ON THE EXISTENCE OF IMMERSIONS AND SUBMERSIONS

BY

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1. **Introduction.** Let M and N be manifolds (always assumed to be smooth, connected, and without boundary) and let $f: M \to N$ be a smooth map. If at each point of M the Jacobian matrix of f has maximal rank, we call f a map of maximal rank. (If $\dim M < \dim N$, then f is an *immersion*, while if $\dim M > \dim N$, f is a *submersion.*)

QUESTION. Which homotopy classes of *continuous* maps $M \to N$ contain a smooth map of maximal rank?

This question has been reduced to a question purely in homotopy theory by M. Hirsch (for immersions) and Phillips (for submersions). (See [7], [17].) Their results are as follows.

We will use the following notation. For any vector bundle ξ over a complex X we let (ξ) denote the stable equivalence class determined by ξ. We will say that a stable bundle (ξ) has geometric dimension $\leqq n$ (for some positive integer n) if there is an n-plane bundle over X which is stably isomorphic to ξ. For a smooth manifold V we let τ_V denote the tangent bundle and ν_V the stable normal bundle; i.e. $\nu_V = -(\tau_V)$.

THEOREM OF HIRSCH. *Let $f: M \to N$ be a continuous map between manifolds, where* $\dim M < \dim N$. *Then f is homotopic to an immersion if, and only if, the stable bundle*

$$f^*(\tau_N) + \nu_M$$

has geometric dimension $\leqq \dim N - \dim M$.

A dual result holds for submersions.

THEOREM OF PHILLIPS. *Let M be an open manifold and $f: M \to N$ a continuous map, where* $\dim M > \dim N$. *Then f is homotopic to a submersion if, and only if, the stable bundle*

$$(\tau_M) + f^*\nu_N$$

has geometric dimension $\leqq \dim M - \dim N$.

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We need here to remark that since M is open, the bundle τ_M is stable over M. (Because M has the homotopy type of an $(m-1)$-complex, $m=\dim M$. See [8, §3.2].)

For a simple application of these theorems, suppose that M and N are π-manifolds (i.e. each has stably trivial tangent bundle) and that $\dim M \neq \dim N$. Since each also has stably trivial normal bundle, it follows by the above theorems that every continuous map $M \to N$ is homotopic to a smooth map of maximal rank.

In the following two sections we use the theorems of Hirsch and Phillips to study more general manifolds M and N, using in part results from [20] and [21].

2. **Immersion of manifolds.** We will use the following notation. M^m and N^n will denote smooth connected manifolds with respective dimensions m and n, $m<n$. We define the *codimension* of a continuous map $M \to N$ to be the positive integer $n-m$. By the basic theorem of Whitney [23] every map of codimension m (i.e. $n=2m$) is homotopic to an immersion, and so we consider here the case $n<2m$.

For any bundle ξ over a complex X we let $w_i\xi \in H^i(X; Z_2)$ denote the ith Stiefel-Whitney class of ξ, $i \geqq 0$. For a manifold V, we set

$$w_i(V) = w_i(\tau_V), \qquad \bar{w}_i(V) = w_i(\nu_V).$$

Suppose now that M and N are manifolds and f: $M \to N$ a continuous map. Set

$$\nu_f = f^*(\tau_N) + \nu_M.$$

We say that f is *orientable* if

$$f^*w_1(N) = w_1(M),$$

i.e. the stable bundle ν_f is orientable.

By Hirsch (see §1), it follows that if f is homotopic to an immersion then

$$w_i(\nu_f) = 0, \qquad i > n-m,$$
$$\delta w_{n-m}(\nu_f) = 0, \qquad n-m \text{ even}, f \text{ orientable}.$$

(Here δ denotes the Bockstein coboundary associated with the exact sequence $Z \to Z \to Z_2$.) Thus, in what follows we will be mainly concerned with *sufficient* conditions for f to be homotopic to an immersion.

Codimension $f=m-1$, $m \geqq 4$.

