# ERGODIC THEORY AND BOUNDARIES

BY

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1. **Introduction.** Let $T$ be a conservative positive contraction on the $L_1$ space of a finite measure space $(X, \mathscr{F}, \mu)$. A theorem of Chacon [5], [2] shows that $T$ defines a sub $\sigma$-field $\mathscr{I}$ of $\mathscr{F}$, consisting of invariant subsets of $X$. The ratio ergodic limits are measurable with respect to $\mathscr{I}$ [5], [2] and the class of these limits contains $L_\infty(X, \mathscr{I}, \mu)$, which can be considered as the invariant functions of the adjoint transformation [2]. The main purpose of the present paper is to show that any positive contraction on $L_1(X, \mathscr{F}, \mu)$ behaves, asymptotically, like a conservative transformation (Theorems 3 and 4) and that the invariant functions of the adjoint transformation can be approximated by the ratio ergodic limits.

Intuitively, a ratio ergodic limit corresponds to the result of an averaging process of different values of a function. It is then natural to consider these limits as functions that are smooth with respect to the asymptotic behaviour of the transformation. This leads (Theorem 6) to a Martin-Doob type representation [12], [8] of invariant functions as the $L_\infty$ functions of a compact Hausdorff space $\mathscr{M}$ with a Baire measure. The topology on $\mathscr{M}$ is just strong enough to make the ergodic limits to correspond to continuous functions. As an example we consider a transformation of Feller [10] and show that for this case the above representation is identical with the Poisson representation of harmonic functions in the unit disk. We also consider the possibility of joining $X$ and $\mathscr{M}$, convergence of measures to $\mathscr{M}$ in $X \cup \mathscr{M}$ (Theorem 7), and a relation (Lemma 9) between the Feller and Martin-Doob type representations, corresponding to a result of Feldman [9].

2. **Preliminaries.** Let $(X, \mathscr{F}, \mu)$ be a finite measure space and let $L_p = L_p(X, \mathscr{F}, \mu)$, $1 \leqq p \leqq \infty$ be the usual Banach spaces, and $L_p^+$ denote the positive cone of $L_p$. Let $T: L_1 \to L_1$ be a positive linear contraction and $U: L_\infty \to L_\infty$ be its dual. For $\alpha \in L_\infty$ define $T_\alpha: L_1 \to L_1$ as $T_\alpha f = \alpha f + T(1-\alpha)f$, $f \in L_1$, and let $U_\alpha$ be its dual. If $\chi_E$ is the characteristic function of $E \in \mathscr{F}$ we write $T_E$ and $U_E$ instead of $T_{\chi_E}$ and $U_{\chi_E}$.

The following partial ordering of $L_1^+$ is similar to that of Bishop and deLeeuw given in [3].

Definition 1. For $f, g \in L_1^+$, $f \prec g$ if and only if there exist an integer $n \geqq 1$ and $\alpha_1, \ldots, \alpha_n \in L_\infty$ such that $0 \leqq \alpha_i \leqq 1$ for $i = 1, \ldots, n$ and $g = T_{\alpha_n} \cdots T_{\alpha_1} f$.

This relation is reflexive and transitive and $f \prec g$ implies $\|f\|_1 \geqq \|g\|_1$. Also, an

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induction argument shows that if $f \prec g$ then there exists an integer $n \geqq 1$ such that $g \prec T^n f$. Hence $\{g \in L_1^+ \mid g \succ f\}$ is (upward) directed by $\prec$.

DEFINITION 2. For $E \in \mathscr{F}, f \in L_1^+$ let

$$\Psi_E f = \sup_{g \succ f} \int_E g \, d\mu, \quad \Theta_E f = \lim_{g \succ f} \Psi_E g.$$

Note that $\Theta_E f = \lim_{n \to \infty} \Psi_E T^n f$.

LEMMA 1. *The limits* $\psi_E = \lim_{n \to \infty} U_E^n \chi_E$ *and* $\theta_E = \lim_{n \to \infty} U^n \psi_E$ *both exist* (*a.e.*) *and satisfy*

$$\Psi_E f = \int \psi_E f \, d\mu, \quad \Theta_E f = \int \theta_E f \, d\mu.$$

**Proof.** By induction, $U_E^n \chi_E \uparrow$ and $U^n \psi_E \downarrow$, so the limits exist. Now if $f \in L_\infty^+$ satisfies

$$(*) \qquad \chi_E U f \leqq \chi_E f, \quad \chi_{E^c} U f \geqq \chi_{E^c} f$$

with $E^c = X - E$, then for all $\alpha \in L_\infty$, $0 \leqq \alpha \leqq 1$, we have

$$U_\alpha f = \alpha f + (1-\alpha) U f \leqq \chi_E f + \chi_{E^c} U f = U_E f.$$

Since, by induction, $U_E^n \chi_E$ satisfies (*) for all $n \geqq 0$, we get, again by induction, $U_{\alpha_n} \cdots U_{\alpha_1} \chi_E \leqq U_E^n \chi_E$, and hence $\Psi_E f = \int \psi_E f \, d\mu$. The final part follows from the definition (cf. also [4] and [2]).

DEFINITION 3. For $E, F \in \mathscr{F}$, let

$$\psi_{EF} = \psi_E + \psi_F - \psi_{E \cup F}, \quad \theta_{EF} = \theta_E + \theta_F - \theta_{E \cup F}.$$

$\Psi_{EF}$, $\Theta_{EF}$ are the functionals on $L_1$ defined by the $L_\infty$ functions $\psi_{EF}$, $\theta_{EF}$.

We note that $\psi_{EF}$ and $\theta_{EF}$ are monotone and subadditive in each index. This follows easily from the following general result, which will be useful to obtain other relations between these set functions (cf. [7]).

LEMMA 2. *If* $a_i$ *is real and* $A_i \in \mathscr{F}$ *for* $i = 1, \ldots, n$ *and* $A = \bigcup_{i=1}^n A_i$, *then* $\chi_A \sum_{i=1}^n a_i \psi_{A_i} \geqq 0$ *implies* $\sum_{i=1}^n a_i \psi_{A_i} \geqq 0$ *and* $\sum_{i=1}^n a_i \theta_{A_i} \geqq 0$.

