FREE ACTIONS ON $S^n \times S^n$ (1)

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0. **Introduction.** Conner [4], Heller [5], Mann [8], Mann and Su [9], and Su [12] have investigated free actions on a product of spheres. The best result is due to Heller [5]:

PROPOSITION O. *Let $n, m \geqq 1$. Then $Z_p \oplus Z_p \oplus Z_p$ cannot act freely on $S^n \times S^m$.*

To my knowledge, no one has considered actions of nonabelian groups. I will prove:

THEOREM A. *Let n be odd, p an odd prime, $p \nmid n+1$. Then any p-group acting freely on $S^n \times S^n$ is abelian.*

THEOREM B. *Let $n \equiv 1$ (4). Then any 2-group acting freely, preserving orientation, on $S^n \times S^n$, is abelian.*

The method of proof is via a Gysinoid sequence (§2), which holds whenever a finite group acts freely on a manifold M with three nonzero homology groups. The most useful cases are (1) $M = S^n \times S^n$ and (2) M = a surface.

We give a short proof of Mazur's theorem that $H^i(G, Z) \neq 0$ for infinitely many $i > 0$, and of the fact that Z_p^3 cannot act freely on $S^n \times S^n$, preserving orientation. Finally, we discuss the *Conjecture*: If G acts freely on $S^n \times S^n$, $Z_2 \oplus Z_2 \subset G$, then $Z_2 \oplus Z_2 \cap Z(G) \neq 1$. We prove this for some special cases.

1. **Topological preliminaries.** The following discussion is necessary in order to prove Theorems A and B for arbitrary continuous actions.

In E^{n+1}, let $e_0 = (1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$ be a standard basis, Δ^n = convex hull of $\{e_0, \ldots, e_n\}$ = standard n-simplex. If Y is a topological space (assumed arcwise connected), and F is a system of local coefficients, *a la* Steenrod, on Y, we define $H^*(Y, F)$ = singular cohomology of Y with local coefficients F, as follows:

Let $\Sigma_i(Y)$ = all i-simplices of Y. $\Gamma_s^i(Y, F) = \{f : \Sigma_i(Y) \to F \mid f(\sigma) \in F_{\sigma(0)}$, for all $\sigma \in \Sigma_i(Y)\}$. Define $\delta : \Gamma_s^i(Y, F) \to \Gamma_s^{i+1}(Y, F)$ by

$$\delta f(\sigma) = \sum_{i=1}^{n} (-1)^i f(\sigma^i) + \alpha^* f(\sigma^0),$$

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where $\sigma^i = i$th face of $\sigma \in \Sigma_{i+1}(Y)$, and $\alpha = \sigma|(e_0e_1)$. [The point is that $\sigma^i(e_0) = \sigma(e_0)$ for $0 < i \leqq n$, but $\sigma^0(e_0) = \sigma(e_1)$.] Then $\delta^2 = 0$, and we set $H_s^*(Y, F) = H^*(\Gamma_s(Y, F))$.

Let $A = G$-module, $Y =$ a topological space on which G acts freely. Then $Y \xrightarrow{\pi} Y/G$ is a finite covering space. We form $\tilde{A} = Y \times_G A = Y \times A/\sim$, where $(y, a) \sim (gy, ga)$. π induces $\tilde{\pi}\colon \tilde{A} \to Y/A$. *Claim*: $\tilde{A}$ is a local coefficient system over Y/G. For if $x_0, x_1 \in Y/G$, α a path from x_0 to x_1, choose $y_1 \in Y$ such that $\pi(y_1) = x_1$. By properties of covering spaces, there exists a unique $\tilde{\alpha}$: $[0, 1] \to Y$, with $\tilde{\alpha}(1) = y_1$, $\pi\tilde{\alpha} = \alpha$. Set $y_0 = \tilde{\alpha}(0)$. This is uniquely determined by the homotopy class of α rel $\{x_0, x_1\}$. If $u \in \tilde{A}_{x_1}$, $u = [(y_1, a)]$, set $\alpha^* u = [(y_0, a)]$. Taking gy_1 in place of y_1 yields $g\tilde{\alpha}$ in place of $\tilde{\alpha}$, by uniqueness, hence gives gy_0 for y_0. Thus $\alpha^*\colon \tilde{A}_{x_1} \to \tilde{A}_{x_0}$ is well defined, and clearly $(\alpha_1 \circ \alpha_2)^* = \alpha_2^* \circ \alpha_1^*$. Thus $\tilde{A}$ is a local coefficient system. The following is well known:

LEMMA 1.1. $\mathrm{Hom}_G(S(Y), A) \approx \Gamma_S(Y/G, \tilde{A})$, *as chain complexes.*

Proof. We will define reciprocal isomorphisms ϕ (left to right), and ψ (right to left). If $f \in \mathrm{Hom}_G(S_i(Y), A)$, let $\bar{\sigma} \in \Sigma_i(Y/G)$. Given y, such that $\pi(y) = \bar{\sigma}(0)$, there exists a unique lifting $\sigma_y\colon \Delta^i \to Y$ such that $\sigma_y(0) = y$. By uniqueness, $g\sigma_y = \sigma_{gy}$. Thus the pair $(y, f(\sigma_y))$ defines an element of $\tilde{A}_{\bar{\sigma}(0)}$, independent of choice of y.

