THE C^k-CLASSIFICATION OF CERTAIN OPERATORS IN L_p(1)

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Introduction. We investigate in this paper the one-parameter family of operators

$$T_\alpha = M + \alpha J$$

acting in $L_p(0, 1)$ $(1 \leqq p < \infty)$, where $\alpha \in \boldsymbol{C}$ (the complex field), $M: f(x) \to xf(x)$ and $J: f(x) \to \int_0^x f(t)\, dt$.

Extending a result of Sakhnovič [8] from the case $p=2$ to the case $1<p<\infty$, Kalisch [4] established recently that T_α is similar to M if $\operatorname{Re} \alpha = 0$.

The situation becomes quite different if $\operatorname{Re} \alpha \neq 0$; thus, T_α is *not* similar to M if $|\operatorname{Re} \alpha| \geqq 1$, and more generally, T_α is not similar to T_β if $[\operatorname{Re} \alpha] \neq [\operatorname{Re} \beta]$ (Proposition 13). This result will follow from our discussion of the C^k-operational calculus (both "global" and "local", in the sense of [5], [6], [7]) for the operators T_α. We recall briefly the terminology, restricting ourselves to bounded operators $T: X \to X$ (X a Banach space) with *real* spectrum $\sigma(T)$.

Fix a compact interval $\Delta \subset \boldsymbol{R}$ (the real line) which contains $\sigma(T)$. For $n = 0, 1, 2, \ldots$, let $C^n(\Delta)$ denote the Banach algebra of all complex valued functions of class C^n on Δ, with the norm

$$\|\varphi\|_{n,\Delta} = \sum_{j=0}^{n} \sup_{\Delta} |\varphi^{(j)}|/j!$$

(We shall write $\|\varphi\|_n$ when $\Delta = [0, 1]$.) We say that T is of class C^n (and we write $T \in (C^n)$) if there exists a continuous representation $\varphi \to T(\varphi)$ of $C^n(\Delta)$ on X such that $T(\varphi) = I$ (the identity operator) for $\varphi(t) \equiv 1$ and $T(\varphi) = T$ for $\varphi(t) \equiv t$. Such a representation is unique when it exists, and is called the C^n-operational calculus for T. For example, $T_0 = M$ is of class C $(= C^0)$, and its C-operational calculus is $\varphi \to M(\varphi)$, where

$$M(\varphi): f(x) \to \varphi(x) f(x), f \in L_p(0, 1).$$

If $W \subset X$ is a linear manifold, we denote by $\boldsymbol{T}(W)$ the algebra of all linear transformations of X with domain W and range contained in W. If W is invariant for T, a C^n-operational calculus for T *on* W is an algebra homomorphism $\varphi \to T(\varphi)$ of $C^n(\Delta)$ into $\boldsymbol{T}(W)$ with the following properties:

(i) $T(\varphi) = I/W$ for $\varphi(t) \equiv 1$;

(ii) $T(\varphi) = T/W$ for $\varphi(t) \equiv t$;

(iii) for each $x \in W$, the mapping $\varphi \to T(\varphi)x$ of $C^n(\Delta)$ into X is continuous.

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For each $n=0, 1, 2, \ldots$, there exists a maximal invariant linear manifold $W_n(T)$ on which T admits a C^n-operational calculus, and the latter is uniquely determined on $W_n(T)$. In fact, $W_n(T)$ is the set of all $x \in X$ for which

$$|x|_n = \sup \{\|\varphi(T)x\|;\ \varphi \text{ a polynomial with } \|\varphi\|_{n,\Delta} \leqq 1\} \text{ is finite.}$$

The "semisimplicity manifold" $W_0(T)$ is particularly important (cf. Theorem 2.1 in [6]); it contains trivially the eigenvectors of T. We shall write $W_n(T_\alpha; p)$ instead of $W_n(T_\alpha)$ when it will be necessary to specify the L_p space under consideration.

We may now describe the main results of this paper (for $1<p<\infty$):

1°. T_α is of class C^n if $|\text{Re}\,\alpha| \leqq n$ and only if $|\text{Re}\,\alpha| < n+1$ (Theorem 6); the C^n-operational calculus for T_α ($|\text{Re}\,\alpha| \leqq n$) is given in Theorems 8 and 9.

2°. The W_k manifolds of T_α are dense for $\text{Re}\,\alpha<0$ and trivial for $\text{Re}\,\alpha \geqq 1$ and $k<[\text{Re}\,\alpha]$ (Theorems 10 and 11).

3°. T_α is not spectral (in Dunford's sense) for $|\text{Re}\,\alpha| \geqq 1$ (it is of course spectral for $\text{Re}\,\alpha=0$, by the Kalisch-Sakhnovič result).