**Proof.** If $f \in L_1^+$ and $E \subset F$, $E, F \in \mathscr{F}$, then by induction: $\chi_{F^c} T_F^n f \leqq \chi_{F^c} T_E^n f$. Hence

$$0 \leqq \int_{F^c} \psi_E T_F^n f \, d\mu \leqq \int_{F^c} \psi_E T_E^n f \, d\mu \leqq \int_{E^c} \psi_E T_E^n f \, d\mu$$
$$\leqq \int \psi_E T_E^n f \, d\mu - \int_E \psi_E T_E^n f \, d\mu \leqq \Psi_E f - \int_E T_E^n f \, d\mu \to 0$$

as $n \to \infty$. Now

$$\begin{aligned}\int \psi_E f\,d\mu &= \int \psi_E \chi_F f\,d\mu + \int \psi_E \chi_{F^c} f\,d\mu = \int \psi_E \chi_F f\,d\mu + \int \psi_E T_E \chi_{F^c} f\,d\mu \\ &= \int \psi_E \chi_F f\,d\mu + \int \psi_E T \chi_{F^c} f\,d\mu = \int \psi_E T_F f\,d\mu \\ &= \int \psi_E T_F^n f\,d\mu = \int_F \psi_E T_F^n f\,d\mu + \int_{F^c} \psi_E T_F^n f\,d\mu,\end{aligned}$$

and hence $\lim_{n\to\infty} \int_F \psi_E T_F^n f\,d\mu = \int \psi_E f\,d\mu$. Using this for the case of $A_i \subset A$, $i = 1, \dots, n$, we get

$$0 \leqq \sum_{i=1}^{n} a_i \int_A \psi_{A_i} T_A^n f\,d\mu \to \sum_{i=1}^{n} a_i \int \psi_{A_i} f\,d\mu$$

as $n \to \infty$, which proves the first assertion. Since $U$ is positive, the remainder follows.

Lemma 3. *If $\chi_E \theta_E \geqq \alpha \chi_E$ then $\theta_E \geqq \alpha \psi_E$.*

**Proof.** From the proof of the previous lemma we have that, for $f \in L_1^+$,

$$\lim_{n\to\infty} \int_{E^c} \theta_E T_E^n f\,d\mu = 0.$$

Hence

$$\begin{aligned}\int \theta_E f\,d\mu &= \int \theta_E T_E^n f\,d\mu = \lim_{n\to\infty} \int_E \theta_E T_E^n f\,d\mu \\ &\geqq \alpha \lim_{n\to\infty} \int_E T_E^n f\,d\mu \geqq \alpha \int \psi_E f\,d\mu.\end{aligned}$$

Finally we prove the following.

Lemma 4. *For $E, F \in \mathscr{F}$, $\|\theta_E\|_\infty = \|\chi_E \theta_E\|_\infty = 0$ or $1$ and $\|\theta_{EF}\|_\infty = \|\chi_E \theta_{EF}\|_\infty = \|\chi_F \theta_{EF}\|_\infty = 0$ or $1$.*

**Proof.** For $g \in L_1^+$, as $n \to \infty$, $0 \leqq \Theta_E(\chi_{E^c} T_E^n g) \leqq \Psi_E(\chi_{E^c} T_E^n g) \to 0$ as in the proof of Lemma 2. Hence the decomposition $\Theta_E g = \Theta_E T_E^n g = \Theta_E(\chi_E T_E^n g) + \Theta_E(\chi_{E^c} T_E^n g)$ shows that $\|\theta_E\|_\infty = \|\chi_E \theta_E\|_\infty$. Now, for $n, m \geqq 1$,

$$\begin{aligned}\Theta_E g &= \Theta_E T_E^n T^m g = \lim_{m\to\infty} \lim_{n\to\infty} \Theta_E(\chi_E T_E^n T^m g) \leqq \lim_{m\to\infty} \lim_{n\to\infty} \|\theta_E\|_\infty \|\chi_E T_E^n T^m g\|_1 \\ &= \lim_{m\to\infty} \|\theta_E\|_\infty \Psi_E T^m g = \|\theta_E\|_\infty \Theta_E g\end{aligned}$$

which completes the proof of the first part, since $\|\theta_E\|_\infty \leqq 1$. For the second part, we have, if $g \in L_1^+$, $0 \leqq \Theta_{EF}(\chi_{E^c} T_E^n g) \leqq \Theta_E(\chi_{E^c} T_E^n g) \to 0$ as $n \to \infty$ which shows that $\|\theta_{EF}\|_\infty = \|\chi_E \theta_{EF}\|_\infty$. Now

$$\Theta_E g - \Theta_E(\chi_{E^c} T_E^n g) = \Theta_E(\chi_E T_E^n g) \leqq \Theta_{E\cup F}(\chi_E T_E^n g) \leqq \|\chi_E T_E^n g\|_1 \leqq \Psi_E g;$$

thus, $\Theta_E g \leqq \lim_{n\to\infty} \Theta_{E\cup F}(\chi_E T_E^n g) \leqq \Psi_E g$. Replacing $g$ by $T^m g$ and letting $m \to \infty$ we get

$$\Theta_E g = \lim_{m\to\infty} \lim_{n\to\infty} \Theta_{E\cup F}(\chi_E T_E^n T^m g).$$

Next, consider

$$\Theta_{EF}(\chi_E T_E^n T^m g) = (\Theta_E + \Theta_F - \Theta_{E\cup F})(\chi_E T_E^n T^m g)$$

and let $n \to \infty$ to get

$$\Theta_{EF} g = \Theta_E g + \lim_{n\to\infty} \Theta_F(\chi_E T_E^n T^m g) - \lim_{n\to\infty} \Theta_{E\cup F}(\chi_E T_E^n T^m g).$$

Now, letting $m \to \infty$ we have

$$\Theta_{EF} g = \lim_{m\to\infty} \lim_{n\to\infty} \Theta_F(\chi_E T_E^n T^m g).$$

But

$$\begin{aligned} \Theta_{EF} g &\leqq \|\theta_{EF}\|_\infty \lim_{m\to\infty} \lim_{n\to\infty} \|\chi_E T_E^n T^m g\|_1 \\ &\leqq \|\theta_{EF}\|_\infty \lim_{m\to\infty} \Psi_E T^m g \\ &\leqq \|\theta_{EF}\|_\infty \Theta_E g. \end{aligned}$$

Hence

$$\begin{aligned} \Theta_{EF} g &= \Theta_{EF}(T_E^n T^m g) \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \Theta_{EF}(\chi_E T_E^n T^m g) \\ &\leqq \|\theta_{EF}\|_\infty \lim_{m\to\infty} \lim_{n\to\infty} \Theta_F(\chi_E T_E^n T^m g) \\ &\leqq \|\theta_{EF}\|_\infty \Theta_{EF} g. \end{aligned}$$

This completes the proof, since $\|\theta_{EF}\|_\infty \leqq 1$.

DEFINITION 4. $\Sigma = \{E \in \mathscr{F} \mid \Theta_{EE^c} = 0\}$.

LEMMA 5. *$\Sigma$ is a field.*

**Proof.** Let $E, F \in \Sigma$ and $G = E \cap F$. Then

$$0 \leqq \theta_{GG^c} = \theta_{G(E^c \cup F^c)} \leqq \theta_{GE^c} + \theta_{GF^c} \leqq \theta_{EE^c} + \theta_{FF^c} = 0.$$

Thus $G \in \Sigma$.

DEFINITION 5. $\mathscr{A}$ is the $L_\infty$-closure of the class of $\Sigma$-simple functions.

We note that $\mathscr{A}$ is a sub-Banach space of $L_\infty$.