THEOREM 2.1. *Let M^m and N^{2m-1} be manifolds, $m \geqq 4$, and let f: $M \to N$ be a continuous map. If m is odd, assume that f is orientable. Then f is homotopic to an immersion if, and only if,*

$$w_m(\nu_f) = 0, \quad m \text{ even}, \qquad \delta w_{m-1}(\nu_f) = 0, \quad m \text{ odd}.$$

The proof of the theorem follows at once from classical obstruction theory [18], as will be shown in §5. (In the case m odd we can omit the hypothesis that f is orientable if we use local coefficients.)

If M and N are orientable manifolds, then every map $f\colon M \to N$ is orientable. Since $H^m(M; Z) \approx Z$, we then obtain from 2.1

COROLLARY 2.2. *Let M^{2q+1} and N^{4q+1} be orientable manifolds, $q \geqq 2$. Then every map $f\colon M \to N$ is homotopic to an immersion.*

Codimension $f = m-2$, $m \geqq 5$.

THEOREM 2.3. (a) *Let M^{4q+1} and N^{8q} be manifolds, $q \geqq 1$, and $f\colon M \to N$ a continuous map. Then f is homotopic to an immersion if $w_{4q}(\nu_f) = 0$.*

(b) *Let M^{4q+2} and N^{8q+2} be manifolds, $q \geqq 1$, and let $f\colon M \to N$ be an orientable map. Suppose that*

$$\delta w_{4q}(\nu_f) = 0 \quad \text{and} \quad w_{4q+2}(\nu_f) = 0.$$

If M is closed, suppose also that M is orientable and that

$$f^* w_2(N) = 0, \qquad w_{4q}(\nu_f) \cdot w_2(M) = 0.$$

Then f is homotopic to an immersion.

(c) *Let M^{4q+3} and N^{8q+4} be manifolds, $q \geqq 1$, and let $f\colon M \to N$ be an orientable map. Suppose that $w_{4q+2}(\nu_f) = 0$. If M is open or if M is closed, orientable, and either $f^* w_2(N) \neq 0$ or*

$$f^* w_2(N) = 0 \quad \text{and} \quad w_{4q+1}(\nu_f) \cdot w_2(M) = 0,$$

then f is homotopic to an immersion.

The proof uses the results of [20], and will be given in §5.

We say that an orientable manifold M is a *spin* manifold if $w_2(M) = 0$; we say that an orientable map $f\colon M \to N$ is a *spin map* if $f^* w_2 N = w_2 M$.

Codimension $f = m-3$, $m \geqq 5$.

THEOREM 2.4. *Let $f\colon M^m \to N^{2m-3}$ be an orientable map, with $m \geqq 5$ and $m \not\equiv 0 \bmod 4$. If $m \equiv 1 \bmod 4$, assume that $H^{m-1}(M; Z_2) = 0$. If $m \equiv 2 \bmod 4$, assume that either M is open or that M is a closed spin manifold and f is a spin map. If $m \equiv 3 \bmod 4$, assume that M is a closed spin manifold and f is a spin map. Then f is homotopic to an immersion if*

$$\begin{aligned} \delta w_{m-3}(\nu_f) &= 0, & & m \equiv 1 \mod 4, \\ w_{m-2}(\nu_f) &= 0, & & m \equiv 2 \mod 4, \\ \delta w_{m-3}(\nu_f) &= 0, & w_{m-1}(\nu_f) = 0, \quad & m \equiv 3 \mod 4. \end{aligned}$$

The proof will be given in §5.

Codimension $f = m-4$, $m \geqq 11$.

THEOREM 2.5. *Let M^{8q+3} and N^{16q+2} be manifolds, $q \geqq 1$, and let $f\colon M \to N$ be a spin map. Suppose that M is a closed spin manifold. If $w_{8q}(\nu_f) = 0$, then f is homotopic to an immersion.*

The proof will be given in §5.

REMARK. If one takes the manifold N to be R^n, then $\nu_f=\nu_M$ and one can obtain stronger results than those given in 2.1–2.4 by using [14]. Note, for example, [6], [11] and [21].

