GENERALIZATIONS OF
WEAKLY COMPACT OPERATORS[1]

BY

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1. Introduction. This paper is concerned with developing generalizing concepts of weakly compact operators analogous to the generalization of compact operators by strictly singular operators. The latter class is given by:

1 DEFINITION. $T \in [X, Y]$[2] is said to be strictly singular if whenever T has a bounded inverse on M, a closed subspace of X, M is finite dimensional [6].

We introduce two definitions below which give rise to distinct classes of operators containing properly (in general) the class of weakly compact operators. The discussion includes conditions for these operators to fall into more familiar classes and the effect of conjugation on their properties.

2 DEFINITION. Let $T \in [X, Y]$. T is said to be almost weakly compact (a.w.c.) if, whenever T has a bounded inverse on a closed subspace, M, then M is reflexive.

It is clear that the class of almost weakly compact operators includes both the strictly singular and the weakly compact operators.

We now give an example to show that these classes are distinct.

3 EXAMPLE. Let $X = l_2 \times l_1$ and $Y = l_2 \times c_0$. $U = I \times T: X \to Y$ where I is the identity map and $T((\alpha_i)) = ((\sum_{n=i}^{\infty} \alpha_n))$. Clearly U is not strictly singular, and, since the image of the unit vectors in l_1, under T, has no weakly convergent subsequence, it follows that U is not weakly compact. To prove U is a.w.c. it suffices, since $l_2 \times l_1$ is weakly sequentially complete, to show that every nonreflexive subspace of $l_2 \times c_0$ contains a copy of c_0, which is not weakly sequentially complete [1]. To this end, let M be one such subspace. We define c^n to be those elements belonging to c which have zeros after the nth coordinate. Since M is not reflexive, and $l_2 \times c^n$ for all n is, it follows that for no n does the projection $P_n: M \to l_2 \times c^n$ have a bounded inverse. Hence, there exists in M a sequence $\{m_n\}$ such that $\|m_n\| = 1$ and

$$\text{(I)} \qquad \|P_n m_n\| < \frac{1}{2^n}.$$

Let the norm in $l_2 \times c_0$ be such that if $m = (x, y)$, then $\|m\| = \|x\|_{l_2} + \|y\|_{c_0}$ we consider the following subsequence:

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$$m_1' = m_1 = (x_1, y_1) \quad \text{where } x_1 \in l_2, \; y_1 = (y_1^1, y_1^2, \ldots) \in c_0$$

$$m_2' = m_{k_2} = (x_{k_2}, y_{k_2}) \quad \text{where } \sup_{j \geq k_2} |y_1^{(j)}| < \frac{1}{2}$$

(II)

$$\vdots \qquad\qquad \vdots$$

$$m_j' = m_{k_j} = (x_{k_j}, y_{k_j}) \quad \text{where } \sup_{j \geq k_n} |y_{k_n-1}^{(j)}| < \frac{1}{2^{n-1}}.$$

$$\vdots \qquad\qquad \vdots$$

We now divide each y_{k_j} into three parts. We write $y_{k_j} = y_{k_j}^+ + y_{k_j}^* + y_{k_j}^-$ where $P_{k_j} m_{k_j} = (x_{k_j}, y_{k_j})$. According to (I) y_{k_j} has the following properties:

(III)
$$y_{k_j} = (\underbrace{y_{k_j}^1, y_{k_j}^2, \ldots, y_{k_j}^{k_j-1}}_{y_{k_j}^+}, \underbrace{\ldots, y_{k_j}^{k_j}}_{y_{k_j}^*}, \underbrace{\ldots}_{y_{k_j}^-}).$$

Starting with $j=4$ via (I) and (II), we have $\|y_{k_j}^*\| \geq 3/4$ and by (III) for different j the $y_{k_j}^*$ are disjoint. We now show that, after renumbering, $T: ((\alpha_j)) \to \sum_{j=1}^{\infty} \alpha_j m_j'$ is an isomorphism of c_0 onto a subspace of M.

