mod p, precisely one is divisible by p, say $k_0 i + j \equiv 0$ (p). Thus $x_r = 1$ for $r \neq k_0$, hence

$$x_1 \otimes \cdots \otimes x_p = 1 \otimes 1 \otimes \cdots \otimes x_{k_0} \otimes \cdots \otimes 1,$$

with $x_{k_0} = e$ or f. Thus our element lies in C_0 or in C_1. It is easy to find the structure of these modules. If $\Lambda = ZG$, then $C_0 = \Lambda(e \otimes 1 \otimes \cdots \otimes 1)$. As $B(e \otimes \cdots \otimes 1) = e \otimes \cdots \otimes 1$, we see that $C_0 \cong Z(G/\langle B \rangle)$. (Here, if $\pi \subset \Pi$, a subgroup, then $Z(\Pi/\pi) = \{x\pi \mid x \in \Pi\}$ = left coset space, as a left Π-module.) C_1 has elements of two sorts: $\zeta^i e \otimes 1 \otimes \cdots \otimes 1 \otimes \zeta^j e \otimes 1 \otimes \cdots \otimes 1$, and $f \otimes 1 \otimes \cdots \otimes 1$. An element of the first type generates a G-module $\cong \Lambda$, and as before, $\Lambda(f \otimes 1 \otimes \cdots \otimes 1) \cong ZG/\langle B \rangle$. Thus $C_1 \cong ZG/\langle B \rangle \oplus F$, with F G-free.

Now suppose the $B^iC^jx_1 \otimes \cdots \otimes x_p$ are all distinct. If the $A^kB^iC^jx_1 \otimes \cdots \otimes x_p$ are all distinct, then $\Lambda x_1 \otimes \cdots \otimes x_p \cong \Lambda$. The only $x_1 \otimes \cdots \otimes x_p$ for which this might fail must have all $x_i = e$ or all $x_i = f$, i.e. must lie in C_{p-1} or C_{2p-1} (whence, for $i \neq 0, 1, p-1, 2p-1$, C_i is G-free). Consider $x = \zeta^{a_1}e \otimes \zeta^{a_2}e \otimes \cdots \otimes \zeta^{a_p}e \in C_{p-1}$. Note that on such elements A acts with sign $+1$ (this is essential to our calculation, and depends on the fact that p is odd). Thus if the $A^kB^iC^jx$ are not all distinct, then $A^kB^iC^jx = x$, some k, i, j. Thus, $(A^kB^iC^j)^r x = x$. (Note $k \neq 0$.) $(A^kB^iC^j)^r = A^{kr}B^{ir}C^{r(r-1)ki/2+jr}$. Choose $r \ni kr \equiv 1$ (p). In other words, we may suppose $k = 1$. $AB^iC^jx = x$ implies $BAB^{i-1}C^jBx = Bx$ or $AB^iC^{j+1}Bx = Bx$. So replacing x by $B^{-j}x$ let us assume $j = 0$, or $AB^ix = x$. x can be multiplied by a power of C, and the relation still holds. So we can suppose $a_1 = 0$. Writing our equation as $B^ix = A^{p-1}x$, we see that

$$e \otimes \zeta^{a_2+i}e \otimes \zeta^{a_3+2i}e \otimes \cdots \otimes \zeta^{a_p+(p-1)i}e = \zeta^{a_2}e \otimes \zeta^{a_3}e \otimes \cdots \otimes \zeta^{a_p}e \otimes e$$

implies

$$a_2 = 0,\ a_3 = i,\ a_4 = 3i, \ldots, a_p = C_{p-1,2}i,$$

or

$$x = e \otimes e \otimes \zeta^i e \otimes \zeta^{3i}e \otimes \cdots \otimes \zeta^{C_{p-1,2}i}e.$$

For $i = 0, 1, \ldots, p-1$, we get just p nonfree components of C_{p-1}, so

$$C_{p-1} \cong ZG/\langle A \rangle \oplus ZG/\langle AB \rangle \oplus \cdots \oplus ZG/\langle AB^{p-1} \rangle \oplus F, \quad F\ G\text{-free}.$$

The case of C_{2p-1} is entirely similar. At last we can compute. Consider

$$0 \leftarrow Z \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_{2p-1} \leftarrow Z \leftarrow 0,$$

and let X, Y, V, W, and U be the image-kernels at C_0, C_1, C_{p-2}, C_{p-1}, and C_{2p-1}, respectively. Applying Tate cohomology to the resulting exact sequences, and using $\hat{H}^i(\Pi, Z(\Pi/\pi)) \approx \hat{H}^i(\pi, Z)$, [3, p. 196], we get, for i odd, four long exact sequences, plus two-dimensional shifts, for every i.

(i odd):

(6.1)
$$0 \longrightarrow H^2(G, Z) \longrightarrow H^{i+1}(G, X) \longrightarrow H^{i+1}(\langle B\rangle, Z) \xrightarrow{\text{Cor}} H^{i+1}(G, Z) \longrightarrow H^{i+2}(G, X) \longrightarrow 0,$$

(6.1)
$$0 \longrightarrow H^i(G, X) \longrightarrow H^{i+1}(G, Y) \longrightarrow H^{i+1}(\langle B\rangle, Z) \xrightarrow{\varphi_{i+1}} H^{i+1}(G, X) \longrightarrow H^{i+2}(G, Y) \longrightarrow 0,$$

(6.1)″
$$0 \longrightarrow H^i(G, V) \longrightarrow H^{i+1}(G, W) \xrightarrow{\psi_{i+1}} \sum_{t=0}^{p-1} H^{i+1}(\langle AB^t\rangle, Z) \longrightarrow H^{i+1}(G, V) \longrightarrow H^{i+2}(G, W) \longrightarrow 0,$$

(6.1)‴
$$0 \longrightarrow H^i(G, U) \longrightarrow H^{i+1}(G, Z) \xrightarrow{\text{“Res”}} \sum_{t=0}^{p-1} H^{i+1}(\langle AB^t\rangle, Z) \longrightarrow H^{i+1}(G, U) \longrightarrow H^{i+2}(G, Z) \longrightarrow 0,$$

$$H^i(G, Y) \cong H^{i+p-3}(G, V), \qquad H^i(G, W) \cong H^{i+p-1}(G, U), \qquad \text{all } i.$$

Here
$$\text{“Res” maps } \xi \to (\text{Res}_{L_0}(\xi), \text{Res}_{L_1}(\xi), \ldots, \text{Res}_{L_{p-1}}(\xi)),$$

where $L_t = \langle AB^t\rangle$. Observe that for i odd, $H^{i+1}(\langle B\rangle, Z) \cong H^{i+1}(L_t, Z) \cong Z_p$. So with $\alpha_j = |H^j(G, Z)|$ we get

$$\begin{aligned}|H^{i+2}(G, Z)| &\leqq |H^{i+1}(G, U)| = |H^{i-p+2}(G, W)| \leqq p^p|H^{i-p+1}(G, V)| \\ &= p^p|H^{i-2p+4}(G, Y)| \leqq p^p|H^{i-2p+3}(G, X)| \\ &\leqq p^{p+1}|H^{i-2p+2}(G, Z)|, \quad (i \text{ odd}), \text{ or } \alpha_{i+2} \leqq p^{p+1}\alpha_{i-2p+2}.\end{aligned}$$

In exactly the same way we show that this holds for i even. Hence

(6.2)
$$\alpha_{j+2p} \leqq p^{p+1}\alpha_j \quad \text{for all } j.$$

For $j > 0$, this is an equality. However, I have not been able to prove this directly. It is easy to see the following:

(1) Cor $= 0$ in positive dimensions,
(2) $\varphi_{i+1} = 0$ for i odd, $i > 0$,
(3) “Res” is onto for $i \geqq 2p-3$, i odd.

The trouble is that I have not been able to show that ψ_{i+1} is onto in a suitable range. As I have no further use for the above sequences, I will limit myself to a discussion of the map “Res”, and will estimate $|\text{Im}(\text{“Res”})|$.

PROPOSITION 6.1. *If $\alpha, \beta \in H^2(G, Z)$ are the inflations of maximal generators $\chi_A \in H^2(\langle A\rangle, Z)$, $\chi_B \in H^2(\langle B\rangle, Z)$, respectively, then $\alpha^n, \alpha^{n-1}\beta, \dots, \beta^n \in H^{2n}(G, Z)$ are linearly independent, for $1 \leqq n \leqq p$.*

Proof. Let $L_t = \langle AB^t\rangle$, $(0 \leqq t < p)$. Set $L_p = \langle B\rangle$. As $H^2(G, Z) \cong^{\delta} \mathrm{Hom}\,(G, Q/Z)$, we will take α, β corresponding to $\alpha\colon A \to 1/p$, $B \to 0$, $\beta\colon A \to 0$, $B \to 1/p$. Let $\chi_t \in H^2(L_t, Z) \cong_{\delta} \mathrm{Hom}\,(L_t, Q/Z)$ correspond to $AB^t \to 1/p$ if $0 \leqq t < p$, and to $B \to 1/p$ if $t = p$. χ_t is a maximal generator. Let $a_t = \alpha - t^{-1}\beta$, $t \neq 0$, $a_0 = \beta$, $a_p = \alpha$.

$$\begin{aligned} \mathrm{Res}_{L_j} a_p = \mathrm{Res}_{L_j}(\alpha) &= \chi_j \quad j \neq p, \\ &= 0 \quad j = p, \\ \mathrm{Res}_{L_j}(a_0) = \mathrm{Res}_{L_j}(\beta) &= j\chi_j \quad j \neq p, \\ &= \chi_p \quad j = p. \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{Res}_{L_j}(a_t) &= (1 - jt^{-1})\chi_j \quad j \neq p, t \neq 0, p, \\ &= -t^{-1}\chi_p \qquad j = p, t \neq 0, p. \end{aligned}$$

Thus we see that $\mathrm{Res}_{L_j}(a_t) = 0$ if and only if $j = t$, for all j, t. Recall that the powers of a maximal generator are again maximal generators. Let $I \subset \{0, 1, \dots, p\}$ be a subset $\ni |I| = n$, $1 \leqq n \leqq p$. Form $a_I = \prod_{i \in I} a_i$. We have: $\mathrm{Res}_{L_j}(a_I) = 0$ if and only if $j \in I$. Choosing $I_j = \{0, \dots, \hat{j}, \dots, n\}$ clearly $\{a_{I_0}, \dots, a_{I_n}\}$ is a set of $n+1$ linearly independent elements of $H^{2n}(G, Z)$. As it is a subset of the span of $\{\alpha^n, \alpha^{n-1}\beta, \dots, \beta^n\}$, the latter set must consist of linearly independent elements. Q.E.D.