THEOREM 1. *For a real valued function $f \in L_\infty$, the following conditions are equivalent*:

(i) $f \in \mathscr{A}$,

(ii) $\lim_{g \succ g_0} \int fg\, d\mu$ *exists for all* $g_0 \in L_1^+$,

(iii) *for all real numbers* $\alpha$ *and* $\varepsilon > 0$,

$$\theta_{EF} = 0 \quad \text{where} \quad E = \{x \mid f(x) \leqq \alpha\}, \quad F = \{x \mid f(x) \geqq \alpha + \varepsilon\}.$$

**Proof.** (i)$\Rightarrow$(ii). If $E \in \Sigma$ then $\theta_E + \theta_{E^c} = \theta_X$; thus, for a real valued $g_0 \in L_1^+$,

$$\limsup_{g \succ g_0} \int_E g\, d\mu = \limsup_{g \succ g_0} \int g\, d\mu - \limsup_{g \succ g_0} \int_{E^c} g\, d\mu$$
$$= \liminf_{g \succ g_0} \int_E g\, d\mu.$$

Therefore $\lim_{g \succ g_0} \int \chi_E g\, d\mu$ exists for all $E \in \Sigma$.

Hence it exists for all $\Sigma$-simple functions, and thus for all $f \in \mathscr{A}$.

(ii)$\Rightarrow$(iii). Suppose $E$ and $F$ are as in (iii) but that $\theta_{EF} \neq 0$. Then $\|\theta_{EF}\|_\infty = 1$ and for all $\delta > 0$ there exists $g_0 \in L_1^+$ with $\|g_0\|_1 = 1$ and $\int \theta_{EF} g\, d\mu \geqq 1-\delta$. Hence $\Theta_E g_0 \geqq 1-\delta$ and $\Theta_F g_0 \geqq 1-\delta$. Thus $\limsup_{g \succ g_0} \int fg\, d\mu \geqq (1-\delta)(\alpha+\varepsilon)$ and $\liminf_{g \succ g_0} \int fg\, d\mu \leqq (1-\delta)\alpha + \delta\|f\|_\infty$. If $\delta$ is chosen sufficiently small we see that $\lim_{g \succ g_0} \int fg\, d\mu$ does not exist.

(iii)$\Rightarrow$(i). Let $a_1 < a_2 < \cdots < a_n$ be $n$ numbers and let $E_i = \{x \mid f(x) \leqq a_i\}$. Now

$$\sum_{i=1}^n \theta_{E_i E_i^c} = \sum_{i=1}^n (\theta_{E_i} + \theta_{E_i^c} - \theta_X)$$
$$\leqq \sum_{i=2}^n (\theta_{E_{i-1}} + \theta_{E_i^c} - \theta_{E_{i-1} \cup E_i^c}) + (\theta_{E_n} + \theta_{E_1{}^c} - \theta_X)$$
$$\leqq 1.$$

Hence if $E_a = \{x \mid f(x) \leqq a\}$ then $\theta_{E_a E_a^c} \neq 0$ for only countably many $a$'s, and so $f \in \mathscr{A}$.

## 3. Invariant functions.

Definition 6. $\mathscr{H} = \{f \mid f \in L_\infty, f = Uf\}$ is the class of invariant functions of $U$. We assume $\mathscr{H} \neq \{0\}$.

Note that $\mathscr{H}$ is a sub-Banach space of $L_\infty$. Also, if $h \in \mathscr{H}$ and $g' \succ g \in L_1^+$, then $\int hg'\, d\mu = \int hg\, d\mu$ and hence $\lim_{g' \succ g} \int hg'\, d\mu$ exists. Thus $\mathscr{H} \subset \mathscr{A}$.

If $f \in \mathscr{A}$, then $\lim_{n\to\infty} \int fT^n g\, d\mu = \lim_{n\to\infty} \int U^n f g\, d\mu$ exists for all $g \in L_1(X, \mathscr{F}, \mu)$. Hence the bounded sequence $U^n f$, $n = 1, 2, \ldots$ has a limit $\pi(f)$ in the $w^*$-topology of $L_\infty$. Obviously the limit lies in $\mathscr{H}$, so $\pi\colon \mathscr{A} \to \mathscr{H}$ is a positive linear contraction.

Definition 7. $\mathscr{A}_0 = \ker \pi = \{f \in \mathscr{A} \mid w^* - \lim U^n f = 0\}$. Hence $\mathscr{A}/\mathscr{A}_0 \cong \mathscr{H}$ is a canonical, isometric isomorphism.

Now $\mathscr{A}$ is a $C^*$-algebra with the usual operations. We show that $\mathscr{A}_0$ is a closed ideal.

Theorem 2. *$\mathscr{A}_0$ is a closed ideal in $\mathscr{A}$.*

**Proof.** Let $f \in \mathscr{A}_0$ and assume that $f$ is real. Choose $\varepsilon > 0$ and set $E = \{x \mid f(x) \geqq \varepsilon\}$. We may assume $E \in \Sigma$. Suppose $\theta_E \neq 0$; then for all $\delta > 0$, there is a $g \in L_1^+$ such that $\|g\|_1 = 1$ and $\Theta_E g \geqq 1-\delta$. Hence:

$$0 = \lim_{n\to\infty} \int U^n f \cdot g\, d\mu = \lim_{n\to\infty} \int fT^n g\, d\mu$$
$$\geqq \varepsilon \lim_{n\to\infty} \int_E T^n g\, d\mu - \|f\|_\infty \lim_{n\to\infty} \int_{E^c} T^n g\, d\mu$$
$$\geqq \varepsilon(1-\delta) - \|f\|_\infty\, \delta.$$

Clearly, this fails for small $\delta$, and so $\theta_E=0$. Thus if $E=\{x \mid |f(x)|>\varepsilon\}$, we have $\theta_E=0$.

Now if $h \in \mathscr{A}$, $h \neq 0$, set $F=\{x \mid |f(x)h(x)| \geqq \varepsilon\}$. Since $F \subset \{x \mid |f(x)| \geqq \varepsilon/\|h\|_\infty\}$, we have $\theta_F=0$. Hence

$$\left|\lim_{n\to\infty} \int U^n(fh)g\, d\mu\right| \leqq \varepsilon \lim_{n\to\infty} \int_F T^n g\, d\mu + \varepsilon\|g\|_1 \qquad \text{if } g \in L_1^+$$
$$\leqq \varepsilon\|g\|_1 \qquad \text{for all } \varepsilon > 0.$$

Hence $fh \in \mathscr{A}_0$.

As a result of the lemma, we have given $\mathscr{A}/\mathscr{A}_0$, and hence $\mathscr{H}$ the structure of a $C^*$-algebra. Thus $\mathscr{H}$ has a representation as the set of complex valued continuous functions on its maximal ideal space. This corresponds to Feller's representation [10] of the invariant functions of certain Markov processes, and we shall refer to $\mathscr{H}$'s maximal ideal space as the Feller boundary.