$$\left\| \sum_1^\infty \alpha_j m_j \right\| = \left\| \sum_1^\infty \alpha_j x_{k_j} \right\|_{l_2} + \left\| \sum_1^\infty \alpha_j y_{k_j} \right\|_{c_0}$$

(IV)
$$\leq \left\| \sum_1^\infty \alpha_j x_{k_j} \right\|_{l_2} + \left\| \sum_1^\infty \alpha_j y_{k_j}^+ \right\|_{c_0} + \left\| \sum_1^\infty \alpha_j y_{k_j}^* \right\|_{c_0} + \left\| \sum_1^\infty \alpha_j y_{k_j}^- \right\|_{c_0}$$

$$\leq \sup |\alpha_j| \left(\sum_1^\infty \frac{1}{2}j + 1 + \sum_1^\infty \frac{1}{2}j \right) \leq 3 \sup |\alpha_j|, \qquad 1 \leq j < \infty,$$

also

$$\left\| \sum_1^\infty \alpha_j m_j \right\| \geq \left\| \sum_1^\infty \alpha_j y_{k_j}^* \right\|_{c_0} - \left(\left\| \sum_1^\infty \alpha_j x_{k_j} \right\|_{l_2} + \left\| \sum_1^\infty \alpha_j y_{k_j}^+ \right\|_{c_0} + \left\| \sum_1^\infty \alpha_j y_{k_j}^- \right\|_{c_0} \right)$$

(V)
$$\geq \sup |\alpha_j| \cdot \tfrac{3}{4} - \sup |\alpha_j| \cdot \tfrac{1}{8} - \sup |\alpha_j| \tfrac{1}{8}$$

$$\geq \tfrac{1}{2} \sup |\alpha_j|, \qquad 1 \leq j < \infty,$$

where we have used the fact that $\{y_{k_j}^*\}$ are disjoint. Combining (IV) and (V) we are done.

2. **Characterizations and algebraic properties.** Since we have here a generalization of weakly compact operators very much like the generalization of compact operators by strictly singular operators, one might expect a similar characterizing theorem, i.e., T is a.w.c. if and only if for all M it follows that there exists $N \subset M$ such that $T|_N$ is weakly compact. This is not quite the case.

We will need the following theorem due to Kato and Goldberg [6, III.1.9].

1   THEOREM. *Let X and Y be Banach spaces and suppose $T \in [X, Y]$ is such that T does not have a bounded inverse on any subspace of finite codimension, then there exists $M \subset X$, M infinite dimensional and closed such that T_M (the restriction of T to M) is compact.*

2   THEOREM. *If T is a.w.c. and M is an infinite dimensional closed subspace of X; there is a closed infinite dimensional subspace $N \subset M$ such that T_N is weakly compact.*

Proof. *Case* I. *M* is reflexive.

In this case, every bounded operator on *M* is weakly compact [6].

Case II. *M* is not reflexive. Then, no subspace of finite codimension is reflexive. Hence, *T* does not have a bounded inverse on any such subspace. By Theorem 1, it follows that there exists $N \subset M$ such that T_N is compact, hence, weakly compact.

The reverse implication is not true as is shown in the next example.

3   DEFINITION. A quasi-reflexive space is one where dim $(X^{**}/JX) < \infty$ where JX is the canonical image of *X* in X^{**}.

4   EXAMPLE. It is shown in [9] that if *M* is an infinite dimensional subspace of a quasi-reflexive space, then $M \supset N$ infinite dimensional and reflexive. Hence, by taking the identity operator on any such space, we show that a.w.c. operators are not characterized by their restrictions to weakly compact operators.

R. Whitley has suggested the following definition and has proven Theorem 6 independently.

5   DEFINITION. $T \in [X, Y]$ is said to have property *R* whenever *T* has a bounded inverse on *M*, infinite dimensional and closed, it follows that $M \supset N$ infinite dimensional and reflexive.

This time we do get a characterization.

6   THEOREM. *T has property R if and only if for all infinite dimensional closed $M \subset X$ it follows that there exists $N \subset M$, N infinite dimensional and closed such that T_N is weakly compact.*

Proof. Suppose that *T* has property *R*, but that for every infinite dimensional subspace *N* of *M*, T_N is not weakly compact. Thus, as in Theorem 2, it follows that there exists $W \subset M$ such that T_W is compact; hence, a contradiction.