For smoothness of exposition, we will invoke the following theorem:

THEOREM 6.2 (KUO). *If G is a group of odd order, $l \geqq 0$, e = exponent of $H^{2l+1}(G, Z)$, then e^2 divides the order of G.*

Proof. Tze-Nan Kuo, Thesis, University of Chicago, 1966.

Applying this result to our group of order p^3, we get $pH^{2l+1}(G, Z) = 0$, $l \geqq 0$.

PROPOSITION 6.3. *$H^3(G, Z) \cong Z_p \oplus Z_p$, $H^4(G, Z) \cong Z_p \oplus Z_p \oplus Z_p \oplus Z_p$.*

Proof. By Proposition 6.1, $H^2(G, Z) \xrightarrow{\alpha} H^4(G, Z)$ is mono. Using the second form of the Res-Cor sequences, we get

$$0 = H^1(G, Z) \xrightarrow{\alpha} H^3(G, Z) \xrightarrow{\rho} T^3 \xrightarrow{\tau} H^2(G, Z) \xrightarrow{\alpha} H^4(G, Z)$$

implies $H^3(G, Z) \approx T^3$. Also (with $K = \langle A\rangle$),

$$0 \to H^2(H, Z)_K \to T^3 \to H^3(H, Z)^K \to 0.$$

As $H^*(H, Z) = P[\beta, \gamma] \otimes E(\mu)$, where $\beta\colon B \to 1/p$, $C \to 0$, $\gamma\colon B \to 0$, $C \to 1/p$, with $k\beta = \beta$, $k\gamma = \gamma + \beta$, $k\mu = \mu$, k being the automorphism of $H^*(H, Z)$ induced by A, we find that $H^2(H, Z)_K = Z\gamma$, $H^3(H, Z)^K = Z\mu$, both of order p, implies $p^2 = |T^3| = |H^3(G, Z)|$, and $\mathrm{Res}\colon H^3(G, Z) \to H^3(H, Z)$ is epi. Choose $\mu \in H^3(G, Z)$, Res $\mu=\mu$ (convenient abuse of notation), $\nu \in H^3(G, Z)$, Res $\nu=0$. Clearly $p\nu=0$. Miss Kuo's Theorem implies that $p\mu=0$. Hence $H^3(G, Z)\approx Z_p \oplus Z_p$, with basis $\{\mu, \nu\}$. As $p\mu=$ Cor Res $\mu=$ Cor μ, we see that Cor: $H^3(H, Z) \to H^3(G, Z)$ is zero.

Claim $|H^4(G, Z)|=p^4$. For using the first form of Res-Cor sequences:

$$H^3(G,Z) \xrightarrow{\text{Res}} H^3(H,Z) \longrightarrow H^3(X) \longrightarrow H^4(G,Z) \xrightarrow{\text{Res}} H^4(H,Z).$$

The first Res is epi, the second has image $Z\beta^2$, of order p. Hence $|H^4(G, Z)| =p|H^3(G, X)|$. But

$$H^2(H, Z) \xrightarrow{\text{Cor}} H^2(G, Z) \longrightarrow H^3(G, X) \longrightarrow H^3(H, Z) \xrightarrow{\text{Cor}} H^3(G, Z),$$

exact. By the formula for Cor in §1, we see that Cor$=0$ in dimension 2.

We noted above that Cor$=0$ in dimension 3. As $|H^2(G, Z)|=p^2$, $|H^3(H, Z)|=p$, so $|H^3(G, X)|=p^3$, and $|H^4(G, Z)|=p^4$. By Proposition 6.1, α^2, $\alpha\beta$, β^2 are independent. Let $\chi \in H^4(G, Z)$ be a further generator.

Claim $p\chi=0$. For if $p\chi=a\alpha^2+b\alpha\beta+c\beta^2$, then $p\chi\alpha=0$. This implies that $a\alpha^3+b\alpha^2\beta+c\alpha\beta^2=0$. As $p\geqq 3$, Proposition 6.1 implies that $a=b=c=0$. So $p\chi=0$ and hence $H^4(G, Z)\approx Z_p \oplus Z_p \oplus Z_p \oplus Z_p$, with basis $\{\alpha^2, \alpha\beta, \beta^2, \chi\}$. Q.E.D.

REMARK 1. The author checked Proposition 6.3 by calculating with an explicit resolution. It was a horrible experience.

REMARK 2. Theorem 6.2 was needed only to show that $pH^3(G, Z)=0$. We will prove this by a different method below, so that Theorem 6.2 is not really necessary in proving Proposition 6.3.

One can prove a few more facts using the Res-Cor sequences. From the Swan spectral sequence [12] arising from the action $A(z, w)=(\zeta z, w)$, $B(z, w)=(z, \zeta w)$, $\zeta=e^{2\pi i/p}$, of G on $S^1 \times S^1$, one gets various sequences and estimates, none of which seem any good, *except* as a means of checking the results received via the spectral sequences of group extensions. To these we now turn.

6.3. Set $D=\langle C\rangle$, $G/D=L\approx Z_p \oplus Z_p$. We will study the spectral sequence of the extension $1 \to D \to G \to L \to 1$, following (more or less) the footsteps of Evens. We will refer to this as the *first spectral sequence*. $E_2^{i,j} \cong H^i(L, H^j(D, Z))$. As $D=Z(G)$, the action of L on $H^j(D, Z)$ is trivial.

Let $\gamma_0 \in H^2(D, Z)$ be a maximal generator, and let $\gamma \in E_2^{0,2}$ correspond to γ_0. [For clarity, we will write $Z_p \oplus Z_p$ for L, and Z_p for D.] From an additive point of view,

$$E_2^{*,2j} \cong H^*(Z_p \oplus Z_p, Z_p) \approx H^*(Z_p, Z_p) \otimes H^*(Z_p, Z_p),$$

by Künneth. However, the multiplication on the right-hand side differs from that in the spectral sequence. In fact, it induces a "horizontal" multiplication o: $E_2^{i,2j} \otimes E_2^{k,2j} \to E_2^{i+k,2j}$, each $j>0$.

LEMMA 6.4. *Cup-product* $\gamma: H^i(L, H^j(D, Z)) \xrightarrow{\approx} H^i(L, H^{j+2}(D, Z))$ *is induced by cup-product* $\gamma_0: H^j(D, Z) \xrightarrow{\approx} H^{j+2}(D, Z)$.

LEMMA 6.5. *If* $\eta \in E_2^{i,2j}$, $\xi \in E_2^{k,2j}$, *then* $\eta\xi = \gamma^j \cdot (\eta \circ \xi)$. *In other words,*

$$H^*(L, H^*(D, Z)) \cong H^*(L, Z_p) \otimes H^*(D, Z),$$

as rings, for $j > 0$.

LEMMA 6.6. $\gamma: E_2^{i,j} \to E_2^{i,j+2}$ *is mono,* $j \geqq 0$, *and iso,* $j > 0$.

The proofs of these lemmas are routine, so we omit them (see [3, Chapter XII, §6]).

We may now calculate some of the structure of E_2. Let $H^*(L, Z) = P[\alpha, \beta] \otimes E(\delta)$, α, β as usual, $\alpha, \beta \in E_2^{2,0}$. Let $\mu, \nu \in E_2^{1,2}$ be independent generators, and $\chi = \mu \circ \nu \in E_2^{2,2}$. As $H^2(G, Z) = Z\alpha \oplus Z\beta$, we must have $d_3(\gamma) = s\delta$, $s \in Z_p^*$. By Proposition 6.1, α^2, $\alpha\beta$, β^2 are linearly independent, hence $d_3\mu = d_3\nu = 0$. Thus in degree 3, only μ and ν survive, whence $H^3(G, Z) = Z\mu \oplus Z\nu$. Of course, $p\mu = p\nu = 0$. [This fills the gap in the proof of Proposition 6.3.] $d_2(\gamma^2) = 2s\gamma\delta \neq 0$, $d_2(\gamma\alpha) = \delta\alpha \neq 0$, $d_2(\gamma\beta) = \delta\beta \neq 0$. By Proposition 6.3, $H^4(G, Z)$ has order p^4, hence $d_3\chi = 0$, and $H^4(G, Z) = Z\alpha^2 \oplus Z\alpha\beta \oplus Z\beta^2 \oplus Z\chi$.

LEMMA 6.7. *For suitable choice of* μ, ν, δ, $\gamma\delta = \alpha\mu - \beta\nu$ *in* E_2.

Proof. In the doubly periodic resolution for L given in §3, we take cocycles
$\delta: a_{03} \to 0,\ a_{12} \to -1,\ a_{21} \to 1,\ a_{30} \to 0,$
$\mu: a_{01} \to \gamma_0,\ a_{10} \to 0,$
$\nu: a_{10} \to \gamma_0,\ a_{01} \to 0,$
representing classes $\delta \in H^3(Z_p \oplus Z_p, Z)$, $\mu, \nu \in H^1(Z_p \oplus Z_p, H^2(Z_p, Z))$. Thus $\gamma\delta \in H^3(Z_p \oplus Z_p, H^2(Z_p, Z))$ is represented by $f_{\gamma_0} \circ \delta$, where $f_{\gamma_0}: Z \to H^2(Z_p, Z)$ map $1 \to \gamma_0$ (i.e. f_{γ_0} = cup-product with γ_0). As for $\alpha\mu - \beta\nu$, by Lemma 3.4, cup-product with the periodicity cocycles α, β (here $\alpha: a_{10} \to 1$, $a_{01} \to 0$, $\beta: a_{01} \to 1$, $a_{10} \to 0$) is just a shift two units east (resp. north). Thus by inspection, $\alpha\mu - \beta\nu$ is represented by the same cocycle that represents $\gamma\delta$. Q.E.D.