3. **Submersion of manifolds.** In this section we assume that M^m and N^n are smooth connected manifolds with $m>n$. Moreover, *throughout the section we assume that M is open*. Suppose that $n=1$, i.e. $N=R^1$ or S^1. Then, as observed by Phillips [17], every map $M^m\to N^1$ is homotopic to a submersion (since M has the homotopy type of an $(m-1)$-complex). We consider here the case $n=2$. For a map $f\colon M\to N$ set $\sigma_f=(\tau_M)+f^*\nu_N$. We will prove

THEOREM 3.1. *Let $f\colon M^m\to N^2$ be a continuous map, $m\geqq 5$, where M is open. If m is even assume that f is orientable. Then f is homotopic to a submersion if, and only if,*

$$w_{m-1}(\sigma_f) = 0, \quad m \text{ odd}, \qquad \delta w_{m-2}(\sigma_f) = 0, \quad m \text{ even}.$$

Suppose that N is a closed orientable surface. Then the stable normal bundle of N is trivial, and so $w_i(\sigma_f)=w_i(M)$, $i\geqq 0$.

On the other hand suppose that $M=M'-\partial M'$, where M' is a compact orientable manifold with nonempty boundary $\partial M'$. It follows from results of Wu and Massey [24], [12], [13], that

$$w_{m-1}(M) = 0, \quad \text{if } m\equiv 3 \mod 4, \qquad \delta w_{m-2}(M) = 0, \quad \text{if } m \text{ even}.$$

(See [5, §2].) Thus from 3.1 we obtain

COROLLARY 3.2. *Let M' be a compact orientable m-manifold with nonempty boundary $\partial M'$, and let N be a closed orientable surface. Let M denote the open manifold $M'-\partial M'$. If $m\geqq 5$ and $m\not\equiv 1$ mod 4, then every map $M\to N$ is homotopic to a submersion.*

(Note [3] for conditions on an open manifold that it be expressible as $M'-\partial M'$.)

Our results on submersions are much less extensive than the results in §2 on immersions. If $f\colon M^m\to N^n$, with $n>2$, then one can still apply the results of [20], [21] to obtain conditions for σ_f to have codimension $\leqq m-n$. However, the results in general will be expressed in terms of higher order cohomology operations.

4. **Examples.** Let M^m and N^n be manifolds, $m\neq n$. The problem of determining the set of maps from M to N of maximal rank falls into two parts: First, determine the homotopy classes of maps from M to N, $[M, N]$; and second, for each homotopy class of maps, determine whether it contains a map of maximal rank. If M and N fit the hypotheses of one of the theorems in §2 or §3, and if $f\colon M\to N$, then the second step above consists simply in computing the characteristic classes

$w_k(\nu_f)$, if $m<n$, $w_k(\sigma_f)$, if $m>n$. By the Whitney duality formula, these classes are given as follows:

$$w_k(\nu_f) = \sum_{i+j=k} \bar{w}_i(M) \cup f^* w_j(N), \qquad w_k(\sigma_f) = \sum_{i+j=k} w_i(M) \cup f^* \bar{w}_j(N).$$

For an illustration we take N to be the real projective space RP^n (of dim n) and the complex projective space CP^n (of dim $2n$).

EXAMPLE A. $N=RP^n$, $n>1$. Since RP^n is the n-skeleton of the Eilenberg-MacLane space $K(Z_2, 1)$, it follows that if X is a complex of dim $<n$, then $[X, RP^n] = H^1(X; Z_2)$. The correspondence here is given by $[f] \to f^*x$, where x generates $H^1(RP^n; Z_2)$. Since $w(RP^n)=(1+x)^{n+1}$, we have

$$w_k(\nu_f) = \sum_{i+j=k} \binom{n+1}{i} u^i \cup \bar{w}_j(M),$$

where $f\colon M^m \to RP^n$, $m<n$, and $u=f^*x$. The results of §2 can now be used to determine the immersions of M^m in RP^n, for appropriate dimensions m and n. (The difficulty in studying submersions is that in general we do not know how to determine the set $[M^m, RP^n]$, when $m>n$.)