Now let *T* have a bounded inverse on *M* and suppose there exists $N \subset M$ such that *T* is weakly compact. It is readily seen that *N* is then reflexive.

7   REMARK. Example 4 shows that the class of operators with property *R* is strictly larger than the class of a.w.c. operators.

8   THEOREM. *The class of operators with property R taking X into X forms a closed ideal in $[X] \equiv [X, X]$.*

Proof. We first show they form a subspace. Let *M* be contained in *X*, and let *T* and *S* have property *R*. There exists, by Theorem 6, $N \subset M$, *N* an infinite dimensional closed subspace such that T_N is weakly compact. Also, there exists $W \subset N$, *W* an infinite dimensional closed subspace, such that S_W is weakly compact. Since the sum of weakly compact operators is weakly compact [8], $T_W + S_W = (T+S)_W$ is weakly compact and, again by Theorem 6, $T+S$ has property *R*.

The remaining properties follow by standard arguments.

9   THEOREM. *The a.w.c. operators taking X into X form a closed subset of $[X]$.*

Proof. Clear.

We do not know at this time whether the a.w.c. operators form a subspace. However, we do have:

10   THEOREM. *Let T be a.w.c. and let S be compact. Then $T+S$ is a.w.c.*

Proof. Let $(T+S)_M$ have a bounded inverse, M a closed subspace of X. S is compact, by a result of Lacey [6, III.2.3], it follows that there exists $N(\varepsilon) \subset M$ where $N(\varepsilon)$ is of finite codimension in M and $\|S_N\| < \varepsilon$ for any $\varepsilon > 0$. If $\|(T+S)m\| \geqq a\|m\|$ choosing $\varepsilon = a/2$ gives $\|Tn\| \geqq \|(T+S)n\| - \|Sn\| \geqq a/2\|n\|$. Hence, T has a bounded inverse on N which is thus reflexive. Since dim $(M/N) < \infty$, M is reflexive.

3. **When all operators are a.w.c. or have property** R.   In I.1 we showed that every nonreflexive subspace of $l_2 \times c_0$ contains a copy of c_0. An inspection of the proof shows that l_2 may be replaced by an arbitrary reflexive space X.

Hence, we have:

1   THEOREM. *If X is reflexive, then every nonreflexive subspace of $X \times c_0$ contains a copy of c_0.*

Similar methods give:

2   THEOREM. *If X is reflexive, then every nonreflexive subspace of $X \times l_1$ contains a copy of l_1.*

As an application of these theorems, we have conditions under which all bounded maps are almost weakly compact.

3   COROLLARY. *Let Y be weakly sequentially complete and X reflexive. Then if $T \in [X \times c_0, Y]$, T is a.w.c.*

Proof. If T has a bounded inverse on a nonreflexive subspace, then by Theorem 1, Y contains a copy of c_0.

By [1], c_0 is not weakly sequentially complete. We have a contradiction since every closed subspace of a weakly sequentially complete space is weakly sequentially complete.

4   COROLLARY. *If X^* is a separable conjugate space and Y is reflexive, then all $T \in [X^*, Y \times c_0]$ are a.w.c.*

Proof. As above, in the contrary case, X^* contains a copy of c_0. By [2, Theorem 4], no separable conjugate space can contain c_0.

5   COROLLARY. *If X is reflexive and Y has a separable conjugate space, then all $T \in [X \times l_1, Y]$ are a.w.c.*

Proof. Suppose not, i.e., T has a bounded inverse on M and M is not reflexive. By Theorem 2, M, and hence TM, contain a copy of l_1. We now use the fact that if

$W \subset Y$ and Y^* is separable, then so is W^* (if $\{y_n^*\}$ dense in Y^* consider $\{y_n^*|_W\}$). But the conjugate space of l_1 is m [1], which is not separable.

6 DEFINITION. A Banach space X is said to be somewhat reflexive if every infinite dimensional closed subspace contains an infinite dimensional reflexive subspace [9].