REMARK 1. One easily checks that the above α, β correspond to the inflations (previously called α, β) of the standard generators of $H^2(\langle A \rangle, Z)$, resp. $H^2(\langle B \rangle, Z)$. We tacitly assumed this in the proof.

LEMMA 6.8. $\delta\mu = -\beta\chi$, $\delta\nu = -\alpha\chi$, $\delta\chi = 0$, $\mu\chi = \nu\chi = 0$, $\mu^2 = \nu^2 = \chi^2 = 0$.

Proof. The projection $J: Z \to Z_p$ induces

$$J_*: H^i(Z_p \oplus Z_p, H^j(Z_p, Z)) \to H^i(Z_p \oplus Z_p, H^j(Z_p, Z_p)).$$

J_* is mono, $j \geqq 0$, and iso, j even, > 0. As in §3, we have

$H^*(Z_p \oplus Z_p, Z_p) \approx H^*(Z_p, Z_p) \otimes H^*(Z_p, Z_p) = E(u_1) \otimes P(v_1) \otimes E(u_2) \otimes P(v_2)$,

with $\Delta u_i = v_i$, $i = 1, 2$.

We take as representative cocycles $u_1\colon a_{10}\to 1$, $a_{01}\to 0$, $u_2\colon a_{01}\to 1$, $a_{10}\to 0$, $v_1\colon a_{20}\to 1$, $a_{11}\to 0$, $a_{02}\to 0$, $v_2\colon a_{02}\to 1$, $a_{11}\to 0$, $a_{20}\to 0$. Let $\bar\gamma=J_*(\gamma)$. Note that J_* preserves products. We now observe $J_*(\delta)=v_1u_2-v_1u_1$ $(=\Delta(u_1u_2))$, $J_*(\alpha)=v_1$, $J_*(\beta)=v_2$, $J_*(\mu)=\bar\gamma u_2$, $J_*(\nu)=\bar\gamma u_1$, $J_*(\chi)=\bar\gamma u_1u_2$. Thus

$$J_*(\delta\mu) = (v_1u_2-v_2u_1)\bar\gamma u_2 = -v_2u_1u_2\bar\gamma = -J_*(\beta\chi).$$

J_* mono implies that $\delta\mu=-\beta\chi$. Likewise, $\delta\nu=-\alpha\chi$. The other relations follow in the same way. Q.E.D.

We observe also that $\mu\nu=\gamma\chi$ (by Lemma 6.5). Let us now insist that $\gamma_0\in H^2(D,Z)$ be the maximal generator given by $\gamma_0\colon C\to 1/p$. As in the proof of Proposition 6.3, if $H=\langle B, C\rangle$, $H^*(H,Z)=P[\beta,\gamma]\otimes E(\mu)$, k=automorphism induced by A, then $k\beta=\beta$, $k\gamma=\gamma+\beta$, $\mathrm{Res}_D(\gamma)=\gamma_0$, $\mathrm{Res}_D(\beta)=0$.

Lemma 6.9. *$\gamma^p\in E_2^{0,2p}$ is a universal cycle. [N.B. There are two distinct γ's, one $\gamma\in E_2^{0,2}$, the other $\gamma\in H^2(H,Z)$.]*

Proof. By the double coset formula for $\mathcal{N}$ [5], we see that

$$\mathrm{Res}_H\,\mathcal{N}(\gamma) = \prod_{i=0}^{p-1}(\gamma-i\beta) = \gamma^p-\gamma\beta^{p-1}.$$

Hence $\mathrm{Res}_D\,\mathcal{N}(\gamma)=\mathrm{Res}_D\,\gamma^p=\gamma_0^p$, i.e. $=\gamma^p$, via $H^{2p}(D,Z)\approx E_2^{0,2p}$. By Corollary II, §1, γ^p is a universal cycle. Q.E.D.

Lemma 6.6 now shows that $\gamma^p\colon E_2^{i,j}\to E_2^{i,j+2p}$ is an isomorphism for $j>0$.

Lemma 6.10. *$E_2^{2n,0}=\sum_{i+j=n}Z\alpha^i\beta^j$, $E_2^{2n+1,0}=\sum_{i+j=n-1}Z\delta\alpha^i\beta^j$, $E_2^{*,2m+1}=0$ $(m>0)$, $E_2^{2n,2}=\sum_{i+j=n-1}Z\chi\alpha^i\beta^j\oplus\sum_{i+j=n}Z\gamma\alpha^i\beta^j$, $E_2^{2n+1,2}=\sum_{i+j=n}Z\mu\alpha^i\beta^j\oplus Z\nu\alpha^i\beta^j$. The other terms are given by periodicity $E_2^{*,2}\approx^{\gamma}E_2^{*,4}\approx\cdots$.*

Proof. By inspection, in view of our previous results. Q.E.D.

Clearly $E_2=E_3$, as odd rows are zero. We have seen that $\alpha,\beta,\mu,\nu,\chi,\gamma^p$ are universal cycles. To find E_4, we must get d_3. Recall that $d_3(\gamma)=s\delta$, $s\not\equiv 0\ (p)$.

Lemma 6.11. *$E_4^{2n,2m}=0$ if $m\not\equiv 0\ (p)$, $m>0$, $n\neq 1$, $E_4^{2,2n}=Z\chi\gamma^{m-1}$, $m>0$, $E_4^{2n,0}=\sum_{i+j=n}Z\alpha^i\beta^j$, $E_4^{1,2m}=0$ if $m\not\equiv 1\ (p)$, $m>1$, $E_4^{2n+1,2n}=0$ if $m\not\equiv -1\ (p)$, $m\neq 1$, $n>0$, $E_4^{2n+1,2(p-1)}=\sum_{i+j=n-1}Z\alpha^i\beta^j\gamma^{p-1}\delta$, $E_4^{2n+1,2}=\sum_{i+j=n}Z\alpha^i\beta^j\mu\oplus Z\alpha^i\beta^j\nu\oplus\sum_{i+j=n-1}Z\alpha^i\beta^j(\alpha\mu-\beta\nu)$. The other terms are given by periodicity $E_4^{*,j}\approx^{\gamma^p}E_4^{*,j+2p}$, $j>0$.*

Proof. $E_3^{2n,2n}=\sum_{i+j=n}Z\alpha^i\beta^j\gamma^m\oplus\sum_{i+j=n-1}Z\chi\alpha^i\beta^j\gamma^{m-1}$ and also $E_3^{2n+1,2n}=\sum_{i+j=n}Z\mu\alpha^i\beta^j\gamma^{m-1}\oplus Z\nu\alpha^i\beta^j\gamma^{m-1}$. Applying d_3, we see: $d_3(\alpha^i\beta^j\gamma^m)=ms\alpha^i\beta^j\gamma^{m-1}\delta$, $d_3(\chi\alpha^i\beta^j\gamma^{m-1})=(m-1)s\alpha^i\beta^j\gamma^{m-2}\chi\delta=0$, as $\chi\delta=0$.

$$d_3(\mu\alpha^i\beta^j\gamma^{m-1}) = (m-1)s\mu\alpha^i\beta^j\gamma^{m-2}\delta = (m-1)s\alpha^i\beta^{j+1}\chi\gamma^{m-2},$$

as $\mu\delta=-\delta\mu=\beta\chi$. Likewise, $d_3(\nu\alpha^i\beta^j\gamma^{m-1})=(m-1)s\alpha^{i+1}\beta^j\chi\gamma^{m-2}$. We deduce: The terms $\alpha^i\beta^j\gamma^m$ perish if $m \not\equiv 0$ (p), as do the $\chi\alpha^i\beta^j\gamma^{m-1}$ if $m \not\equiv 0$ (p) and $i>0$ or $j>0$. The $\mu\alpha^i\beta^j\gamma^{m-1}$ and $\alpha^i\beta^j\gamma^{m-1}$ perish for $m \not\equiv 1$ (p), but

$$\mu\alpha^i\beta^j\gamma^{m-1} - \nu\alpha^{i-1}\beta^{j+1}\gamma^{m-1} = \alpha^{i-1}\beta^j\gamma^{m-1}(\alpha\mu - \beta\nu) = \alpha^{i-1}\beta^j\gamma^m\delta$$

goes to zero under d_3 $(\delta^2=0)$. $d_3(\gamma^{m+1}\delta) = (m+1)s\gamma^m\delta$, so if $m \not\equiv -1$ (p), the $\alpha^{i-1}\beta^j\gamma^m\delta$ perish because they are boundaries. Putting these facts together and examining a few special cases, gives the result. Q.E.D.

Corollary 6.12. *$\alpha\mu = \beta\nu$ in $H^*(G, Z)$, whence $\alpha\mu\nu = \beta\mu\nu = 0$.*

Proof. $d_3(\gamma^2) = 2\gamma\delta = 2(\alpha\mu - \beta)$, so $\alpha\mu = \beta$ in E_4. But $E_4^{i,j} = 0$, $i+j=5$, $(i,j) \neq (3,2)$. The first statement is now clear. As $\mu^2 = \nu^2 = 0$ $H^*(G, Z)$, μ and ν being of odd degree, the second assertion follows. Done.

We have actually shown that $H^5(G, Z) = Z\alpha\mu \oplus Z\alpha\nu \oplus Z\beta\mu$, so that $\alpha\mu = \beta\nu$ is the only relation in dimension 5. Let a, λ be the automorphisms of G given by $a\colon A \to B$, $B \to A$, $\lambda\colon B \to B$, $A \to AB$, respectively. $a^2 = 1$, $\lambda^p = 1$. Clearly

$$\alpha^a = \beta, \quad \beta^a = \alpha; \quad \alpha^\lambda = \alpha, \quad \beta^\lambda = \beta + \alpha. \tag{6.13}$$

Only the second part needs checking. $\alpha\colon A \to 1/p$, $B \to 0$, $\beta\colon A \to 0$, $B \to 1/p$. By the definition of §1, $\alpha^\lambda = \alpha \circ \lambda$, $\beta^\lambda = \beta \circ \lambda$. So $\alpha^\lambda(A) = \alpha(AB) = 1/p$. $\alpha^\lambda(B) = \alpha(B) = 0$. $\beta^\lambda(A) = \beta(AB) = 1/p$. $\beta^\lambda(B) = \beta(B) = 1/p$. Done.

a and λ act on $H^3(G, Z)$. Suppose $\mu^a = d\mu + e\nu$, $\nu^a = f\mu + g\nu$. $\alpha\mu = \beta\nu$ implies $\beta\mu^a = \alpha\nu^a$, or $\beta(d\mu + e\nu) = \alpha(f\mu + g\nu)$, or $d\beta\mu = (f-e)\alpha\mu + g\alpha\nu$, implying $d = g = 0$, $f = e$. $\mu^a = e\nu$. $a^2 = 1$ so $e^2 = 1$, hence $e = \pm 1$ (everything is mod p, of course).

