ON A CLASS OF STOCHASTIC PROCESSES AND ITS RELATIONSHIP TO INFINITE PARTICLE GASES(1)

BY

D. PAUL JOHNSON

Introduction. Consider the right continuous sample paths $w\colon [0, +\infty) \to E = \{+1, -1\}$ with coordinates $x(t) = x_t = x_t(w) \in E$ together with the σ-algebras $\mathscr{M}_t$ generated by $x(s)$, $s \leqq t$, and assume that for each probability measure f on E and $e \in E$, there exist probability measures P_f and $P_{f|e}$ on $\mathscr{M}_\infty$ for which

(0.A) $P_{f|e}(\cdot) = P_f(\cdot \mid x(0) = e)$,

(0.B) $P_f(x(0) = e) = f(e)$,

(0.C) $P_{f|e}(x(t+h) \in A \mid \mathscr{M}_t) = P_{f_t|x_t}(x(h) \in A)$, [a.e. $P_{f|e}$],

where A is a set of points in E and $f_t(A) = P_f(x(t) \in A)$. Such a stochastic process will be called a K-process. The expression $P_{f|e}(\Lambda)$ is to be thought of as the probability that, starting with $x(0)$ distributed according to f, the event Λ will take place conditional on $x(0) = e$. A K-process is a temporally homogeneous Markov process if and only if $P_{f|e}$ is independent of f, as the reader can easily check. If

$$\gamma_e(u) = \frac{d}{dt} P_{f|e}(x(t) = 1)\bigg|_{t=0}, \quad u = f(+1),$$

then when γ_{+1} and γ_{-1} are real analytic on the closed interval [0, 1], they uniquely determine the distribution of the K-process $x(t)$.

In this paper, I shall construct a model of an infinite particle gas with velocities ± 1 in which the motion of a tagged particle is a K-process with specific $\gamma_{\pm 1}$ and in which the sample paths of any two particles are independent. This will be accomplished by constructing a gas of n like particles, each of which has velocities ± 1, and then letting $n \to \infty$. Each of the n particle gases will be a Markov jump process in which one waits an exponential holding time and then picks an index i according to the uniform distribution $1/n$ and lets the corresponding particle collide with one or more of the remaining particles. The effect of a collision between a single particle and a set of particles will be a change of state only for the single particle. H. P. McKean [3] has carried out this construction for the case when $\gamma_{\pm 1}(u) = \pm(u-1)$. In this paper, we generalize his results to the case when $-\gamma_{+1}$

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and γ_{-1} are both positive on the open interval (0, 1) and real analytic on the closed interval [0, 1].

The paper is arranged as follows. In §1 we derive a few basic properties of K-processes, including the fact that the functions γ_{+1} and γ_{-1} uniquely determine the distribution of the process. In §2, we define the n-molecule gases. In §3, we introduce the basic notions of convergence and equivalence that will be used throughout the paper, and we calculate the limit as $n\to\infty$ of the generators of the n-molecule gases. In §4 we state the main theorems and finally give their proofs in §5.

1. K-processes.

Let x_t, $t\in[0,\infty)$ be a K-process. Then, defining

$$P_{f|e}(t;A) = P_{f|e}(x_t\in A),$$

we get a formula for the probabilities of joint observations reminiscent of the case of temporally homogeneous Markov processes.

THEOREM 1. *If x_t is a K-process, then for* $0<t_1<\cdots<t_n<\infty$,

$$P_{f|e}[x(t_1)\in A_1,\ldots,x(t_n)\in A_n] = \int_{A_1} P_{f|e}(t_1;d\xi_1)\int_{A_2} P_{f_{t_1}|\xi_1}(t_2-t_1;d\xi_2)\cdots\int_{A_n} P_{f_{t_{n-1}}|\xi_{n-1}}(t_n-t_{n-1};d\xi_n).$$

Proof. This is immediate from (0.C).

COROLLARY. *If x_t is a K-process, then*

$$P_{f|e}(s+t;A) = \int P_{f|e}(t;d\xi)P_{f_t|\xi}(s;A).$$

If $\gamma(u) = u\gamma_{+1}(u)+(1-u)\gamma_{-1}(u)$, then we have the following theorem.

THEOREM 2. *If x_t is a K-process and if $P_{f|e}(t;+1)$ is differentiable in $t\geqq 0$, then*

$$\frac{d}{dt}f_t(+1) = \gamma[f_t(+1)], \tag{2.A}$$

$$\frac{d}{dt}P_{f|e}(t;+1) = P_{f|e}(t;+1)\gamma_{+1}[f_t(+1)]+P_{f|e}(t;-1)\gamma_{-1}[f_t(+1)]. \tag{2.B}$$

Proof. Taking the equation in the Corollary to Theorem 1 and differentiating both sides with respect to s and letting $s=0$, we get (2.B). (2.A) follows from (2.B) if we notice that

$$f_t(+1) = \int f(de)P_{f|e}(t;+1).$$

Equation (2.A) of Theorem 2, which is in general nonlinear, has a unique solution bounded by 0 and 1 if γ is real analytic in the closed interval [0, 1] and if $\gamma(0)\geqq 0$ and $\gamma(1)\leqq 0$. Once the solution of (2.A) is known, (2.B) becomes a linear differential equation for $P_{f|e}(t;+1)$. This equation, in turn, has a unique solution bounded by 0 and 1 if γ_{+1} and γ_{-1} are continuous and $\mp\gamma_{\pm} \geqq 0$. Having uniquely determined the transition function $P_{f|e}$, we can construct the K-process by defining probabilities on the cylinder sets in the manner suggested in Theorem 1:

$$P[x(t_1) \in A_1, x(t_2) \in A_2, \ldots, x(t_n) \in A_n]$$
$$= \int_{A_1} P_{f|e}(t_1; d\xi_1) \int_{A_2} P_{f_{t_1}|\xi_1}(t_2 - t_1; d\xi_2) \cdots \int_{A_n} P_{f_{t_{n-1}}|\xi_{n-1}}(t_n - t_{n-1}; d\xi_n),$$

where $0 < t_1 < \cdots < t_n < \infty$. Thus to each pair of functions $-\gamma_{+1}$ and γ_{-1} which are positive on the open interval (0, 1) and real analytic on the closed interval [0, 1], there corresponds a unique K-process.

2. ***n*-molecule gases.** To define the n-molecule gases; let E be the set of integers ± 1, let L be a fixed positive constant and let $C_N^e(k)$, $e \in E$ and $k \leqq N$ be a family of nonnegative real numbers with the property that when $C_N = \max_{k \leqq N, e = \pm 1} C_N^e(k)$ one has

$$\sum_{N=1}^{\infty} N^P C_N \leqq P!\, L^P, \qquad P \geqq 1.$$

We construct the n-molecule gases as follows. Let $X_n(t) = [x_1^n(t), \ldots, x_n^n(t)]$ be a Markov jump process on the n-dimensional space E^n with holding time distribution, in the state $[e_1, \ldots, e_n]$, equal to

$$\exp\left[-t \sum_{N=1}^{n-1} n^{-N} \sum_{i, j_1, \ldots, j_N}^{(n)} C(e_i | e_{j_1}, \ldots, e_{j_N})\right]$$

where $\sum_{j_0, j_1, \ldots, j_N}^{(n)}$ denotes the sum taken over all sequences $(j_0, \ldots, j_N)$, $1 \leqq j_k \leqq n$, $j_k \neq j_p$ when $k \neq p$; and where $C(e|e_1, \ldots, e_N) = C_N$ (number of $+1$'s in the set $e_1, \ldots, e_N$). Starting at the state $[e_1, \ldots, e_n]$, the probability that the first jump is to the state $[e_1, \ldots, e_{i-1}, -e_i, e_{i+1}, \ldots, e_n]$ will be given by

$$\frac{\sum_{N=1}^{n-1} n^{-N} \sum_{j_1, \ldots, j_N}^{(n)} C(e_i | e_{j_1}, \ldots, e_{j_N})}{\sum_{N=1}^{n-1} n^{-N} \sum_{k, j_1, \ldots, j_N}^{(n)} C(e_k | e_{j_1}, \ldots, e_{j_N})}.$$

The generator of the n-molecule gas will therefore be

$$G_n\phi(e_1, \ldots, e_n) = \sum_{N=1}^{n-1} n^{-N} \sum_{i, j_1, \ldots, j_N}^{(n)} C(e_i | e_{j_1}, \ldots, e_{j_N})\, \Delta_i\phi$$

where $\Delta_i\phi = \phi(\ldots, -e_i, \ldots) - \phi(\ldots, e_i, \ldots)$ or zero, depending on whether ϕ depends on the variable e_i or not.