Set $\phi(f)(\bar{\sigma}) = [(y, f(\sigma_y))]$. Given $\gamma \in \Gamma_s^i(Y/G, \tilde{A})$, we wish to define a G-morphism $\psi(\gamma)\colon S_i(Y) \to A$. As $S_i(Y)$ is G-free, it suffices to define $\psi(\gamma)$ on $\Sigma_i(Y)$. Let $\sigma \in \Sigma_i(Y)$, $\bar{\sigma} = \pi(\sigma)$. Then define $\psi(\gamma)(\sigma)$ by $\gamma(\bar{\sigma}) = [(\sigma(0), \psi(\gamma)(\sigma))]$. $\psi(\gamma)$ is clearly a G-morphism. $\phi\psi = 1$, $\psi\phi = 1$ are obvious. We leave to the reader the simple-minded task of checking that ψ and ϕ are chain maps. Q.E.D.

LEMMA 1.2. *Let X be an arcwise connected, paracompact, topological n-manifold. If F is any system of local coefficients on X, then $H_S^i(X, F) = 0$, for $i > n$.*

Proof. Such a well-known result needs no proof.

LEMMA 1.3 (HELLER). *Let Y be an n-manifold, G a finite group acting freely on Y. $S(Y) =$ singular complex of Y, and $B_n(Y) = \mathrm{Im}\{\partial\colon S_{n+1}(Y) \to S_n(Y)\}$. Then B_n is G-projective.*

Proof. B_n is G-projective if and only if for every G-module A, $\mathrm{Ext}^i(B_n, A) = 0$, $i > 0$. $\mathrm{Ext}_G^i(B_n, A) \approx H^{n+1+i}(\mathrm{Hom}_G(S(Y), A))$, since $\cdots \to S_{n+2} \to S_{n+1} \to B_n \to 0$ is a free G-resolution of B_n. By Lemma 1.1, this is $\approx$ to $H_S^{n+1+i}(Y/G, \tilde{A})$, and by Lemma 1.2, the latter is zero. Q.E.D.

COROLLARY 1.4. Under the hypotheses of Lemma 1.3, we have

$$0 \longrightarrow Z \xrightarrow{\text{mono}} S_n/B_n \longrightarrow S_{n-1} \longrightarrow \cdots$$
$$\longrightarrow S_1 \longrightarrow S_0 \xrightarrow{\text{epi}} Z \longrightarrow 0$$

exact at S_0, with S_n/B_n G-cohomologically trivial.

Application:

PROPOSITION 1.5 (MAZUR). *If G is a finite group, then $H^i(G, \mathbf{Z}) \neq 0$, for infinitely many $i > 0$.*

Proof. Imbed $G \subset U(n)$, as a subgroup, suitable n. G acts freely on $U(n)$ by left multiplication. As $\pi_0(U(n)) = 0$, each $g \in G$ has $L_g \simeq 1$. Hence G acts trivially on $H^*(U(n), \mathbf{Z}) = E(u_1) \otimes E(u_3) \otimes \cdots \otimes E(u_{2n-1})$ (torsion free). By Corollary 1.4, $U(n)$ has a finite G-complex

$$C: 0 \to \mathbf{Z} \to C_{n^2} \to \cdots \to C_0 \to \mathbf{Z} \to 0$$

with all C_i cohomologically trivial. The cohomology of $C^* = \mathrm{Hom}\,(C, \mathbf{Z})$ are either 0, or finite sums of $\mathbf{Z}$, trivial G-action. The result follows by splicing C^* into short exact sequences, and applying Tate cohomology and dimension-shifting. Q.E.D.

2. **The exact sequence, and applications.** Let Y be an n-manifold, G a finite group acting freely on Y. Suppose Y has precisely three nonzero homology groups, in dimensions 0, l, and n: $H_0(Y) = \mathbf{Z}$, $H_l(Y) \neq 0$, $H_n(Y) = \mathbf{Z}'$. [Note that these are G-modules. If G has no subgroup of index 2, $\mathbf{Z}' = \mathbf{Z}$ with trivial action. In any case, G acts trivially on $H_0(Y)$.]

