SOME SPECTRAL PROPERTIES OF AN OPERATOR ASSOCIATED WITH A PAIR OF NONNEGATIVE MATRICES(1)

BY

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Abstract. An operator—in general nonlinear—associated with a pair of nonnegative matrices, is defined and some of its spectral properties studied. If the pair of matrices are a square matrix A and the identity matrix of the same order, the operator reduces to the linear operator A. The results obtained include generalizations of one of the principal conclusions of the theorem of Perron-Frobenius.

1. **Introduction.** Let $A_{m\times n}$ and $B_{m\times n}$ be two nonnegative matrices, i.e., matrices whose entries are nonnegative real numbers. *It is assumed that no row of A or column of B consists entirely of zeros.* $r^{(m)}=\{r_1,\dots,r_m\}$ and $c^{(n)}=\{c_1,\dots,c_n\}$ are sets of positive numbers. When will there exist diagonal matrices $D_{m\times m}$ and $E_{n\times n}$, and a positive number θ, such that DAE has its row-sums equal to the r_i and θDBE has its column-sums equal to the c_j?

This question can be reformulated as follows: Let $\mathcal{N}$ denote the first orthant of real Euclidean n-space, $\mathcal{M}$ that of real Euclidean m-space, and $\mathcal{N}^0$ the subset of $\mathcal{N}$ consisting of points all of whose coordinates are positive. Let $x\in\mathcal{N}^0$, $x=(x_1,\dots,x_n)$. Regarding x as a column vector, denoting by $(Ax)_i$ the ith element of Ax, and letting u stand for $(r_1/(Ax)_1,\dots,r_m/(Ax)_m)$, we see that $x\to u$ is a mapping of $\mathcal{N}^0$ into $\mathcal{M}$ and $u\to(c_1/(B^Tu)_1,\dots,c_n/(B^Tu)_n)$ is a mapping of u into $\mathcal{N}^0$, and hence we obtain $x\xrightarrow{T}(c_1/(B^Tu)_1,\dots,c_n/(B^Tu)_n)$ as a mapping $T=T(A,r^{(m)};B,c^{(n)})$ of $\mathcal{N}^0$ into $\mathcal{N}^0$. We extend this map to a map T of $\mathcal{N}$ into $\mathcal{N}$ by continuity, at those points where it cannot be defined as above. The question asked in the preceding paragraph can now be rephrased as follows: Under what conditions does T have a positive eigenvector x associated with a positive eigenvalue θ? For if such a θ and such an x exist, then on taking E to be diag $(x_1,\dots,x_n)$ and D to be diag $(u_1,\dots,u_m)$, we see that DAE has row-sums equal to the r_i and θDBE has column-sums equal to the c_j. In this paper some of the spectral properties of T, and particularly the question posed above, are studied.

The operator T was introduced by us in [3], and it was shown that if either A or B were positive, then $Tx=\theta x$ regarded as an equation in x and θ, $x\in\mathcal{N}$, $x\neq 0$, $\theta\geq 0$, has one and only one solution $Tx_0=\theta_0x_0$, given by $x_0\in\mathcal{N}^0$ and $\theta_0>0$.

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That this conclusion holds in the case $A=B$, under the added and *necessary* assumption that there exists a matrix of the same pattern as A, and with row-sums equal to the r_i and column-sums equal to the c_j, was shown in [4] (see also §6). The approach to the problem was the 'matrix-reduction approach' of the first paragraph.

The yet more special case when A and B are not only equal but are also square matrices was treated in [6] using the 'matrix-reduction approach', and in [1] using the 'nonlinear operator approach' of the second paragraph. We refer the reader to [1] for references to other work related to that of this paper.

2. **Notation.** The definitions and notations A, B, T, etc., of §1 will be used throughout the paper. $x, y, \ldots$ will, unless stated to the contrary, stand for elements of $\mathcal{N}$. Since $T=T(A, r^{(m)}; B, c^{(n)})=T(\tilde{A}, 1^{(m)}; \tilde{B}, 1^{(n)})$, where $1^{(m)}$ and $1^{(n)}$ are m- and n-vectors consisting entirely of unit elements, and $\tilde{A}=\operatorname{diag}(1/r_1, \ldots, 1/r_m)A$ and $\tilde{B}=B\operatorname{diag}(1/c_1, \ldots, 1/c_n)$, we may, and shall, assume in all sections except the last one, that $r^{(m)}=1^{(m)}$ and $c^{(n)}=1^{(n)}$. With this assumption in mind, we write $T=T(A; B)$.