To calculate the joint probabilities of M molecules in the gas, notice that if

$$T_t\phi(e_1, \ldots, e_n) = E[\phi(X_t^n) \mid X_0^n = (e_1, \ldots, e_n)]$$

is the semigroup associated with the Markov process X^n, then

$$T_t\phi(e_1,\dots,e_n) = \exp[tG_n]\phi(e_1,\dots,e_n).$$

Thus for any two functions ϕ and ψ on E^n and $\xi \in E^n$, we have

$$\begin{aligned} E[\phi(X_s^n)\psi(X_{s+t}^n) \mid X_0^n = \xi] &= E[\phi(X_s^n)E[\psi(X_{s+t}^n) \mid X_s^n] \mid X_0^n = \xi] \\ &= E[\phi(X_s^n)T_t\psi(X_s^n) \mid X_0^n = \xi] \\ &= T_s(\phi T_t\psi)(\xi) \\ &= \exp[sG_n]\phi \exp[tG_n]\psi. \end{aligned}$$

Using this argument, it is easily shown that if the molecules are initially independent and indentically distributed with distribution f and if $E_k = (e_1^k,\dots,e_M^k, E, E, \dots, E)$, then the joint distribution of the first M particles is given by

$$\begin{aligned} &P[X^n(t_1)\in E_1,\dots,X^n(t_M)\in E_M] \\ (1)\quad &= \int f(d\xi_1)\cdots f(d\xi_n)\exp[t_1G_n]\chi_{E_1}\exp[(t_2-t_1)G_n]\chi_{E_2}\cdots\exp[(t_M-t_{M-1})G_n]\chi_{E_M}, \end{aligned}$$

where χ_A is the indicator function of the set A.

3. **Preliminaries and notation.** We wish to calculate the limit of (1) as n goes to infinity. To do this, one might calculate $\lim_{n\to\infty} G_n\phi$ when ϕ has a finite number of variables. However, as is evident from (1), this is not necessary as we are only interested in the behavior of $G_n\phi$ modulo an integration. Instead, we introduce the following notion of equivalence and convergence.

DEFINITION 3. Let I be the set of all indices ij where i and j are nonnegative integers. Suppose that ϕ and ψ are functions whose variables are indexed by indices in I and suppose that $J\subset I$. Then we define

(3.A) If f is a probability measure on E, then

$$\int_J f^\infty\phi = \int\prod_{\alpha\in J} f(d\xi_\alpha)\phi.$$

(3.B) $\phi\equiv\psi \bmod J$ if and only if there exist functions ϕ_j and ψ_j, whose variables have indices in I, and one to one mappings δ_j of J onto J such that

$$\phi = \sum\phi_j,\quad \psi = \sum\psi_j$$

and $\phi_j^\delta = \psi_j$ where ϕ_j^δ is ϕ_j with the variables $e_{\delta(\alpha)}$ replacing e_α for $\alpha\in J$.

(3.C) $\|\phi\|$ is the sup norm of ϕ.

(3.D) $\|\phi\|_J = \inf_{\psi\equiv\phi \bmod J}\|\psi\|$.

(3.E) $\phi_n\to\phi \bmod J$ as $n\to\infty$ if and only if $\|\phi_n-\phi\|_J\to 0$ as $n\to\infty$.

One should note that if $\phi\equiv\psi \bmod J$, then

$$\int_J f^\infty\phi = \int_J f^\infty\psi.$$

Similarly, if $\phi_n \to \phi$ mod J, then

$$\int_J f^\infty \phi_n \to \int_J f^\infty \phi.$$

It is also easily verified that if $\phi_1 \equiv \psi_1$ mod J, $\phi_2 \equiv \psi_2$ mod J and ϕ has variables whose indices all lie in J^c, then

$$\phi_1 + \phi_2 \equiv \psi_1 + \psi_2 \mod J$$

and

$$\phi\phi_1 \equiv \phi\psi_1 \mod J.$$

These facts will be used throughout the paper without further comment.

Now suppose that ϕ is a function of M variables, M finite, which are indexed by an index set J of positive integers. We wish to evaluate $\lim_{n\to\infty} G_n\phi$ modulo J^c. Clearly we have

$$G_n\phi(e_1, \ldots, e_n) = \sum_{N=1}^{n-1} n^{-N} \sum_{i,j_1,\ldots,j_N}^{(n)} C(e_i | e_{j_1}, \ldots, e_{j_N})\, \Delta_i \phi$$

$$= \sum_{N=1}^{n-1} n^{-N} \sum_{i,j_1,\ldots,j_N; j_k \in J^c}^{(n)} C(e_i | e_{j_1}, \ldots, e_{j_N})\, \Delta_i \phi \tag{2}$$

$$+ \sum_{N=1}^{n-1} n^{-N} \sum_{i,j_1,\ldots,j_N; \text{some } j_k \in J}^{(n)} C(e_i | e_{j_1}, \ldots, e_{j_N})\, \Delta_i \phi. \tag{3}$$

Since we are only interested in evaluating these sums modulo J^c, we may rename the indices of variables which are not contained in J. Using the new indices $1j$, and adopting the convention that the indices $0i$ and i are the same, we find that the first sum (2) is equivalent to

$$\sum_{N=1}^{n-M} n^{-N}(n-M)\cdots(n-M-N+1) \sum_i C(e_{0i} | e_{11}, \ldots, e_{1N})\, \Delta_{0i}\phi \tag{4}$$

modulo J^c since

$$\sum_{i,j_1,\ldots,j_N; j_k \in J^c}^{(n)} C(e_i | e_{j_1}, \ldots, e_{j_N})\, \Delta_i \phi$$

$$\equiv (n-M)(n-M-1)\cdots(n-M-N+1) \sum_i C(e_{0i} | e_{11}, \ldots, e_{1N})\, \Delta_{0i}\phi \ (\text{mod } J^c),$$

$$M+N \leqq n,$$

$$\equiv 0 \text{ otherwise.}$$

As $n \to \infty$, (4) converges absolutely since it is clearly bounded by

$$2M\|\phi\| \sum_{N=1}^{\infty} C_N.$$

The second sum (3) is bounded by

$$2M^2\|\phi\|n^{-1}\sum_{N=1}^{\infty} NC_N \leqq 2M^2\|\phi\|n^{-1}L$$

and thus, using the dominated convergence theorem, we see that as $n\to\infty$, $G_n\phi(e_1,\dots,e_N)$ converges modulo J^c to

$$\sum_{N=1}^{\infty}\sum_{i} C(e_{0i}|e_{11},\dots,e_{1N})\,\Delta_{0i}\phi.$$

Now let the pairs ij of nonnegative integers be indices for variables $e_{ij}\in E$ and let ψ be a function of a subset of these variables. We shall call i the order of the index ij and the order of ψ will be defined as the maximum of the orders of its variables. Let ψ be a function of finite order less than or equal to p. Then we define

$$D_p\psi = \sum_{N=1}^{\infty}\sum_{i=0}^{p}\sum_{j=1}^{\infty} C(e_{ij}|e_{p+1,1},\dots,e_{p+1,N})\,\Delta_{ij}\psi.$$

Clearly the operator D_p introduces a new set of variables of order $p+1$. Remembering the convention that the indices $0i$ and i are the same, we have established for any function ϕ whose variables are indexed by a finite set of positive integers J that

$$G_n\phi \to D_p\phi \quad \mathrm{mod}\, J^c$$

for $p\geqq 0$.

In general, we will write $D\phi$ instead of $D_p\phi$ with the understanding that D always adds variables whose indices are of order at least one higher than the order of the function on which it is operating. If ϕ has infinite order, then we will always be able to write $\phi=\sum\phi_k$ where ϕ_k has finite order and then let $D\phi=\sum D\phi_k$. Finally, we let

$$\exp\,[tD]\phi = \sum_{n=0}^{\infty}\frac{t^n}{n!}D^n\phi.$$

4. **Main theorems.** Our first theorem is the following:

THEOREM 4. *If $\phi_1,\dots,\phi_p$ are bounded functions whose variables have indices in a finite set J of M indices, then for $8LM(t_1+\cdots+t_p)<1$ we have*

$$\exp\,[t_pG_n]\phi_p\cdots\exp\,[t_1G_n]\phi_1 \to \exp\,[t_pD]\phi_p\cdots\exp\,[t_1D]\phi_1 \quad \text{modulo } J^c,$$

where the expression on the right is well defined and finite.