7 PROPOSITION. *Let either X or Y be somewhat reflexive, then any $T \in [X, Y]$ has property R.*

Proof. The conclusion is obvious from the definition of somewhat reflexive spaces.

4. Relation to strictly singular and weakly compact operators. If we restrict our attention to certain spaces we find that the operators above fall into more familiar classes.

1 REMARK. Let X be a Banach space with a Schauder basis, $\{x_n\}$ [4]. A sequence $\{z_n\}$, $\|z_n\| = 1$ is said to be a block basis with respect to $\{x_n\}$ if $z_n = \sum_{\alpha_n+1}^{\alpha_n+1} a_i x_i$ where $\alpha_1 < \alpha_2 < \cdots$. It is the case [2], that $\{z_n\}$ is a basic sequence if $\{x_n\}$ is.

Two basic sequences $\{x_i\}$ and $\{y_i\}$ are said to be equivalent if $\sum t_i x_i$ converges if and only if $\sum t_i y_i$ converges. In this case $[x_i]$ ($= \mathrm{sp}\,\{x_i\}$ = the closed linear span of the x_i) is isomorphic to $[y_i]$ under $A: (\sum t_i x_i) = \sum t_i y_i$ (p. 71 of [5]).

A space is said to be block homogeneous if every block basis is equivalent to the original basis. The spaces l_p, $1 \leq p < \infty$, and c_0 are block homogeneous. To see this in l_p let $\{z_n\}$ be a block basis, $z_n = \sum_{\alpha_n+1}^{\alpha_n+1} a_i x_i$, hence,

$$\left\| \sum_1^N t_n z_n \right\|_{l_p} = \left\{ |t_1|^p \sum_{\alpha_1+1}^{\alpha_2} |a_i|^p + \cdots + |t_N|^p \sum_{\alpha_N+1}^{\alpha_{N+1}} |a_i|^p \right\}^{1/p}$$

$$= \{|t_1|^p + \cdots + |t_N|^p\}^{1/p} \qquad \text{since } \|z_n\| = 1.$$

The result is now obvious. It is an open question as to whether or not these are the only block homogeneous spaces.

We will be using the following theorem of Bessaga and Pełczyński [2, Theorem 3].

2 THEOREM. *Let $\{x_n\}$ be a basis of X. If $\{y_n\} \subset X$ satisfies*

(a) $\inf \|y_n\| = \varepsilon > 0$,

(b) $f_i(y_n) \to 0$, $i = 1, 2, \ldots$, [3]

Then there exists $\{y_{n_k}\}$ which is a basic sequence equivalent to a block basis with respect to the $\{x_n\}$.

It is clear that by using the null spaces of the $\{f_i\}$ that we may, in any infinite dimensional space, find such a sequence [2, C.2].

3 THEOREM. *If either X or Y is block homogeneous, then T a.w.c. (T has property R) implies T is strictly singular or T is weakly compact.*

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[3] $\{f_i\} \subset X^*$ is the biorthogonal sequence. If $x = \sum a_i x_i$ then $a_i \equiv f_i(x)$ and the f_i are continuous [16, p. 207].

Proof. If X or Y is reflexive, all bounded maps $T: X \to Y$ are weakly compact [5]. We may now suppose that Y is block homogeneous and not reflexive. We will show T is strictly singular. To do this, it suffices to show that no infinite dimensional closed subspace, M, is reflexive. Suppose to the contrary. By the remark following Theorem 2, M contains a basic sequence $\{y_{n_k}\}$, i.e., $[y_{n_k}] \subset M$ and is thus reflexive. Using the fact that Y is block homogeneous we see that there exist isomorphisms A, B, such that $[y_{n_k}] \overset{A}{\to} [z_k] \overset{B}{\to} \{x_k\}$ where $\{z_k\}$ is the equivalent block basis of Theorem 2 and $\{x_k\}$ is the basis of Y. Thus Y is reflexive which is a contradiction.

We now consider a somewhat weaker property. Let X be a Banach space, not necessarily with a basis, and suppose that every separable subspace is contained in a subspace isomorphic to a block homogeneous space. We shall denote this property by (**). It is shown in [7, p. 336] that any separable subspace of $l_p(S)$, $1 \leq p < \infty$, S an uncountable set, is contained in a block homogeneous subspace isometrically isomorphic to l_p. The same is true of $c_0(S)$.