PROPOSITION 2.1. *The sequence* (2.2) *is exact* ($i \in \mathbf{Z}$):

$$\begin{aligned} \cdots \to \hat{H}^{i+n-l}(G, \mathbf{Z}') \to \hat{H}^{i-l-1}(G, \mathbf{Z}) \to \hat{H}^i(G, H_l(Y)) \\ \to \hat{H}^{i+1+n-l}(G, \mathbf{Z}') \to \cdots . \end{aligned} \tag{2.2}$$

Proof. By Corollary 1.4, Y has a finite G-complex C:

$$0 \to \mathbf{Z}' \to C_n \to \cdots \to C_l \to \cdots \to C_0 \to \mathbf{Z} \to 0,$$

exact except at C_l, where the homology is $H_l(Y)$. Each C_i is cohomologically trivial. Splice

$$(*) \qquad 0 \to \mathbf{Z}' \to C_n \to \cdots \to C_{l+1} \to B_l \to 0 \quad \text{exact},$$

$$(**) \qquad 0 \to Z_l \to C_l \to \cdots \to C_0 \to Z \to 0 \quad \text{exact}.$$

$0 \to B_l \to Z_l \to H_l(Y) \to 0$. The long exact sequence implies

$$(\bigstar) \qquad \cdots \to \hat{H}^i(G, B_l) \to \hat{H}^i(G, Z_l) \to \hat{H}^i(G, H_l(Y)) \to \hat{H}^{i+1}(G, B_l) \to \cdots,$$

(*), (**) imply $\hat{H}^i(G, B_l) \approx \hat{H}^{i+n-l}(G, \mathbf{Z})'$, $\hat{H}^i(G, Z_l) \approx \hat{H}^{i-l-1}(G, \mathbf{Z})$. Substituting these values in ($\bigstar$) gives the result. Q.E.D.

COROLLARY 2.3. *In the above situation, if G preserves orientation, and if $H_l(Y)$ is cohomologically trivial, then G is periodic of period $n+1$.*

Proof. For then $\mathbf{Z}' = \mathbf{Z}$, and $\hat{H}^{n+1}(G, \mathbf{Z}) \approx \hat{H}^0(G, \mathbf{Z}) \approx \mathbf{Z}/g\mathbf{Z}$, $g = |G|$. By [3, Chapter XII, §11], done. (This implies n is odd, $\Rightarrow n = 3$.)

REMARK 1. Poincaré duality and universal coefficient theorem imply either $n=2l$ or $n=3$, $l=1$.

REMARK 2. If Y is a cell complex and if G acts freely, cellularly, then the cells afford a finite G-complex for Y. Hence the argument goes through and Proposition 2.1 holds in this case also.

Case 1. $Y=$ a closed compact surface. $n=2$, $l=1$. Therefore (2.2) is

$$\cdots \to \hat{H}^i(G, \mathbf{Z}) \to \hat{H}^{i-3}(G, \mathbf{Z}) \to \hat{H}^{i-1}(G, H_1(Y)) \to \hat{H}^{i+1}(G, \mathbf{Z}) \to \cdots.$$

Suppose G preserves orientation. Then $\mathbf{Z}'=\mathbf{Z}$. Set $i=1$. As

$$H^2(G, \mathbf{Z}) \approx \operatorname{Hom}(G/[G, G], \mathbf{Q}/\mathbf{Z}) = (G/[G, G])^{\wedge} \quad (\approx G/[G, G]),$$

and $H^1(G, \mathbf{Z})=0$, $\hat{H}^{-i}(G, \mathbf{Z}) \approx \hat{H}^i(G, \mathbf{Z})$, we get $\hat{H}^{-2}(G, \mathbf{Z}) \approx H_1(G, \mathbf{Z}) \approx G/[G,G]$,

$$0 \to G/[G, G] \to \hat{H}^0(G, H_1(Y)) \to (G/[G, G])^{\wedge} \to 0. \tag{2.4}$$

Sad to say, this striking result seems almost useless. The sequence is not split in general. If $Y=S^1 \times S^1=$ torus, then $H_1(Y)=\mathbf{Z} \oplus \mathbf{Z}$ and using the Lefschetz formula it is easy to see that G acts trivially on $\mathbf{Z} \oplus \mathbf{Z}$ (see below). Thus

$$\hat{H}^0(G, \mathbf{Z} \oplus \mathbf{Z}) \approx \mathbf{Z}_g \oplus \mathbf{Z}_g, \qquad g = |G|.$$