Thus as $n\to\infty$, (1) converges to

$$\int f^{\infty}\exp\,[t_1D]\chi_{E_1}\exp\,[(t_2-t_1)D]\chi_{E_2}\cdots\exp\,[(t_M-t_{M-1})D]\chi_{E_M}.$$

This limiting distribution can be used to define a combined motion of M molecules for which:

I. the paths of any two molecules are independent for $16Lt<1$, and

II. the distribution of a tagged particle is that of a K-process with

$$\gamma_{\pm 1}(u) = \mp \sum_{N=1}^{\infty} \sum_{k=0}^{N} C_N^{\pm 1}(k)\binom{N}{k} u^k(1-u)^{N-k}.$$

Finally, we are able to show that

III. any k-process for which $-\gamma_+$ and γ_- are positive on the open interval (0, 1) and real analytic on the closed interval [0, 1] can be constructed as the motion of a tagged particle in an infinite particle gas as described above.

The proof of I is contained in the following theorem.

THEOREM 5. *If $a_\nu, b_\nu \in E$, $\nu=1,\ldots,m$ and if $0<t_1<\cdots<t_m$ are real and $16Lt_m<1$, then for $i\neq j$ and any initial probability measure f on E,*

$$\lim_{n\to\infty} P[x_i^n(t_\nu) = a_\nu,\ x_j^n(t_\nu) = b_\nu,\ 1 \leqq \nu \leqq m]$$
$$= \lim_{n\to\infty} P[x_i^n(t_\nu) = a_\nu,\ 1 \leqq \nu \leqq m] \lim_{n\to\infty} P[x_j^n(t_\nu) = b_\nu,\ 1 \leqq \nu \leqq m].$$

The proof of this and the following theorem is based on the fact that if ϕ and ψ have variables whose indices form disjoint sets whose union is J, then their product $\phi\psi$, written $\phi \otimes \psi$ when the indices of their variables are disjoint, is such that

$$D(\phi \otimes \psi) \equiv \phi \otimes D\psi + \psi \otimes D\phi \mod J^c.$$

Letting m be the maximum order of ϕ and ψ, this equation is easily extended to

$$(5)\quad D^p(\phi \otimes \psi) = \sum_{k=0}^{p} \binom{p}{k} (D_{p-1+m}\cdots D_{k+m}\phi) \otimes (D_{k-1+m}\cdots D_m\psi) \mod J^c$$

(see M. Kac [2] for the terminology and another instance of this phenomenon).

If we let

$$(6)\quad \begin{aligned} \chi_a^i(\ldots, e_i, \ldots) &= 1 \quad \text{if } e_i = a, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

$$f_t(e) = \lim_{n\to\infty} P[x_n^1(t) = e] = \int f^\infty \exp[tD]\chi_e^1$$

and

$$P_{f|a}(t, b) = \lim_{n\to\infty} P[x_n^1(t) = b \mid x_n^1(0) = a] = \int_{\{1\}^c} f^\infty \exp[tD]\chi_b^1(a, \ldots),$$

then II is proved in the following theorem which establishes the limiting motion of a fixed molecule in the gas as a K-process for which

$$\gamma_{\pm 1}(u) = \mp \sum_{N=1}^{\infty} \sum_{k=0}^{N} C_N^{\pm 1}(k)\binom{N}{k} u^k(1-u)^{N-k}.$$

THEOREM 6. *For* $0<t_1<\cdots<t_m$ *with* $8Lt_m<1$ *and* $e_0,\ldots,e_m\in E$, *we have*

$$\int_{\{1\}^c} f^\infty \exp[t_1D]\chi^1_{e_1}\cdots\exp[(t_m-t_{m-1})D]\chi^1_{e_m}(e_0)$$
$$= P_{f|e_0}(t_1,e_1)P_{f_{t_1}|e_1}(t_2-t_1,e_2)\cdots P_{f_{t_{m-1}}|e_{m-1}}(t_m-t_{m-1},e_m)$$

and

$$\gamma_{\pm 1}(u) = \frac{d}{dt}P_{f|\pm 1}(t;1)|_{t=0} = \mp\sum_{N=1}^{\infty}\sum_{k=0}^{N} C_N^{\pm 1}(k)\binom{N}{k}u^k(1-u)^{N-k}.$$

COROLLARY. *For* $8Lt<1$,

$$f_t(+1) = \int f^\infty \exp[tD]\chi^1_{+1}$$

is the solution of the differential equation

$$(d/dt)f_t(+1) = \gamma[f_t(+1)]$$

where $\gamma(u)=u\gamma_{+1}(u)+(1-u)\gamma_{-1}(u)$.

To prove III, we introduce the following definition.

DEFINITION 7. Let H be the class of functions γ, mapping [0, 1] into the real numbers, for which there exist positive real numbers $C_N(k)$ and L such that if $C_N=\max_{k\leqq N} C_N(k)$,

$$\gamma(u) = \sum_{N=1}^{\infty} B_N(u)$$

where

$$B_N(u) = \sum_{k=0}^{N} C_N(k)\binom{N}{k}u^k(1-u)^{N-k}$$

and

$$\sum_{N=1}^{\infty} N^pC_N \leqq p!\,L^p, \qquad p\geqq 1.$$

Thus if $-\gamma_{+1}$ and γ_{-1} are both in H, we can construct an associated K-process for small t by taking the limiting motion of one coordinate in an n-dimensional Markov chain as $n\to\infty$. The class H can be described more simply as follows.

THEOREM 8. $F\in H$ *if and only if* F *is positive on the open interval* (0, 1) *and real analytic on the closed interval* [0, 1].

The proofs of Theorems 4 through 8 follow.

5. **Proofs.**

Proof of Theorem 4.

LEMMA 9. *If* $\tau_k(i_1,\ldots,i_p)$ *is the number of integers* $i_1,\ldots,i_p$ *equal to* k *and if*

$$A_p = \sum_{i_p=1}^{p}\sum_{i_{p-1}=1}^{p-1}\cdots\sum_{i_1=1}^{1}\tau_p(i_1,\ldots,i_p)!\cdots\tau_1(i_1,\ldots,i_p)!,$$

then $A_p=1\cdot 3\cdot\cdots\cdot(2p-1)\leqq 2^pp!$.

Proof. Use induction on p. The lemma is certainly true for $p=1$. Suppose that it also holds for p. Then

$$
\begin{aligned}
A_{p+1} &= \sum_{i_{p+1}=1}^{p+1} \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \tau_{p+1}(i_1,\ldots,i_{p+1})! \cdots \tau_1(i_1,\ldots,i_{p+1})! \\
&= \sum_{i_{p+1}=1}^{p+1} \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \tau_{p}(i_1,\ldots,i_{p+1})! \cdots \tau_1(i_1,\ldots,i_{p+1})! \\
&= \sum_{i_{p+1}=1}^{p+1} \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \tau_{p}(i_1,\ldots,i_{p})! \cdots \tau_{i_{p+1}+1}(i_1,\ldots,i_{p})! \\
&\qquad \cdot [1+\tau_{i_{p+1}}(i_1,\ldots,i_p)]!\tau_{i_{p+1}-1}(i_1,\ldots,i_p)! \cdots \tau_1(i_1,\ldots,i_p)! \\
&= \sum_{i_{p+1}=1}^{p+1} \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} [1+\tau_{i_{p+1}}(i_1,\ldots,i_p)]\tau_p(i_1,\ldots,i_p)! \cdots \tau_1(i_1,\ldots,i_p)! \\
&= (p+1) \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \tau_p(i_1,\ldots,i_p)! \cdots \tau_1(i_1,\ldots,i_p)! \\
&\quad + \sum_{i_{p+1}=1}^{p+1} \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \tau_{i_{p+1}}(i_1,\ldots,i_p)\tau_p(i_1,\ldots,i_p)! \cdots \tau_1(i_1,\ldots,i_p)! \\
&= (p+1)A_p + \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \left[\sum_{i_{p+1}=1}^{p+1} \tau_{i_{p+1}}(i_1,\ldots,i_p) \right] \tau_p(i_1,\ldots,i_p)! \\
&\qquad \cdots \tau_1(i_1,\ldots,i_p)! \\
&= (p+1)A_p + \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} p\tau_p(i_1,\ldots,i_p)! \ldots \tau_1(i_1,\ldots,i_p)! \\
&= (p+1)A_p + pA_p = (2p+1)A_p = 1\cdot 3\cdot 5\cdot \cdots \cdot(2p-1)(2p+1)
\end{aligned}
$$

and the lemma is proved.