1. **Five lemmas.** Let $\{J^\zeta;\ \text{Re}\,\zeta>0\}$ be the Riemann-Liouville holomorphic semigroup, acting in $L_p(0, 1)$ $(1 \leqq p<\infty)$:

$$(J^\zeta f)(x) = (1/\Gamma(\zeta)) \int_0^x (x-t)^{\zeta-1} f(t)\, dt$$

($f \in L_p(0, 1)$, $x \in [0, 1]$, $\text{Re}\,\zeta>0$). It is known (cf. [3] for $p=2$, and [4] for $1<p<\infty$) that if $1<p<\infty$, the semigroup $\{J^\zeta\}$ admits a strongly continuous boundary group $\{J^{i\gamma};\ \gamma \in \boldsymbol{R}\}$ of bounded operators, and

$$\|J^{i\gamma}\| \leqq e^{\pi|\gamma|/2} \qquad (\gamma \in \boldsymbol{R})$$

(see also the estimates at the end of Kalisch's paper [4]). The operator J^ζ ($\text{Re}\,\zeta>0$) is one-to-one in $L_p(0, 1)$ ($p \geqq 1$); its inverse, with domain $\mathscr{D}_{-\zeta}=\mathscr{R}(J^\zeta)$ (the range of J^ζ), is a closed operator, which we denote by $J^{-\zeta}$.

For $1<p<\infty$, note that $\mathscr{R}(J^{\beta+i\gamma})=\mathscr{R}(J^\beta)$ ($\beta>0$, $\gamma \in \boldsymbol{R}$), since $J^{\beta+i\gamma}=J^\beta J^{i\gamma}$ and $J^{i\gamma}$ is nonsingular.

For $p=1$ and $\gamma \in \boldsymbol{R}$, we define $J^{i\gamma}$ as follows. Its domain is $\mathscr{D}_{i\gamma}=U\{\mathscr{R}(J^\zeta);\ \text{Re}\,\zeta>0\}$; if $f \in \mathscr{D}_{i\gamma}$, say $f=J^\zeta h$ for some $h \in L_1(0, 1)$ and $\zeta \in \boldsymbol{C}$ with $\text{Re}\,\zeta>0$, then $J^{i\gamma}f=J^{\zeta+i\gamma}h$. One verifies easily that $J^{i\gamma}$ is well defined.

We shall use the notation $\mathscr{D}_\zeta$ also for $\text{Re}\,\zeta>0$, in which case $\mathscr{D}_\zeta=L_p(0, 1)$ (the domain of J^ζ!); similarly $\mathscr{D}_{i\gamma}=L_p(0, 1)$ for $1<p<\infty$.

LEMMA 1. *For any $\zeta \in \boldsymbol{C}$, $\mathscr{D}_\zeta$ is invariant under M, and the following (trivially equivalent) identities are valid on $\mathscr{D}_\zeta$:*

(1) $MJ^\zeta - J^\zeta M = \zeta J^{\zeta+1}$,

(2) $J^\zeta M = T_{-\zeta} J^\zeta$,

(3) $MJ^\zeta = J^\zeta T_\zeta$.

Proof. If $\operatorname{Re}\zeta>0$ ($\operatorname{Re}\zeta\geqq 0$ if $1<p<\infty$), the first statement is trivial, since $\mathscr{D}_\zeta=L_p(0,1)$. For $g\in L_p(0,1)$ and $\operatorname{Re}\zeta>0$,

$$\begin{aligned}(MJ^\zeta-J^\zeta M)g(x) &= \Gamma(\zeta)^{-1}\int_0^x (x-t)^{\zeta-1}(x-t)g(t)\,dt\\ &= \Gamma(\zeta)^{-1}\Gamma(\zeta+1)J^{\zeta+1}g(x) = \zeta J^{\zeta+1}g(x).\end{aligned}$$

This proves the first, and hence all three identities of the lemma (for $1\leqq p<\infty$ and $\operatorname{Re}\zeta>0$). By (3), $M\mathscr{R}(J^\zeta)\subset\mathscr{R}(J^\zeta)$, i.e. $\mathscr{D}_{-\zeta}$ is invariant under M. Apply $J^{-\zeta}$ to both sides of (3):

$$J^{-\zeta}MJ^\zeta = T_\zeta,$$

i.e., $J^{-\zeta}M=T_\zeta J^{-\zeta}$ on $\mathscr{D}_{-\zeta}$. This proves (2) (and hence the lemma) for $\operatorname{Re}\zeta<0$.

Let $\mathscr{D}=U\{\mathscr{D}_\zeta;\ \operatorname{Re}\zeta<0\}$. $\mathscr{D}$ is M-invariant, and dense in $L_p(0,1)$ ($1\leqq p<\infty$). Let $g\in\mathscr{D}$; then $g=J^\zeta h$ for some $\zeta\in\boldsymbol{C}$ with $\operatorname{Re}\zeta>0$ and some $h\in L_p(0,1)$. Using (3) twice (for $\operatorname{Re}\zeta>0$), we obtain (for $\gamma\in\boldsymbol{R}$):

$$\begin{aligned}MJ^{i\gamma}g &= MJ^{\zeta+i\gamma}h = J^{\zeta+i\gamma}T_{\zeta+i\gamma}h\\ &= J^{i\gamma}J^\zeta(T_\zeta+i\gamma J)h = J^{i\gamma}(MJ^\zeta+i\gamma J^{\zeta+1})h\\ &= J^{i\gamma}(M+i\gamma J)J^\zeta h = J^{i\gamma}T_{i\gamma}g.\end{aligned}$$

If $p=1$, this finishes the proof of the lemma for $\operatorname{Re}\zeta=0$. If $1<p<\infty$, this shows that (3) is valid on the dense subset $\mathscr{D}$ of $L_p(0,1)$ (for $\zeta=i\gamma$); since $J^{i\gamma}$ is a bounded operator, (3) is valid everywhere on $L_p(0,1)$.