This implies $|G/[G, G]|=g$, implying $[G, G]=1$, i.e. G is abelian. Thus our sequence (2.4) is just $0 \to G \to \mathbf{Z}_g \oplus \mathbf{Z}_g \to G \to 0$. Hence $G=$ sum of two cyclic groups. If $G=\mathbf{Z}_n \oplus \mathbf{Z}_n$ with obvious action on $S^1 \times S^1$, then $g=n^2$ and $0 \to \mathbf{Z}_n \oplus \mathbf{Z}_n \to \mathbf{Z}_{n^2} \oplus \mathbf{Z}_{n^2} \to \mathbf{Z}_n \oplus \mathbf{Z}_n \to 0$ is not split. We have proved:

PROPOSITION 2.5. *If G acts freely, preserving orientation, on the torus, then G is the sum of two cyclic groups.*

REMARK 1. This is an easy exercise in covering spaces. That is the way L. E. J. Brouwer first proved it in 1919 [2].

REMARK 2. It is simple to prove that any finite group acts freely on a 2-sphere with handles Y (see [1]). In case G preserves orientation, if $r=$ the number of handles, then $H_1(Y) \approx \mathbf{Z}^{2r}$, $r \equiv 1$ (g), by Hurwitz formula, and $\operatorname{tr} g=2$ for all $g \in G-\{1\}$, by Lefschetz formula.

REMARK 3. Using the Swan spectral sequence [13], one can explicitate the maps in (2.2) via the multiplication structure. Thus, in the case of a surface, the map $\hat{H}^i(G, \mathbf{Z}) \to \hat{H}^{i-3}(G, \mathbf{Z})$ is cup product with a class $\sigma \in \hat{H}^{-3}(G, \mathbf{Z})$. As we have no application for the product structure, we omit details.

REMARK 4. In case of a surface, (2.2) was noted (independently) by Kawada and Tate [6].

Case 2. $Y=S^n \times S^n$. (i) Suppose n odd. *Claim*: G acts trivially on $H_n(Y)=\mathbf{Z} \oplus \mathbf{Z}$ (if G preserves orientation). For, if $g \in G$, $g \neq 1$, $0=\operatorname{tr}_0 g-\operatorname{tr}_n g+\operatorname{tr}_{2n} g=1-\operatorname{tr}_n g+1$, by Lefschetz formula, implying $\operatorname{tr}_n g=2$. $g^r=1$, some r implies $\det g= \pm 1$. Suppose $\det g=1$. Then g has characteristic polynomial $g^2-2g+1=(g-1)^2=0$, implying

g has a nonzero fixed point, $\phi=(a, b)$, say. We assume g.c.d. $(a, b)=1$. If $ac+bd=1$, then $\psi=(c, d)$ is a complement for ϕ. In the $\{\psi, \phi\}$ basis,

$$g \sim \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}.$$

$\operatorname{tr} g=2 \Rightarrow \alpha=1$. $g^r=1 \Rightarrow r\beta=0 \Rightarrow \beta=0$. That is,

$$g \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e. g acts trivially.

Thus $\det g=1$ implies g acts trivially. If $\det g=-1$, then $\det g^2=1 \Rightarrow g^2=1$ (as transformation). The characteristic polynomial is $g^2-2g-1=0$, $g^2=1 \Rightarrow g=0$. Contradiction. Hence $\det g=1$ always, so G acts trivially as asserted.

(2.2) now becomes

$$\text{(2.6)} \qquad \begin{aligned} \cdots \to H^i(G, Z \oplus Z) \to H^{i+n+1}(G, Z) \to H^{i-n}(G, Z) \\ \to H^{i+1}(G, Z \oplus Z) \to \cdots. \end{aligned}$$

COROLLARY 2.7. *Suppose n odd, and G acts freely on $S^n \times S^n$, p.o.* (*preserving orientation*). *Then*

$$0 \to H^n(G, Z) \to H^{n+1}(G, Z) \to Z_g \oplus Z_g \to H^{n+1}(G, Z) \to H^n(G, Z) \to 0$$

is exact. (*Set $i=0$ in* (2.6). *We have identified $H^{-m}(G, Z) \approx H^m(G, Z)$.*)

PROPOSITION 2.8 (CONNER). *$Z_p \oplus Z_p \oplus Z_p$ cannot act freely, p.o. on $S^n \times S^n$.*

Proof. We will see below that $1, Z_2, Z_4, Z_2 \oplus Z_2$ are the only groups acting freely on $S^n \times S^n$, for n even. If n odd, apply Corollary 2.7. $g=p^3$. We know that $pH^*(Z_p \oplus Z_p \oplus Z_p, Z)=0$ in positive dimensions implies we have $Z_p^a \to Z_{p^3} \oplus Z_{p^3} \to Z_p^b$, exact, some a, b. Contradiction. Q.E.D.