EXAMPLE B. $N=CP^n$, $n \geqq 1$. Now CP^n is the $(2n+1)$-skeleton of the Eilenberg-MacLane space $K(Z, 2)$, and so if a complex X has dimension $\leqq 2n$, then $[X, CP^n] = H^2(X; Z)$, the correspondence being given by $[f] \to f^*y$, where y generates $H^2(CP^n; Z)$. Let M^m be a manifold and $f\colon M^m \to CP^n$ a map, $m \leqq 2n$. Since $w(CP^n)=(1+y)^{n+1}$ mod 2, we have

$$w_{2k}(\nu_f) = \sum_{i+j=k} \binom{n+1}{i} v^i \cup \bar{w}_{2j}(M), \qquad \delta w_{2k}(\nu_f) = \sum_{i+j=k} \binom{n+1}{i} v^i \cup \delta \bar{w}_{2j}(M),$$

where $v=f^*y$. The results of §2 can now be used to determine the immersions of M^m in CP^n for appropriate m and n. Take M to be CP^q, for example. Since $H^2(CP^q; Z) \approx Z$, we have $[CP^q, CP^n]=Z$, $q \leqq n$, and so each homotopy class of map $f\colon CP^q \to CP^n$ is characterized by an integer, called the *degree* of the map. (See Feder [4].) By 2.3(b) one can show:

(4.1) *Let q be a positive integer. Then for each integer d there is an immersion of CP^{2q+1} in CP^{4q+1} of degree d.*

REMARK. (4.1) suggests the following general problem. Let q and n be integers, $0<q<n$. *Determine the integers d for which there is an immersion of CP^q in CP^n of degree d.* By Whitney [23], if $n \geqq 2q$ all integers d can occur. By Feder [4], if $n \leqq [3q/2]-1$, only $d=\pm 1$ can occur. (In [22] we show that for $q=2$, $n=3$, only $d=\pm 1$ can occur, while if $q=3$, $n=4$, then d can occur if, and only if, there is an integer e such that $5d^2=e^2+4$. Note also [4, Theorem 8.3].)

5. **Proofs of theorems.** For a topological group G let BG denote the classifying space for G constructed by Milnor [15]. Let $O(n)$, $n \geqq 1$, denote the orthogonal group of rank n, and let O denote the stable orthogonal group [2]. If X is a complex

then a stable vector bundle over X can be regarded as a map $X \to BO$. Now the natural inclusion $O(n) \subset O$ induces a map $p_n\colon BO(n) \to BO$, and a stable bundle ξ over X has geometric dimension $\leqq n$ if, and only if, there is a map $\eta\colon X \to BO(n)$ such $p_n \circ \eta = \xi$. Up to homotopy type the map p_n can be regarded as a fiber map [1], with fiber $V_n = O/O(n)$.

By Stiefel (see [18]), V_n is $(n-1)$-connected and (for $n \geqq 3$),

$$\begin{aligned} \pi_n(V_n) &= Z, \qquad n \text{ even}, \\ &= Z_2, \qquad n \text{ odd}. \end{aligned}$$

Thus by standard obstruction theory (e.g. see [18], [10], or [19]), if X has dim $\leqq n+1$ then a stable bundle ξ over X has geometric dim $\leqq n$ if, and only if,

$$(*) \qquad w_{n+1}(\xi) = 0, \quad n \text{ odd}, \qquad \delta w_n(\xi) = 0, \quad n \text{ even},$$

assuming that ξ is orientable in the case n even. This proves Theorem 2.1. Furthermore, $(*)$ proves 2.3(a) (since $\pi_{4q}(V_{4q-1}) = 0$, see [16]) and also proves 2.4 in the case $m \equiv 1 \bmod 4$ (since $\pi_{4q+1}(V_{4q-2}) = 0$, [16]). Finally, since an open m-manifold has the homotopy type of an $(m-1)$-complex, $(*)$ also proves 3.1, and 2.3–2.4 in the cases M is open.