If U is any continuous operator on $\mathcal{N}$ into $\mathcal{N}$, and $x \in \mathcal{N}$ is given, then the greatest nonnegative number λ for which $Ux \geqq \lambda x$ holds will be denoted by $\Lambda(x)$. $\Lambda(x)$ thus depends on U even though this is not explicitly indicated in the symbol. As in [2], by the term maximal eigenvalue of U we mean that positive eigenvalue, if any such exists, which is not less than any other positive eigenvalue. We write eigenvector or eigenvalue to mean, in general, a real nonnegative eigenvector or a real, nonnegative eigenvalue.

If X is any $m \times n$ matrix, and $1 \leqq i_1, \ldots, i_r \leqq m$ and $1 \leqq j_1, \ldots, j_s \leqq n$, then $X[i_1, \ldots, i_r \mid j_1, \ldots, j_s]$ stands for the submatrix of X determined by the rows $i_1, \ldots, i_r$ and the columns $j_1, \ldots, j_s$.

3. **General properties of T.** The following theorem states some obvious properties that $T=T(A; B)$ possesses. We recall that A is assumed to have no zero rows and B to have no zero columns.

THEOREM 3.1. *T maps $\mathcal{N}$ continuously into itself and $\mathcal{N}^0$ into itself. T is homogeneous of degree one and is monotonically increasing. If $x \in \mathcal{N}^0$, then $x_i > y_i$, for all i, implies that $(Tx)_i > (Ty)_i$ for all i.*

The proof of the next result is also easy, and is given in [5].

THEOREM 3.2. *Let U be a continuous operator on $\mathcal{N}$ into $\mathcal{N}$ which is homogeneous of degree one and which is such that $Ux \neq 0$ if $x \neq 0$. Then there exists a positive eigenvalue associated with a nonnegative eigenvector.*

The next theorem is proved in [1].

THEOREM 3.3. *Let U be a continuous operator on $\mathcal{N}$ into $\mathcal{N}$. Then $\Lambda(x)$ is upper semicontinuous on $\mathcal{N}-0$. If U is also homogeneous and there exists an x such that $(Ux)_i > 0$, for all i, then there exists a positive number ρ and $u \in \mathcal{N}$, $u \neq 0$, such that $\rho = \sup\{\Lambda(x) \mid x \neq 0\} = \Lambda(u)$.*

THEOREM 3.4. *Let U be a continuous, monotonically increasing operator on $\mathcal{N}$ into $\mathcal{N}$ which is also homogeneous of degree one. Let $Ux=\sigma x$ and $Uy \geqq \delta y$, and let $y_i=0$ whenever $x_i=0$. Then $\sigma \geqq \delta$.*

Proof. There exists a positive number α such that $\alpha y \leqq x$, and $\alpha y_i = x_i \neq 0$, for some i. But $\alpha y \leqq x \Rightarrow U\alpha y \leqq Ux \Rightarrow \alpha \delta y_i \leqq \sigma x_i \Rightarrow \delta \leqq \sigma$.

COROLLARY 1. *Eigenvectors of U with the same pattern of zero and nonzero elements have the same eigenvalues. Hence the number of eigenvalues is finite.* (*This corollary is also contained in Theorem* 2 *of* [5].)

COROLLARY 2. *Any positive eigenvector has its eigenvalue not less than the eigenvalue associated with any nonnegative eigenvector.*

COROLLARY 3. *If a positive eigenvector x exists with associated eigenvalue σ, then* $\sigma = \rho = \sup\{\Lambda(x) \mid x \neq 0\}$.

Because of Theorem 3.1, we see that the preceding two theorems hold for T.

THEOREM 3.5. *If either A or B is a positive matrix, then T has a unique nonnegative eigenvalue ρ and a unique nonnegative eigenvector u. Both of these are indeed positive. Further* $\Lambda(u) = \sup\{\Lambda(x) \mid x \neq 0\} = \rho$.

The proof of this result is essentially contained in [3]. All but the last conclusion follows from Theorem 3 of [5] also.

COROLLARY. *If A has no zero row and B has no zero column, then $T(A; B)$ has eigenvalue the positive number* $\rho = \sup\{\Lambda(x) \mid x \neq 0\}$. *The associated eigenvector u is nonnegative and* $\Lambda(u) = \rho$.