LEMMA 10. *If p, q and M are positive integers and $L \geqq 0$, then*

$$\sum_{N_1,\ldots,N_p} (M+N_1+\cdots+N_p)^q C_{N_p} \cdots C_{N_1} \leqq q!(2L)^q \exp[M/L]2^p.$$

Proof. If B_n^m equals the number of ways of picking $0 \leqq k_m \leqq \cdots \leqq k_1 \leqq n$, then $B_n^m = B_{n-1}^m + B_n^{m-1}$ and hence $B_n \leqq 2^{m+n}$. Thus

$$
\begin{aligned}
&\sum_{N_1,\ldots,N_p} (M+N_1+\cdots+N_p)^q C_{N_p} \cdots C_{N_1} \\
&\quad \leqq \sum_{k=0}^{q} \binom{q}{k} \left[\sum_{N_1,\ldots,N_{p-1}} C_{N_{p-1}} \cdots C_{N_1}(M+N_1+\cdots+N_{p-1})^k \right] \left[\sum_{N_p=1}^{\infty} N_p^{q-k} C_{N_p} \right] \\
&\quad \leqq \sum_{k=0}^{q} \frac{q!}{k!} L^{q-k} \sum_{N_1,\ldots,N_{p-1}} C_{N_{p-1}} \cdots C_{N_1}(M+N_1+\cdots+N_{p-1})^k \\
&\quad \leqq \sum_{k_1=0}^{q} \frac{q!}{k_1!} L^{q-k_1} \sum_{k_2=0}^{k_1} \frac{k_1!}{k_2!} L^{k_1-k_2} \cdots \sum_{k_p=0}^{k_{p-1}} \frac{k_{p-1}!}{k_p!} L^{k_{p-1}-k_p} M^{k_p} \\
&\quad = q!L^q \sum_{k_1=0}^{q} \sum_{k_2=0}^{k_1} \cdots \sum_{k_p=0}^{k_{p-1}} \frac{L^{-k_p}}{k_p!} M^{k_p} \\
&\quad \leqq q!L^q \exp[M/L] B_{q+1}^{p-1} \leqq q!(2L)^q \exp[M/L]2^p.
\end{aligned}
$$

LEMMA 11. *There exists a sequence a_n with $\lim_{n\to\infty} a_n=0$ for which*

$$\sum_{N=n}^{\infty} N^p C_N \leqq (p+2)!L^{p+2}a_n, \qquad p \geqq 0.$$

Proof. Let

$$a_n(p) = \frac{\sum_{N=n}^{\infty} N^p C_N}{(p+2)!L^{p+2}}$$

and

$$a_n = \sum_{q=1}^{\infty} a_n(q) < \infty.$$

We need only check that $\lim_{n\to\infty} a_n=0$. But this follows from the monotone convergence theorem, since the functions $a_n(\cdot)$ are nonnegative monotone decreasing with $\lim_{n\to\infty} a_n(p)=0$.

Let $E_N^j=(e_{i_1},\ldots,e_{i_N})$ be an N-tuple of variables for which $i_k\neq i_p$ when $k\neq p$; the N-tuples may be different for different j. Using this notation, we have

$$G_n\phi_p\cdots G_n\phi_1 = \sum_{N_p=1}^{n-1} n^{-N_p} \sum^{(n)} C(e_{i_p}|E_{N_p}^p)\,\Delta_{i_p}\phi_p \sum_{N_{p-1}=1}^{n-1} n^{-N_{p-1}} \sum^{(n)} C(e_{i_{p-1}}|E_{N_{p-1}}^{p-1})\,\Delta_{i_{p-1}}\cdots\;\Delta_{i_1}\phi_1.$$

We can break this sum into two parts; the first part having each choice of $E_{N_1}^1,\ldots,E_{N_p}^p$ such that no two sets have a variable in common, and the second being bounded by

$$2^p\|\phi_1\|\cdots\|\phi_p\| \sum_{\max(N_1,\ldots,N_p)\leqq n-1} n^{-N_1-\cdots-N_p} \cdot M(M+N_1)\cdots(M+N_1+\cdots+N_p)WC_{N_1}\cdots C_{N_p} \tag{7}$$

where $M(M+N_1)\cdots(M+N_1+\cdots+N_p)$ is the number of ways of choosing $e_{i_1},\ldots,e_{i_p}$ and

$$W = N_1(N_2+\cdots+N_p)n^{N_1+\cdots+N_p-1}+N_2(N_3+\cdots+N_p)n^{N_1+\cdots+N_p-1} +\cdots+N_{p-1}N_pn^{N_1+\cdots+N_p-1};$$

$N_k(N_{k+1}+\cdots+N_p)n^{N_1+\cdots+N_p-1}$ bounds the number of ways of having one of the paired variables in E_k and one in $E_{k+1}\cup\cdots\cup E_p$. Clearly we have

$$|W| \leqq (M+N_1+\cdots+N_p)^2n^{N_1+\cdots+N_p-1}.$$

Using this bound for $|W|$ in (7) and using Lemma 10, we have (7) bounded by

$$n^{-1}2^p\|\phi_1\|\cdots\|\phi_p\| \sum_{N_1,\ldots,N_p} (M+N_1+\cdots+N_p)^{p+2}C_{N_1}\cdots C_{N_p} \leqq n^{-1}\|\phi_1\|\cdots\|\phi_p\| \exp[M/L](p+2)!(8L)^{p+2}. \tag{8}$$

In the first sum, the sets of variables $E^1_{N_1}, \ldots, E^p_{N_p}$ are such that no two sets have a variable in common. Thus, since we will be integrating over all of the variables whose indices are outside of J, we shall rename those variables by letting $E^i_{N_i} = (e_{i1}, \ldots, e_{iN_i})$. Furthermore, we shall rename the variables whose indices are in J as $e_{01}, e_{02}, \ldots, e_{0M}$. With this in mind, we have modulo J^c,

$$\text{(9)}\quad \begin{aligned} G_n\phi_p \cdots G_n\phi_1 + \text{error} = \sum_{N_p=1}^{n-1} \sum_{i_{p-1}, j_{p-1} \leqq n} C(e_{i_{p-1}j_{p-1}} | E^p_{N_p}) \\ \cdot \Delta_{i_{p-1}j_{p-1}}\phi_p \sum_{N_{p-1}=1}^{n-1} \cdots \sum_{N_1=1}^{n-1} \sum_{i_0, j_0 \leqq n} C(e_{i_0 j_0} | E^1_{N_1})\, \Delta_{i_0 j_0}\phi_1. \end{aligned}$$

If (9) converges as $n \to \infty$, it clearly converges to $D\phi_p \cdots D\phi_1$. To see that it does converge, we use Lemma 9 to show that the tail of the series (9) is bounded by

$$\sum_{q=1}^{p} \sum_{N_p=1}^{\infty} \cdots \sum_{N_q=n}^{\infty} \cdots \sum_{N_1=1}^{\infty} 2^p \|\phi_1\| \cdots \|\phi_p\| M(M+N_1) \cdots (M+N_1+\cdots+N_p) C_{N_1} \cdots C_{N_p}$$

$$= \sum_{q=1}^{p} (2M)^p M \|\phi_1\| \cdots \|\phi_p\| \sum_{N_p=1}^{\infty} \cdots \sum_{N_q=n}^{\infty} \cdots \sum_{N_1=1}^{\infty} \left(\sum_{i_p=1}^{p} N_{i_p} \right) \left(\sum_{i_{p-1}=1}^{p-1} N_{i_{p-1}} \right) \cdots \left(\sum_{i_1=1}^{1} N_{i_1} \right) C_{N_1} \cdots C_{N_p}$$

$$= (2M)^p M \|\phi_1\| \cdots \|\phi_p\| \sum_{q=1}^{p} \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \sum_{N_p=1}^{\infty} \cdots \sum_{N_q=n}^{\infty} \cdots \sum_{N_1=1}^{\infty} N_{i_1} \cdots N_{i_p} C_{N_1} \cdots C_{N_p}$$

$$\text{(10)}\quad \begin{aligned} = (2M)^p M \|\phi_1\| \cdots \|\phi_p\| \sum_{q=1}^{p} \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \left(\sum_{N_p=1}^{\infty} N_p^{\tau_p(i_1,\ldots,i_p)} C_{N_p} \right) \\ \cdots \left(\sum_{N_q=n}^{\infty} N_q^{\tau_q(i_1,\ldots,i_p)} C_{N_q} \right) \cdots \left(\sum_{N_1=1}^{\infty} N_1^{\tau_1(i_1,\ldots,i_p)} C_{N_1} \right) \end{aligned}$$

$$\leqq (2M)^p M \|\phi_1\| \cdots \|\phi_p\| a_n \sum_{q=1}^{p} \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \tau_p(i_1, \ldots, i_p)!\, L^{\tau_p(i_1,\ldots,i_p)} \cdots [\tau_q(i_1, \ldots, i_p)+2]!\, L^{\tau_q(i_1,\ldots,i_p)+2} \cdots \tau_1(i_1, \ldots, i_p)!\, L^{\tau_1(i_1,\ldots,i_p)}$$

$$\leqq a_n (2ML)^p ML^2 \|\phi_1\| \cdots \|\phi_p\| \sum_{q=1}^{p} (p+2)(p+1) \sum_{i_p=1}^{p} \cdots \sum_{i_1=1}^{1} \tau_p(i_1, \ldots, i_p)! \cdots \tau_1(i_1, \ldots, i_p)!$$

$$\leqq a_n (4ML)^p ML^2 \|\phi_1\| \cdots \|\phi_p\| \sum_{q=1}^{p} (p+2)!$$

$$\leqq a_n \|\phi_1\| \cdots \|\phi_p\| (p+3)!(4LM)^{p+3}.$$

Thus combining the inequalities (8) and (10), we have

$$\|G_n\phi_p\cdots G_n\phi_1 - D\phi_p\cdots D\phi_1\|_{J^c} \leqq (n^{-1}+a_n)\|\phi_1\|\cdots\|\phi_p\| \exp[M/L](p+3)!(8LM)^{p+3}. \tag{11}$$