THEOREM A. *If n odd, p odd prime, $p \nmid n+1$ and if G is a p-group acting freely on $S^n \times S^n$, then G is abelian.*

Proof. It suffices to show that G contains no minimal nonabelian p-groups. Rédei [11] found these to be of two sorts (p odd).

Type 1. $G=(A, B\colon A^{p^u}=B^{p^v}=1, B^A=B^{1+p^{v-1}})$, $u \geqq 1$, $v \geqq 2$.

Type 2. $G=(A, B\colon A^{p^u}=B^{p^v}=C^p=1, AC=CA, BC=CB, B^A=BC)$, $u \geqq v \geqq 1$.

A group G of Type 1 is a split extension: $1 \to Z_{p^v} \to G \to Z_{p^u} \to 1$ (*). Apply [7, Proposition 5.1], with $r=p^v$, $s=p^u$, $t=1+p^{v-1}$, $a=(n+1)/2$. Corollary 2.7 implies $\alpha_{n+1}=g\alpha_n$. Here $g=p^{u+v}$, $n+1=2a$, $n=2a-1$. By [7, Proposition 5.1]

$$H^{2a}(G, Z) \approx Z_s \oplus \sum_{1 \leqq i < a} Z_{q_i} \oplus Z_{h_a}, \qquad H^{2a-1}(G, Z) \approx \sum_{1 \leqq i \leqq a-1} Z_{q_i},$$

where $h_i=(t^i-1, r)$, $k_i=(\sum_{j=0}^{s-1} t^{ij}, r)$, $q_i=h_ik_i/r$. Thus $\alpha_{2a}=sh_a\alpha_{2a-1}\Rightarrow sh_a=p^{u+v}$, or $h_a=p^v$. But calculation shows

$$\begin{aligned} h_i &= p^{v-1} \quad p\nmid i, \\ &= p^v \qquad p\mid i. \end{aligned}$$

As $p\nmid a$, we have a contradiction. Hence no Type 1 group can act freely on $S^n\times S^n$ if $p\nmid n+1$.

If G is a Type 2 group acting freely on $S^n\times S^n$, by Proposition 2.8, $Z_p\oplus Z_p \oplus Z_p \not\subset G$. If $u>1$, then $A^{p^{k-1}}\neq 1\in Z(G)$, so that the group $\langle C, B, A^{p^{k-1}}\rangle$ is abelian and contains a subgroup $\approx Z_p\oplus Z_p\oplus Z_p$. Contradiction. Hence $u=1, \Rightarrow v=1\Rightarrow G$ is the group of order p^3 treated in [7, Section 6]. By [7, Corollary 6.27] (with $n+1$ in place of n), we see $pH^{n+1}(G, Z)=0$. $g=p^3$, so Corollary 2.7 implies $Z_p^a\to Z_{p^3}\oplus Z_{p^3}\to Z_p^a$ exact. Contradiction. Thus no Type 2 group can act freely on $S^n\times S^n$ if $p\nmid n+1$. This proves Theorem A. Q.E.D.

PROPOSITION 2.9. *Suppose n odd, G acting freely, p.o., on $S^n\times S^n$. If G is periodic, then period* $(G)\mid n+1$.

Proof. Corollary 2.7 implies that $\alpha_{n+1}=\alpha_n g$. As $\alpha_n=1$, this implies $\alpha_{n+1}=g$. With $i=n$ in (2.6) we get: $0\to H^0(G, Z)\to H^{n+1}(G, Z)\oplus H^{n+1}(G, Z)$. As $H^0(G, Z)\approx Z_g$, this means that $H^{n+1}(G, Z)\approx Z_g\Rightarrow G$ is periodic, and $pd(G)\mid n+1$ (by [3, Chapter XII, Proposition 11.1]).

THEOREM B. *Let $n\equiv 1$ (4). Then a 2-group acting freely, p.o. on $S^n\times S^n$ is abelian.*

Proof. Rédei found three kinds of minimal nonabelian 2-groups: Type 1, Type 2 (as above) and the quaternion group Q of order 8. Let G be our 2-group. If $Q\subset G$, then as period $Q=4$, Proposition 2.9 implies $4\mid n+1$, or $n\equiv -1$ (4). Contradiction. Hence $Q\not\subset G$. A 2-group of Type 2, not containing $Z_2\oplus Z_2\oplus Z_2$, must be dihedral of order 8, D. A glance at Even's result [7, (4.0)], shows that $2H^{n+1}(D, Z)=0$, so Corollary 2.7 again gives a contradiction, implying $D\not\subset G$. If a Type 1 group $\supset G$, then as before, $h_a=2^v$. $a=(n+1)/2$. But $4\nmid n+1$ implies a is odd $\Rightarrow h_a=2^{v-1}$. Contradiction. Hence G must be abelian. Q.E.D.