LEMMA 2. *For $\beta,\gamma\in\boldsymbol{R}$ arbitrary, the operators $T_{\beta+i\gamma}$ and T_β acting in $L_p(0,1)$ ($1<p<\infty$) are similar, with $J^{i\gamma}$ implementing the similarity*:

$$J^{i\gamma}T_{\beta+i\gamma}J^{-i\gamma} = T_\beta,$$

(*for $\beta=0$ and $p=2$, this is due to Sakhnovič* [8]).

Proof. By (3),

$$J^{i\gamma}T_{\beta+i\gamma}J^{-i\gamma}=J^{i\gamma}(T_{i\gamma}+\beta J)J^{-i\gamma} = M+\beta J = T_\beta.$$

REMARK. For any $1\leqq p<\infty$ and any complex numbers α and ζ, T_α is *unboundedly* similar to T_ζ (in particular, T_α is unboundedly similar to M). Indeed,

$$J^{-(\zeta-\alpha)}T_\alpha J^{\zeta-\alpha} = T_\zeta \quad \text{on } \mathscr{D}_{\zeta-\alpha},$$

where everything makes sense by Lemma 1. This suggests considering the map $\varphi\to J^{-\alpha}M(\varphi)J^\alpha$ as a "possible" operational calculus for T_α.

LEMMA 3. *For any integer $n\geqq 0$, and $1\leqq p<\infty$, the operator T_n acting in $L_p(0,1)$ belongs to $(C^n)-(C^{n-1})$, and the C^n-operational calculus for T_n is given by*

$$T_n(\varphi) = J^{-n}M(\varphi)J^n,\quad \varphi\in C^n[0,1]. \tag{4}$$

REMARK. (1) $(C^{-1})=\Phi$ by convention. (2) $T_n(\varphi)$ is well defined, since $\mathscr{R}(J^n)$ is a $C^n[0, 1]$-module.

Proof. By Leibnitz' formula,

$$T_n(\varphi) = \sum_{j=0}^{n} \binom{n}{j} M(\varphi^{(j)})J^j, \qquad \varphi \in C^n[0, 1]. \tag{5}$$

Thus $T_n(\varphi)$ is a bounded operator on $L_p(0, 1)$. In fact, since $\|J^j\| \leqq 1/j!$ (cf. [3, p. 664]), we have

$$\|T_n(\varphi)\| \leqq \binom{n}{[n/2]} \|\varphi\|_n, \qquad \varphi \in C^n[0, 1]. \tag{6}$$

The map $\varphi \to T_n(\varphi)$ is clearly an algebra homomorphism of $C^n[0, 1]$ into the bounded operators on $L_p(0, 1)$, which is continuous by (6). If $\varphi(x) \equiv 1$, $T_n(\varphi)$ is trivially the identity operator. If $\varphi(x) \equiv x$, $T_n(\varphi) = T_n$ by (3). Thus, $T_n \in (C^n)$ and its C^n-operational calculus is given by (4). Finally, apply (5) to the functions

$$\varphi_t(x) = e^{itx} \qquad (x, t \in R).$$

Thus

$$(e^{itT_n}g)(x) = \sum_{j=0}^{n} \binom{n}{j} (it)^j e^{itx} (J^j g)(x),$$

and consequently $\|e^{itT_n}\| \neq O(|t|^{n-1})$ $(n=1, 2, \ldots)$. Therefore $T_n \notin (C^{n-1})$ $(n \geqq 1)$ by Lemma 2.11 in [5].

LEMMA 4. *Let $n \geqq 0$ be an integer. Then, for φ of class C^n and $g \in \mathscr{R}(J^n)$ (or vice versa, or for both φ and g in $\mathscr{R}(J^n)$), the following identity is valid*:

$$J^n(\varphi J^{-n} g) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j J^j (g J^{-j} \varphi).$$

REMARK. Here, as usual, $\mathscr{R}(J^n)$ denotes the range of J^n in $L_p(0, 1)$, with p arbitrary $(1 \leqq p < \infty)$. Note the analogy with Leibnitz' formula (in fact, the latter may be used to prove the lemma).

Proof. We use induction on n. The lemma is trivial for $n=0$. Suppose it is true for $n=k$. Let φ and g be as required for $n=k+1$. Write $g' = J^{-1}g$. Then φ and g' satisfy the hypothesis for $n=k$. Therefore

$$\begin{aligned} J^{k+1}(\varphi J^{-(k+1)} g) &= J J^k (\varphi J^{-k} g') \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} J^j J (g' J^{-j} \varphi). \end{aligned}$$

An integration by parts shows that

$$J(g' J^{-j} \varphi) = g J^{-j} \varphi - J(g J^{-(j+1)} \varphi).$$

Thus

$$J^{k+1}(\varphi J^{-(k+1)}g) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\{J^j(gJ^{-j}\varphi)-J^{j+1}(gJ^{-(j+1)}\varphi)\}$$
$$= g\varphi+\sum_{j=1}^{k}(-1)^j\left[\binom{k}{j}+\binom{k}{j-1}\right] J^j(gJ^{-j}\varphi)$$
$$+(-1)^{k+1}J^{k+1}(gJ^{-(k+1)}\varphi).$$

Since

$$\binom{k}{j}+\binom{k}{j-1} = \binom{k+1}{j},$$

we obtain the correct identity for $n=k+1$, Q.E.D.