The proof is along the same lines as that used for the reducible linear operator in [2, p. 66] and we merely sketch it. Let $T_\varepsilon = T(A_\varepsilon; B)$ where A_ε differs from A only in that all the zero entries of A are replaced by a positive number ε. As $\varepsilon \to 0$, $T_\varepsilon x \to Tx$, uniformly for all x for which $\sum x_i = 1$. Further, in finding the supremum of $\Lambda(x)$ over all $x \neq 0$, it is sufficient, from the homogeneity of T, to take into account all x such that $\sum x_i = 1$. Now, by the preceding theorem, T_ε has the unique maximal eigenvalue ρ_ε, and one shows easily that $\rho_\varepsilon \to \rho$ as $\varepsilon \to 0$, and if u_ε is the eigenvector of T_ε associated with ρ_ε, then as $\varepsilon \to 0$, u_ε has a limit point u.

N.B. If for each $\varepsilon > 0$, $\rho_\varepsilon \gtrless$ a constant, then $\rho \gtrless$ the same constant.

4. **Theorem of Perron-Frobenius.** *Let us observe that if A is a square matrix of order n and I is the identity matrix of the same order then $T(A; I)$ reduces to the linear operator* (*represented by*) *A.*

One of the principal conclusions of the theorem of Perron-Frobenius is that when A, assumed in this section to be a square matrix, is irreducible, it has a unique positive eigenvector associated with a unique positive eigenvalue, the latter being the eigenvalue of maximum modulus. Now, in general, conclusions about the spectral characteristics of A must take into account the magnitudes of the entries of A. The theorem of Perron-Frobenius shows that some conclusions can be reached taking into account merely the pattern of A. However, more light is thrown on the concept of irreducibility if one looks upon it *not solely as a statement about the pattern of* A, *but also as one about the pattern of* A *vis à vis that of the identity matrix* I (cf. (2) below). Indeed, as is easily verified, the following statements are equivalent:

(1) A is irreducible.

(2) There does not exist $1 \leqq i_1, \ldots, i_r \leqq m$ such that for every $1 \leqq j_1, \ldots, j_s \leqq n$ the following statement holds: $I[i_1, \ldots, i_r \mid j_1, \ldots, j_s]$ is a zero matrix implies $A[i_1, \ldots, i_r \mid j_1, \ldots, j_s]$ is a zero matrix.

(3) There does not exist a vector $x=(x_1, \ldots, x_n)$ such that $x_{i_1}=\cdots=x_{i_r}=0$ and the other $x_i \neq 0$ implies $(Ax)_{i_1}=\cdots=(Ax)_{i_r}=0$.

Thus (2) may be taken to be the definition of the irreducibility of A, and provides the source for our definition of the irreducibility of a matrix with respect to another given in the next section.

Finally, we observe that the imposition of the *condition of irreducibility in the theorem of Perron-Frobenius is meant precisely to ensure that* (3) *holds*. It is condition (3) that enables one to reach the conclusions of the theorem about the existence of a positive eigenvalue and an associated positive eigenvector. For, an exceedingly simple argument using the fixed-point theorem (cf. [5] or [3]) shows that A has a nonnegative eigenvector associated with a positive eigenvalue. But (3) guarantees that such an eigenvector must be positive.

The foregoing considerations motivate the next section.

5. **Reducibility of one matrix with respect to another.** Two $m \times n$ matrices are said to have the same pattern if either of them has a zero in any position when and only when the other has a zero in that position. The composite pattern of a set of $m \times n$ matrices is the pattern of that $m \times n$ matrix which has a zero in any position if and only if all the matrices of the set have zeroes in that position. The pattern of a matrix is subordinate to that of another if the first has zero entries in any position if the second one does. Viewing a row of an $m \times n$ matrix as a $1 \times n$ matrix, we speak of the pattern of a row, and of the composite pattern of a set of rows, etc.

Let $\mathscr{P}$ be the set of composite patterns of all possible collections of the rows of B. (Here, the word 'pattern' could be taken, for instance, to mean a $1 \times n$ matrix whose entries are either zeroes or ones.) For $p \in \mathscr{P}$, we define $f(p)$ to be the composite pattern of *all* those rows of B for which the *corresponding* rows of A have pattern subordinate to p. If there are no rows of A with patterns subordinate to p, we define $f(p)$ to be the pattern of the $1 \times n$ matrix whose entries are all zeroes.

Definition. A is reducible with respect to B if there exists $p \in \mathscr{P}$ such that $f(p)=p$, but p is not the pattern either of the $1 \times n$ matrix consisting solely of zeroes or of that consisting solely of ones.

If A is not reducible with respect to B, then A is said to be irreducible with respect to B. We say that $T(A; B)$ is irreducible if A is irreducible with respect to B.

N.B. Clearly, if $m=n$, A has no zero columns and I is the identity matrix of order n, then A is reducible with respect to I if and only if A is reducible.