Using (10) with $n=1$, we see that

$$\exp[t_m D]\phi_m\cdots\exp[t_1 D]\phi_1$$

converges since

$$\begin{aligned}
&\|\exp[t_m D]\phi_m\cdots\exp[t_1 D]\phi_1\| \\
&= \left\|\sum_{p_m=0}^{\infty}\cdots\sum_{p_1=0}^{\infty}\frac{t_1^{p_1}\cdots t_m^{p_m}}{p_1!\cdots p_m!}D^{p_m}\phi_m\cdots D^{p_1}\phi_1\right\| \\
&\leqq \sum_{p_m=0}^{\infty}\cdots\sum_{p_1=0}^{\infty}\frac{t_1^{p_1}\cdots t_m^{p_m}}{p_1!\cdots p_m!}a_1\|\phi_1\|\cdots\|\phi_m\|(p_1+\cdots+p_m+3)!(4LM)^{p_1+\cdots+p_m+3} \\
&= a_1\|\phi_1\|\cdots\|\phi_m\|(4LM)^3\sum_{q=0}^{\infty}\sum_{p_1+\cdots+p_m=q}(4LMt_1)^{p_1}\cdots(4LMt_m)^{p_m}\frac{(q+3)!}{p_1!\cdots p_m!} \\
&= a_1\|\phi_1\|\cdots\|\phi_m\|(4LM)^3\sum_{q=0}^{\infty}(q+3)(q+2)(q+1)(4LMt_1+\cdots+4LMt_m)^q \\
&= 6a_1\|\phi_1\|\cdots\|\phi_m\|(4LM)^3(1-4LMt_1-\cdots-4LMt_m)^{-4}
\end{aligned}$$

for $|4LMt_1+\cdots+4LMt_m|<1$.

Finally, using (11) we have

$$\begin{aligned}
&\|\exp[t_m G_n]\phi_m\cdots\exp[t_1 G_n]\phi_1 - \exp[t_m D]\phi_m\cdots\exp[t_1 D]\phi_1\|_{J^c} \\
&\leqq \sum_{p_1=0}^{\infty}\cdots\sum_{p_m=0}^{\infty}\frac{t_1^{p_1}\cdots t_m^{p_m}}{p_1!\cdots p_m!}\|G_n^{p_m}\phi_m\cdots G_n^{p_1}\phi_1 - D^{p_m}\phi_m\cdots D^{p_1}\phi_1\|_{J^c} \\
&\leqq (n^{-1}+a_n)\|\phi_1\|\cdots\|\phi_m\|\exp[M/L]\sum_{p_1=0}^{\infty}\cdots\sum_{p_m=0}^{\infty}\frac{t_1^{p_1}\cdots t_m^{p_m}}{p_1!\cdots p_m!} \\
&\qquad\cdot(p_1+\cdots+p_m+3)!(8LM)^{p_1+\cdots+p_m+3} \\
&\leqq (n^{-1}+a_n)\|\phi_1\|\cdots\|\phi_m\|e^{M/L}(1-8LMt_1-\cdots-8LMt_m)^{-4}.
\end{aligned}$$

Letting $n\to\infty$, the theorem is proved.

Proof of Theorem 5. Using equation (5), we have

$$\begin{aligned}
&\lim_{n\to\infty} P[x_i^n(t_\nu)=a_\nu,\ x_j^n(t_\nu)=b_\nu,\ 1\leqq\nu\leqq m] \\
&= \lim_{n\to\infty}\int f^n\exp[t_1 G_n]\chi_{a_1}^i\chi_{b_1}^j\cdots\exp[(t_m-t_{m-1})G_n]\chi_{a_m}^i\chi_{b_m}^j \\
&= \lim_{n\to\infty}\sum\frac{t_1^{p_1}\cdots(t_m-t_{m-1})^{p_m}}{p_1!\cdots p_m!}\int f^n G_n^{p_1}\chi_{a_1}^i\chi_{b_1}^j\cdots G_n^{p_m}\chi_{a_m}^i\chi_{b_m}^j
\end{aligned}$$

$$= \sum \frac{t_1^{p_1} \cdots (t_m - t_{m-1})^{p_m}}{p_1! \cdots p_m!} \int f^\infty D^{p_1} \chi^i_{a_1} \chi^j_{b_1} \cdots D^{p_m} \chi^i_{a_m} \chi^j_{b_m}$$

$$= \sum \frac{t_1^{p_1} \cdots (t_m - t_{m-1})^{p_m}}{p_1! \cdots p_m!} \int f^\infty D^{p_1} \chi^i_{a_1} \chi^j_{b_1} \cdots D^{p_{m-1}} \chi^i_{a_{m-1}} \chi^j_{b_{m-1}}$$

$$\sum_{k_m=0}^{p_m} \binom{p_m}{k_m} (D_{k_m} \cdots D_1 \chi^i_{a_m}) \otimes (D_p \cdots D_{k_m+1} \chi^j_{b_m})$$

$$= \sum \frac{t_1^{p_1} \cdots (t_m - t_{m-1})^{p_m}}{p_1! \cdots p_m!} \sum_{k_m=0}^{p_m} \binom{p_m}{k_m} \int f^\infty D^{p_1} \chi^i_{a_1} \chi^j_{b_1}$$

$$\cdots D^{p_{m-1}} (\chi^i_{a_{m-1}} D^{k_m} \chi^i_{a_m}) \otimes (\chi^j_{b_{m-1}} D^{p_m - k_m} \chi^j_{b_m})$$

$$= \sum \frac{t_1^{p_1} \cdots (t_m - t_{m-1})^{p_m}}{p_1! \cdots p_m!} \sum_{k_m=0}^{p_m} \cdots \sum_{k_1=0}^{p_1} \binom{p_m}{k_m} \cdots \binom{p_1}{k_1}$$

$$\cdot \int f^\infty (D^{k_1} \chi^i_{a_1} \cdots D^{k_m} \chi^i_{a_m}) \otimes (D^{p_1 - k_1} \chi^j_{b_1} \cdots D^{p_m - k_m} \chi^j_{b_m})$$

$$= \sum \frac{t_1^{p_1} \cdots (t_m - t_{m-1})^{p_m}}{k_1! \cdots k_m! (p_1 - k_1)! \cdots (p_m - k_m)!}$$

$$\cdot \int f^\infty (D^{k_1} \chi^i_{a_1} \cdots D^{k_m} \chi^i_{a_m}) \otimes (D^{p_1 - k_1} \chi^j_{b_1} \cdots D^{p_m - k_m} \chi^j_{b_m})$$

$$= \left(\int f^\infty \exp [t_1 D] \chi^i_{a_1} \cdots \exp [(t_m - t_{m-1}) D] \chi^i_{a_m} \right)$$

$$\cdot \left(\int f^\infty \exp [t_1 D] \chi^j_{b_1} \cdots \exp [(t_m - t_{m-1}) D] \chi^j_{b_m} \right)$$

$$= \lim_{n \to \infty} P[x_i^n(t_\nu) = a_\nu, 1 \leqq \nu \leqq m] \lim_{n \to \infty} P[x_j^n(t_\nu) = b_\nu, 1 \leqq \nu \leqq n].$$