To prove the remaining theorems in §2 (assuming now that M is a *closed* manifold) we need some results from [20], and [21]. In [20] we do not deal with *stable* bundles, and so we will need the following relationship between n-plane bundles and stable bundles.

LEMMA 5.1. *Let X be a complex of* dim *n and let ξ be an oriented stable vector bundle over X such that $w_n(\xi) = 0$. Then there is an oriented n-plane bundle η over X such that η is stably equivalent to ξ and $\chi(\eta) = 0$ (where $\chi(\eta)$ denotes the Euler class of η). Moreover, ξ has geometric dimension $\leqq k$ (where $k < n$) if, and only if, η has $n-k$ linearly independent cross-sections.*

The proof is standard and is left to the reader.

Proof of 2.3(b). Let η be an n-plane bundle over M corresponding to the stable bundle ν_f. Thus by 5.1 and by the hypotheses of 2.3(b),

$$w_2(\eta) = w_2(M), \qquad w_{4q}(\eta) \cdot w_2(M) = 0, \qquad \delta w_{4q}(\eta) = 0, \qquad \chi(\eta) = 0,$$

and so by Theorem 7.3 of [20], η has 2 linearly independent cross sections. Thus, by 5.1, ν_f has geometric dimension $\leqq 4q$ and so by Hirsch, f is homotopic to an immersion.

Before proving 2.3(c) we need a preliminary result. Let ξ be a vector bundle (stable or otherwise) over a complex X. Define a homomorphism

$$\alpha_\xi\colon H^i(X; Z_2) \to H^{i+2}(X; Z_2), \qquad i \geqq 0,$$

by

$$x \to Sq^2(x) + x \cdot w_2(\xi).$$

Suppose that X is a closed manifold M of dim m, and let ξ, η be two bundles over M. Then

$$\alpha_\xi = \alpha_\eta: H^{m-2}(M; Z_2) \to H^m(M; Z_2)$$

if, and only if, $w_2(\xi) = w_2(\eta)$, as may be seen by using Poincaré duality. In particular if we take ξ to be the tangent bundle of M, then by Wu [24] $\alpha_\xi H^{m-2}(M; Z_2) = 0$, provided M is orientable, and so we have:

LEMMA 5.2. *Let η be a bundle over a closed orientable m-manifold M, $m \geqq 2$. If $w_2(\eta) \neq w_2(M)$, then*

$$\alpha_\eta H^{m-2}(M; Z_2) = H^m(M; Z_2).$$

Proof of 2.3(c). The first obstruction to ν_f pulling back to $BO(4q+1)$ is the class $w_{4q+2}(\nu_f)$, which vanishes by hypothesis. The second (and final) obstruction is a coset in $H^{4q+3}(M; Z_2)$ of the subgroup $\alpha_{\nu_f} H^{4q+1}(M; Z_2)$. (See [9], [10], [20].) Now if $f^* w_2(N) \neq 0$ then $w_2(\nu_f) \neq w_2(M)$, and so by (5.2), $\alpha_{\nu_f} H^{4q+1}(M; Z_2) = H^{4q+3}(M; Z_2)$, since M is closed. Thus the second obstruction contains zero and hence vanishes, which completes the proof of 2.3(c) in this case.

Suppose on the other hand that

$$f^* w_2(N) = 0, \qquad w_{4q+1}(\nu_f) \cdot w_2(M) = 0.$$

Then the theorem follows, as above, by using 5.1 and applying 7.3 of [20]. We omit the details.

Proof of 2.4. We have already done the case $m \equiv 1 \bmod 4$. If $m \equiv 2 \bmod 4$, we use Theorem 1.3 of [21] (applied to the bundle ν_f), while if $m \equiv 3 \bmod 4$ we use 5.1 above together with Theorem 1.1 of [21]. We leave the details to the reader.

Proof of 2.5. This follows at once from [21, Theorem 1.3] applied to the bundle ν_f.

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