The 'only if' part of this statement is obvious by Theorem 5.1. To prove the 'if' part, suppose that R and R_1 are subsets of $\{1, \ldots, n\}$, $R \subset R_1$, $R \neq R_1$, with the property that if $x_i = 0$, $i \in R$ and $x_i \neq 0$, $i \notin R$, then $(Ax)_i = 0$, $i \in R_1$, and $(Ax)_i \neq 0$, $i \notin R_1$. Because of the assumption that no column of A consists entirely of zeroes, it follows that $R_1 \neq \{1, \ldots, n\}$. Now, consider y, with $y_i = 0$ if and only if $i \in R_1$. Then there exists $R_2 \supset R_1$ such that $(Ay)_i = 0$ if and only if $i \in R_2$. If $R_2 = R_1$ the proof is complete. If R_2 properly contains R_1, we proceed as above and obtain, in a finite number of steps, a set R_u with the following properties. R_u is a proper subset of $\{1, \ldots, n\}$ and if z is such that $z_i = 0$ if and only if $i \in R_u$ then $(Az)_i = 0$ if and only if $i \in R_u$.

The definition of the irreducibility of A with respect to I is thus a slight weakening of that of the irreducibility of A. We recall that $T(A; I)$ is the linear operator A.

THEOREM 5.1. *The following statements are equivalent*:

(1) *A is irreducible with respect to B.*

(2) *$T(A; B)$ has the property that for no proper subset S of $\{1, \ldots, n\}$ is it true that $x_i = 0$, $i \in S$, $x_i \neq 0$, $i \notin S$, implies that $(Tx)_i \neq 0$, $i \in S$ and $(Tx)_i \neq 0$, $i \notin S$.*

Proof. If A is reducible with respect to B, there exists $p \in \mathscr{P}$ such that $f(p) = p$, and p is the pattern neither of the $1 \times n$ matrix of all zeroes nor of that of all ones. Let x have a pattern which is the complement of p. Then, clearly, $x_i = (Tx)_i = 0$ if and only if $i \in S$, where S is a proper subset of $\{1, \ldots, n\}$. The proof is completed by reversing the argument.

N.B. The analogue of statement (2) of §4 is *not* equivalent to the preceding statements as is shown by the following examples: Let A and B be 3×3 matrices with $a_{13} = a_{23} = b_{13} = 0$, the remaining elements being positive. Then $B[1|3]$ is the only zero matrix of B. $A[1|3]$ is also a zero submatrix of A. But A is irreducible with respect to B.

On the other hand, let A and B be 3×3 matrices with $a_{13} = a_{23} = a_{32} = b_{13} = b_{21} = b_{23} = 0$, the remaining elements being positive. Then $B[1, 2|3]$ is the only zero submatrix of B with elements chosen from the first and second rows of B. $A[1, 2|3]$ is also a zero matrix. Here, A is reducible with respect to B.

Examples of classes of matrix-pairs, A and B, where A is irreducible with respect to B are:

(1) A is irreducible, $B = I_{n \times n}$.

(2) Either A or B is positive.

Examples of classes of matrix-pairs A and B, where A is reducible with respect to B are:

(1) A is reducible, has no zero columns, and $B=I_{n\times n}$.

(2) A and B are of the same pattern, and A—and therefore also B,—has at least one zero element.

THEOREM 5.2. *Let T be irreducible. Then there exists one and only one nonnegative eigenvector u and one and only one nonnegative eigenvalue σ. u and σ are both positive and $\sigma=\rho$.*

Proof. By (2) of Theorem 5.1, any eigenvectors that exist must be positive, and by Corollary 1 to Theorem 3.4, they must have the same eigenvalue. But, the corollary to Theorem 3.5 assures us that there exists the positive eigenvalue ρ. Hence, in order to complete the proof, we need to show that if $Tx=\rho x$ and $Ty=\rho y$, x, y, >0, then y is a multiple of x.

We denote in what follows, the sets $\{1,\dots,m\}$ and $\{1,\dots,n\}$ by M and N respectively. M_1, M_2 will stand for subsets of M and N_1, N_2 for subsets of N. M_1' will be the complement of M_1 with respect to M and similar meanings will hold for M_2', N_1', N_2'.