Proof of Theorem 6. Clearly

$$\exp [tD] \phi \otimes \psi \equiv \left(\sum \frac{t^p}{p!} D_{2p} D_{2p-2} \cdots D_2 \phi \right) \otimes \left(\sum \frac{t^p}{p!} D_{2p-1} D_{2p-3} \cdots D_1 \psi \right)$$

modulo the indices of variables of ϕ and ψ and thus $\exp [tD] \phi \otimes \psi$ converges modulo the indices of variables of ϕ and ψ whenever $\exp [tD]\phi$ and $\exp [tD]\psi$ converge. Using this fact, we have for $8Lt < 1$ and χ^i_e defined as in (6),

$$\int_{\{1\}^c} f^\infty \exp [tD] \chi^1_{\xi_1} \otimes \cdots \otimes \chi^m_{\xi_m}(e_0, \ldots)$$

$$= \sum_{p=0}^\infty \frac{t^p}{p!} \int_{\{1\}^c} f^\infty D^p \chi^1_{\xi_1} \otimes \cdots \otimes \chi^m_{\xi_m}(e_0, \ldots)$$

$$= \sum_{p=0}^\infty \frac{t^p}{p!} \int_{\{1\}^c} f^\infty \sum_{k=0}^p \binom{p}{k} [D_p \cdots D_{k+1} \chi^1_{\xi_1}(e_0, \ldots)] \otimes D_k \cdots D_1 \chi^2_{\xi_2} \cdots \chi^m_{\xi_m}$$

$$(12)\qquad = \sum_{p=0}^{\infty}\frac{t^p}{p!}\sum_{k=0}^{p}\binom{p}{k}\left[\int_{\{1\}^c} f^{\infty} D_p\cdots D_{k+1}\chi^1_{\xi_1}(e_0,\ldots)\right]\int f^{\infty} D_k\cdots D_1\chi^2_{\xi_2}\cdots\chi^m_{\xi_m}$$

$$= \sum_{p=0}^{\infty}\sum_{k=0}^{p}\frac{t^k t^{p-k}}{k!(p-k)!}\left[\int_{\{1\}^c} f^{\infty} D^{p-k}\chi^1_{\xi_1}(e_0,\ldots)\right]\int f^{\infty} D^k\chi^2_{\xi_2}\cdots\chi^m_{\xi_m}$$

$$= \left[\int_{\{1\}^c} f^{\infty}\exp[tD]\chi^1_{\xi_1}(e_0,\ldots)\right]\int f^{\infty}\exp[tD]\chi^2_{\xi_2}\cdots\chi^m_{\xi_m}$$

$$= \left[\int_{\{1\}^c} f^{\infty}\exp[tD]\chi^1_{\xi_1}(e_0,\ldots)\right]\prod_{k=2}^{m}\int f^{\infty}\exp[tD]\chi^k_{\xi_k}.$$

Now suppose that ϕ is a function on the product space E^m. Then for $a, b \in E$ and $8Lt<1$ we have, using (12),

$$\int_{\{1\}^c} f^{\infty}\exp[tD]\chi^1_b\phi(a,\ldots)$$

$$= \sum_{\xi_1,\ldots,\xi_m}\left[\int_{\{1\}^c} f^{\infty}\exp[tD]\chi^1_{\xi_1}\otimes\cdots\otimes\chi^m_{\xi_m}(a,\ldots)\right] \cdot\chi^1_b(\xi_1,\ldots,\xi_m)\phi(\xi_1,\ldots,\xi_m)$$

$$(13)\qquad = \sum_{\xi_1,\ldots,\xi_m}\left[\int_{\{1\}^c} f^{\infty}\exp[tD]\chi^1_{\xi_1}(a,\ldots)\right]\left(\prod_{k=2}^{m}\int f^{\infty}\exp[tD]\chi^k_{\xi_k}\right) \cdot\chi^1_b(\xi_1,\ldots,\xi_m)\phi(\xi_1,\ldots,\xi_m)$$

$$= \sum_{\xi_1}\left[\int_{\{1\}^c} f^{\infty}\exp[tD]\chi^1_{\xi_1}(a,\ldots)\right]\int_{\{1\}^c} f_t^{\infty}\chi^1_b(\xi_1,\ldots)\phi(\xi_1,\ldots)$$

$$= \int_{\{1\}^c} f^{\infty}\exp[tD]\chi^1_b(a,\ldots)\int_{\{1\}^c} f_t^{\infty}\phi(b,\ldots).$$

We can now easily prove the theorem if we notice that (13) holds whenever ϕ can be written as an absolutely converging series $\sum\phi_k$, where each ϕ_k has a finite number of variables. Since $\phi=\exp[t_2D]\chi^1_{e_2}\cdots\exp[t_m-t_{m-1}D]\chi^1_{e_m}$ is such a function, we have

$$\int_{\{1\}^c} f^{\infty}\exp[t_1D]\chi^1_{e_1}\cdots\exp[(t_m-t_{m-1})D]\chi^1_{e_m}$$

$$= \left[\int_{\{1\}^c} f^{\infty}\exp[t_1D]\chi^1_{e_1}(e_0,\ldots)\right]\int_{\{1\}^c} f^{\infty}_{t_1}\exp[(t_2-t_1)D]\chi^1_{e_2} \cdots\exp[(t_m-t_{m-1})D]\chi^1_{e_m}(e_1,\ldots)$$

$$= \left[\int_{\{1\}^c} f^{\infty}\exp[t_1D]\chi^1_{e_1}(e_0,\ldots)\right]\left[\int_{\{1\}^c} f^{\infty}_{t_1}\exp[(t_2-t_1)D]\chi^1_{e_2}(e_1,\ldots)\right]$$

$$\cdot\int_{\{1\}^c}(f_{t_1})^{\infty}_{t_2-t_1}\exp[(t_3-t_2)D]\chi^1_{e_3}\cdots\exp[(t_m-t_{m-1})D]\chi^1_{e_m}(e_2,\ldots).$$

Proceeding in this manner and noting that $(f_t)_s = f_{t+s}$, we prove the first part of Theorem 6.

Finally,

$$
\begin{aligned}
\gamma_{\pm 1}(u) &= \frac{d}{dt} P_{f|\pm 1}(t;1)\Big|_{t=0} \\
&= \frac{d}{dt}\int_{\{1\}^c} f^{\infty} e^{tD}\chi^1_{+1}(\pm 1,\ldots)\Big|_{t=0} \\
&= \int_{\{1\}^c} f^{\infty} D\chi^1_{+1}(\pm 1,\ldots) \\
&= \int_{\{1\}^c} f^{\infty} \sum_{N=1}^{\infty} C(\pm 1|e_{2,1},\ldots,e_{2,N})[\chi_{+1}(\mp 1)-\chi_{+1}(\pm 1)] \\
&= \mp \sum_{N=1}^{\infty}\sum_{k=0}^{N} C_N^{\pm 1}(k)\binom{N}{k}u^k(1-u)^{N-k},
\end{aligned}
$$

the differentiation being justified by the absolute convergence of the resulting sum. This completes the proof.

Proof of Theorem 8. A necessary condition for a function to be in H is that it be real analytic, as the following lemma demonstrates.

LEMMA 12. *If $F \in H$, then F has derivatives of all orders and*

$$|(d/du)^p F(u)| \leqq p!(2L)^p.$$

Proof.

$$
\begin{aligned}
\left|\left(\frac{d}{du}\right)^p F(u)\right| &= \left|\left(\frac{d}{du}\right)^p \sum_{N=1}^{\infty}\sum_{k=0}^{N} C_N(k)\binom{N}{k}u^k(1-u)^{N-k}\right| \\
&= \left|\sum_{q=0}^{p}\binom{p}{q}\sum_{N=p-q}^{\infty}\sum_{k=q}^{N-p+q} C_N(k)\binom{N}{k}k\cdots(k-q+1)u^{k-q}\right. \\
&\qquad \left.\cdot(N-k)\cdots(N-k-p+q+1)(-1)^{p-q}(1-u)^{N-k-p+q}\right| \\
&= \left|\sum_{q=0}^{p}\binom{p}{q}\sum_{N=p-q}^{\infty}\sum_{k=0}^{N-p} C_N(k+q)\binom{N}{k+q}(k+q)\cdots(k+1)\right. \\
&\qquad \left.\cdot(N-k-q)\cdots(N-k-p+1)(-1)^{p-q}u^k(1-u)^{N-p-k}\right| \\
&\leqq \sum_{q=0}^{p}\binom{p}{q}\sum_{N=p-q}^{\infty}\frac{N!}{(N-p)!}C_N\sum_{k=0}^{N-p}\binom{N-p}{k}u^k(1-u)^{N-p-k} \\
&\leqq \sum_{q=0}^{p}\binom{p}{q}\sum_{N=p-q}^{\infty}N^pC_N \leqq \sum_{q=0}^{p}\binom{p}{q}p!L^p = p!(2L)^p.
\end{aligned}
$$

Notice that the term-wise differentiation is justified by the convergence of the resulting sums.