LEMMA 5. *For any integer $n\geqq 0$, and for $1\leqq p<\infty$, the operator T_{-n} acting in $L_p(0, 1)$ belongs to $(C^n)-(C^{n-1})$, and the C^n-operational calculus for T_{-n} is given by*

$$T_{-n}(\varphi) = \sum_{j=0}^{n}\binom{n}{j}(-1)^jJ^jM(\varphi^{(j)}), \qquad \varphi\in C^n[0, 1]. \tag{7}$$

Proof. The map $\varphi\to T_{-n}(\varphi)$ of $C^n[0, 1]$ into the bounded operators on $L_p(0, 1)$ is clearly linear and continuous, in fact,

$$\|T_{-n}(\varphi)\| \leqq \binom{n}{[n/2]}\|\varphi\|_n.$$

For $g\in\mathscr{R}(J^n)$, we have by Lemma 4:

$$T_{-n}(\varphi)g = J^n(\varphi J^{-n}g). \tag{8}$$

Therefore $\mathscr{R}(J^n)$ is invariant under $T_{-n}(\varphi)$ (for all $\varphi\in C^n[0, 1]$) and $T_{-n}(\varphi\psi)= T_{-n}(\varphi)T_{-n}(\psi)$ on $\mathscr{R}(J^n)$ (for all $\varphi, \psi\in C^n[0, 1]$). Since $\mathscr{R}(J^n)$ is dense in $L_p(0, 1)$ and $T_{-n}(\varphi)$ is continuous, it follows that $T_{-n}(\cdot)$ is multiplicative on $C^n[0, 1]$. The relations $T_{-n}(\varphi)=I(T_{-n})$ for $\varphi(x)\equiv 1$ $(\equiv x)$ are trivial on $\mathscr{R}(J^n)$ by (8) and Lemma 1; by density, they are true throughout $L_p(0, 1)$.

Finally, one verifies that $T_{-n}\notin(C^{n-1})$ just as in Lemma 3.

2. **Global classification** ($1<p<\infty$).

THEOREM 6. *The operator T_α acting in $L_p(0, 1)$ $(1<p<\infty)$ is of class C^n $(n= 0, 1, 2, \ldots)$ if $|\mathrm{Re}\,\alpha|\leqq n$ and only if $|\mathrm{Re}\,\alpha|<n+1$.*

In other words, T_α is of class C^n in the strip $|\mathrm{Re}\,\alpha|\leqq n$ and is *not* of class C^n outside the strip $|\mathrm{Re}\,\alpha|<n+1$.

The theorem is an immediate corollary of Lemmas 3 and 5, together with the following

LEMMA 7. *Suppose that, for some integer $n\geqq 0$ and some $\alpha_0\in\boldsymbol{C}$, the operator T_{α_0} is of class C^n (when acting in $L_p(0, 1)$, $1<p<\infty$). Then T_α is of class C^n for all α in the strip $-n\leqq\mathrm{Re}\,\alpha\leqq\mathrm{Re}\,\alpha_0$ if $\mathrm{Re}\,\alpha_0\geqq 0$ ($\mathrm{Re}\,\alpha_0\leqq\mathrm{Re}\,\alpha\leqq n$ if $\mathrm{Re}\,\alpha_0\leqq 0$).*

Proof. To fix the ideas, suppose $\operatorname{Re}\alpha_0=\beta_0\geqq 0$. Write $\alpha=\beta+i\gamma$ $(\beta,\gamma\in\boldsymbol{R})$. Fixing a polynomial φ and elements $f\in L_p(0,1)$, $g\in L_q(0,1)$ $(p^{-1}+q^{-1}=1)$, we define

$$\Phi(\alpha) = \langle e^{\pi\alpha^2}\varphi(T_\alpha)f, g\rangle, \qquad \alpha\in\boldsymbol{C}.$$

Since $|e^{\pi\alpha^2}|\leqq e^{\pi\beta^2}$, and since $\varphi(T_\alpha)$ is a polynomial in α (with operator coefficients), we have $|\Phi(\alpha)|=O(e^{\varepsilon|\gamma|})$ (for $|\gamma|\to\infty$) in the strip $-n\leqq\beta\leqq\beta_0$, for any $\varepsilon>0$.

By Lemma 2 and the estimate $\|J^{i\gamma}\|\leqq e^{\pi|\gamma|/2}$, we have:

$$\begin{aligned}|\Phi(\beta+i\gamma)| &\leqq \exp\pi(\beta^2-\gamma^2+|\gamma|)\cdot\|f\|_p\|g\|_q\,\|\varphi(T_\beta)\|\\ &\leqq \exp\pi(\beta^2+1/4)\cdot\|f\|_p\,\|g\|_q\,\|\varphi(T_\beta)\|\end{aligned}$$

for all $\beta,\gamma\in R$.