Let us assume temporarily, that A has no zero columns. Now, there exists $c>0$ such that $cy_i<x_i$, $i\in N_1$ and $cy_i=x_i$, $i\in N_1'$, where N_1' is nonnull. If $N_1'=N$, there is nothing left to prove. Therefore assume that N_1' is a proper subset of N. Let M_1 consist of all the elements of M for which $A[M_1|N_1]=0$. Because it has been assumed that A has no zero columns, we have $M_1\neq M$. If $M_1=\varnothing$, the fact that B has no zero columns will mean that $(Tcy)_i<(Tx)_i$ for each $i\in N_1'$, i.e., $c\rho y_i<\rho x_i$, $i\in N_1'$. But $cy_i=x_i\neq 0$, $i\in N_1'$. Thus we have a contradiction. Hence M_1 is a proper subset of M.

If $(Tcy)_i<(Tx)_i$ for any $i\in N_1'$, then we arrive at a contradiction, as in the preceding paragraph. But $(Tcy)_i=(Tx)_i$ for each $i\in N_1'$ only if $B[M_1'|N_1']=0$. Suppose, then, that this is the case.

Now, however, there exists $d>0$ such that $dy_i>x_i$, $i\in N_2$ and $dy_i=x_i$, $i\in N_2'$. Then $N_1'\subset N_2$, and $N_2\neq N$. Let M_2 be the subset consisting of all elements of M for which $A[M_2|N_2]=0$. Since A has no zero rows $M_2\cap M_1=\varnothing$ and hence $M_2'\supset M_1$. It may also be assumed that M_2 is a proper subset of M.

Now if $B[M_2'|N_2']\neq 0$, then there arises a contradiction as before. Suppose, then, that $B[M_2'|N_2']=0$. This implies that $B[M_1|N_2']=0$. If we now recall the facts that $N_2'\subset N_1$ and that $A[M_1|N_1]=0$, it follows easily that there is a composite pattern p from among rows i of B, $i\in M_2'$, such that $f(p)=p$, and furthermore p is not the pattern of a vector consisting solely of zeroes or of one consisting solely of ones. (The definition of $f(\dot{})$ is given early in this section.) We have thus arrived at the conclusion that T is reducible, contrary to our hypothesis. The proof that y is a multiple of x is now complete for the case where A has no zero columns.

If some of the columns of A consist entirely of zeroes we may, without loss of generality, assume that these are the last $n-p$ columns, where $0<p<n$. Let A_1 and B_1 be the matrices obtained from A and B respectively, by omitting the last $n-p$

columns for each. $T_1=T(A_1; B_1)$ is irreducible since T is. T_1 has just been shown to possess a unique positive eigenvector $(x_1, \ldots, x_p)$ corresponding to a unique positive eigenvalue, which is clearly ρ.

Consider a vector $x=(x_1, \ldots, x_p, x_{p+1}, \ldots, x_n)$. If x is an eigenvector for T with eigenvalue ρ, we have in particular, $(Tx)_i=\rho x_i$, $i>p$. For any such i, $(Tx)_i$ is a function only of $x_1, \ldots, x_p$ and hence, is uniquely determined. Thus x_i is uniquely determined for $i>p$.

The theorem is now fully proved.

As an obvious corollary to the theorem we have

COROLLARY. *If A is irreducible with respect to B, there exists a row-stochastic matrix A_1, a column-stochastic matrix A_2, a positive number θ, and two diagonal matrices D and E with positive diagonal entries such that $DAE=A_1$ and $\theta DBE=A_2$. A_1, A_2 and θ are uniquely determined. D and E are also uniquely determined up to a scalar multiple.*

REMARK. In [5], a continuous, monotone increasing operator U on $\mathcal{N}$ into $\mathcal{N}$ which is homogeneous of degree one is called indecomposable if the following condition is satisfied: The relations $x_i=y_i$, $i\in R$, where $R\subset\{1, \ldots, n\}$ and $x_i<y_i$, $i\notin R$, imply that there exists at least one $i\in R$ for which $(Ux)_i<(Uy)_i$.

Taking U to be the operator $T(A; B)$ we see that indecomposability implies irreducibility. That the reverse implication need not hold is seen by considering the following example: Let A and B be 2×2 matrices whose only zero elements are a_{12} and b_{21}. Let $R=\{1\}$.

6. When $T(A; B)$ is reducible, general results about its spectrum cannot normally be obtained without taking into account the magnitudes of the elements of A and B, and not merely their patterns. The case $A=B$ is, however, an exception to this statement. In a joint paper, Professor Hans Schneider and the author of this paper have obtained necessary and sufficient conditions that $A, r_1, \ldots, r_m, c_1, \ldots, c_n$ have to satisfy in order that $T(A, r^{(m)}; B, c^{(n)})$ should have a positive eigenvalue associated with a positive eigenvector. This along with other results will appear elsewhere.

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