The diagram (obtained from the spectral sequence) commutes:

$$\begin{array}{ccccccc} H^1(Z_p \oplus Z_p, H^2(Z_p, Z)) & \approx & E_2^{1,2} & \approx & E_\infty^{1,2} & \approx & H^3(G, Z) \\ \downarrow a & & \downarrow a & & a \downarrow & & a \downarrow \\ H^1(Z_p \oplus Z_p, H^2(Z_p, Z)) & \approx & E_2^{1,2} & \approx & E_\infty^{1,2} & \approx & H^3(G, Z). \end{array}$$

Hence by the definition of μ, ν we see that $\mu^a = -\nu$. (Since $c^a = c^{-1}$ implies that $a\colon H^2(Z_p, Z) \to H^2(Z_p, Z)$ is multiplication by -1.) Thus

$$\mu^a = -\nu, \qquad \nu^a = -\mu. \tag{6.14}$$

A similar argument yields

$$\mu^\lambda = \mu + \nu, \qquad \nu^\lambda = \nu. \tag{6.15}$$

Let $H_t = \langle AB^t, C\rangle$. $H = H_p = \langle B, C\rangle$.

Lemma 6.16. $\mathrm{Res}_H\, \mu \neq 0$, $\mathrm{Res}_H\, \nu = 0$.

Proof. Observe $\lambda(C) = C$. Thus λ = identity on H. Thus

$$\begin{array}{ccc} H^3(G, Z) & \xrightarrow{\lambda} & H^3(G, Z) \\ \mathrm{Res}_H \downarrow & & \downarrow \mathrm{Res}_H \\ H^3(H, Z) & \xrightarrow{\text{id.}} & H^3(H, Z) \end{array} \qquad \text{commutes.}$$

We saw in the proof of Proposition 6.3 that Res_H is onto. Let $d\mu+e\nu\neq 0$ be an element such that $\mathrm{Res}_H\,(d\mu+e\nu)=0$. Apply λ, get $\mathrm{Res}_H\,(d(\mu+\nu)+e\nu)=0$. Subtract, then $d\,\mathrm{Res}_H\,(\nu)=0$. If $d\neq 0$, then $\mathrm{Res}_H\,\nu=0$. If $d=0$, then $e\neq 0$, and so $\mathrm{Res}_H\,\nu=0$ again. Thus $\mathrm{Res}_H\,\nu=0$. As $\mathrm{Res}_H\neq 0$, so $\mathrm{Res}_H\,\mu\neq 0$. Q.E.D.

Observe that $H_t=H_0^{\lambda^t}$, $H_0=H^a$. Applying a to Lemma 6.16 gives $\mathrm{Res}_{H_0}\,\nu\neq 0$. $\mathrm{Res}_{H_0}\,\mu=0$.

Claim. $\mathrm{Res}_{H_t}\,(\mu)\neq 0$ if and only if $t\neq 0$. Indeed, apply λ^t to $\mathrm{Res}_{H_0}\,(\mu)=0$. Get $\mathrm{Res}_{H_t}\,(\mu+t\nu)=0$ $(0<t<p)$. If $\mathrm{Res}_{H_t}\,(\mu)=0$, then $\mathrm{Res}_{H_t}\,(\nu)=0$, implying $\mathrm{Res}_{H_t}=0$. Contradiction. Hence $\mathrm{Res}_{H_t}\,(\mu)\neq 0$ if $0<t<p$. By Lemma 6.16 this holds also for $t=p$. Finally, observe that in $H^*(H_t, Z)$, no element of even degree is a zero-divisor. Thus, if $\xi\in H^*(G, Z)$, of even degree, $\mathrm{Res}_{H_t}\,(\xi)=0$ if and only if $\mathrm{Res}_{H_t}\,(\xi\mu)=0$, for $t\neq 0$.

Proposition 6.17. *The elements $\alpha^i\mu, \alpha^{i-1}\beta\mu, \ldots, \beta^i\mu, \alpha^2\nu$ in $H^{2i+3}(G, Z)$ are linearly independent for $0\leqq i<p$.*

Proof. Exactly as in Proposition 6.1. Obtain $a_t\in H^2(G, Z)$, $\mathrm{Res}_{H_j}\,(a_t)=0$ if and only if $t=j$. Let $I\subset\{1,\ldots,p\}$ consist of i elements. Form $a_I=\prod_{t\in I}a_t$. Then we see that $\{A_I\mu\}$ contains at least $i+1$ linearly independent elements. As the A_I lie in the span of $\{\alpha^i\mu,\ldots,\beta^i\mu\}$, the latter set of $i+1$ elements is linearly independent. As for $\alpha^i\nu$, we saw that $\mathrm{Res}_{H_0}\,(\alpha^i\nu)\neq 0$, $\mathrm{Res}_{H_0}\,(\alpha^k\beta^l\mu)=0$. Hence the enlarged set $\{\alpha^i\mu,\ldots,\beta^i\mu,\alpha^i\nu\}$ is still independent. Q.E.D.

Lemma 6.18. *$\gamma^i\chi\in E_4^{2,2i+2}$ survives to E_∞ for $0\leqq i<p-1$.*

Proof. As the fibre terms are zero, or universal cycles, need only show $\gamma^i\chi$ are universal cycles. $d_{2i+1}: E_4^{2,2i+2}\to E_4^{2(i+1)+1,2}$ is the only possibly nonzero differential.

As $E_4^{2(i+1)+1,2}$ is generated by the $\alpha^{i+1}\mu,\ldots,\beta^{i+1}\mu,\alpha^{i+1}\nu$ and $i+1\leqq p$, Proposition 6.17 implies that $d_{2i+1}(\gamma^i\chi)=0$. Q.E.D.

Thus, in degrees $\leqq 2p$, $E_4=E_\infty$. Inspection reveals:

$$(6.19)\quad \alpha_{2i}=p^{i+2},\quad 2\leqq i<p;\quad \alpha_{2i+1}=p^{i+1},\quad 1\leqq i<p-1;\quad \alpha_{2p}=p^{p+3}.$$

Lemma 6.20. $\alpha_{2p+1}=p^{p+1}$, $\alpha_{2p+2}=p^{p+3}$, $\alpha\beta^p=\beta\alpha^p$, $\alpha^p\mu=\beta^p\nu$.

Proof. As $\alpha_1=1$, $\alpha_2=p^2$, inequality (6.2) gives $\alpha_{2p+1}\leqq p^{p+1}$, $\alpha_{2p+2}\leqq p^{p+3}$. The $p+1$ elements $\alpha^{p-1}\mu,\ldots,\beta^{p-1}\mu,\alpha^{p-1}\nu\in E_4^{2p-1,2}$ are linearly independent and survive to E_∞ (Proposition 6.17). This implies $\alpha_{2p+1}=p^{p+1}$. This shows also that $d_{2p-1}(\gamma^{p-1}\delta)\neq 0$. In degree $2p+2$, there are independent generators $\gamma^p\alpha$, $\gamma^p\beta$, $\gamma^{p-1}\chi\in E_4^{2,2p}$, and the $\alpha^{p+1},\ldots,\beta^{p+1}$, in $E_4^{2p+2,0}$. As $d_{2p-1}(\gamma^{p-1}\delta)\neq 0$, there is precisely one nontrivial relation, $f(\alpha,\beta)=0$. Thus, as $\gamma^p\alpha$, $\gamma^p\beta$ survive to E_∞, we get $\alpha_{2p+2}\geqq p^{p+3}$, implying that $\alpha_{2p+2}=p^{p+3}$. Thus $\gamma^{p-1}\chi$ perishes, so $d_{2p-1}(\gamma^{p-1}\chi)\neq 0$. This says that there is precisely one relation among the elements $\alpha^p\mu, \alpha^{p-1}\beta\mu, \ldots,$

$\beta^p\mu$, $\alpha^p\nu$. To find the relations, we apply the automorphisms a and λ. Thus, $f(\alpha, \beta)=0$ implies $f(\beta, \alpha)=0$, and $f(\alpha, \alpha+\beta)=0$.

Let

$$f(\alpha, \beta) = \sum_{i=0}^{p+1} c_i\alpha^i\beta^{p+1-i},$$

$$f(\alpha, \alpha+\beta) = \sum_{i=0}^{p+1} c_i\alpha^i(\alpha+\beta)^{p+1-i} = \sum_{0\leqq i\leqq p+1;\, 0\leqq j\leqq p+1-i} c_i\binom{p+1-i}{j}\alpha^{j+i}\beta^{p+1-j-i}$$

$$= \sum_{i'=0}^{p+1}\sum_{0\leqq i\leqq i'} c_i\binom{p+1-i}{i'-i}\alpha^{i'}\beta^{p+1-i'}.$$

By uniqueness of the relation, $f(X, X+Y)=ef(X, Y)$, some integer e. Iteration p times yields $e^p=1$, whence $e=1$ (everything is mod p). Thus, we may equate coefficients:

$$c_0\binom{p+1}{r}+c_1\binom{p}{r-1}+c_2\binom{p-1}{r-2}+\cdots+c_r\binom{p-r+1}{0} = c_r$$

$(0\leqq r\leqq p+1)$. This plus symmetry implies $c_0=c_2=\cdots=c_{p-1}=c_{p+1}=0$, and $c_1+c_p=0$. We may take $c_1=1$. Thus $f(\alpha, \beta)=\alpha\beta^p-\beta\alpha^p$. The relation $\alpha^p\mu=\beta^p\nu$ is proved similarly. Q.E.D.