As is known [8], [11], the Feller boundary is larger than it need be. In the next section, we obtain some properties of ratio ergodic limits, and use them to define a sub $C^*$-algebra $\mathscr{G}$ of $\mathscr{H}$, with a maximal ideal space $\mathscr{M}$, smaller than the Feller boundary, but large enough to represent $\mathscr{H}$ as a function algebra on $\mathscr{M}$. This corresponds to the Martin-Doob representation [12], [8], [11] for some classes of functions, and $\mathscr{M}$ will be referred to as the Martin-Doob boundary.

4. **Properties of ratio ergodic limits.** In [6] Chacon and Ornstein proved that for any $f, g \in L_1$, with $g>0$, the limit:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} T^k f}{\sum_{k=1}^{n} T^k g}$$

exists a.e. We denote the limit function by $(f/g)$. It is also known [5], [4], [1], that if $\alpha \leqq (f/g) \leqq \beta$ a.e. on $E \in \mathscr{F}$, then $\alpha \leqq \Psi_E(f)/\Psi_E(g) \leqq \beta$.

THEOREM 3. *If $f, g \in L_1^+$ with $g>0$, and*

$$E = \{x \mid (f/g)(x) \leqq a\},$$
$$F = \{x \mid (f/g)(x) \geqq a+\varepsilon\},$$

*then $\theta_{E,F}=0$, for all $a \geqq 0$ and $\varepsilon>0$.*

**Proof.** If $\theta_{E,F} \neq 0$ then $\|\chi_E\theta_{E,F}\|_\infty=1$. Let $\delta>0$ and set $E_\delta=\{x \mid \theta_{E,F}(x) \geqq 1-\delta\} \cap E$, and similarly for $F_\delta$. Then $\|\chi_{E-E_\delta}\theta_{E-E_\delta}, F\|_\infty \leqq \|\chi_{E-E_\delta}\theta_{E,F}\|_\infty \leqq 1-\delta$. Hence $\theta_{E-E_\delta,F}=0$, and so $\theta_{E_\delta,F} \leqq \theta_{E,F} \leqq \theta_{E_\delta,F}+\theta_{E-E_\delta,F}=\theta_{E_\delta,F}$ which implies $\theta_{E_\delta,F}=\theta_{E,F}$. Now $\psi_{E_\delta} \geqq \theta_{E_\delta F_\delta} \geqq 1-\delta$ on $E_\delta \cup F_\delta$. Hence $\psi_{E_\delta} \geqq (1-\delta)\psi_{F_\delta}$ on $E_\delta \cup F_\delta$, which by Lemma 2 yields $\psi_{E_\delta} \geqq (1-\delta)\psi_{F_\delta}$, and $\psi_{F_\delta} \geqq (1-\delta)\psi_{E_\delta}$. Now $(f/g) \leqq a$ on $E_\delta$ yields $\Psi_{E_\delta}f/\Psi_{E_\delta}g \leqq a$. Similarly $(f/g) \geqq a+\varepsilon$ on $F_\delta$ implies $\Psi_{F_\delta}f/\Psi_{F_\delta}g \geqq a+\varepsilon$. These relations yield $a\,\Psi_{E_\delta}g \geqq (1-\delta)^2(a+\varepsilon)\Psi_{E_\delta}g$ which is false for small $\delta$ if $\Psi_{E_\delta}(g) \neq 0$. Hence $\theta_{E,F}=0$.

COROLLARY. *If $f, g \in L_1$ and $(f/g) \in L_\infty$, then $(f/g) \in \mathscr{A}$.*

REMARK. If $T$ is conservative, then Theorem 3 corresponds to the fact that $(f/g)$ is measurable with respect to the $\sigma$-field of invariant sets (cf. [5], [2]).

THEOREM 4. *If $(f/g) \in L_\infty$, and $h \in \mathscr{A}$, then $\int \pi(h)\cdot f\,d\mu = \int \pi(h(f/g))g\,d\mu$.*

**Proof.** Recall that $\int \pi(h(f/g))g\,d\mu = \lim_{n\to\infty} \int h(f/g)T^n g\,d\mu$. We may assume $f$ and $h$ are real. Choose $\varepsilon > 0$, and let $E_{ij}$, $1 \leqq i, j \leqq k$ be a $\sum$ partition of $X$ such that

$$\Big\| h - \sum_{ij} h_i \chi_{E_{ij}} \Big\|_\infty < \varepsilon, \quad \Big\| (f/g) - \sum_{ij} \alpha_j \chi_{E_{ij}} \Big\|_\infty < \varepsilon$$

for suitable real $h_i$, $\alpha_j$ with $|h_i| \leqq \|h\|_\infty$, $|\alpha_j| \leqq \|(f/g)\|_\infty$. Now

$$\Big| \lim_{n\to\infty} \int h(f/g)T^n g\,d\mu - \sum_{ij=1}^{k} h_i\alpha_j \lim_{n\to\infty} \int_{E_{ij}} T^n g\,d\mu \Big|$$
$$= \Big| \lim_{n\to\infty} \int h(f/g)T^n g\,d\mu - \sum_{ij} h_i\alpha_j \Theta_{E_{ij}}(g) \Big| \leqq \varepsilon \|g\|_1 (\|h\|_\infty + \|(f/g)\|_\infty).$$

Let $\delta > 0$ be fixed and set $E'_{ij} = \{x \mid \theta_{E_{ij}}(x) \geqq 1-\delta\} \cap E_{ij}$. Then, as before, $\theta_{E'_{ij}} = \theta_{E_{ij}}$, and from Lemma 3, $\theta_{E'_{ij}} \geqq (1-\delta)\psi_{E'_{ij}}$. Now $|\alpha_j - (f/g)| \leqq \varepsilon$ on $E'_{ij}$ implies that $|\alpha_j - \Psi_{E'_{ij}} f / \Psi_{E'_{ij}} g| \leqq \varepsilon$. [Here we consider only those $E_{ij}$'s with $\theta_{E_{ij}} \neq 0$.] Hence:

$$\Big| \sum_{ij} h_i\alpha_j \Theta_{E_{ij}} g - \sum_{\theta_{E_{ij}} \neq 0} h_i \frac{\Psi_{E'_{ij}} f}{\Psi_{E'_{ij}} g} \Theta_{E_{ij}} g \Big| \leqq \varepsilon \|h\|_\infty \|g\|_1.$$

Also:

$$\Big| \sum_{\theta_{E_{ij}} \neq 0} h_i \frac{\Psi_{E'_{ij}} f}{\Psi_{E'_{ij}} g} \Theta_{E_{ij}} g - \sum_{ij} h_i \Psi_{E'_{ij}} f \Big| \leqq \|g\|_1 \|h\|_\infty (\|(f/g)\|_\infty + \varepsilon) k^2 \delta.$$

Finally,

$$\Big| \sum_{ij} h_i \Psi_{E'_{ij}} f - \sum_{ij} h_i \Theta_{E_{ij}} f \Big| \leqq \|h\|_\infty \|f\|_1 k^2 \delta$$

and

$$\Big| \sum_{ij} h_i \Theta_{E_{ij}} f - \lim_{n\to\infty} \int h T^n f\,d\mu \Big| \leqq \varepsilon \|f\|_1.$$

Putting together all these inequalities, we conclude the result.