4 REMARK. To show any space X is reflexive, it suffices to show that any separable subspace is reflexive.

5. THEOREM. *If X has (**) and T has property R, then T is a.w.c.*

Proof. Let T_M have a bounded inverse and let N be a separable subspace of M, then it suffices by the above remark to show N is reflexive. By hypothesis, $N \subset W \subset X$, where W is isomorphic to B and B is block homogeneous. T_N has a bounded inverse so $N \supset Z$ reflexive. Let A be the isomorphism between W and B. Then $A(Z)$ is a reflexive subspace of B. By the proof of Theorem 3, B is then reflexive. Hence, so is W and then so is N.

6 DEFINITION. $T \in [X, Y]$ is said to be completely continuous if T maps weakly convergent sequences into norm convergent sequences.

7 THEOREM. *If T is a.w.c. or has property R, then T, completely continuous, implies T is strictly singular.*

Proof. Let T have a bounded inverse on M, infinite dimensional and closed, and suppose T has property R. Then $M \supset N$, infinite dimensional and reflexive, hence, $S(N)$, the unit ball in N, is weakly compact and by the Eberlein-Šmuljan theorem weakly sequentially compact. Let $\{x_n\} \subset S(N)$ then $\{x_{n_k}\}$ converges weakly to some x. By hypothesis, $Tx_{n_k} \to Tx$ in norm. Since T has a bounded inverse, $x_{n_k} \to x$ in norm, i.e., $S(N)$ is compact. Hence, W is finite dimensional, a contradiction. Thus M is finite dimensional and T is strictly singular. The obvious modifications yield the proof for the a.w.c. case.

If CC denotes the completely continuous operators, K the strictly singular, and AW the almost weakly compact. We have shown AW $\cap$ CC $\subseteq K$. We do not have equality, for consider the injection $T: l_p \to l_q$, $1 < p < q < \infty$. T is weakly, hence almost weakly compact, and is strictly singular by [7]. But T is obviously not completely continuous.

In [12] the following is proven:

8   THEOREM. *Let S be a compact Hausdorff space and $T: C(S) \to Z$ then the following are equivalent*:

(i) *T is strictly singular.*

(ii) *T does not have a bounded inverse on a subspace is isomorphic to c_0.*

(iii) *T is weakly compact.*

By c.l.i. we shall mean continuous linear image.

9   COROLLARY. *Let E be c.l.i. of $C(S)$, S a compact Hausdorff space. If T is a.w.c. or has property R, then T is weakly compact.*

Proof. We have the following

$$C(S) \xrightarrow{\;I\;} E$$
$$\downarrow T$$
$$Z$$

If T is not weakly compact, then since I is onto $T \cdot I$ is not weakly compact (if U is the unit ball in $C(S)$, then $I(U)$ is a neighborhood of 0 in E by the open mapping theorem). By Theorem 8 there exists $X \subset C(S)$ where X is isomorphic to c_0 and $(T \cdot I)_X$ has a bounded inverse. Thus, $I(X)$ is isomorphic to c_0 and T has a bounded inverse there. The proof is completed by noting that c_0 is not reflexive nor does it contain any infinite dimensional reflexive subspaces (since then by p. 194 of [1], c_0 itself would be reflexive).

We note that the theorem is not true for maps into $C(S)$. For let J be the quasi-reflexive space of R. C. James. J has a basis and is thus separable. Hence, by [1] there exists an isometry, $T: J \to C[0, 1]$. T has property R by I.3.7 but T is not weakly compact since J would then be reflexive.

10   REMARK. If T maps $C(S)$ into Z then, by Theorem 8, T a.w.c. or has property R implies T strictly singular.

5. **Almost weakly compact compactness and property R under conjugation.**   We now ask what properties of a.w.c. operators with property R are carried over to their conjugates and vice versa?

In general, it seems that we can say nothing as is seen in the examples below. However, if we restrict our attention to certain types of spaces we do obtain results.