We now have the E_∞ term $(=E_{2p})$. Inspection shows that

(6.21) $$\alpha_{j+2p} = p^{p+1}\alpha_j, \qquad j > 0,$$

as previously asserted. The elements $\chi, \gamma\chi, \ldots, \gamma^{p-2}\chi$ are in E_∞. How do they multiply? To settle this we use the Res-Cor sequences associated with $1\to H\to G \to \langle A\rangle \to 1$. We know that $H^*(H, Z)=P[\beta, \gamma]\otimes E(\mu)$. $\mathrm{Res}_H\,\beta=\beta$, $\mathrm{Res}_H\,\mu=\mu$, and $\mathrm{Res}_H\,\mathcal{N}(\gamma)=\gamma^p-\gamma\beta^{p-1}$ (proof of Lemma 6.9). Let $\zeta=\gamma^p-\gamma\beta^{p-1}$. As $k\beta=\beta$, $k\gamma=\gamma+\beta$, we find: $H^{2i}(H, Z)^k=Z\beta^i$, $0<i<p$. $H^{2i+1}(H, Z)^k=Z\mu\beta^{i-1}$, $0<i<p-1$, $H^{2p}(H, Z)^k=Z\beta^p\oplus Z\zeta$. $H^{2i}(H, Z)_k=Z\gamma^i$, $1\leqq i<p$, $H^{2i+1}(H, Z)_k=Z\mu\gamma^{i-1}$, $0<i\leqq p$.

LEMMA 6.22. Cor $\gamma=0$, Cor $\gamma^i\neq 0$, $1\leqq i\leqq p$.

Proof. First observe Cor: $H^{2i+1}(H, Z)\to H^{2i+1}(G, Z)$ is 0, $0\leqq i<p$. Indeed, the formula Cor (Res $a\cdot b)=a\cdot$ Cor b [3, Chapter XII, §8], Res $\alpha=0$, imply α Cor $\xi=0$ for $\xi\in H^*(H, Z)$. If $\xi\in H^{2i+1}(H, Z)$, then Cor $\xi\in H^{2i+1}(G, Z)$ lies in the span of $\alpha^i\mu, \ldots, \beta^i\mu, \alpha^i\nu$. As α annihilates no nonzero combination of these (Proposition 6.17), so α Cor $\xi=0$ implies Cor $\xi=0$. $H^{2i}(H, Z)^k=Z\beta^i$, $H^{2i+1}(H, Z)^k =Z\mu\beta^{i-1}$, show that the image of Res: $H^{2i}(G, Z)\to H^{2i}(H, Z)$ has order p $(1\leqq i<p)$, and order p^2 if $i=p$, since $H^{2p}(H, Z)^k=Z\beta^p\oplus Z\zeta$. By the Res-Cor sequences:

$$0\to Z_p\to H^{2i}(H, Z)\to H^{2i}(G, X)\to H^{2i+1}(G, Z)\to Z_p\to 0, \text{ exact } (1\leqq i<p),$$

$$0\to Z_p\oplus Z_p\to H^{2p}(H, Z)\to H^{2p}(G, X)\to H^{2p+1}(G, Z)\to Z_p\to 0, \text{ exact.}$$

$|H^{2i}(H, Z)| = p^{i+1}$, so by (6.19) and Lemma 6.20, we get: $|H^{2i}(G, X)| = p^{2i}$, $2 \leqq i < p$, $|H^{2p}(G, X)| = p^{2p-1}$. If $\mathrm{Cor}_{2i} = 0$, $2 \leqq i < p$, then

$$0 \to H^{2i-1}(G, Z) \to H^{2i}(G, X) \to H^{2i}(H, Z) \to 0$$

were exact. This implies $|H^{2i}(G, X)| = p^i \cdot p^{i+1} = p^{2i+1}$, a contradiction. Therefore $\mathrm{Cor}_{2i} \neq 0$, and hence $\mathrm{Cor}\,(\gamma^i) \neq 0$, $2 \leqq i < p$.

Res Cor $\gamma^{p-1} = N_K \gamma^{p-1} = -\beta^{p-1}$ implies Res (Cor $\beta\gamma^{p-1}$) $= -\beta^p \neq 0$. Thus if Cor $\gamma^p = 0$, then

$$0 \to H^{2p-1}(G, Z) \to H^{2p}(G, X) \to H^{2p}(H, Z) \to Z_p \to 0$$

were exact. This implies $p|H^{2p}(G, X)| = p^p \cdot p^{p+1} = p^{2p+1}$, or $|H^{2p}(G, X)| = p^{2p}$. A contradiction. Hence Cor $\gamma^p \neq 0$. As for Cor $\gamma = 0$, we may check this directly using the formula in §1. Q.E.D.

COROLLARY 6.23. *$\mathcal{N}(\gamma)$ has order p^2.*

Proof #1. As $\mathrm{Res}_H \mathcal{N}(\gamma) = \gamma^p - \gamma\beta^{p-1}$, apply Cor, and note Cor $\gamma = 0$. Get $p\mathcal{N}(\gamma) = \mathrm{Cor}\,\gamma^p \neq 0$, by previous result. Done.

Proof #2. Glance back at (6.1)–(6.1)‴. Clearly Cor: $\hat{H}^0(H, \mathbf{Z}) \to \hat{H}^0(G, \mathbf{Z})$ has cokernel cyclic of order p^2. That is, $H^1(G, X) \approx Z_{p^2}$. We see that

$$\begin{aligned} |H^{2p}(G, \mathbf{Z})| &\leqq p^p|H^{2p-1}(G, U)| = p^p|H^p(G, W)| \\ &\leqq p^p|H^{p-1}(G, V)| = p^p|H^2(G, Y)| \leqq p^{p+1}|H^1(G, X)| = p^{p+3}. \end{aligned}$$

But we know that $\alpha_{2p} = p^{p+3}$, whence, equality holds at every stage. In particular, $H^p(G, V) \xrightarrow{\text{epi}} H^p(G, W)$ is actually an isomorphism. So

$$\begin{aligned} \mathbf{Z}_{p^2} = H^1(G, X) &\xrightarrow{\text{mono}} H^2(G, Y) \approx H^{p-1}(G, V) \approx H^p(G, W) \\ &\approx H^{2p-1}(G, U) \to H^{2p}(G, \mathbf{Z}). \end{aligned}$$

Thus $H^{2p}(G, \mathbf{Z})$ contains an element of order p^2. Now the spectral sequence, Res Cor $\gamma^p = N_K\gamma^p = 0$, and Cor $\gamma^p \neq 0$, plus p Cor $\gamma^p = 0$, and $H^{2p}(G, \mathbf{Z}) = \mathbf{Z}\mathcal{N}(\gamma) \oplus \mathbf{Z}\,\mathrm{Cor}\,\gamma^p \oplus \sum_{i+j=p} \mathbf{Z}\alpha^i\beta^j$, show that $\mathcal{N}(\gamma)$ is of order p^2. Q.E.D.

REMARK 1. α Cor $\gamma^p = 0$, Res Cor $\gamma^p = 0$ are what tell us that Cor $\gamma^p \notin \sum \mathbf{Z}\alpha^i\beta^j$. For if so, then applying Res shows that Cor γ^p lies in $\mathbf{Z}\alpha\beta^{p-1} \oplus \cdots \oplus \mathbf{Z}\alpha^p$. As $\alpha\beta^p = \alpha^p\beta$ is the only relation in degree $2p+2$, we see that α Cor $\gamma^p = 0$ implies Cor $\gamma^p = 0$. Contradiction.

Let $\chi_i = \mathrm{Cor}\,\gamma^{i+1}$, $1 \leqq i < p-2$. Note $\chi_i^\lambda = \chi_i$, $\alpha\chi_i = 0$. As in Remark 1, we see that χ_i is not a polynomial in α and β. The spectral sequence implies that $H^{2i+2}(G, \mathbf{Z}) = \mathbf{Z}\chi_i \oplus \sum_{k+l=i+1} \mathbf{Z}\alpha^k\beta^l$, $(1 \leqq i < p-2)$.

LEMMA 6.24. *$\chi_i\chi_j = 0$, $1 \leqq i$, $j < p-2$, $\alpha\chi_i = \beta\chi_i = \mu\chi_i = \nu\chi_i = 0$, $\chi_i^\lambda = \chi_i$, $\chi_i^a = \varepsilon_i\chi_i$, $\varepsilon_i = \pm 1$. $\mathrm{Res}_N\,(\chi_i) = 0$, for all proper subgroups N of G.*

Proof. The fact that $\mathrm{Res}_H \chi_i = N_k \chi^{i+1} = 0$ is easily checked. The formula $\mathrm{Cor}(\mathrm{Res}\, a\cdot b) = a\cdot \mathrm{Cor}\, b$ now implies $\chi_i\chi_j = 0$, $\alpha\chi_i = 0$, $\nu\chi_i = 0$. $(\alpha\chi_i)^a = \beta\chi_i^a = 0$, $(\nu\chi_i)^a = \mu\chi_i^a = 0$. So we need specify χ_i^a. Let $\chi_i^a = \varepsilon_i\chi_i + f_i(\alpha, \beta)$, implying $\chi_i = \varepsilon_i^2\chi_i + \varepsilon_i f_i(\alpha, \beta) + f_i(\beta, \alpha)$ which in turn implies that $\varepsilon_i = \pm 1$. $\beta\chi_i^a = 0$ implies $0 = \varepsilon_i\beta\chi_i + \beta f_i(\alpha, \beta)$. $\beta\chi_i$ is λ-invariant, as $\alpha\chi_i = 0$ and $\chi_i^\lambda = \chi_i$. Hence so is $\beta f_i(\alpha, \beta)$. But powers of α are the only λ-invariant polynomials in degree $< 2p$. Hence $f_i(\alpha, \beta) = 0$ or $\chi_i^a = \varepsilon_i\chi_i$. This proves all but the last assertion.

We may assume N is of index p, i.e. $N = \langle A^iB^j, C\rangle$, some i, j. There exists $f(\alpha, \beta)$ = polynomial in α, β such that $\mathrm{Res}_N f(\alpha, \beta) \neq 0$ (can take f linear). Thus $\mathrm{Res}_N f(\alpha, \beta)\, \mathrm{Res}_N \chi_i = \mathrm{Res}_N (f(\alpha, \beta)\chi_i) = \mathrm{Res}_N (0) = 0$. Hence $\mathrm{Res}_N \chi_i = 0$. Q.E.D.