LEMMA 13. *If $F, G \in H$, then $FG \in H$.*

Proof. Let

$$F(u) = \sum_{N=1}^{\infty} \sum_{k=0}^{N} C_N(k) \binom{N}{k} u^k (1-u)^{N-k}$$

$$G(u) = \sum_{N=1}^{\infty} \sum_{k=0}^{N} d_N(k) \binom{N}{k} u^k (1-u)^{N-k}.$$

Then

$$F(u)G(u) = \sum_{N=1}^{\infty} \sum_{k=0}^{N} e_N(k) \binom{N}{k} u^k (1-u)^{N-k}$$

where

$$e_N(k) = \binom{N}{k}^{-1} \sum_{N_1+N_2=N} \sum_{k_1+k_2=k;\, k_1 \leqq N_1;\, k_2 \leqq N_2} C_{N_1}(k_1) d_{N_2}(k_2) \binom{N_1}{k_1} \binom{N_2}{k_2}.$$

Letting $e_N = \max_{k \leqq n} e_N(k)$, we have

$$\begin{aligned}
\sum_{N=1}^{\infty} N^p e_N &\leqq \sum_{N=1}^{\infty} \sum_{k=0}^{N} N^p e_N(k) \\
&= \sum_{N=1}^{\infty} \sum_{k=0}^{N} N^p \binom{N}{k}^{-1} \sum_{N_1+N_2=N} \sum_{k_1+k_2=k;\, k_1 \leqq N_1;\, k_2 \leqq N_2} C_{N_1}(k_1) d_{N_2}(k_2) \binom{N_1}{k_1} \binom{N_2}{k_2} \\
&= \sum_{N_1, N_2} \sum_{k_1 \leqq N_1;\, k_2 \leqq N_2} (N_1+N_2)^p C_{N_1}(k_1) d_{N_2}(k_2) \binom{N_1+N_2}{k_1+k_2}^{-1} \binom{N_1}{k_1} \binom{N_2}{k_2} \\
&\leqq \sum_{N_1, N_2} \sum_{k_1 \leqq N_1;\, k_2 \leqq N_2} (N_1+N_2)^p C_{N_1}(k_1) d_{N_2}(k_2) \\
&\leqq \sum_{N_1, N_2} N_1 N_2 \sum_{q=0}^{p} \binom{p}{q} N_1^q N_2^{p-q} C_{N_1} d_{N_2} \\
&\leqq \sum_{q=0}^{p} \binom{p}{q} \left(\sum N_1^{q+1} C_{N_1}\right) \left(\sum N_2^{p-q+1} d_{N_2}\right) \\
&\leqq \sum_{q=0}^{p} \binom{p}{q} (q+1)!\, L^{q+1} (p-q+1)!\, L^{p-q+1} \leqq (4L)^{p+2} p!.
\end{aligned}$$

LEMMA 14. *If $F \in H$, then $\exp(F) \in H$.*

Proof. Let

$$F(u) = \sum_{N=1}^{\infty} \sum_{k=0}^{N} C_N(k) \binom{N}{k} u^k (1-u)^{N-k}.$$

Then

$$\exp[F(u)] = \sum_{M=1}^{\infty} \sum_{j=0}^{M} \binom{M}{j} \hat{C}_M(j) u^j (1-u)^{M-j},$$

where

$$\hat{C}_M(j) = \binom{M}{j}^{-1} \sum_{n=0}^{\infty} (n!)^{-1} \sum_{N_1+\cdots+N_n=M} \sum_{k_1+\cdots+k_n=j} \binom{N_1}{k_1} \cdots \binom{N_n}{k_n} C_{N_1}(k_1)\cdots C_{N_n}(k_n);$$

in this last expression, $k_1 \leqq N_1, \ldots, k_n \leqq N_n$. Thus

$$\begin{aligned}
\sum_{M=1}^{\infty} M^p \hat{C}_M &\leqq \sum_{M=1}^{\infty}\sum_{j=0}^{M} M^p \hat{C}_M(j) \\
&= \sum_{M=1}^{\infty}\sum_{j=0}^{M} M^p \binom{M}{j}^{-1} \sum_{n=0}^{\infty} (n!)^{-1} \sum_{N_1+\cdots+N_n=M} \sum_{k_1+\cdots+k_n=j} \binom{N_1}{k_1} \\
&\qquad \cdots \binom{N_n}{k_n} C_{N_1}(k_1)\cdots C_{N_n}(k_n) \qquad (k_1 \leqq N_1, \ldots, k_n \leqq N_n) \\
&= \sum_{N_1,\ldots,N_n} \sum_{k_1\leq N_1,\ldots,k_n\leq N_n} \sum_{n=0}^{\infty} (n!)^{-1}(N_1+\cdots+N_n)^p \\
&\qquad \cdot \binom{N_1+\cdots+N_n}{k_1+\cdots+k_n}^{-1} \binom{N_1}{k_1}\cdots\binom{N_n}{k_n} C_{N_1}(k_1)\cdots C_{N_n}(k_n) \\
&\leqq \sum_{N_1,\ldots,N_n} \sum_{k_1\leq N_1,\ldots,k_n\leq N_n} \sum_{n=0}^{\infty} (n!)^{-1}(N_1+\cdots+N_n)^p C_{N_1}\cdots C_{N_n} \\
&\leqq \sum_{N_1,\ldots,N_n} \sum_{n=0}^{\infty} (n!)^{-1} N_1\cdots N_n (N_1+\cdots+N_n)^p C_{N_1}\cdots C_{N_n} \\
&= \sum_{n=0}^{\infty} (n!)^{-1} \sum_{N_1,\ldots,N_n} \sum_{s_1+\cdots+s_n=p} \frac{p!}{s_1!\cdots s_n!} \\
&\qquad \cdot N_1^{s_1}\cdots N_n^{s_n} N_1\cdots N_n C_{N_1}\cdots C_{N_n} \\
&= \sum_{n=0}^{\infty} (n!)^{-1} \sum_{s_1+\cdots+s_n=p} p! L^p (s_1+1)\cdots(s_n+1) \\
&= \sum_{n=0}^{\infty} (n!)^{-1} L^p 2n(2n+1)\cdots(2n+p-1) \\
&\leqq (2L)^p p! \sum_{n=0}^{\infty} (n!)^{-1} \binom{n+p-1}{p} \\
&\leqq p!(2L)^p \sum_{n=0}^{\infty} (n!)^{-1} 2^n 2^{p-1} \leqq p!(4L)^p e^2.
\end{aligned}$$

LEMMA 15 (HAUSDORFF [1]). *If a polynomial F is positive* (>0) *on the open interval* (0, 1), *then it can be expressed as*

$$F(u) = \sum_{m=0}^{N} a_m \chi_{N,m}(u), \qquad a_m \geqq 0$$

where

$$\chi_{N,m}(u) = \binom{N}{m} u^m (1-u)^{N-m},$$

provided that N is sufficiently large.

LEMMA 16. *If F is a complex valued function on a complex disc $|z| \leqq 1+\delta$, $\delta>0$, real on the real numbers and analytic on the closed disc $|z| \leqq 1$; then for any sufficiently large real constant C, $F(z)+C \in H$.*

Proof. Let F be real on the real line and analytic in the closed disc $|z| \leqq 1$. Then there exists a $\delta>0$ such that $F(z)=\sum_{N=0}^{\infty} \alpha_N z^N$ for $|z| \leqq 1+\delta$ where

$$|\alpha_N| = |F^{(N)}(0)|/N! \leqq (1+\delta)^{-N}A, \quad A \text{ a positive constant.}$$

Now let

$$\begin{aligned} b_N &= \alpha_N \text{ if } \alpha_N \geqq 0, \\ &= 0 \quad \text{otherwise,} \end{aligned} \qquad \begin{aligned} a_N &= -\alpha_N \text{ if } \alpha_N < 0, \\ &= 0 \quad \text{otherwise,} \end{aligned} \qquad C = d-b_0+\sum_{N=0}^{\infty} a_N, \qquad d \geqq 0.$$

Then

$$F(z)+C = d+\sum_{N=1}^{\infty} b_N(z^N)+\sum_{N=1}^{\infty} a_N(1-z^N).$$

But

$$1-z^N = \sum_{k=0}^{N-1} \binom{N}{k} z^k(1-z)^{N-k}, \qquad N \geqq 1,$$

and hence, letting

$$B_N(z) = \sum_{k=0}^{N} C_N(k)\binom{N}{k} z^k(1-z)^{N-k}$$

where $C_1(0)=a_1+d$, $C_1(1)=b_1+d$ and

$$\begin{aligned} C_N(k) &= a_N \quad \text{for } 0 \leqq k < N, \\ &= b_N \quad \text{for } k = N, \end{aligned} \qquad N > 1,$$

we get

$$F(z)+C = \sum_{N=1}^{\infty} B_N(z).$$

Thus we have represented $F+C$ as a sum of Bernstein polynomials when $C \geqq -b_0+\sum_{N=0}^{\infty} a_N$. We therefore need only show that there exists an $L>0$ such that

$$\sum_{N=1}^{\infty} N^p C_N \leqq p!L^p, \qquad p \geqq 1.$$

But

$$C_N = \max_{k \leqq N} C_N(k) \leqq (1+\delta)^{-N}A.$$

Hence

$$\sum_{N=0}^{\infty} N^p C_N \leqq A \sum_{N=0}^{\infty} N^p \exp[-N \ln(1+\delta)]$$

which corresponds to

$$\int_0^{\infty} t^p \exp[-t \ln(1+\delta)]\, dt = \frac{p!}{[\ln(1+\delta)]^{p+1}} \leqq p!\, L^p$$

for a suitable L.