Since T_{-n} and T_{β_0} are of class C^n (by Lemma 5, the hypothesis and Lemma 2), there exists a constant K (depending only on n, β_0 and p) such that

$$\|\varphi(T_{-n})\| \leqq K\|\varphi\|_n \quad\text{and}\quad \|\varphi(T_{\beta_0})\| \leqq K\|\varphi\|_n.$$

Hence

$$|\Phi(-n+i\gamma)| \leqq M\|f\|_p\,\|g\|_q\,\|\varphi\|_n$$

and

$$|\Phi(\beta_0+i\gamma)| \leqq M\|f\|_p\,\|g\|_q\,\|\varphi\|_n$$

for all real γ, where $M=K\exp\pi(\delta^2+1/4)$ and $\delta=\max(n,\beta_0)$. By the Phragmèn-Lindelöf principle (cf. [9, p. 180]), it follows that $|\Phi(\alpha)|\leqq M\|f\|_p\,\|g\|_q\,\|\varphi\|_n$ for $-n\leqq\operatorname{Re}\alpha\leqq\beta_0$. Hence, for such α,

$$\|\varphi(T_\alpha)\| \leqq M\exp\pi(\gamma^2-\beta^2)\|\varphi\|_n,$$

and the lemma follows.

The next two theorems give explicitly the C^n-operational calculus for T_α $(|\operatorname{Re}\alpha|\leqq n)$ acting in $L_p(0,1)$, *with* $1<p<\infty$.

THEOREM 8. *Let n be a nonnegative integer. Then for $0\leqq\operatorname{Re}\alpha\leqq n$ and $\varphi\in C^n[0,1]$ the range of J^α (i.e., the domain of $J^{-\alpha}$) is invariant under $M(\varphi)$, and the C^n-operational calculus for T_α is given by*

$$T_\alpha(\varphi) = J^{-\alpha}M(\varphi)J^\alpha, \qquad \varphi\in C^n[0,1].$$

Proof. By Lemma 1 (3),

$$\varphi(M)J^\alpha = J^\alpha\varphi(T_\alpha) \tag{9}$$

for any *polynomial* φ. In particular,

$$\varphi(M)\mathscr{R}(J^\alpha) \subset \mathscr{R}(J^\alpha), \qquad \varphi = \text{a polynomial}. \tag{10}$$

Let $\varphi \in C^n[0, 1]$, and choose polynomials φ_k which converge to φ in $C^n[0, 1]$. In particular, $\varphi_k \to \varphi$ uniformly in $[0, 1]$, and therefore

$$\mathscr{D}_{-\alpha} \ni \varphi_k J^\alpha g \to \varphi J^\alpha g \tag{11}$$

in $L_p(0, 1)$, for any $g \in L_p(0, 1)$. By (9), we have:

$$J^{-\alpha}(\varphi_k J^\alpha g) = \varphi_k(T_\alpha) = T_\alpha(\varphi_k) \to T_\alpha(\varphi) \tag{12}$$

in the uniform operator topology, since $T_\alpha \in (C^n)$ by Theorem 6 (for $|\text{Re } \alpha| \leqq n$ and $1 < p < \infty$) and $\varphi_k \to \varphi$ in $C^n[0, 1]$. Since $J^{-\alpha}$ is a closed operator, it follows from (11) and (12) that $\varphi J^\alpha g \in \mathscr{D}_{-\alpha}$ and

$$J^{-\alpha}(\varphi J^\alpha g) = T_\alpha(\varphi), \qquad \text{Q.E.D.}$$

We consider next the range $-n \leqq \text{Re } \alpha < 0$ $(n = 1, 2, \ldots)$. Note that $\text{Re } (\alpha + n) \geqq 0$. The notation $T_{-n}(\varphi)$ is that of Lemma 5.

THEOREM 9. *Let n be a nonnegative integer. Then for $-n \leqq \text{Re } \alpha < 0$ and $\varphi \in C^n[0, 1]$, the range of $J^{\alpha+n}$ (i.e., $\mathscr{D}_{-(\alpha+n)}$) is invariant under $T_{-n}(\varphi)$, and the C^n-operational calculus for T_α is given by*

$$T_\alpha(\varphi) = J^{-(\alpha+n)} T_{-n}(\varphi) J^{\alpha+n}, \qquad \varphi \in C^n[0, 1].$$

Proof. By (10), $\mathscr{R}(J^{\alpha+n})$ is invariant for $M(\varphi)$ for any polynomial φ; it is therefore invariant for the operator

$$T_{-n}(\varphi) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j J^j M(\varphi^{(j)}), \qquad \varphi = \text{a polynomial}.$$

Thus, for any polynomial φ, the operator

$$S_\alpha(\varphi) = J^{-(\alpha+n)} T_{-n}(\varphi) J^{\alpha+n}$$

is everywhere defined. Being closed, it is *continuous* by the Closed Graph Theorem.

Let $g \in \mathscr{D}_\alpha = \mathscr{R}(J^{-\alpha})$, say $g = J^{-\alpha} h$ with $h \in L_p(0, 1)$. By Lemma 1,

$$\begin{aligned} S_\alpha(\varphi) g &= J^{-(\alpha+n)} T_{-n}(\varphi) J^n h = J^{-(\alpha+n)} J^n \varphi(M) h \\ &= J^{-\alpha} \varphi(M) h = \varphi(T_\alpha) J^{-\alpha} h \\ &= \varphi(T_\alpha) g \end{aligned}$$

for any polynomial φ.