1   EXAMPLE. Let $T: l_1 \to c_0$ be an epimorphism, (note: such a T exists by Mazur's theorem [6, II.4.5]. T is a.w.c., since T is strictly singular [7]. But T^* is then a monomorphism of l_1 into m. Thus T^* is not a.w.c. nor does it have property R. (l_1 contains no infinite dimensional reflexive subspaces [1, p. 194].)

2   EXAMPLE. James [10] shows the existence of a space B such that $B^{(iv)}$, the fourth conjugate, is separable and $B^{(iv)} = E \oplus V$ where E is isomorphic to l_1. In [13] Pełczyński points out that if T is the epimorphism from l_1 to B (B is separable) then T^{**} is strictly singular. Thus, if we let $A = T^*$, we see that A is a

monomorphism of B^* (into m) and $A^*=T^{**}$ is strictly singular. Since B is not reflexive, A is not a.w.c., but A^*, being strictly singular, is a.w.c.

We leave open the question of the existence of a T such that T does not have property R but T^* does.

3 DEFINITION. A Banach space X is said to be subprojective if given M, an infinite dimensional subspace of X, it follows there exists $N \subseteq M$, N infinite dimensional such that $X = N \oplus W$ [15].

4 THEOREM. *Let $T: X \to Y$ where Y is subprojective. If T^* is a.w.c. then T has property* R.

Proof. We proceed as in [15]. Suppose not. Let T have a bounded inverse and suppose M contains no infinite dimensional reflexive subspaces. There exists $W \subseteq TM$ such that $Y = W \oplus V$. $W = TN$ for some N contained in M. Let $P: Y \to TN$, then P^* takes $(TN)^*$ isomorphically into Y^*. Let $V = P^*((TN)^*)$. As in [15], T_V^* has a bounded inverse, thus W is reflexive.

5 DEFINITION. Let $\{x_n\}$ be a Schauder basis for the Banach space X. The basis is said to be unconditional if there exists $\varepsilon > 0$ such that $\|\sum_1^n a_{p_i} x_{p_i} + \sum_1^m a_{q_i} x_{q_i}\| \geqq \varepsilon \|\sum_1^n a_{p_i} x_{p_i}\|$ for any set of nonoverlapping indices $\{p_i\}$, $\{q_i\}$. See [11] for various equivalences.

Gurevič [8] has shown the following:

6 LEMMA. *If $\{x_n\}$ is an unconditional basis for a Banach space X, then for $\{\alpha_i\} \in m$ there exists K, K dependent only on $\sup |\alpha_i|$ such that $\|\sum_1^n \alpha_i a_i x_i\| \leq K \|\sum_1^n a_i x_i\|$.*

7 LEMMA. *If X is a Banach space with an unconditional basis $\{x_n\}$, then every nonreflexive subspace of X contains*
  (i) *a complemented copy of c_0 or*
  (ii) *a complemented copy of l_1.*

Proof. We divide the proof into two cases. Let M be a nonreflexive subspace of X.

Case I. M is not weakly sequentially complete. In [3] it is shown that any such subspace contains a copy of c_0. Now since X is separable, by a result of Sobczyk [14] it follows that this subspace is complemented in X.

Case II. M is weakly sequentially complete. Since M is not reflexive, $S(M)$ is not weakly compact. By the Eberlein-Šmuljan theorem, there is a sequence $\{y_j\}$ in the unit ball of M such that $\{y_j\}$ has no weak Cauchy subsequence.

Let $\{g_j\}$ be the coordinate functionals, i.e., if $x \in X$, it follows $x = \sum g_j(x) x_j$ and suppose $\|x_j\| = 1$ for all j. By the Cantor process we may extract a subsequence of the $\{y_j\}$ which converges for all $\{g_j\}$. We assume, without loss, that $\{y_j\}$ is the desired subsequence. Since $\{y_j\}$ is not weak Cauchy, there exists f, $\|f\| = 1$, and $\varepsilon > 0$ such that for all N, there exists $n, m > N$ such that