## 5. A representation for $\mathscr{H}$.

DEFINITION 8. $\mathscr{G}$ is the sub-$C^*$-algebra of $\mathscr{H}$ generated by the class $\{\pi(l/1) \mid l \in L_\infty\}$.

Let $\mathscr{M} \subset \mathscr{G}^*$ be the maximal ideal space of $\mathscr{G}$ with the $w^*$ topology induced from $\mathscr{G}^*$. Let $\mathscr{B}$ be the $\sigma$-field of Baire sets of $\mathscr{M}$.

Note that $\mathscr{G}$ contains the unit $\pi(1)$ of $\mathscr{H}$, and that $g \in \mathscr{G}$ is invertible in $\mathscr{G}$ if and only if it is invertible in $\mathscr{H}$.

The $C^*$-algebra $\mathscr{C}(\mathscr{M})$ of continuous complex valued functions on $\mathscr{M}$ is isometrically* isomorphic to $\mathscr{G}$ under the Gelfand mapping $\sigma\colon \mathscr{G}\to\mathscr{C}(\mathscr{M})$. This mapping is order preserving. To see this first we need a few lemmas.

LEMMA 6. *If $f\in\mathscr{H}$ then $\pi|f|^2\geqq|f|^2$.*

**Proof.** We can assume that $f$ is real. Let $g\in L_1^+$ with $\|g\|_1=1$. Then

$$\left|\int fg\,d\mu\right| = \left|\int f\cdot Tg\,d\mu\right| \leqq \left|\int |f|^2Tg\,d\mu\right|^{1/2}\left|\int Tg\,d\mu\right|^{1/2}.$$

Hence $|\int fg\,d\mu|^2\leqq\int U|f|^2g\,d\mu$. If $|f|^2>|Uf|^2$ on a set of positive measure, then there exist $E\in\mathscr{F}$, $\mu(E)>0$, $a\geqq 0$ and $\varepsilon>0$ such that $|f|\geqq a+\varepsilon$ and $U|f|^2\leqq a^2$ on $E$. Take $g=f\chi_E/|f|\mu(E)$. Then

$$(a+\varepsilon)^2 \leqq \left|\int fg\,d\mu\right|^2 \leqq \int U|f|^2g\,d\mu \leqq a^2$$

which is a contradiction. Hence $U|f|^2\geqq|f|^2$ and $\pi|f|^2\geqq|f|^2$.

There is a canonical map $j\colon L_1\to\mathscr{G}^*$ defined by $(jf)(g)=\int fg\,d\mu, f\in L_1, g\in\mathscr{G}$. We now show that

LEMMA 7. *$\mathscr{M}$ is contained in the $w^*$-closure* of $jL_1^+$ *in $\mathscr{G}^*$.*

**Proof.** Choose $m\in\mathscr{M}$ and suppose that the $w^*$ neighborhood $\{F\mid |Fg_i-mg_i|<\varepsilon, i=1,\dots,n\}$ of $m$ defined by $g_1,\dots,g_n\in\mathscr{G}$, $\varepsilon>0$ is disjoint of $jL_1^+$. Let

$$u = \sum_{i=1}^{n}\pi[(g_i-1mg_i)\overline{(g_i-1mg_i)}].$$

Now, let $f\in L_1^+$, $\|f\|_1=1$. Then

$$\begin{aligned}(jf)u &= \sum_{i=1}^{n}\int\pi|g_i-1mg_i|^2f\,d\mu\\ &\geqq \sum_{i=1}^{n}\int|g_i-1mg_i|^2f\,d\mu\\ &\geqq \sum_{i=1}^{n}\left|\int(g_i-1mg_i)f\,d\mu\right| \geqq \varepsilon^2.\end{aligned}$$

Hence $u\geqq\varepsilon^2$ a.e. and hence $u$ is invertible in $L_\infty$. This implies that $u$ is invertible in $\mathscr{G}$. But this is impossible since $mu=0$.

COROLLARY. *$jL_1$ is dense in $\mathscr{G}^*$ in the $w^*$-topology.*

THEOREM 5. *The Gelfand mapping $\sigma\colon\mathscr{G}\to\mathscr{C}(\mathscr{M})$ is positive.*

**Proof.** If $g\geqq 0$ a.e. then $g^{**}\geqq 0$ on $jL_1^+$ where $g\to g^{**}$ is the canonical embedding of $\mathscr{G}$ into $\mathscr{G}^{**}$. Since $jL_1^+$ is dense in $\mathscr{M}$ and $g^{**}$ is continuous, $g^{**}\geqq 0$ on $\mathscr{M}$. Hence $\sigma g=g^{**}|_{\mathscr{M}}\geqq 0$.

Now we would like to extend $\sigma$ to $\mathscr{H}$. First note that, by the Riesz representation theorem, any $F \in \mathscr{G}^*$ can be represented by a measure $\mu_F$ on $(\mathscr{M}, \mathscr{B})$. In particular, let $\tilde{\mu} = \mu_{j1}$. From the order-preserving property of the Riesz representation one can see that for any $f \in L_1$, $\mu_{jf}$ is absolutely continuous with respect to $\tilde{\mu}$. In fact we can obtain $d\mu_{jf}/d\tilde{\mu}$ as follows. First, considering only $L_\infty$ functions we have

LEMMA 8. *If $f \in L_\infty$ then $\mu_{jf} \ll \tilde{\mu}$ and $d\mu_{jf}/d\tilde{\mu} = \sigma\pi(f/1)$.*

**Proof.** For any $g \in \mathscr{G}$,

$$\int_{\mathscr{M}} \sigma g \cdot \sigma\pi(f/1)\, d\tilde{\mu} = \int_{\mathscr{M}} \sigma\pi[g \cdot \pi(f/1)]\, d\tilde{\mu} = \int_X \pi[g\pi(f/1)]\, d\mu$$
$$= \int_X \pi[g(f/1)]\, d\mu = \int_X gf\, d\mu,$$

where the last equality follows from Theorem 4.

DEFINITION 9. Let $\tau f = \sigma\pi(f/1), f \in L_\infty$.

Note that the linear mapping $f \to \tau f$ defines a positive contraction $L_\infty(X, \mathscr{F}, \mu) \to L_\infty(\mathscr{M}, \mathscr{B}, \tilde{\mu})$. But it is also a contraction for the corresponding $L_1$ norms; hence it is a contraction for all $L_p$ norms, $1 \leqq p \leqq \infty$. We can then extend this mapping to $L_p(X, \mathscr{F}, \mu) \to L_p(\mathscr{M}, \mathscr{B}, \tilde{\mu})$ with the property that $\int_X gf\, d\mu = \int_{\mathscr{M}} \sigma g \tau f\, d\tilde{\mu}$ for all $g \in \mathscr{G}, f \in L_p$.

We can now prove a representation theorem for $\mathscr{H}$.