This shows that the continuous operators $S_\alpha(\varphi)$ and $\varphi(T_\alpha) = T_\alpha(\varphi)$ coincide on the dense subset $\mathscr{D}_\alpha$ of $L_p(0, 1)$. Thus, for every polynomial φ,

$$T_\alpha(\varphi) = J^{-(\alpha+n)} T_{-n}(\varphi) J^{\alpha+n}. \tag{13}$$

Let $\varphi \in C^n[0, 1]$, and let φ_k be polynomials converging to φ in $C^n[0, 1]$. Since T_α and T_{-n} are of class C^n (by Theorem 6), we have (in the uniform operator topology):

$$T_\alpha(\varphi_k) \to T_\alpha(\varphi); \quad T_{-n}(\varphi_k) \to T_{-n}(\varphi) \tag{14}$$

for any $k \to \infty$.

Fix $g \in L_p(0, 1)$. Then $T_{-n}(\varphi_k)J^{\alpha+n}g \in \mathscr{D}_{-(\alpha+n)}$ (cf. beginning of the proof) and

$$T_{-n}(\varphi_k)J^{\alpha+n}g \to T_{-n}(\varphi)J^{\alpha+n}g$$

for $k \to \infty$ (by (14)). Moreover

$$J^{-(\alpha+n)}[T_{-n}(\varphi_k)J^{\alpha+n}g] = T_\alpha(\varphi_k)g \to T_\alpha(\varphi)g$$

by (13) and (14). Since $J^{-(\alpha+n)}$ is closed, it follows that $T_{-n}(\varphi)J^{\alpha+n}g \in \mathscr{D}_{-(\alpha+n)} = \mathscr{R}(J^{\alpha+n})$ and $J^{-(\alpha+n)}T_{-n}(\varphi)J^{\alpha+n}g = T_\alpha(\varphi)g$, Q.E.D.

3. **The local C^k-operational calculus.** Note first that the results of §2 are also relevant to the case $p=1$, in the sense of the *local* C^k-operational calculus. Let $L = \bigcup_{1<p<\infty} L_p(0, 1)$. This is a dense linear manifold in $L_1(0, 1)$, which is invariant under T_α for all $\alpha \in \mathbf{C}$. Let $n \geqq 0$ be an integer, and let $|\mathrm{Re}\,\alpha| \leqq n$. If $f \in L$, say $f \in L_p(0, 1)$ for some $1<p<\infty$, then the mapping $\varphi \in C^n[0, 1] \to T_\alpha(\varphi)f \in L_1(0, 1)$ is continuous ($T_\alpha(\cdot)$ is given by Theorems 8 and 9) because $\|T_\alpha(\varphi)f\|_1 \leqq \|T_\alpha(\varphi)f\|_p \leqq \|T_\alpha(\cdot)\|_p \|f\|_p \|\varphi\|_n$, where $\|T_\alpha(\cdot)\|_p$ denotes the norm of the C^n-operational calculus for T_α acting in $L_p(0, 1)$. Thus $W_n(T_\alpha; 1) \supset L$ for $|\mathrm{Re}\,\alpha| \leqq n$, and the C^n-operational calculus for T_α on L is provided by Theorems 8 and 9.

In the next two theorems, we study the manifolds $W_k(T_\alpha; p)$ for $k < |\mathrm{Re}\,\alpha|$ (they coincide with the whole space for $k \geqq |\mathrm{Re}\,\alpha|$, at least for $1<p<\infty$, by §2). It turns out that the situation is totally different in the right and left half-planes.

THEOREM 10. *For $\alpha \in \mathbf{C}$ with* $\mathrm{Re}\,\alpha < 0$ *and* $1<p<\infty$,

$$W_k(T_\alpha; p) \supset \mathscr{D}_{\alpha+k}, \qquad 0 \leqq k < |\mathrm{Re}\,\alpha|,$$

and the C^k-operational calculus for T_α on $\mathscr{D}_{\alpha+k}$ is given by

$$T_\alpha(\varphi) = J^{-(\alpha+k)}T_{-k}(\varphi)J^{\alpha+k}, \quad \varphi \in C^k[0, 1], \tag{15}$$

(*where $T_{-k}(\varphi)$ is defined in Lemma* 5).

Proof. Fix p, α and k as in the theorem, and define $T_\alpha(\cdot)$ by (15). One verifies easily that the mapping $\varphi \to T_\alpha(\varphi)$ is an algebra homomorphism of $C^k[0, 1]$ into $\mathbf{T}(\mathscr{D}_{\alpha+k})$ which sends the functions $\varphi(x) \equiv 1$ and $\varphi(x) \equiv x$ respectively to $I|\mathscr{D}_{\alpha+k}$ and $T_\alpha|\mathscr{D}_{\alpha+k}$ (cf. Lemma 1). Moreover, for each $g \in \mathscr{D}_{\alpha+k}$, the mapping $\varphi \to T_\alpha(\varphi)g$ of $C^k[0, 1]$ into $L_p(0, 1)$ is continuous, since

$$T_\alpha(\varphi)g = J^{-\alpha-k}T_{-k}(\varphi)h$$

for $g = J^{-\alpha-k}h$ with $h \in L_p(0, 1)$. Q.E.D.