It is well known that if G acts freely on S^n, n odd (necessarily p.o., by Lefschetz formula), then G is periodic, and $pd(G)\mid n+1$. This shows that if G acts freely on S^n, S^m such that g.c.d. $(n+1, m+1)=2$, then $pd(G)=2$ implies G cyclic, [since $\hat{H}^0(G, Z)\approx\hat{H}^2(G, Z)$ says $Z_g\cong G/[G, G]\Rightarrow[G, G]=1$, $Z_g\cong G$].

CONJECTURE 1. If G acts freely, p.o., on $S^n\times S^n$, and $S^m\times S^m$, with $(n+1, m+1)=2$, then G is abelian (therefore the sum of two cyclic groups, by Proposition 2.8).

I have only been able to prove the anemic

PROPOSITION 2.10. *If G acts freely, p.o. on $S^n\times S^n$ and $S^{n+2}\times S^{n+2}$, (n odd), then G is abelian, hence the sum of two cyclic groups.*

Proof. The jumps in (2.6) are $\cdots +(n+1), +(n+1), -(2n+1), +(n+1), +(n+1)\cdots$. Writing indices in place of the groups in our sequences, we get $1, n+2, -n+1, (2)^2, n+3, -n+2, (3)^2, n+4, -n+3, \ldots, -3, (n-2)^2, 2n-1, -2, (n-1)^2, 2n, -1$. Recall $\alpha_1 = \alpha_{-1} = 1$. Thus for $n=3$, get $(\cdot)$ $\alpha_6 = \alpha_2\alpha_5$. For $n \geqq 5$

$$\text{(i)} \qquad \alpha_2\alpha_4\cdots\alpha_{n-1}\alpha_{n+2}\cdots\alpha_{2n-1} = \alpha_3\alpha_5\cdots\alpha_{n-2}\alpha_{n+3}\cdots\alpha_{2n}.$$

Corollary 2.7 implies $(\colon)$ $\alpha_{n+1} = g\alpha_n$. $-1, (n)^2, 2n+1, 0, (n+1)^2, 2n+2, 1$ gives $(\colon)$ $\alpha_{2n+2} = g\alpha_{2n+1}$. (i) for n and $n+2$ yields

$$\frac{\alpha_2\alpha_4\cdots\alpha_{n+1}\alpha_{n+3}\cdots\alpha_{2n+1}\alpha_{2n+3}}{\alpha_2\alpha_4\cdots\alpha_{n-1}\alpha_{n+2}\cdots\alpha_{2n-1}} = \frac{\alpha_3\cdots\alpha_n\alpha_{n+5}\cdots\alpha_{2n+4}}{\alpha_3\cdots\alpha_{n-2}\alpha_{n+3}\cdots\alpha_{2n}} \qquad (n \geqq 3)$$

implying

$$(*) \qquad \frac{\alpha_{n+1}\alpha_{2n+1}\alpha_{2n+3}}{\alpha_{n+2}} = \frac{\alpha_n\alpha_{2n+2}\alpha_{2n+4}}{\alpha_{n+3}}.$$

For $n+2$, equation $(\colon)$ implies $\alpha_{n+3} = g\alpha_{n+2}$. Therefore $g^2\alpha_{2n+1}\alpha_{2n+3} = \alpha_{2n+2}\alpha_{2n+4}$, implying $g\alpha_{2n+3} = \alpha_{2n+4}$. Consider

$$0 \to H^{n+2}(G, Z \oplus Z) \to H^{2n+3}(G, Z) \to H^2(G, Z)$$

$$\begin{tikzcd}[column sep=small] \to H^{n+3}(G, Z \oplus Z) \arrow[dr] \arrow[rr] & & H^{2n+4}(G, Z), \\ & Y \arrow[ur] \arrow[dr] & \\ 0 \arrow[ur] & & 0 \end{tikzcd}$$

$\alpha_{n+2}^2\alpha_2|Y| = \alpha_{2n+3}\alpha_{n+3}^2 = \alpha_{2n+3}g^2\alpha_{n+2}^2$, or $\alpha_2|Y| = g^2\alpha_{2n+3} = g\alpha_{2n+4}$, or $\alpha_2/g = \alpha_{2n+4}/|Y| \geqq 1$. Hence $\alpha_2 \geqq g$. But $\alpha_2 = |G/[G, G]| \leqq g \Rightarrow \alpha_2 = g \Rightarrow G$ abelian, as desired. Q.E.D.