(I) $$|f(y_n - y_m)| > \varepsilon.$$

Pick N and then n_1 and m_1. Let $z_1 = y_{n_1} - y_{m_1}$. Choose N_1 such that $\|\sum_{N_1}^\infty a_i^{(1)} x_i\|$ $< \varepsilon C/4$ where $C < 1$ and $z_1 = \sum_1^\infty a_i^{(1)} x_i$. By (I) choose $n_2, m_2 > N_1$ such that $|f(y_{n_2} - y_{m_2})| > \varepsilon$ and $\|\sum_1^{N_1} a_i^{(2)} x_i\| < \varepsilon C/4$ where $z_2 = y_{n_2} - y_{m_2} = \sum_1^\infty a_i^{(2)} x_i$. Note that such a z_2 may be found since the $\{y_j\}$ converge for each g_j and $a_i^{(2)} = g_i(y_{n_2} - y_{m_2})$. Continuing we get

$$\text{(II)} \qquad z_j = \sum_1^\infty a_i^{(j)} x_i, \qquad \left\| \sum_1^{N_j} a_i^{(j+1)} x_i \right\| < \frac{\varepsilon}{2^{j+1}} C$$

and

$$\left\| \sum_{N_{j+1}+1} a_i^{(j+1)} x_i \right\| < \frac{\varepsilon}{2^{j+1}} C.$$

Let

$$w_j = \sum_{N_j+1}^{N_{j+1}} a_i^{(j+1)} x_i.$$

Choosing C appropriately, as in [2], we see that $\{z_j\}$ is a basic sequence equivalent to the block basis sequence $\{w_j\}$. We note now, that $C < 1$, $|f(w_j)| > \varepsilon/2$. We will now show that $[w_j]$ is isomorphic to l_1. Since f is of norm 1, $2 \geq \|z_j\| \geq \varepsilon$. It may then be seen via (II) that $3 \geq \|w_j\| \geq \varepsilon/2$ for all j. Therefore, we may assume that $\|w_j\| = 1$ and there exists $\delta > 0$ such that $|f(w_j)| > \delta$ for all j.

Let $w = \sum a_i w_i \in [w_j]$ and let $\|w\| = 1$. Let $a_j f(w_j) = r_j e^{-i\theta_j}$ applying Lemma 6 we see that there exists K such that,

$$\left\| \sum_1^n a_j w_j \right\| \geq K \left\| \sum_1^n e^{-i\theta_j} a_j w_j \right\| \geq K \left| f \sum_1^n e^{-i\theta_j} a_j w_j \right| \geq K\delta \sum_1^n |a_j|.$$

Since this holds for all n, we have $\|w\| \geq K\delta \sum_1^\infty |a_j|$ also $\|w\| \leq \sum_1^\infty |a_j|$. Hence, $[w_j]$ is isomorphic to l_1 under $A(a_j) = \sum_1^\infty a_j w_j$. Now since $[z_j]$ is equivalent to $[w_j]$ we have shown M contains a copy of l_1. This result has been shown in a different manner in [3]. Applying C.7 of [2], we obtain the desired result.

Now we can prove

8   THEOREM. *If $T: X \to Y$ where Y is a space with an unconditional basis, then T^* a.w.c. implies T is a.w.c.. Further, if the basis is shrinking [4], then T^* has property R, implies T is a.w.c.*

Proof. Suppose not. Then T has a bounded inverse on M where M is not reflexive. Then, by Lemma 7, TM contains $W = TN$ isomorphic to l_1 or c_0 and complemented in Y. As in Theorem 4, we obtain $V \subset Y^*$ such that T_V^* has a bounded inverse. Here, however, V is isomorphic to m or l_1, neither of which is reflexive. Hence, if T^* is a.w.c., then we have a contradiction. If the basis of Y is shrinking, then Y does not contain any copy of l_1 [4]. Hence, V is isomorphic to l_1 which contains no reflexive subspaces. Thus, in this case, it suffices to assume T^* has property R.

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Note added in proof. The problem concerning block homogeneity mentioned at the beginning of paragraph 4 has been solved in the affirmative by M. Zippin, *On perfectly homogeneous bases in Banach spaces*, Israel J. Math. **4** (1966), 265–272.

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