THEOREM 6. *There is a positive isometric * isomorphism between $\mathscr{H}$ and $L_\infty(\mathscr{M}, \mathscr{B}, \tilde{\mu})$.*

**Proof.** Let $h \in \mathscr{H}$ and define $\phi_h \in \mathscr{G}^*$ by

$$\phi_h(g) = \int_X \pi(gh)\, d\mu.$$

Note that if $h \in \mathscr{G}$ then $\phi_h$ is represented by the measure $\sigma(h) \cdot d\tilde{\mu}$ on $\mathscr{M}$. Let $\gamma_h$ be the representing measure of $\phi_h$, $h \in \mathscr{H}$. Then, for any nonnegative continuous function $\sigma g$ $(g \in \mathscr{G}^+)$ on $\mathscr{M}$

$$\left| \int_{\mathscr{M}} \sigma g\, d\gamma_h \right| = \left| \int_X \pi(gh)\, d\mu \right| \leqq \|h\|_\infty \int_X g\, d\mu = \|h\|_\infty \int_{\mathscr{M}} \sigma g \cdot d\tilde{\mu}$$

which shows that $\gamma_h$ is absolutely continuous with respect to $\tilde{\mu}$ and has a density function bounded by $\|h\|_\infty$. We denote this density function by $\sigma h$, noting that it is actually an extension of $\sigma$, and $\|\sigma h\|_\infty \leqq \|h\|_\infty$. Furthermore, if $l \in L_\infty$ then

$$\int_{\mathscr{M}} \sigma(h)\tau(l)\, d\tilde{\mu} = \int_{\mathscr{M}} \sigma(h)\sigma\pi(l/1)\, d\tilde{\mu} = \int_X \pi(h \cdot \pi(l/1))\, d\mu$$
$$= \int_X \pi(h \cdot (l/1))\, d\mu = \int_X hl\, d\mu.$$

Hence $|\int_X hl\, d\mu| \leqq \|\sigma h\|_\infty \|\tau l\|_1 \leqq \|\sigma h\|_\infty \cdot \|l\|_1$, so $\|h\|_\infty \leqq \|\sigma h\|_\infty$. Thus the extended $\sigma$ is also an $L_\infty$-norm isometry. To show that $\sigma\mathscr{H} = L_\infty(\mathscr{M}, \mathscr{B}, \tilde{\mu})$, first note that, if $h \in \mathscr{H}$, $l \in L_\infty(X, \mathscr{F}, \mu)$ then

$$\left|\int_X hl\, d\mu\right| = \left|\int_{\mathscr{M}} \sigma(h)\tau(l)\, d\tilde{\mu}\right| \leqq \|\sigma h\|_1 \|\tau l\|_\infty \leqq \|\sigma h\|_1 \|l\|_\infty,$$

hence $\|h\|_1 \leqq \|\sigma h\|_1$. Thus $\sigma^{-1}: \sigma\mathscr{H} \to \mathscr{H}$ is an $L_1$-contraction onto $\mathscr{H}$. Now if $\sigma h_n$ is an a.e. monotone sequence in $\sigma\mathscr{H}$ converging a.e. to a function $l$ in $L_\infty(\mathscr{M}, \mathscr{B}, \tilde{\mu})$ then $h_n$ is an a.e. bounded and monotone sequence in $\mathscr{H}$. If the limit function is $g$, one can easily see that $g \in \mathscr{H}$ and $\sigma g = l$. Since $\sigma\mathscr{H}$ contains the continuous functions, this shows that $\sigma\mathscr{H} = L_\infty(\mathscr{M}, \mathscr{B}, \tilde{\mu})$. Now we want to show that

$$\int_X \pi(hf)\, d\mu = \int_{\mathscr{M}} \sigma(h)\sigma(f)\, d\tilde{\mu},$$

for all $h, f \in \mathscr{H}$. In fact, for a fixed $h \in \mathscr{H}$, let $\mathscr{N} \subset \mathscr{H}$ be the class of functions $f$ for which this relation holds. Then $\sigma\mathscr{N}$ contains the continuous functions of $\mathscr{M}$, and one can show, as before, that $\sigma\mathscr{N}$ is closed under a.e. monotone limits. Hence $\sigma\mathscr{N} = L_\infty(\mathscr{M}, \mathscr{B}, \tilde{\mu})$.

Finally, we show that extended $\sigma$ is multiplicative, i.e. $\sigma(h_1)\cdot\sigma(h_2) = \sigma(\pi(h_1h_2))$ for all $h_1, h_2 \in \mathscr{H}$. First note that if $f \in L_\infty(\mathscr{M}, \mathscr{F}, \tilde{\mu})$ and $\int_{\mathscr{M}} f\tau(l)\, d\tilde{\mu} = 0$ for all $l \in L_\infty(X, \mathscr{F}, \mu)$ then $\sigma^{-1}f = 0$, hence $f = 0$. Now for $h \in \mathscr{H}$, $g \in \mathscr{G}$, $l \in L_\infty(X, \mathscr{F}, \mu)$,

$$\begin{aligned}\int_{\mathscr{M}} \sigma(h)\sigma(g)\tau(l)\, d\tilde{\mu} &= \int_{\mathscr{M}} \sigma(h)\sigma(g)\sigma(\pi(l/1))\, d\tilde{\mu} \\ &= \int_{\mathscr{M}} \sigma(h)\sigma\pi(g\pi(l/1))\, d\tilde{\mu} = \int_X \pi(h\pi(g\pi(l/1)))\, d\mu \\ &= \int_X \pi(hg(l/1))\, d\mu = \int_X \pi(hg)l\, d\mu = \int_{\mathscr{M}} \sigma\pi(hg)\tau(l)\, d\tilde{\mu},\end{aligned}$$

hence $\sigma(h)\cdot\sigma(g) = \sigma\pi(hg)$.

Now suppose that $h_1, h_2 \in \mathscr{H}$, $l \in L_\infty(X, \mathscr{F}, \mu)$. Then

$$\begin{aligned}\int_{\mathscr{M}} \sigma(h_1)\sigma(h_2)\tau(l)\, d\tilde{\mu} &= \int_{\mathscr{M}} \sigma(h_1)\sigma\pi(h_2\pi(l/1))\, d\tilde{\mu} \\ &= \int_X \pi(h_1\pi(h_2\pi(l/1)))\, d\mu = \int_X \pi(h_1h_2)l\, d\mu \\ &= \int_{\mathscr{M}} \sigma\pi(h_1h_2)\tau(l)\, d\tilde{\mu}\end{aligned}$$

which shows that $\sigma(h_1)\sigma(h_2) = \sigma\pi(h_1h_2)$, and completes the proof of the theorem.