PROPOSITION 2.11. *If G acts freely on S^n, n odd, and g.c.d. $(g, n+1) = 1$ or 2, then G is abelian.*

Proof. We know G periodic, $pd(G) \mid n+1$, $G_p =$ cyclic, (p odd), cyclic or generalized quaternion, ($p=2$) [3, Chapter XII].

Case 1. $g = |G|$ is odd: By Swan's formula for p-periods [7, Theorem 3.1], as now $(g, n+1) = 1$, and $pd(G) = \text{l.c.m. } 2|N(G_p) : C(G_p)| \mid n+1$, we must have $N(G_p) = C(G_p)$, for all prime $p|g$. G is a Z-group, therefore by [15] is of form $1 \to Z_r \to G \to Z_s \to 1$ $(r, s) = 1$, and therefore, split, by Schur-Zassenhaus. If $p|r$, then $(Z_r)_p = G_p$ is characteristic in Z_r, and hence normal in G. That is, $N(G_p) = G$ implies $G_p \subset Z(G)$, so G is cyclic.

Case 2. g even: Here $(g, n+1) = 2$ (by hypothesis). As generalized quaternion groups have period 4, [3, Chapter XII, Section 11] we must have $G_2 =$ cyclic. So G is a Z-group such that every subgroup of odd order is cyclic.

Much as in Case 1, it follows that G is cyclic (use Milnor's Theorem: If a group acts freely on S^n, then every involution lies in the center [10]). Q.E.D.

REMARK. Milnor's Theorem cannot be dispensed with. For if, say, Σ_3 acted freely on S^3, the hypotheses of Proposition 2.11 would hold: $(6, 4)=2$.

PROPOSITION 2.12. *If n is even, >0, only 1 and Z_2 act freely, p.o. on $S^n \times S^n$; only $1, Z_2, Z_2 \oplus Z_2, Z_4$ act freely on $S^n \times S^n$.*

Proof. Suppose G acts freely, p.o. If $g \in G-\{1\}$, then $\operatorname{tr}_0 g+\operatorname{tr}_n g+\operatorname{tr}_{2n} g =2+\operatorname{tr}_n g=0$ or $\operatorname{tr}_n g=-2$, on $H_n(S^n \times S^n)=\boldsymbol{Z} \oplus \boldsymbol{Z}$. If $g^p=1$, p odd, then $\det g=1 \Rightarrow$characteristic polynomial is $g^2+2g+1=(g+1)^2=0 \Rightarrow g\phi=-\phi$, some $\phi \neq 0$, $g^p=1 \Rightarrow$contradiction. Hence G is a 2-group. If $g \in G$ has order 4, $\det g=\pm 1$ $\Rightarrow \det g^2=1$. As above, this gives $(g^2+1)^2=0$, or $g^4+2g^2+1=2+2g^2=0$, or $g^2=-1$. The characteristic polynomial of g is $g^2+2g+\det g=-1+\det g+2g=0$. $\det g=1 \Rightarrow g=0$, no go. If $\det g=-1$, then $g=1$, contradicting $g^2=-1$. Hence G has only elements of order 2. Therefore G is abelian elementary. $Z_2 \oplus Z_2 \oplus Z_2 \not\subset G$, implying $G=1, Z_2$ or $Z_2 \oplus Z_2$. Suppose $G=Z_2 \oplus Z_2=\{1, x, y, xy\}$. $\operatorname{tr} x=\operatorname{tr} y =\operatorname{tr} xy=-2$. If $N=1+x+y+xy$ were zero, taking traces implies $2-2-2-2=0$. Contradiction. Hence $\psi=N\psi_0 \neq 0$, some ψ_0. Clearly $x\psi=\psi$. Now $x^2+2x\pm 1=0$ and $x^2=1$, hence $x=0$ or $x=-1$. The first is nonsense, and the second implies $-\psi=\psi \Rightarrow \psi=0$. Contradiction! Hence only $G=1$ or Z_2 is permissible. Thus if G reverses orientation, only $1, Z_2, Z_4, Z_2 \oplus Z_2$ are permissible. Q.E.D.

Note. It is easy to see that each of these can occur.

PROPOSITION 2.13. *Let $H \lhd G$, $V=$fix-point free complex representation of H. Then $\boldsymbol{C}G \otimes_{\boldsymbol{C}H} V$, a G-representation, is again H-free.*

Proof. Trivial.