In particular, $W_k(T_\alpha; p)$ is dense in $L_p(0, 1)$ for $\operatorname{Re} \alpha < 0$ and $k \geqq 0$ arbitrary. For $\operatorname{Re} \alpha \geqq 1$, we get the "other" extreme.

THEOREM 11. *For* $\alpha \in \boldsymbol{C}$ *with* $\operatorname{Re} \alpha \geqq 1$ *and* $1 < p < \infty$, $W_k(T_\alpha; p) = (0)$ *if* $k < [\operatorname{Re} \alpha]$. *The same is true for* $p = 1$ *if* α *is an integer.*

Proof. If α is an integer, this is a trivial consequence of Lemma 3 and Leibnitz' formula.

Suppose then that $1 < p < \infty$, and that $f \in W_k(T_\alpha; p)$ for some fixed $k < m = [\operatorname{Re} \alpha]$. As in the proof of Lemma 7, we apply the Phragmèn-Lindelöf principle in the strip $0 \leqq \operatorname{Re} \zeta \leqq \operatorname{Re} \alpha$ to the function $\Phi(\zeta) = \langle e^{\pi\zeta^2} \varphi(T_\zeta) f, g \rangle$ where φ is a polynomial and $g \in L_q(0, 1)$ (both fixed). We then obtain that $f \in W_k(T_\zeta; p)$ for all ζ in the strip, hence in particular for $\zeta = m$. Since $k < m$, we conclude that f is the null function.

4. Similarity and spectrality.

LEMMA 12. *Let* $\alpha \in \boldsymbol{C}$ *and* $1 < p < \infty$. *Then every* $s \in [0, 1) = \sigma(T_\alpha) \backslash \{1\}$ *is an eigenvalue of* $T_{-\alpha}$ *for* $\operatorname{Re} \alpha \geqq 1$ ($\operatorname{Re} \alpha > 1$ *or* $\alpha = 1$ if $p = 1$), *while* T_α *has no eigenvalue for* $\operatorname{Re} \alpha \geqq 0$ ($\operatorname{Re} \alpha > 0$ *or* $\alpha = 0$ *if* $p = 1$).

Proof. Let C_s denote the characteristic function of the interval $[s, 1]$, $0 \leqq s < 1$. One verifies easily that C_s is an eigenvector of T_{-1} corresponding to the eigenvalue s (for $1 \leqq p < \infty$).

By Lemma 1, (3),

$$T_{-\alpha} J^{\alpha-1} C_s = J^{\alpha-1} T_{-1} C_s = s J^{\alpha-1} C_s,$$

i.e. $J^{\alpha-1} C_s$ (which is in $L_p(0, 1)$ for α as in the first statement of the lemma) is an eigenvector of $T_{-\alpha}$ corresponding to the eigenvalue s.

Next, suppose $T_\alpha g = \lambda g$ for $g \in L_p(0, 1)$ and $\lambda \in \boldsymbol{C}$. If $\operatorname{Re} \alpha \geqq 0$ ($\operatorname{Re} \alpha > 0$ or $\alpha = 0$ if $p = 1$), we may apply J^α on both sides of this equation; by Lemma 1,(3), we obtain

$$M J^\alpha g = \lambda J^\alpha g.$$

Since M has no eigenvector $\neq 0$ and J^α is one-one, it follows that g is the zero element.

Let $\alpha, \beta \in \boldsymbol{C}$. By Lemma 2, T_α and T_β are similar if $\operatorname{Re} \alpha = \operatorname{Re} \beta$ (and $1 < p < \infty$). On the other hand, since the C^k-classification and the point spectrum are similarity invariants, it follows from Lemmas 3, 5 and 12 that T_α and T_β are *not* similar if α and β are distinct integers (for $1 \leqq p < \infty$).

Conjecture. For $1 < p < \infty$ and $\alpha, \beta \in \boldsymbol{C}$, T_α and T_β are similar if and only if $\operatorname{Re} \alpha = \operatorname{Re} \beta$. (By Lemma 2, it would suffice to verify that T_α and T_β are not similar if α and β are distinct *real* numbers.)

PROPOSITION 13. *Let* $\alpha, \beta \in \boldsymbol{C}$ *and* $1 < p < \infty$. *Then* T_α *and* T_β (*acting in* $L_p(0, 1)$) *are not similar if* $[\operatorname{Re} \alpha] \neq [\operatorname{Re} \beta]$.

Proof. Assume, without loss of generality, that $\operatorname{Re}\alpha < \operatorname{Re}\beta$. If either $0 \leqq \operatorname{Re}\alpha$ or $\operatorname{Re}\beta \leqq 0$, this follows from Theorem 6 and the similarity invariance of the C^k-classification. If $\operatorname{Re}\alpha < 0 < 1 \leqq \operatorname{Re}\beta$, $W_0(T_\alpha)$ is dense in $L_p(0, 1)$ (Theorem 10) while $W_0(T_\beta) = (0)$ (Theorem 11). Thus T_α and T_β are not similar.