We can now give sufficient conditions that F be contained in H.

LEMMA 17. *If F is analytic on the closed disc $|z| \leqq 1$, real on the reals and positive on* $[0, 1]$, *then* $F \in H$.

Proof. Since F is analytic on $|z| \leqq 1$, real on the reals and positive on $[0, 1]$, it can be written as $\exp[-C](z-z_1)(z-z_1^*)\cdots(z-z_n)(z-z_n^*)\exp[C+G(z)]$ where z^* is the conjugate of z, G is analytic on $|z| \leqq 1$ and real on the reals. For C sufficiently large, $C+G(z) \in H$ and hence $\exp[C+G(z)] \in H$. Since, according to Hausdorff's lemma, $(z-z_1)(z-z_1^*)\cdots(z-z_n)(z-z_n^*) \in H$ and since H is closed under products, the proof is complete.

LEMMA 18. *If $F, G \in H$ and G is a polynomial, $0 < G(u) < 1$ on $(0,1)$, then $F[G(\cdot)] \in H$.*

Proof. Let

$$F(u) = \sum_{N=1}^{\infty} \sum_{k=0}^{N} C_N(k)\binom{N}{k} u^k(1-u)^{N-k}$$

$$G(u) = \sum_{p=0}^{M} d(p)u^p(1-u)^{M-p}$$

$$1-G(u) = \sum_{q=0}^{M} e(q)u^q(1-u)^{M-q}.$$

Then

$$F[G(u)] = \sum_{N=1}^{\infty} \sum_{k=0}^{N} C_N(k)\binom{N}{k}\left[\sum_{p=0}^{M} d(p)u^p(1-u)^{M-p}\right]^k\left[\sum_{q=0}^{M} e(q)u^q(1-u)^{M-q}\right]^{N-k}$$

$$= \sum_{N=1}^{\infty} \sum_{j=0}^{NM} \hat{C}_{NM}(j)\binom{NM}{j} u^j(1-u)^{MN-j},$$

where

$$\hat{C}_{NM}(j) = \binom{NM}{j}^{-1} \sum_{k=0}^{N} C_N(k)\binom{N}{k} \sum_{p_1+\cdots+p_k+q_1+\cdots+q_{N-k}=j;\,0\leqq p,q\leqq M} \cdot d(p_1)\cdots d(p_k)e(q_1)\cdots e(q_{N-k}).$$

Using Stirling's formula, we have

$$\binom{NM}{j}^{-1} \sim (2\pi MN)^{1/2}\left(\frac{j}{MN}\right)^{1/2}\left(1-\frac{j}{MN}\right)^{1/2}\left(\frac{j}{MN}\right)^{j}\left(1-\frac{j}{MN}\right)^{MN-j}$$

$$\leqq 2\pi MN\left(\frac{j}{MN}\right)^{j}\left(1-\frac{j}{MN}\right)^{MN-j}$$

and thus

$$\sum_{N=1}^{\infty} (NM)^p \hat{C}_{NM} \leqq \sum_{N=1}^{\infty} (NM)^p \sum_{j=0}^{NM} \hat{C}_{NM}(j)$$

$$= \sum_{N=1}^{\infty} (NM)^p \sum_{j=0}^{NM} \sum_{k=0}^{N} C_N(k)\binom{N}{k}\binom{NM}{j}^{-1} \sum_{p_1+\cdots+p_k+q_1+\cdots+q_{N-k}=j;\,0\leqq p,q\leqq M} \cdot d(p_1)\cdots d(p_k)e(q_1)\cdots e(q_{N-k})$$

$$\leqq \sum_{N=1}^{\infty} (NM)^p \sum_{j=0}^{NM} \sum_{k=0}^{N} C_N(k) \binom{N}{k} 2\pi MN \sum_{p_1+\cdots+p_k+q_1+\cdots+q_{N-k}=j;\, 0\leqq p,q\leqq M} \cdot d(p_1)\cdots d(p_k) e(q_1)\cdots e(q_{N-k}) \left(\frac{j}{NM}\right)^j \left(1-\frac{j}{NM}\right)^{NM-j}$$

$$\leqq 2\pi M^{p+1} \sum_{N=1}^{\infty} N^{p+1} C_N \sum_{m=0}^{MN} \sum_{k=0}^{N} \binom{N}{k} \sum_{j=0}^{NM} \sum_{p_1+\cdots+p_k+q_1+\cdots+q_{N-k}=j;\, 0\leqq p,q\leqq M} \cdot d(p_1)\cdots d(p_k) e(q_1)\cdots e(q_{N-k}) \left(\frac{m}{NM}\right)^j \left(1-\frac{m}{NM}\right)^{NM-j}$$

$$= 2\pi M^{p+1} \sum_{N=1}^{\infty} N^{p+1} C_N \sum_{m=0}^{MN} \sum_{k=0}^{N} \binom{N}{k} \left[G\left(\frac{m}{NM}\right)\right]^k \left[1-G\left(\frac{m}{NM}\right)\right]^{N-k}$$

$$= 2\pi M^{p+2} \sum_{N=1}^{\infty} N^{p+2} C_N$$

$$\leqq 2\pi M^{p+2}(p+2)!L^{p+2} \leqq p!\hat{L}^p$$

for suitable $\hat{L}$.

We are now in a position to prove Theorem 8. If $F \in H$, then F is certainly positive on the open interval (0, 1) and, by Lemma 12, it is real analytic on the closed interval [0, 1]. Therefore assume that F is positive and real analytic on the closed interval [0, 1]; if F had roots at 0 or 1, we could divide through by them. Then we can find a domain D, symmetric about and containing the interval [0, 1] on which F is analytic. If there exists a polynomial G mapping D conformally onto a domain containing the unit disc; and if G is real on the reals with $G(0)=0$ and $G(1)=1$; then $F[G^{-1}(w)]$ is analytic on the closed unit disc; real on the reals and positive on [0, 1]. Thus, by Lemma 17, $F[G^{-1}(\cdot)] \in H$ and by Lemma 18 $F[G^{-1}(G(\cdot))] = F(\cdot) \in H$. Therefore, to complete the proof we need only show that G exists.

Let $G_1(z)$ be the unique conformal mapping of D onto the disc $|z|<1$, where $G_1(0)=0$ and $G_1'(0)>0$. Since D is symmetric about the interval [0, 1], $[G_1(z^*)]^*$ also maps D conformally onto $|z|<1$ and hence $[G_1(z^*)]^* = G_1(z)$ and G_1 is real on the real axis. Let $G_2(z) = [G_1(1)]^{-1}G_1(z)$. Then G_2 maps D conformally onto the disc $|z| < [G_1(1)]^{-1}$. Clearly G_2 is real on the real axis with $G_2(0)=0$ and $G_2(1)=1$. Let $G_3(z)$ be a polynomial for which $G_3(0)=0$ and $|G_2(z)-G_3(z)|<\varepsilon$ for all $z \in D$, and define $G(z)$ as $a[G_3(z)+G_3(z^*)^*]$ where $a=[G_3(1)+G_3(1)^*]^{-1}$. Clearly we have $(2+2\varepsilon)^{-1} \leqq |a| \leqq (2-2\varepsilon)^{-1}$. If D' is the inverse image of the disc

$$|w| < (2-2\varepsilon)^{-1}[1+[G_1(1)]^{-1}-2\varepsilon]-\varepsilon(1-\varepsilon)^{-1}$$

under the mapping G, and C is the inverse image of the circle $|w| = 2^{-1}[1+[G_1(1)]^{-1}]$ under the mapping G_2; then $G_2 \neq 0, \infty$ on C and

$$2aG_2(z)-w = G(z)-w+h(z)$$

where $|G(z)-w|>\varepsilon(1-\varepsilon)^{-1}>|h(z)|$ for all $z \in C$ and $w \in G(D')$. Thus, by Rouché's theorem, $G(z)$ takes on the value w only once. Therefore, since $G(D')$ is a disc containing the unit disc for ε sufficiently small and since G is a 1-1 mapping of D' onto the disc $G(D')$ with $G(0)=0$, $G(1)=1$ and G real on the real axis; the proof is complete.

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University of California,
Riverside, California