REMARK 2. As $\langle c\rangle = Z(G)$ is characteristic, and α, β are inflations, it follows that the subring $\boldsymbol{Z}[\alpha, \beta]$ is Aut (G)-invariant. The proof of Proposition 6.1 shows that χ_i is the only element (up to constant multiple) of $H^{2i+2}(G, \boldsymbol{Z})$ which has $\mathrm{Res}_N = 0$ for all proper N. Thus $\boldsymbol{Z}\chi_i$ is also Aut (G)-invariant, as is the decomposition of $H^{2i+2}(G, \boldsymbol{Z})$.

If $p > 3$, then χ_2 is defined, and differs from $\mu\nu$ by a constant multiple (both are λ-invariant). Here $(\mu\nu)^a = \mu^a\nu^a = \nu\mu = -\mu\nu$, so in this case $\varepsilon_2 = -1$. We now define $\chi_{p-2} = \mathrm{Cor}\, \gamma^{p-1} + \beta^{p-1}$. As $\mathrm{Res}\,\mathrm{Cor}\, \gamma^{p-1} = -\beta^{p-1}$ so $\mathrm{Res}\, \chi_{p-2} = 0$,

$$\chi_{p-2} \in H^{2p-1}(G, \boldsymbol{Z}).$$

LEMMA 6.25. $\alpha\chi_{p-2} = \alpha\beta^{p-1}$, $\beta\chi_{p-2} = \beta\alpha^{p-1}\cdot \chi_{p-2}^a = \chi_{p-2}$, $\chi_{p-2}^\lambda = \chi_{p-2} + (\beta+\alpha)^{p-1} - \beta^{p-1}$, $\mu\alpha^{p-1} = \mu\chi_{p-2}$, $\nu\beta^{p-1} = \nu\chi_{p-2}$.

Proof. $\alpha\, \mathrm{Cor}\, \gamma^{p-1} = 0$, so $\alpha\chi_{p-2} = \alpha\beta^{p-1}$ (*). Likewise, $\nu\chi_{p-2} = \nu\beta^{p-1}$. We check easily that $\chi_{p-2} \neq$ a polynomial in α, β. Hence

$$H^{2p-2}(G, \boldsymbol{Z}) = \boldsymbol{Z}\chi_{p-2} \oplus \sum_{k+l=p-1} \boldsymbol{Z}\alpha^k\beta^l.$$

Let $\chi_{p-2}^a = \varepsilon\chi_{p-2} + g(\alpha, \beta)$. Apply a to (*). Get $\beta\chi_{p-2}^a = \beta\alpha^{p-1}$, or $\varepsilon\beta\chi_{p-2} + \beta g(\alpha, \beta) = \beta\alpha^{p-1}$. Thus $\beta\chi_{p-2} = \varepsilon^{-1}(\beta\alpha^{p-1} - \beta g(\alpha, \beta)) = f(\alpha, \beta)$. But $\beta\chi_{p-2} = \beta^p + \beta\, \mathrm{Cor}\, \gamma^{p-1} = \beta^p + \mathrm{Cor}\, \beta\gamma^{p-1}$. As $\mathrm{Cor}\, \beta\gamma^{p-1}$ is λ-invariant, we get $f(\alpha, \beta) = \alpha^p + f(\alpha, \beta)$. $\mathrm{Res}\, \chi_{p-2} = 0$. Hence $f(\alpha, \beta)$ has no β^p-term. A small calculation now gives $f(\alpha, \beta) = \beta\alpha^{p-1}$, i.e. $\beta\chi_{p-2} = \beta\alpha^{p-1}$. A further, smaller calculation proves $\varepsilon = 1$. Finally, applying a to $\nu\beta^{p-1} = \nu\chi_{p-2}$ yields $\mu\alpha^{p-1} = \mu\chi_{p-2}$. Q.E.D.

Observe that Lemma 6.25 implies that $\alpha\beta^p = \beta\alpha^p$, and $\alpha^p\mu = \beta^p\nu$. Note also that cup-product with $\mathcal{N}(\gamma)$ is a monomorphism (by the spectral sequence). We can summarize our conclusions in one statement:

THEOREM 6.26. *The cohomology ring of*

$$G = (A, B : A^p = B^p = [A, B]^p = [A, [A, B]] = [B, [A, B]] = 1),$$

for p odd, is as follows: $H^*(G, \boldsymbol{Z}) = \boldsymbol{Z}[\alpha, \beta, \mu, \nu, \zeta, \chi_1, \chi_2, \ldots, \chi_{p-2}]$, $\deg \alpha = \deg \beta = 2$, $\deg \mu = \deg \nu = 3$, $\deg \zeta = 2p$, $\deg \chi_i = 2i+2$, *with relations*: (0) $p\alpha = p\beta = p\mu$

$=p\nu=p\chi_i=p^2\zeta=0$, (1) $\alpha\mu=\beta\nu$, (2) $\alpha^p\mu=\beta^p\nu$, (3) $\mu^2=\nu^2=0$, (4) $\chi_i\chi_j=\alpha\chi_i=\beta\chi_i=\mu\chi_i=\nu\chi_i=0$, $1\leqq i$, $j<p-2$, (5) $\chi_i\chi_{p-2}=0$, $1\leqq i<p-2$, $\chi_{p-2}^2=\alpha^{p-1}\beta^{p-1}$, (6) $\alpha\chi_{p-2}=\alpha\beta^{p-1}$, $\beta\chi_{p-2}=\beta\alpha^{p-1}$, (7) $\mu\alpha^{p-1}=\mu\chi_{p-2}$, $\nu\beta^{p-1}=\nu\chi_{p-2}$, (8) $\alpha\beta^p=\beta\alpha^p$.

If $p>3$, *then* $\chi_2=d\mu\nu$, *for some* $d\in Z_p^*$; *if* $p=3$, *then* $p\zeta=e\mu\nu$, *some* $e\in Z_p^*$. a, λ *act as follows*:

(i) $\alpha^a=\beta$, $\mu^a=-\nu$, $\chi_i^a=\varepsilon_i\chi_i$, $\varepsilon_i=\pm 1$, $\varepsilon_{p-2}=1$, $\varepsilon_2=-1$ *if* $p>3$,

(ii) $\alpha^\lambda=\alpha$, $\beta^\lambda=\beta+\alpha$, $\nu^\lambda=\mu+\nu$, $\chi_i^\lambda=\chi_i$, $1\leqq i<p-2$, $\chi_{p-2}^\lambda=\chi_{p-2}+(\beta+\alpha)^{p-1}-\beta^{p-1}$, $\zeta^\lambda=\zeta$.

Here $a,\lambda: G\to G$ *are*: a: $A\to B$, $B\to A$, λ: $B\to B$, $A\to AB$. *If* $H=\langle B, C\rangle$ (*where* $C=[A,B]=B^{-1}A^{-1}BA$, $\gamma\in H^2(H,Z)$ *corresponding to* $C\to 1/p$, $B\to 0$), *we may take* $\chi_i=\operatorname{Cor}\gamma^{i+1}$, $1\leqq i<p-2$, $\chi_{p-2}=\operatorname{Cor}\gamma^{p-1}+\beta^{p-1}$, *and* $\zeta=\mathcal{N}(\gamma)$.

Proof. Almost all done in the lemmas, except for a few easy exercises. Q.E.D.

REMARKS. (1) Of course, the above relations are not mutually independent. Thus, for example, (6) implies (8); (6), (7) imply (2).

COROLLARY 6.27.

$$\text{exponent } H^n(G,Z) = p \quad \text{if } 2p\nmid n,$$
$$= p^2 \quad \text{if } 2p\mid n.$$

6.4. It is of interest to calculate the spectral sequence of

$$(*) \qquad 1\to H\to G\to\langle A\rangle\to 1 \quad \text{(split)}, \qquad H=\langle B, C\rangle.$$

We will call it the *second spectral sequence*. Set $\mathcal{A}=H^*(H,Z)=P[\beta,\gamma]\otimes E(\mu)$, and $\zeta=\gamma^p-\gamma\beta^{p-1}$. As before, $k\beta=\beta$, $k\gamma=\gamma+\beta$, $k\mu=\mu$. Note $k\zeta=\zeta$. We will often write Z_p for $\langle A\rangle$, $Z_p\oplus Z_p$ for H, and will identify $E_2^{0,j}\cong H^j(Z_p\oplus Z_p, Z)^k$.

LEMMA 6.28.

(1) $$E_2^{i,j}\approx Z_p \quad i,j>0,\ j\not\equiv 0\ (p-1) \text{ and even},$$
$$\approx 0 \quad i,j>0,\ j\equiv 0\ (p-1) \text{ and even}.$$

(2) $$\mathcal{A}^k = P[\beta,\zeta]\otimes E(\mu).$$

(3) $$N\mathcal{A} = \beta^{p-1}\mathcal{A}^k.$$

(4) $$_N\mathcal{A} = \sum_{i=0}^{p=1}\gamma^i\mathcal{A}^k.$$

(5) $$\mathcal{A}^{k-1} = \beta_N\mathcal{A}.$$

Proof.

$$E_2^{i,j}\cong H^i(Z_p, H^j(Z_p\oplus Z_p, Z)) = H^0(\langle k\rangle, H^j(Z_p\oplus Z_p, Z)), \quad i \text{ even}, >0,$$
$$= H^{-1}(\langle k\rangle, H^j(Z_p\oplus Z_p, Z)), \quad i \text{ odd}, >0.$$

Thus (2), (3), (4), (5) imply (1), by inspection. Obviously, $\mathcal{A}^k\supset P[\beta,\zeta]\otimes E(\mu)$.