If $\operatorname{Re}\alpha \leqq -1 < 0 < \operatorname{Re}\beta$, every $s \in [0, 1)$ is an eigenvalue of T_α, while T_β has no eigenvalue (Lemma 12), and the conclusion follows from the similarity invariance of the point spectrum. Q.E.D.

We next discuss the spectrality of T_α in Dunford's sense [1].

LEMMA 14. *Let T be a bounded spectral operator with real spectrum, acting in the Banach space X. Let $T = S + N$ be its canonical decomposition* (cf. [1]). *Then*:

(a) *If $W_k(T)$ is dense in X for some integer $k \geqq 0$, then T is of finite type $\leqq k$ (i.e., $N^{k+1} = 0$).*

(b) *If T is of finite type k, then $W_j(T) \neq (0)$ for all $j \geqq 0$; in fact, $W_j(T) \supset \mathscr{R}(N^{k-j})$ for $j = 0, \ldots, k-1$, and trivially $W_j(T) = X$ for $j \geqq k$.*

Proof. Fix a compact interval $\Delta \supset \sigma(T)$. Let $S(\cdot)$ be the C-operational calculus for S (defined on $C(\Delta)$), and let $\|S(\cdot)\|$ be its norm.

(a) Let $x \in W_k(T)$. The function $e^{izN}x$ ($z \in \boldsymbol{C}$) is entire of order one and minimal type (since N is a quasi-nilpotent operator). For $z = t \in \boldsymbol{R}$, we have:

$$\|e^{itN}x\| = \|e^{-itS}e^{itT}x\| \leqq \|S(\cdot)\| \, \|e^{itT}x\| \leqq \|S(\cdot)\| \, |x|_k \, \|\varphi_t\|_{k,\Delta},$$

where $\varphi_t(s) = e^{its}$, $t, s \in \boldsymbol{R}$.

Thus $\|e^{itN}x\| = O(|t|^k)$, and therefore $e^{izN}x$ is a polynomial of order $\leqq k$ by Theorem 3.13.8 in [3]. Hence $N^{k+1}x = 0$ for each $x \in W_k(T)$, and it follows that $N^{k+1} = 0$ since $W_k(T)$ is dense in X.

(b) We have $N^{k+1} = 0$ and $N^k \neq 0$. The analytic operational calculus for T takes the form (cf. [1]):

$$T(\varphi) = \sum_{m=0}^{k} S(\varphi^{(m)})N^m/m!$$

If $x \in \mathscr{R}(N^{k-j})$, say $x = N^{k-j}y$ with $y \in X (0 \leqq j < k)$, then

$$T(\varphi)x = \sum_{m=0}^{j} S(\varphi^{(m)})N^m y/m!$$

In particular, $\|p(T)x\| \leqq \|S(\cdot)\| \cdot \max_{0 \leq m \leq j} \|N^m y\| \, \|p\|_{j,\Delta}$ for any polynomial p, i.e. $x \in W_j(T)$, Q.E.D.

For simplicity, we state the following result for $1 < p < \infty$, although part of the conclusion remains valid for $p = 1$.

PROPOSITION 15. *Let $1 < p < \infty$. Then T_α is spectral for* $\operatorname{Re}\alpha = 0$, *and is not spectral for* $|\operatorname{Re}\alpha| \geqq 1$.

Proof. The first statement is a trivial corollary of Lemma 2.

By Theorem 6, T_α is of class C^n if $n \geqq |\mathrm{Re}\ \alpha|$. Thus, if T_α were spectral, it should be of finite type by Lemma 14(a). In particular, its point spectrum should be at most countable by [2, Theorem 1, p. 56]. But this contradicts Lemma 12 if $\mathrm{Re}\ \alpha \leqq -1$. Also all $W_j(T_\alpha)$ $(j \geqq 0)$ should be nontrivial by Lemma 14(b), contradicting Theorem 11 if $\mathrm{Re}\ \alpha \geqq 1$. Thus T_α is not spectral for $|\mathrm{Re}\ \alpha| \geqq 1$.

5. **Remarks.** It is interesting to regard the results of this paper as statements about the operators $\alpha^{-1}T_\alpha = J + \alpha^{-1}M$ $(0 \neq \alpha \in C)$, which are perturbations of J by a scalar operator of arbitrarily small norm. Thus, if α and β are nonzero complex numbers, the following assertions can be made (for $1 < p < \infty$):

(a) If $[|\mathrm{Re}\ \alpha|] \neq [|\mathrm{Re}\ \beta|]$, $J + \alpha^{-1}M$ and $J + \beta^{-1}M$ belong to distinct (C^k)-classes, although they differ only by the scalar operator $(\alpha^{-1} - \beta^{-1})M$, which is of arbitrarily small norm. This shows that the commutativity hypothesis in [5, Corollary 5.6] cannot be replaced by a restriction on the norm of the perturbing scalar operator.

(b) The perturbations $J - \alpha^{-1}M$ and $J + \alpha^{-1}M$ have respectively a dense and a trivial semisimplicity manifold, a "pure" point spectrum (up to the right end point of the spectrum $[0, \alpha^{-1}]$) and no point spectrum.

(c) The perturbations $J + \alpha^{-1}M$ and $J + \beta^{-1}M$ are not similar if $[\mathrm{Re}\ \alpha] \neq [\mathrm{Re}\ \beta]$.

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