We remark that every $f \in L_p(\mathscr{M}, \mathscr{B}, \tilde{\mu})$, $1 \leqq p < \infty$, induces a function $h \in L_p(X, \mathscr{F}, \mu)$, defined by $\int_X hl\, d\mu = \int_{\mathscr{M}} f\tau(l)\, d\tilde{\mu}$ for all $l \in L_q(X, \mathscr{F}, \mu)$, $1/p + 1/q = 1$. Since $\tau$ is an $L_q$-contraction the integral on $\mathscr{M}$ is defined and $h$ satisfies $\int_X hl\, d\mu =$

$\int_X hTl\,d\mu$, for all $l \in L_q(X, \mathscr{F}, \mu)$. The case $p=1$ causes no difficulty. If $f \in L_1(\mathscr{M}, \mathscr{B}, \tilde{\mu})$, $l \in L_\infty(X, \mathscr{F}, \mu)$,

$$\begin{aligned} \left| \int_{\mathscr{M}} f\tau(l)\,d\tilde{\mu} \right| &\leqq \left| \int_{\{|f| \geq n\}} f\tau(l)\,d\tilde{\mu} \right| + \left| \int_{\{|f| < n\}} f\tau(l)\,d\tilde{\mu} \right| \\ &\leqq \|\tau l\|_\infty \int_{\{|f| \geq n\}} |f|\,d\tilde{\mu} + n\|\tau l\|_1 \\ &\leqq \|l\|_\infty \left[ \int_{\{|f| \geq n\}} |f|\,d\tilde{\mu} \right] + n\|l\|_1. \end{aligned}$$

Thus, if $l_k$ is a sequence in $L_\infty$ with $\|l_k\|_1 \to 0$ and $\|l_k\|_\infty \leqq K$ then

$$\lim_k \left| \int f\tau(l_k)\,d\tilde{\mu} \right| \leqq K \int_{\{|f| \geq n\}} |f|\,d\tilde{\mu} \quad \text{for all } n \geqq 1.$$

Hence this limit is zero and the functional $l \to \int f\tau(l)\,d\tilde{\mu}$ on $L$ is induced by an $L_1$-function $h$. In a similar way, any Baire measure on $(\mathscr{M}, \mathscr{B})$ induces what one might call "an invariant functional" on $L_\infty(X, \mathscr{F}, \mu)$.

We also note the following relation between the maximal ideal spaces of $\mathscr{H}$ and $\mathscr{G}$; that is, between the Feller and Martin boundaries (cf. [9]). Since $\mathscr{H}$ is isometrically isomorphic to $L_\infty(\mathscr{M}, \mathscr{B}, \tilde{\mu})$, we state this relation in the following familiar form:

LEMMA 9. *Let $\mathscr{M}$ be a compact Hausdorff space, $\mathscr{B}$ its Baire sets, and $\tilde{\mu}$ a Baire measure on $(\mathscr{M}, \mathscr{B})$ with support $\mathscr{M}$. Let $\mathscr{M}'$ be the maximal ideal space of the $C^*$-algebra $L_\infty(\mathscr{M}, \mathscr{B}, \tilde{\mu})$. Then there is a continuous and onto map $\rho: \mathscr{M}' \to \mathscr{M}$.*

**Proof.** Interpret $\mathscr{M}'$ and $\mathscr{M}$ as classes of homomorphisms and define $\rho: \mathscr{M}' \to \mathscr{M}$ by $\rho(\phi) = \phi|_{\mathscr{C}(\mathscr{M})}$. Then $\rho$ is continuous. We show it is onto. Let $m \in \mathscr{M}$, and consider the ideal generated by $m \cdot L_\infty(\mathscr{M}, \mathscr{B}, \tilde{\mu})$. If it is proper, it can be embedded in a maximal ideal, whose image must then be $m$ under $\rho$. We show it is proper. If not, then $1 = \sum_1^n f_i g_i$ where $f_i \in m$, $g_i \in L_\infty(\mathscr{M}, \mathscr{B}, \tilde{\mu})$. Since $m$ is a maximal ideal, $\exists x_0$ such that $f_i(x_0) = 0$ $i = 1, \ldots, n$. Hence $|f_i| \leqq \varepsilon/h \sup |g_i|$ on some neighborhood $U$ of $x_0$, such that $\mu(U) \neq 0$. Hence $1 = |\sum f_i g_i| \leqq \varepsilon$ on $U$, which is a contradiction.

COROLLARY. *$\mathscr{M}$ is homeomorphic to the quotient space $\mathscr{M}'/\rho$.*

We finish this section by considering the possibility of joining $X$ and $\mathscr{M}$. In general, this cannot be done. If, however, $T$ is induced by a Markov kernel, such that the transform of every point measure is absolutely continuous with respect to $\mu$, then the members of $\mathscr{H}$ can be considered as actual functions on $X$, and the evaluations of these functions at points of $X$ induce bounded linear functionals on $\mathscr{H}$. Hence $X$ can be embedded in $\mathscr{G}^*$ (possibly in a many to one fashion). We shall denote the image of $X$ under this mapping as $X$ also. Hence $X \subset j(L_1^+(X, \mathscr{F}, \mu))$. Using the method of Lemma 7, $X$ is dense in $\mathscr{M}$, in the $w^*$-topology of $\mathscr{G}^*$.

Let $\bar{X}$ be the $w^*$-closure of $X$ in $\mathscr{G}^*$. Then $\bar{X}$ is a compact Hausdorff space. The following result, stated for the Martin-Doob boundary, is also true for the Feller boundary.

THEOREM 7. *For any* $g \in L_1(X, \mathscr{F}, \mu)$, $T^n g\, d\mu \to \tau(g)\, d\bar{\mu}$ *in the* $w^*$*-topology of Baire measures on* $\bar{X}$.

**Proof.** Let $\mathscr{A}_1$ = the sub-$C^*$-algebra of $\mathscr{A}$, consisting of functions $g' \in \mathscr{A}$ such that $\pi(g') \in \mathscr{G}$.

Let $\mathscr{C} = \{f \in \mathscr{C}(\bar{X}) \mid f|_X \in \mathscr{A}_1\}$, $\mathscr{C}_0 = \{f \in \mathscr{C}(\bar{X}) \mid f|_X \in \mathscr{A}_0\}$. By the Stone-Weierstrass theorem $\mathscr{C} = \mathscr{C}(\bar{X})$. Also, $\mathscr{C}_0$ is a closed ideal in $\mathscr{C}$. Let $\mathscr{N} \subset \bar{X}$ be the closed subset such that $\mathscr{C}_0 = \{f \in \mathscr{C} \mid f(\mathscr{N}) = 0\}$. Then we have

$$\mathscr{C}(\mathscr{N}) \cong^* \mathscr{C}(\bar{X})/\mathscr{C}_0 \cong^* \mathscr{A}_1/\mathscr{A}_0 \cong^* \mathscr{G} \cong^* \mathscr{C}(\mathscr{M}).$$

Hence $\mathscr{C}(\mathscr{N}) \cong^* \mathscr{C}(\mathscr{M})$ is induced by a homeomorphism $\phi: \mathscr{N} \to \mathscr{M}$. Hence $g(s) = g(\phi(s))$ under the above sequence of isomorphisms. But $\mathscr{G}$ separates the points of $\mathscr{G}^*$, so $\phi$ = identity and $\mathscr{N} = \mathscr{M}$.