Using Proposition 2.13, we can construct free actions on $S^{2n-1} \times S^m$ as follows: Let $1 \to H \to K \to 1$, exact, and suppose H acts freely on $\boldsymbol{C}^a$. Then H acts freely on $\boldsymbol{C}G \otimes_H \boldsymbol{C}^a \approx \boldsymbol{C}^{la}$, $l=|G:H|$. We may suppose the action of G is unitary. Therefore G acts on S^{2al-1} such that H acts freely. If now K acts freely on S^m, so G acts freely on $S^{2al-1} \times S^m$ via $g(z, w)=(gz, \bar{g}w)$. This procedure immediately gives many free actions of nonabelian groups on $S^n \times S^n$, n odd. In particular, any metacyclic group so acts (suitable n). We call these *semiproduct actions.*

EXAMPLE. G of Type I: $1 \to Z_{p^2} \to G \to Z_p \to 1$ acts freely on $S^{2p-1} \times S^{2p-1}$. Therefore, the condition of Theorem A is *necessary.*

A nonsemiproduct action of Z_r on $S^{2n-1} \times S^{2n-1}$, which is fix-point free, is as follows: Consider $S^{2n-1} \subset \boldsymbol{C}^n$, the unit sphere. Let $\zeta=e^{2\pi i/r}$. If $z, w \in S^{2n-1}$, set $T_w(z)=z+(\zeta-1)(z, w)w$. T_w is unitary for fixed w. Indeed, $T_w=$identity on $\boldsymbol{C}w$, $T_w=$multiplication by ζ on $\boldsymbol{C}w$. Thus, $T_w^r=1$. Moreover, $T_{\zeta w}=T_w$.

Define $T\colon S^{2n-1} \times S^{2n-1} \to S^{2n-1} \times S^{2n-1}$ by $T(z, w)=(T_w(z), \zeta w)$. Clearly $T^n=1$, and T is fix-point free. If $S(z, w)=(\zeta z, w)$, then $ST=TS$, $S^r=1$, and $\langle S, T\rangle \approx Z_r \oplus Z_r$ acts freely on $S^{2n-1} \times S^{2n-1}$.

In conclusion, we discuss the following

CONJECTURE II: If G acts freely on $S^n \times S^n$, and $Z_2 \oplus Z_2 \subset Z(G)$, then $(Z_2 \oplus Z_2) \cap Z(G) \neq 1$. This has several points in its favor.

(1) It holds for any extension $G: 1 \to H \to G \xrightarrow{\pi} K \to 1$, where H acts freely on S^a, K acts freely on S^b. For let $L = Z_2 \oplus Z_2 = \{1, x, y, xy\}$. H acts freely on S^a implies $L \not\subset H$. If $L \cap H = 1$, then $L \approx \pi(L) \subset K$. Contradiction. Therefore $H \cap L = \{1, x\}$, say. By Milnor's Theorem $x \in Z(H)$ and is the only involution in H. As $\{1, x\}$ is characteristic in H, it is normal in G, hence $x \in Z(G)$. So Conjecture II holds in this case. In particular, it holds for semiproduct actions. A weaker statement, II′, implies II:

(2) II′: If $L = \{1, x, y, xy\} \subset G$, then $G = C(x) \cup C(y) \cup C(xy)$. Indeed, suppose II′ holds. If $L \cap Z(G) = 1$, we can fix $g, h \in G$ such that $gx = xg$, $gy \neq yg$, $hx \neq xh$, $hy = yh$ ($\Rightarrow gh \neq 1$). Consider gh. If $ghx = xgh$, then $hx = xh$; if $ghy = ygh$, then $hy = gh$; contradictions. Hence $ghxy = xygh$, or $gyhx = ygxh$, implying $g^{-1}ygy = xhxh^{-1} = u \neq 1$. Clearly $xu = ux$, $yu = uy$. And $u \notin L$ is evident. $xu = ux$ says that $x(xhxh^{-1}) = xhxh^{-1}x$, or $hxh^{-1}x = xhxh^{-1}$, implying $u^2 = 1$. Thus $L \oplus \langle u \rangle \approx Z_2 \oplus Z_2 \oplus Z_2 \subset G$. Contradiction. Thus $L \cap Z(G) \neq 1$, i.e. II′ implies II.

(3) II holds if G is an extension $1 \to G_0 \to G \to Z_2 \to 1$, such that G_0 is abelian. (Exercise.) Thus in particular, II holds for $S^1 \times S^1$.

(4) II holds if G is a 2-group. (Exercise.)

(5) II implies Milnor's Theorem. For if G acts freely on S^n, $u \in G$, involution, then $Z_2 \oplus Z_2 \subset G \oplus G$ acting freely on $S^n \times S^n$. And $Z(G \oplus G) = Z(G) \oplus Z(G)$. Therefore II implies $Z_2 \subset Z(G)$.

REMARK. The group $G = (a, b: a^8 = b^2 = 1, b^{-1}ab = a^3)$ acts freely on $S^3 \times S^3$. $L = \langle a^4, b \rangle \approx Z_2 \oplus Z_2$ is contained in G, but $L \ntriangleleft G$.

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