$\xi\beta$ or $\xi\mu$ invariant implies ξ invariant. So if $\xi \in \mathscr{A}^k$, we may suppose $\xi=\gamma^n+a_{n-1}\gamma^{n-1}\beta+\cdots+a_0\beta^n$. By Newton's formulas, if $s_r=1^r+2^r+\cdots+(p-1)^r$, we have

(6.29) $$s_r \equiv 0 \ [\text{if } r \not\equiv 0\,(p-1)] \equiv -1 \ [\text{if } r \equiv 0\,(p-1)].$$

$$\xi^k = \sum_{i=0}^{n} a_i(\gamma+\beta)^i\beta^{n-i}[a_n = -1] = \gamma^n+\left(a_{n-1}+\binom{n}{1}\right)\gamma^{n-1}\beta+\cdots = \xi,$$

so equating coefficients gives $n\equiv 0$ (p). Say, $n=pn_0$. Form $\xi-\zeta^{n_0}=\beta\xi'$. This is invariant, implying ξ' is invariant. By induction on degrees, $\xi' \in P[\beta, \zeta]$, hence so is ξ. This proves (2).

$$N\gamma^p = \sum_{j=0}^{p-1} (\gamma+j\beta)^n = \sum_{k=1}^{n} \binom{n}{k} s_k\gamma^{n-k}\beta^k.$$

For $n<p-1$, this is 0; for $n=p-1$, it is $-\beta^{p-1}$. Thus:

(6.30) $$\begin{aligned} N\gamma^i &= 0, & 1 \leqq i < p-1, \\ &= -\beta^{p-1}, & i = p-1. \end{aligned}$$

If $\xi \in \mathscr{A}^k$, then $N(\xi\eta)=\xi N(\eta)$. By (6.30), if $\xi \in \mathscr{A}^k$, $N(\gamma^{p-1}\xi)=-\beta^{p-1}\xi$. Hence $N\mathscr{A}\supset\beta^{p-1}\mathscr{A}^k$. To show the reverse inclusion, it clearly suffices to prove: $\beta^{p-1}|N\gamma^n$ (✡) for all $n>0$. For then $N\gamma^n=\beta^{p-1}\xi_n$, and $N\gamma^n \in \mathscr{A}^k$ implies $\xi_n \in \mathscr{A}^k$. If $n<p$, we are done, by (6.30). If $n>p$, then $\gamma^n=\gamma^{n-p}\zeta+\beta^{p-1}\gamma^{n-p+1}$, implying $n\gamma^n=n(\gamma^{n-p})\zeta+\beta^{p-1}N(\gamma^{n-p+1})$. (✡) now follows by induction. This proves (3).

Next, let $\xi \in {}_N\mathscr{A}$, i.e. $N\xi=0$. We can suppose $\xi=\gamma^n+a_1\gamma^{n-1}\beta+\cdots$. We saw that $\beta^{p-1}|N\eta$ for all η. $0=N\xi=N\gamma^n+a_1N\gamma^{n-1}\cdot\beta+\cdots$, hence $\beta^p|N\gamma^n$. Write $n=c_0+c_1p+c_2p^2+\cdots$, $0\leqq c_i<p$. We know that $N\gamma^n=C_{n,1}s_1\gamma^{n-1}\beta+C_{n,2}s_2\gamma^{n-2}\beta^2+\cdots=-C_{n,p-1}\gamma^{n-p+1}\beta^{p-1}+\text{multiple of }\beta^p$. $\beta^p|N\gamma^n$ implies $C_{n,p-1}=0$ implies $c_0<p-1$ [20, p. 5]. Write $n=c_0+pn_0$. Then $\gamma^{c_0}\zeta^{n_0}=\gamma^n-c_1\gamma^{n-p+1}\beta^{p-1}+\cdots$, and $N(\gamma^{c_0}\zeta^{n_0})=0$, as $\zeta \in \mathscr{A}^k$ *and* $c_0<p-1$, by (6.30). Thus $N\xi=0$ implies $N(\xi-\gamma^{c_0}\zeta^{n_0})=0$; and degree $(\xi-\gamma^{c_0}\zeta^{n_0})<$ degree ξ. By induction $\xi-\gamma^{c_0}\zeta^{n_0} \in \sum_{i=0}^{p-2}\gamma^i\mathscr{A}^k$. Hence so is ξ, since $c_0<p-1$. This proves ${}_N\mathscr{A}\subseteq\sum_{i=0}^{p-2}\gamma^i\mathscr{A}^k$. By (6.30) the opposite inclusion holds as well. This proves (4).

$\gamma^{k-1}=\beta$, $(\gamma^2)^{k-1}=2\gamma\beta+\beta^2,\ldots$. This proves $\gamma^i\beta \in \mathscr{A}^{k-1}$, $0\leqq i<p-1$. As $\mathscr{A}^k\mathscr{A}^{k-1}\subseteq\mathscr{A}^{k-1}$, we see by (4) that $\mathscr{A}^{k-1}\supset\beta_N\mathscr{A}$. For the reverse inclusion, it suffices to show $(\gamma^n)^{k-1} \in \beta_N$ for all $n>0$. If $n<p$, this is easy by (6.30). If $n>p$, $(\gamma^n)^{k-1}=(\gamma^{n-p})^{k-1}\zeta+\beta^{p-1}(\gamma^{n-p+1})^{k-1}$, and we conclude by induction. This proves (5). Q.E.D.

$E_2\approx H^*(Z_p, \mathscr{A})$ as bigraded rings, up to a sign (see §1). The multiplication in $H^*(Z_p, \mathscr{A})$ is given as follows: [3, Chapter XII, §7]. Let $\alpha \in H^2(Z_p, Z)$ be the maximal generator corresponding to $A \to 1/p$. Then α^i: $\mathscr{A}^k/N\mathscr{A} \xrightarrow{\cong} H^{2i}(Z_p, \mathscr{A})$.

$\alpha^i: {}_N\mathscr{A}/\mathscr{A}^{k-1} \xrightarrow{\cong} H^{2i-1}(Z_p, \mathscr{A})$. If $\eta \in H^*(Z_p, \mathscr{A})$, $\xi \in H^*(Z_p, \mathscr{A})$ are homogeneous cohomology classes represented by a (resp. b) $\in \mathscr{A}$, then $\eta\xi$ is represented by

$$(6.31)\qquad \begin{array}{ll} ab & \text{if } \deg\eta \text{ or } \deg\xi \text{ is even,} \\ \sum_{0\leqq i<j<p} a^{k^i}b^{k^j} & \text{if } \deg\eta \text{ and } \deg\xi \text{ are odd.} \end{array}$$

This shows that $\alpha: E_2^{i,j} \to E_2^{i+2,j}$ is iso for $i>0$. It is easy to check that $\zeta: E_2^{i,j} \to E_2^{i,j+2p}$ is iso for $i,j>0$. Let $\chi \in E_2^{1,3}$ correspond to $\mu \in \mathscr{A}^3$, $\nu_i \in E_2^{1,2i}$ to γ^i, $1 \leqq i < p$, and $\sigma \in E_2^{1,2p}$ to ζ.

LEMMA 6.32. *In* E_2,

$$(1)\qquad \beta\nu_i = \beta\chi = \beta\sigma = 0, \quad 1 \leqq i < p-1.$$

$$(2)\qquad \sigma^2 = \nu_i^2 = \chi^2 = \mu\chi = 0,$$

$$(3)\qquad \begin{aligned} \nu_1\chi &= \alpha\beta\mu \quad (p=3), \\ &= 0 \qquad p > 3. \end{aligned}$$

Proof. $\beta\nu_i$ corresponds to $\beta\gamma^i \in \mathscr{A}^{k-1}$ if $i<p-1$. Therefore $\beta\nu_i=0$. $\beta\chi$ corresponds to $\beta\mu \in \mathscr{A}^{k-1}$ implies $\beta\chi=0$. Likewise, $\beta\zeta \in \mathscr{A}^{k-1}$ implies $\beta\sigma=0$. This proves (1). We leave (2) as an exercise, and will prove (3).

Suppose $p>3$. Then as ν_1 corresponds to γ, and χ to μ, using the rule (6.31), $\nu_1\chi$ corresponds to $\sum_{0\leqq i<j<p} \gamma^{k^i}\mu^{k^i} = -s_2\mu\beta$. $p>3$ implies $s_2=0$, $p=3$ implies $s_2=-1$. Thus $p>3$ implies $\nu_1\chi=0$. If $p=3$, since $\nu_1\chi \in E_2^{2,5}$, and $\alpha\beta\mu \in E_2^{2,5}$ also corresponds to $\mu\beta$, so $\nu_1\chi=\alpha\beta\mu$. Q.E.D.

DEFINITION. If u, v are quantities, we say $u \doteq v$ if and only if $u=ev$, $e \in Z_p^*$.

LEMMA 6.33. $d_2(\nu_2) \doteq \alpha\chi$, $d_2(\nu_i) \doteq \alpha\mu\nu_{i-2}$, $3 \leqq i < p-2$, $d_2(\mu\nu_i)=0$, $1 \leqq i < p-1$.

Proof. We know that $\alpha_2=\alpha_3=p^2$, $\alpha_4=p^4$. This shows that α, β, ν_1, μ, χ survive to E_∞. $\alpha_5=p^3$ implies ν_2 perishes. As α^3, $\alpha^2\beta$ are linearly independent, $d_2(\nu_2)=d_3(\nu_2)=0$, $d_5(\nu_2)\neq 0$, or $d_2(\nu_2)=0$, $d_3(\nu_2)\neq 0$ are not possible, implying $d_2(\nu_2)\doteq\alpha\chi$. We now proceed inductively, using (1) our knowledge of the α_i, (2) the linear independence of the $\{\alpha^k\beta^l\}$ and $\{\alpha^k\beta^l\mu\} \cup \{\alpha^{k+l}\nu\}$, in certain ranges. We omit details. Done.

Let $E_\infty^{(n)} = E_\infty^{0,n} \oplus E_\infty^{1,n-1} \oplus \cdots \oplus E_\infty^{n,0}$ in this order.

COROLLARY 6.34. $E_3 = E_\infty$ *in dimensions* $\leqq 2p$.

$$E_\infty^{(4)} = Z\beta^2 \oplus Z\chi \oplus Z\alpha\beta \oplus Z\alpha^2.$$

$$E_\infty^{(2i)} = Z\beta^i + Z\mu\nu_{i-2} \oplus Z\alpha\beta^{i-1} \oplus Z\alpha^2\beta^{i-2} \oplus \cdots \oplus Z\alpha^i, \quad 3 \leqq i \leqq p-1.$$

$$E_\infty^{(2i+1)} = Z\beta^{i-1}\mu \oplus Z\alpha\beta^{i-2}\mu \oplus Z\alpha^2\beta^{i-3}\mu \oplus \cdots \oplus Z\alpha^{i-1}\mu \oplus Z\alpha^{i-1}\nu, \quad 2 \leqq i \leqq p-2.$$

$$E_\infty^{(2p)} = Z\zeta \oplus Z\beta^p + Z\mu\nu_{p-2} \oplus Z\alpha\mu\nu_{p-3} \oplus Z\alpha^2\beta^{p-2} \oplus Z\alpha^3\beta^{p-3} \oplus \cdots \oplus Z\alpha^p.$$

Comparing with Theorem 6.26 and its lemmas, we see that $\chi_i \in H^*(G, Z)$ projects onto $\mu\nu_{i-1}$, $2 \leqq i \leqq p-2$. Notation: $\pi(\chi_i) \doteq \mu\nu_{i-2}$. Also $\pi(\mu)=\mu$, $\pi(\nu) \doteq \nu_1$.