In other words,

$$\{f \in \mathscr{C}(\bar{X}) \mid f|_X \in \mathscr{A}_0\} = \{f \in \mathscr{C}(\bar{X}) \mid f(\mathscr{M}) = 0\}.$$

Thus if $f \in \mathscr{C}(\bar{X})$, $g \in L_1(\bar{X}, \mathscr{F}, \mu)$, then:

$$\int_{\bar{X}} f T^n g\, d\mu = \int_X U^n(f|_X) g\, d\mu \to \int_X \pi(f|_X) g\, d\mu = \int_{\mathscr{M}} \sigma\pi(f|_X)\tau(g)\, d\bar{\mu}$$

and

$$\int_{\mathscr{M}} \sigma\pi(f|_X)\cdot\tau(g)\, d\bar{\mu} = \int_{\mathscr{M}} f|_{\mathscr{M}}\cdot\tau(g)\, d\bar{\mu} = \int_{\bar{X}} f\tau(g)\, d\bar{\mu}.$$

Thus $T^n g\, d\mu \to \tau(g)\, d\bar{\mu}$.

6. **Harmonic functions in the unit disk.** As an example we consider a transformation suggested by Feller in [10].

Let $D = \{z = re^{i\phi} \mid 0 \leqq r < 1,\ -\pi \leqq \phi \leqq \pi\}$ be the unit disk with the (geometric) boundary $C$. Let $\mathscr{F}$ and $\mu$ be the $\sigma$-field of Borel subsets and the Lebesque measure. For every $z \in D$, $E \in \mathscr{F}$, let

$$P(z, E) = \mu(Q_z \cap E)/\mu(Q_z)$$

where $Q_z = \{Z \mid |Z - z| < 1 - |z|\}$. Then $P$ defines a Markov kernel, such that the transformation of a unit mass at $z \in D$ is given by the measure $P(z, \cdot) \ll \mu$. We let $T$ be the induced transformation on $L_1(D, \mathscr{F}, \mu)$. The adjoint $U$ of $T$ is given by

$$(Uf)(z) = \iint f(Z)P(z, dZ), \qquad f \in L_\infty,\ z \in D.$$

It is clear that any bounded harmonic function $h$ belongs to $\mathscr{H}$. The converse is also true, but it seems that no explicit proof of it has been given and we would like to indicate an outline for this proof.

If $R$ is a Borel subset of $[0, 1)$ let $C_R=\{z \mid |z| \in R\}$. One can then obtain the following

LEMMA 10. *Let* $\frac{1}{2} \leqq K < 1$ *and* $R$ *be a Borel subset of* $[K, 1)$. *Then for all* $z \in D$, $\frac{1}{2} \leq |z| \leqq K$,

$$\frac{\mu(Q_z \cap C_R)}{\mu(Q_z \cap C_{[K,1)})} \geqq \frac{1}{16}\left[\frac{\lambda(R)}{1-K}\right]^{3/2}$$

*where* $\lambda$ *is the one dimensional Lebesgue measure.*

COROLLARY. *Let* $E=C_{[0,1/2)} \cup [K, 1)$ *and let* $f \in L_1^+, f=0$ *a.e. on* $C_{[K,1)}$. *Then*

$$\int_{C_R} T_E^n f\, d\mu \geqq \frac{1}{16}\left[\frac{\lambda(R)}{1-K}\right]^{3/2} \int_{C_{[K,1)}} T_E^n f\, d\mu$$

*for all* $n \geqq 0$.

Using this corollary one can see that if a function $h \in \mathscr{H}$ (which is necessarily continuous) has the form $h(\mathrm{re}^{i\phi})=f(r)g(\phi)$ then $\lim_{r\uparrow 1} f(r)$ exists, and that this implies the harmonicity of $h$.

Now if $h$ is any function in $\mathscr{H}$, let $t$ be an irrational number and consider, for a fixed $n$, $-\infty < n < \infty$,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} (\tau^k \cdot z)^{-n} h(\tau^k \cdot z) = F_n$$

where $\tau\colon D \to D$ is given by $\tau z=e^{i2\pi t}z$. This limit $F_n$ exists for all nonzero $z \in D$, depends only on $r=|z|$, and satisfies

$$r^n F_n(r) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{-in\phi}\, h(re^{in\phi})\, d\phi.$$

But, it is clear that

$$e^{in\phi} r^n F_n(r) = \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} e^{-i2\pi knt} h(re^{i\Phi} e^{i2\pi kt})$$

is a function in $\mathscr{H}$, hence $e^{in\phi} r^n F_n(r)$ must be harmonic, which shows that $r^n F_n(r)= C_n r^{|n|}$ and completes the proof of the following

LEMMA 11. *A bounded function belongs to* $\mathscr{H}$ *if and only if it is a harmonic function.*

One then shows that the $C^*$-algebra $\mathscr{H}$ is isometrically $*$-isomorphic to $L_\infty$ of the unit circle. For any bounded measurable function $l$ on $D$, let $\lambda_l$ be the measure on the unit circle obtained by sweeping out $l\, d\mu$ by the Poisson kernel. The harmonic function $\pi(l/1)$ corresponds to $d\lambda_l/d\lambda$, which is continuous. It then follows that the maximal ideal space $\mathscr{M}$ of $\mathscr{G}$ is homeomorphic to the unit circle. Since $T$ is induced by a Markov kernel, $D$ can be imbedded into $\mathscr{G}^*$. Then $D \cup \mathscr{M}$ is homeomorphic to the closed unit disk.

## References

1. M. A. Akcoglu, *An ergodic lemma*, Proc. Amer. Math. Soc. **16** (1965), 388–392.

2. ———, *Pointwise ergodic theorems*, Trans. Amer. Math. Soc. **125** (1966), 296–309.

3. E. Bishop and K. deLeeuw, *The representations of linear functionals by measures on sets of extreme points*, Ann. Inst. Fourier (Grenoble) **9** (1959), 305–331

4. A. Brunel, *Sur un lemme ergodique voisin du lemme de E. Hopf, et sur une des applications*, C. R. Acad. Sci. Paris **256** (1963), 5481–5484.

5. R. V. Chacon, *Identification of the limit of operator averages*, J. Math. Mech. **11** (1962), 961–968.

6. R. V. Chacon and D. S. Ornstein, *A general ergodic theorem*, Illinois J. Math. **4** (1960), 153–160.

7. G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble) **5** (1955), 131–295.

8. J. L. Doob, *Discrete potential theory and boundaries*, J. Math. Mech. **8** (1959), 433–458.

9. J. Feldman, *Feller and Martin boundaries for countable sets*, Illinois J. Math. **6** (1962), 356–366.

10. W. Feller, *Boundaries induced by non-negative matrices*, Trans. Amer. Math. Soc. **83** (1956), 19–54.

11. G. A. Hunt, *Markoff chains and Martin boundaries*, Illinois, J. Math. **4** (1960), 313–340.

12. R. S. Martin, *Minimal positive harmonic functions*, Trans. Amer. Math. Soc. **49** (1941), 137–172.

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