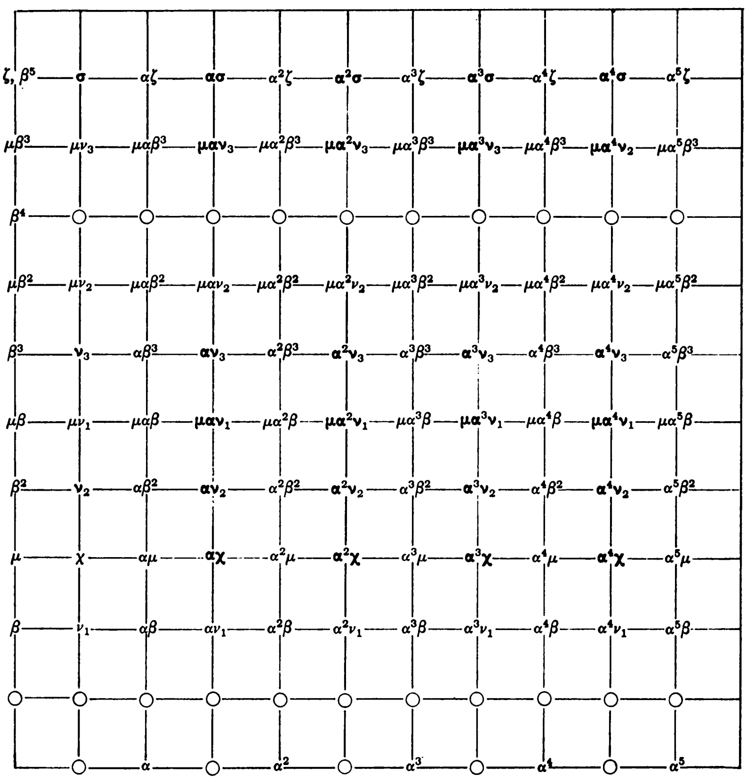
LEMMA 6.34. $d_2(\sigma) \doteq \alpha\mu\nu_{p-2}$.

Proof. As $\pi(\text{Cor } \gamma^p) \doteq \mu\nu_{p-2}$ (easily checked), and $\alpha \text{ Cor } \gamma^p = 0$, so $\alpha\mu\nu_{p-2}$ must be zero in E_∞. But $\alpha\mu\nu_{p-2}$ is a universal cycle. Therefore it must be hit from behind. The only possibility is $d_2(\sigma) \doteq \alpha\mu\nu_{p-2}$. Q.E.D.

It is now easy to see that $E_3 = E_\infty$. At this point we leave the second spectral sequence.

6.5. *Final remark.* Consider the three statements (a) $G \cong H$, (b) $ZG \cong ZH$ as rings, (c) $H^*(G, Z) \cong H^*(H, Z)$ as rings. Does (b) imply (a)? This is called the group ring problem, and is unsolved, although the answer is yes for special classes of groups. Using Yoneda's definition of group cohomology in terms of long exact sequences [see MacLane's *Homology*, Chapter III] one sees that $H^*(G, Z)$ (as ring) is determined by the *augmented ring* $ZG \xrightarrow{\varepsilon} Z$. Moreover, any ring isomorphism $\theta: ZG \xrightarrow{\cong} ZH$ may be modified so as to commute with augmentation. Thus,

E_2 term, second spectral sequence (case $p=5$)



boldface elements do not survive to $E_3 = E_\infty$

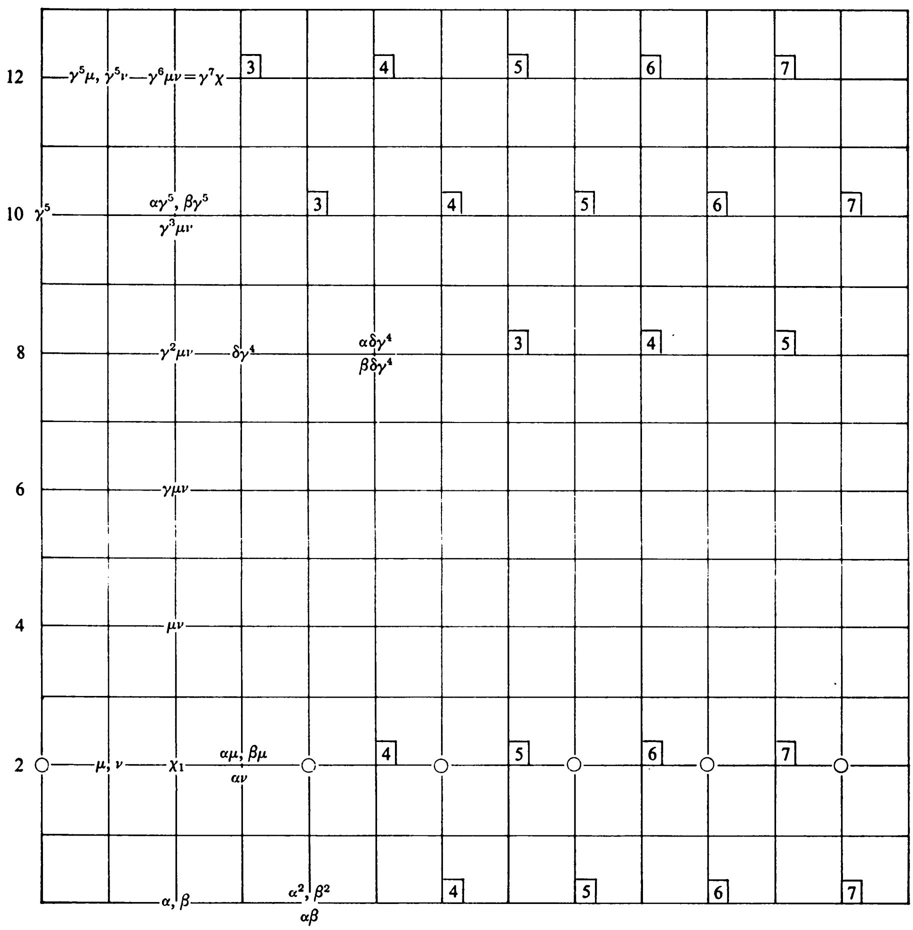
(b) implies (c). For non-p groups, (c) implies (a) is false, as shown by Alperin and Atiyah (unpublished) [essentially because $1 \to N \to G \to K \to 1$, exact, $(|N|, |K|) = 1$, implies $H^*(G, Z) \cong H^*(K, Z) \oplus H^*(N, Z)^k$, as rings]. For p-groups, (c) implies (a) may be true. Certainly our results show that (c) implies (a) for groups of order p^3; in fact, H^2 and H^3 suffice to determine G. Note also that groups of types I and II have the same character table [Burnside, *Theory of groups*, p. 319, Exercise 4], but different cohomology rings.

ACKNOWLEDGEMENTS. I wish to thank the following persons:

My thesis advisor, Professor Alex Heller, for his continued help and encouragement.

Professor Leonard Evens, who kindly showed me his results and conjectures concerning groups of type II, in the case $p=3$. He had calculated the multiplication

first spectral sequence $E_4 = E_5$ term (case $p=5$)



$n\urcorner$ denotes n independent generators and various differentials in the E_2 terms of the first and second spectral sequences (see §6), and had correctly conjectured the relations in the cohomology ring.

Professor R. G. Swan, for helpful conversations. Also, Professor J. Alperin, Miss Tze-Nan Kuo, Professor Melvin Rothenberg, and Professor Norman Steenrod.

BIBLIOGRAPHY

1. M. F. Atiyah, *Characters and cohomology of finite groups*, Inst. Hautes Études Sci. Publ. Math. **9** (1961).

2. H. Cardenas, *The cohomology algebra of* Σ_{p^2}, *coefficients* Z_p, Thesis, Princeton Univ., Princeton, N. J., 1965.

3. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956.

4. L. Evens, *The cohomology ring of a finite group*, Trans. Amer. Math. Soc. **101** (1961), 224–239.

5. ———, *A generalization of the transfer map in cohomology of groups*, Trans. Amer. Math. Soc. **108** (1963), 54–65.

6. ———, *On the Chern classes of representations of finite groups*, Trans. Amer. Math. Soc. **115** (1965), 180–193.

7. P. J. Hilton and D. Rees, *Natural maps of extension functors and a theorem of R. G. Swan*, Proc. Cambridge Philos. Soc. **57** (1961), 489–502.

8. G. P. Hochschild and J.-P. Serre, *Cohomology of group extensions*, Trans. Amer. Math. Soc. **74** (1953), 110–135.

9. M. Nakaoka, *Homology of the infinite symmetric groups*, Ann. of Math. **73** (1961), 229–257.

10. ———, *Note on cohomology algebras of symmetric groups*, J. Math. Osaka City Univ. **13** (1962), 45–55.

11. N. Steenrod and D. B. A. Epstein, *Cohomology operations*, Princeton Univ. Press, Princeton, N. J., 1962.

12. R. G. Swan, *A new method in fixed point theory*, Comment Math. Helv. **34** (1960).

13. ———, *The p-period of a finite group*, Illinois J. Math. **4** (1960), 341–346.

14. ———, *Periodic resolutions for finite groups*, Ann. of Math. **72** (1960), 267–291.

15. J. Tate, *Homology of Noetherian rings and local rings*, Illinois J. Math. **1** (1957), 14–27.

16. C. T. C. Wall, *Resolutions for extensions of groups*, Proc. Cambridge Philos. Soc. **57** (1961), 251–255.

17. H. Zassenhaus, *The theory of groups*, Chelsea, New York, 1949.

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIFORNIA