MEASURES ON F-SPACES(1)

BY

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0. **Introduction.** A partially ordered set $(P, \leqq)$ has the *property* (I) iff for any sequences $\{x_n\}_{n\in\omega}$ and $\{y_n\}_{n\in\omega}$ in P such that $x_n \leqq y_m$ for all n, m, there exists $x \in P$ such that $x_n \leqq x \leqq y_n$ for all n.

THEOREM A. *Let S be a compact Hausdorff space. Then the conditions*

(a) *S is a Stonian space (i.e., disjoint open subsets of S have disjoint closures),*

(a′) *$C_R(S)$ (=set of continuous, real-valued functions on S) is a conditionally complete lattice,*

(a″) *S is totally disconnected and the Boolean algebra $\mathcal{O}$ of open closed subsets of S is complete,*

(b) *S is a σ-Stonian space (i.e., disjoint open subsets of S, at least one of which is an F_σ, have disjoint closures),*

(b′) *$C_R(S)$ is conditionally σ-complete,*

(b″) *S is totally disconnected and $\mathcal{O}$ is σ-complete,*

(c) *S is an F-space (i.e., disjoint open F_σ subsets of S have disjoint closures),*

(c′) *$C_R(S)$ has the property* (I),

(c″) *S is totally disconnected and $\mathcal{O}$ has the property* (I),

are related as follows. (a) $\Leftrightarrow$ (a′) $\Leftrightarrow$ (a″), (b) $\Leftrightarrow$ (b″), (c) $\Leftrightarrow$ (c′) $\Leftarrow$ (c″), *and if S is totally disconnected, then* (c′) $\Rightarrow$ (c″).

The equivalences (a) $\Leftrightarrow$ (a′) $\Leftrightarrow$ (a″) and (b) $\Leftrightarrow$ (b′) $\Leftrightarrow$ (b″) are proved in [12]. The equivalence of (c) and (c′) is the principal result of §1 of this paper. (c″) $\Rightarrow$ (c) is proved in [6, p. 1619]. Since there exist connected F-spaces [4, pp. 374–375], (c) does not imply (c″). Since there exist Boolean algebras which are *σ-complete* but not complete, (b) does not imply (a). A closed subspace of an F-space is again an F-space (corollary to Theorem 2.1). $\beta\omega$, the Stone-Čech compactification of the discrete space ω of nonnegative integers, is the Stone space of the complete Boolean algebra 2^ω and so is Stonian. Thus $\beta\omega \sim \omega$ is an F-space. $\beta\omega \sim \omega$ is the Stone space of the Boolean algebra $\mathcal{B}$ of subsets of ω modulo the finite subsets of ω, and $\mathcal{B}$ is not σ-complete; indeed, if $\{e_n\}_{n\in\omega}$ is an increasing sequence in $\mathcal{B}$ such that $\bigvee_{n\in\omega} e_n$ (=supremum of $\{e_n : n \in \omega\}$) exists, then there exists $N \in \omega$ such that $n \geqq N \Rightarrow e_n = e_N$. Thus (c″) does not imply (b″).

The remainder of the paper is devoted (essentially) to a study of the space $M(S)$

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of regular, complex-valued Borel measures on S when S is an F-space. The principal results of §§2 and 3 are, respectively,

THEOREM B. *Let S be an F-space. Then $\sigma(M(S), C(S))$-convergent sequences are $\sigma(M(S), M(S)^*)$-convergent, and*

THEOREM C. *Let $\mathscr{B}$ be a Boolean algebra which has the property* (I) (*i.e., a Boolean algebra whose Stone space is an F-space*). *Then a subset K of* ba $(\mathscr{B})$ (*=bounded, additive, complex-valued functions on $\mathscr{B}$*) *is bounded* (*with respect to the total-variation norm on* ba $(\mathscr{B})$) *iff* $\sup_{\mu\in K} |\mu(E)| < \infty$ *for all* $E \in \mathscr{B}$.

The paper concludes with the following generalization of the Vitali-Hahn-Saks theorem [3, p. 158] which is obtained by combining Theorems B and C.

THEOREM D. *Let $\mathscr{B}$ be a Boolean algebra which has the property* (I), $\mu \in$ ba $(\mathscr{B})$, *and $\{\mu_n\}_{n\in\omega}$ a sequence in* ba $(\mathscr{B})$ *such that*

(1) *each μ_n is absolutely continuous with respect to μ,*

(2) $\{\mu_n(E)\}_{n\in\omega}$ *converges for all $E \in \mathscr{B}$.*

Then the μ_n are uniformly absolutely continuous with respect to μ.

Theorem B was proved by Grothendieck for S Stonian [5] and by Andô [1] for S σ-Stonian. Our proof is a modification of Grothendieck's. Theorem C was proved by Nikodym [9] for $\mathscr{B}$ σ-complete and the members of K countably additive, and was conjectured by Badé. Theorem D was proved by Andô for $\mathscr{B}$ σ-complete.

I wish to thank Professor W. G. Badé for his assistance in the preparation of this paper. I also wish to acknowledge my debt to Andô's paper [1]. This paper would never have been written (by me, at any rate; however, most of the results of §§2–4 were obtained independently by Haskell Rosenthal) had I not read Andô's paper.

1. A characterization of F-spaces.

NOTATION. For S a set and $f, g: S \to \boldsymbol{R}$ (=reals), $f \wedge g$ and $f \vee g$ are defined by

$$(f\wedge g)(s) = \min (f(s), g(s)), \quad (f\vee g)(s) = \max (f(s), g(s)), \qquad s \in S,$$

and for $A \subset S$, χ_A is the characteristic function of A. If S is a topological space, and $A, B \subset S$, then $A < B$ means that the interior of B contains the closure of A.

LEMMA 1.1. *Let S be a compact Hausdorff space. If $A, B \subset S$ and $A < B$, then there exists an open F_σ subset U of S such that $A < U < B$.*

Proof. Let $f \in C_{\boldsymbol{R}}(S)$ be 0 on A and 1 on $S \sim B$. Set $U = \{s \in S : f(s) < \frac{1}{2}\}$.

LEMMA 1.2. *Let S be a Hausdorff space, D a dense subspace of $\boldsymbol{R}$, and $\{W(r) : r \in D\}$ a set of open subsets of S such that*

(1) $r, s \in D, r < s \Rightarrow W(r) < W(s)$,

(2) $\bigcup_{r\in D} W(r) = S$,

(3) $\bigcap_{r\in D} W(r) = \varnothing$.

Then the function $f: S \to \boldsymbol{R}$ defined by $f(s) = \inf \{r : s \in W(r)\}$, $s \in S$, is continuous.

Proof. [8, p. 114].

THEOREM 1.1. *Let S be a compact Hausdorff space. Then S is an F-space iff $C_R(S)$ has the property* (I).

Proof. Suppose $C_R(S)$ has the property (I). Let U and V be disjoint, open F_σ subsets of S. Let $u, v \in C_R(S)$ be such that $0 \leqq u, v \leqq 1$, $U = u^{-1}[R \sim \{0\}]$, $V = v^{-1}[R \sim \{0\}]$. Set $f_n = (1-v)^n$, $g_n = 1 \wedge nu$. Then $f_n \geqq g_m$ for all n, m. Let $f \in C_R(S)$ be such that $f_n \geqq f \geqq g_n$ for all n. Then f is 1 on U and 0 on V.

Now suppose S is an F-space. Let $\{f_n\}_{n\in\omega}$ and $\{g_n\}_{n\in\omega}$ be sequences in $C_R(S)$ such that $f_n \geqq g_m$ for all n, m. We may assume that for any $s \in S$ and $n \in \omega$, $1 > f_n(s) \geqq f_{n+1}(s) \geqq g_{n+1}(s) \geqq g_n(s) > 0$. Let D be the set of dyadic rationals. For $r \in D$,

$$U(r) = \bigcup_{n\in\omega} \{t \in S : g_n(t) > r\},$$

$$V(r) = \bigcup_{n\in\omega} \{t \in S : f_n(t) < r\},$$

$U(r)$ and $V(r)$ are disjoint, open F_σ sets. Thus for $r \in D$,

$$(*) \qquad V(r) < S \sim U(r).$$

We now construct open sets $W(r)$, $r \in D$, such that

(i) $r \in D \Rightarrow V(r) < W(r) < S \sim U(r)$,

(ii) $r, s \in D, r < s \Rightarrow W(r) < W(s)$.

Lemma 1.2 and (ii) will yield the continuity of the function $f: S \to R$ defined by $f(t) = \inf\{r : t \in W(r)\}$, $t \in S$, and we shall see that (i) will yield $f_n \geqq f \geqq g_n$ for all n. If $r \in D$ and $r \geqq 1$ (resp., $r \leqq 0$), let $W(r)$ be S (resp., $\varnothing$). These $W(r)$ obviously satisfy (i) and (ii). We now construct by induction on n the sets $W(k/2^n)$, $k = 0, \ldots, 2^n$. Suppose $W(k/2^n)$, $k = 0, \ldots, 2^n$ have been chosen such that

$$(\mathrm{i}_n) \qquad V(k/2^n) < W(k/2^n) < S \sim U(k/2^n), \qquad k = 0, \ldots, 2^n,$$

$$(\mathrm{ii}_n) \qquad W(k/2^n) < W((k+1)/2^n), \qquad k = 0, \ldots, 2^n - 1.$$

To find $W((2k+1)/2^{n+1})$ it is enough (Lemma 1.1) to show

$$(0) \quad V((2k+1)/2^{n+1}) \cup W(k/2^n) < W((k+1)/2^n) \cap (S \sim U((2k+1)/2^{n+1})).$$

Note first that

$$(**) \qquad \begin{aligned} r, s \in D, r < s &\Rightarrow S \sim U(r) \subset S \sim U(s), \\ &\Rightarrow V(r) \subset V(s). \end{aligned}$$

From (**) and (ii_n) we obtain

$$V((2k+1)/2^{n+1}) \subset V((k+1)/2^n) < W((k+1)/2^n),$$
$$W(k/2^n) < W((k+1)/2^n)$$

which yields

$$(1) \qquad V((2k+1)/2^{n+1}) \cup W(k/2^n) < W((k+1)/2^n).$$

From (*), (i_n) and (**) we obtain

$$V((2k+1)/2^{n+1}) < S \sim U((2k+1)/2^{n+1}),$$
$$W(k/2^n) < S \sim U(k/2^n) \subset S \sim U((2k+1)/2^{n+1})$$

which yields

$$V((2k+1)/2^{n+1}) \cup W(k/2^n) < S \sim U((2k+1)/2^{n+1}). \tag{2}$$

Combining (1) and (2) we obtain (0). Thus we have

$$(\mathrm{i}_{n+1}) \qquad V(k/2^{n+1}) < W(k/2^{n+1}) < S \sim U(k/2^{n+1}), \qquad k = 0, \dots, 2^{n+1}$$

$$(\mathrm{ii}_{n+1}) \qquad W(k/2^{n+1}) < W((k+1)/2^{n+1}), \qquad k = 0, \dots, 2^{n+1}-1.$$

This completes our induction. We now show that condition (i) implies that $f_n \leqq f \leqq g_n$ for all n. Let $x \in S$, $\varepsilon > 0$. Let $r, s \in D$ be such that $f(x)-\varepsilon < r < f(x) < s < f(x)+\varepsilon$. $f(x) > r \Rightarrow s \notin W(r) \Rightarrow x \notin V(r) \Rightarrow f_n(x) \geqq r > f(x)-\varepsilon$ for all n. $f(x) < s \Rightarrow x \in W(s) \Rightarrow x \notin U(s) \Rightarrow g_n(x) \leqq s < f(x)+\varepsilon$ for all n. Thus $f_n \geqq f \geqq g_n$ for all n.

2. **Sequences of measures on an F-space.** The principal result of this section is the equivalence of sequential $\sigma(M(S), C(S))$-convergence and sequential $\sigma(M(S), M(S)^*)$-convergence whenever S is an F-space. Theorem 2.2 is fundamental in this section in that it enables us to reduce the case of S an F-space to the more tractable (and known) case of S Stonian. The case of S Stonian is an easy consequence of Lemma 2.3 and Theorem 2.3.

Let S be a compact Hausdorff space. $C(S)$ is the space of continuous complex-valued functions on S. $M(S)$ is the space of complex, regular Borel measures on S. $C(S)$ is given the sup norm and $M(S)$ the total-variation norm. For $\mu \in M(S)$ and $\in L^1(\mu)$, we write $\mu(f)$ in place of $\int f\,d\mu$. In this manner we identify $M(S)$ and $C(S)^*$. As is well known, the norm of a member of $M(S)$ is the same whether the member is regarded as a measure or a functional. For $\mu \in M(S)$, $|\mu| \in M(S)$ is defined by $|\mu|(E) = \sup\{\sum_{F\in\mathscr{F}} |\mu(F)| : \mathscr{F}$ is a finite set of pairwise disjoint Borel subsets of $E\}$, E a Borel subset of S. car $(\mu) = S \sim \bigcup\{U : U$ open, $|\mu|(U)=0\}$. Note that if $f, g \in C_R(S)$, $f \geqq g$, and $|\mu|(f) \leqq |\mu|(g)$, then f and g agree on car (μ).

THEOREM 2.1. *Let L and M be lattices, $g: L \to M$ a lattice epimorphism, and let L have the property* (I). *Then M has the property* (I).

Proof. Let $\{x_n\}_{n\in\omega}$ and $\{y_n\}_{n\in\omega}$ be sequences in M such that $x_n \geqq x_{n+1} \geqq y_{n+1} \geqq y_n$ for all n. Let $x_n', y_n' \in L$ be such that $g(x_n')=x_n$, $g(y_n')=y_n$. Let $\{x_n''\}_{n\in\omega}$ and $\{y_n''\}_{n\in\omega}$ be defined by

$$x_0'' = x_0' \vee y_0', \; y_0'' = x_0' \wedge y_0',$$
$$x_{n+1}'' = x_n'' \wedge (x_{n+1}' \vee y_{n+1}' \vee y_n''), \; y_{n+1}'' = y_n'' \vee (x_{n+1}' \wedge y_{n+1}' \wedge x_n'').$$

Then $x_n'' \geqq x_{n+1}'' \geqq y_{n+1}'' \geqq y_n''$, $g(x_n'')=x_n$, and $g(y_n'')=y_n$ for all n. Let $x'' \in L$ be such that $x_n'' \geqq x'' \geqq y_n''$ for all n. Set $x=g(x'')$. Then $x_n \geqq x \geqq y_n$ for all n.

COROLLARY. *A closed subspace of an F-space is an F-space.*

Proof. Let S be an F-space and S_0 a closed subspace of S. Let $g: C_R(S) \to C_R(S_0)$ be the restriction map. g is onto and $C_R(S)$ has the property (I). Thus $C_R(S_0)$ has the property (I) and so S_0 is an F-space.

LEMMA 2.1. *Let S be a compact Hausdorff space, S_0 a closed subspace of S, and let $\kappa: M(S_0) \to M(S)$ be defined by*

$$\kappa(\mu)(f) = \mu(f|S_0), \qquad f \in C(S), \mu \in M(S_0)$$

($f|S_0$ is the restriction of f to S_0). Then κ is an isometry and $\kappa[M(S_0)] = \{\mu \in M(S)$: car $(\mu) \subset S_0\}$, and for $\mu \in M(S_0)$, car $(\mu) =$ car $(\kappa(\mu))$.

THEOREM 2.2. *Let S be an F-space and $\mu \in M(S)$. Then* car (μ) *is Stonian*(3).

Proof. Since car $(\mu) =$ car $(|\mu|)$, we may assume $\mu \geqq 0$. $S_0 =$ car (μ) is an F-space, and by Lemma 2.1 there exists $\mu_0 \in M(S_0)$ such that car $(\mu_0) = S_0$. Thus we may assume $S = S_0$. We first show that $C_R(S)$ is conditionally σ-complete. Let $\{f_n\}_{n\in\omega}$ be a bounded increasing sequence in $C_R(S)$, and let $\{g_n\}_{n\in\omega}$ be a decreasing sequence in the set U of upper bounds of $\{f_n : n \in \omega\}$ such that $\inf_n \mu(g_n) = \inf_{g\in U} \mu(g)$. Let $f \in C_R(S)$ be such that $g_n \geqq f \geqq f_n$ for all n. We assert that $f = \bigvee_{n\in\omega} f_n$. Let $g \in U$. $f \wedge g \in U$ so that $\mu(f \wedge g) \geqq \inf_n \mu(g_n) \geqq \mu(f)$. Thus f and $f \wedge g$ agree on car (μ), i.e. $f \leqq g$.

Now let A be a nonvoid subset of $C_R(S)$ which is bounded above, and let U be the set of upper bounds of A. Let $\{f_n\}_{n\in\omega}$ be a decreasing sequence in U such that $\inf_n \mu(f_n) = \inf_{f\in U} \mu(f)$. Set $f = \bigwedge_{n\in\omega} f_n$. Just as in the above paragraph we see that $f = \bigvee A$.

LEMMA 2.3 (PHILLIPS). *Let $\{\mu_n\}_{n\in\omega}$ be a sequence in* ba (2^ω) *($=$bounded, additive, complex-valued functions on 2^ω) such that $\lim_n \mu_n(A) = 0$ for all $A \subset \omega$. Then*

$$\lim_n \sum_{k\in\omega} |\mu_n(\{k\})| = 0.$$

In particular, $\lim_n \mu_n(\{n\}) = 0$.

Proof. [10, p. 525].

THEOREM 2.3 (GROTHENDIECK). *Let S be a compact Hausdorff space. In order that $K \subset M(S)$ be conditionally $\sigma(M(S), M(S)^*)$-compact, it is necessary and sufficient that for any sequence $\{U_n\}_{n\in\omega}$ of pairwise disjoint open sets, $\lim_n \mu(U_n) = 0$ uniformly for $\mu \in K$.*

Proof. [5, p. 146].

THEOREM 2.4 (GROTHENDIECK-ANDÔ). *Let S be a σ-Stonian space, and let $\{\mu_n\}_{n\in\omega}$ be a $\sigma(M(S), C(S))$-convergent sequence in $M(S)$. Then $\{\mu_n\}_{n\in\omega}$ is $\sigma(M(S), M(S)^*)$-convergent.*

(3) This theorem has also been proved by Professor Kenneth Hoffman.

Proof. We may assume that $\{\mu_n\}_{n\in\omega}$ $\sigma(M(S), C(S))$-converges to 0. We show (via Theorem 2.3) that $\{\mu_n : n \in \omega\}$ is conditionally $\sigma(M(S), M(S)^*)$-compact and from this deduce the $\sigma(M(S), M(S)^*)$-convergence. Suppose, then, that $\{\mu_n : n \in \omega\}$ is not conditionally $\sigma(M(S), M(S)^*)$-compact. Then there is a sequence $\{U_n\}_{n\in\omega}$ of pairwise disjoint open subsets of S such that $\{\mu_n(U_k)\}_{k\in\omega}$ does not converge to 0 uniformly in n. By passing to a subsequence of $\{\mu_n\}_{n\in\omega}$ we may assume the existence of an $\varepsilon > 0$ such that $|\mu_n(U_n)| > \varepsilon$ for all n. By the regularity of the μ_n and the total disconnectedness of S we may assume the U_n to be open and closed. For $n \in \omega$ and $A \subset \omega$, set $\nu_n(A) = \mu_n(\bigvee_{k\in A} U_k)$ ($\bigvee_{k\in A} U_k$ is the supremum of $\{U_k : k \in A\}$ in the Boolean algebra of open closed subsets of S). Clearly, $\nu_n \in \mathrm{ba}\,(2^\omega)$ for all n, and $\lim_n \nu_n(A) = 0$ for all $A \subset \omega$. By Lemma 2.3 $\lim_n \nu_n(\{n\}) = 0$. But $|\nu_n(\{n\})| = |\mu_n(U_n)| > \varepsilon$ for all n. This contradiction establishes the conditional $\sigma(M(S), M(S)^*)$-compactness of $\{\mu_n : n \in \omega\}$. $\{\mu_n\}_{n\in\omega}$ thus has a $\sigma(M(S), M(S)^*)$-cluster point. Since every $\sigma(M(S), M(S)^*)$-cluster point of $\{\mu_n\}_{n\in\omega}$ is also a $\sigma(M(S), C(S))$-cluster point, and since $\{\mu_n\}_{n\in\omega}$ $\sigma(M(S), C(S))$-converges to 0, 0 is the only $\sigma(M(S), M(S)^*)$-cluster point of $\{\mu_n\}_{n\in\omega}$, i.e., $\{\mu_n\}_{n\in\omega}$ $\sigma(M(S), M(S)^*)$-converges to 0.

Theorem 2.4 was proved by Grothendieck for S Stonian [5, p. 168] and by Andô [1, p. 395] for S σ-Stonian. The above proof is a modification of Grothendieck's proof and is completely different from Andô's. That Grothendieck's proof could be so modified has also been observed by Isbell and Semadeni [7, p. 46].

DEFINITION. A *G-space* is a compact Hausdorff space S for which $\sigma(M(S), C(S))$-convergent sequences are $\sigma(M(S), M(S)^*)$-convergent.

THEOREM 2.5. *An F-space is a G-space.*

Proof. Let S be an F-space, and let $\{\mu_n\}_{n\in\omega}$ be a sequence in $M(S)$ which $\sigma(M(S), C(S))$-converges to 0. Set $\mu = \sum_{n\in\omega} 2^{-n}|\mu_n|$, and set $S_0 = \mathrm{car}\,(\mu)$. By Theorem 2.2 S_0 is Stonian. $\mathrm{car}\,(\mu_n) \subset S_0$ for all n so we may regard μ_n as a member of $M(S_0)$ (via Lemma 2.1). $\{\mu_n\}_{n\in\omega}$ $\sigma(M(S_0), C(S_0))$-converges to 0. By Theorem 2.3 $\{\mu_n\}_{n\in\omega}$ $\sigma(M(S_0), M(S_0)^*)$-converges to 0, and this is the same as its $\sigma(M(S), M(S)^*)$-convergence to 0.

COROLLARY. *There exist connected G-spaces.*

Proof. There exist connected F-spaces [4, pp. 374–375].

The above corollary answers a question posed by Isbell and Semadeni [7]. Namely, do connected G-spaces exist?

We conclude this section with an example of a G-space which is not an F-space.

LEMMA 2.4 (GROTHENDIECK). *Let E be a Banach space. Then the following conditions are equivalent.*

(a) *Every $\sigma(E^*, E)$-convergent sequence in E^* is $\sigma(E^*, E^{**})$-convergent.*

(b) *Every continuous linear map of E to a separable Banach space is weakly compact.*

Proof. [5, p. 169].

LEMMA 2.5. *Every infinite, compact Hausdorff space possesses a pair of open, disjoint, nonclosed F_σ subsets.*

Proof([4]). Let $\{s_n\}_{n\in\omega}$ be a discrete sequence of distinct points in S, and let $A=\{s_n : n\in\omega\}^-$. For $x\in A$, let $f_0(x)$ be $(-1)^n/(n+1)$ if $x=s_n$ for some n and 0 if $x\in A\sim\{s_n : n\in\omega\}$. f_0 is continuous on A and so has a continuous extension, say f, to S. Let $U_0=f^{-1}[(0;\infty)]$, $U_1=f^{-1}[(-\infty;0)]$. Clearly, U_0 and U_1 are disjoint open F_σ subsets of S. Let x_k be a cluster point of $\{s_{2n+k}\}_{n\in\omega}$, $k=0, 1$. Then $x_k\in U_k^-\sim U_k$, $k=0, 1$.

THEOREM 2.6 (BADÉ). *Let S be an infinite F-space, U_1 and U_2 disjoint, open, nonclosed F_σ subsets of S, and let $x_i\in U_i^-\sim U_i$, $i=1, 2$. Let S_0 be the space obtained from S by identifying x_1 with x_2. Then S_0 is a G-space which is not an F-space.*

Proof. Let $q\colon S\to S_0$ be the quotient map. Since S is compact, q maps F_σ sets onto F_σ sets. Let $V_i=q[U_i]$, $i=1, 2$. Since S is an F-space, $U_1^-\cap U_2^-=\varnothing$. In particular $x_i\notin U_1\cup U_2$, $i=1, 2$. Thus $V_1\cap V_2=\varnothing$, and $q^{-1}[V_i]=U_i$, $i=1, 2$. Thus V_1 and V_2 are open. $V_i^-\supset q[U_i^-]\supset\{q(x_i)\}$, $i=1, 2$. Thus $q(x_1)\in V_1^-\cap V_2^-$. Therefore S_0 is not an F-space.

We identify $C(S_0)$ with $\{f\in C(S) : f(x_1)=f(x_2)\}$. Let F be a separable Banach space, and let $u\colon C(S_0)\to F$ be a continuous linear map. Let $f_0\in C_{\mathbf{R}}(S)$ be such that $0\leqq f_0\leqq 1$, $f_0(x_1)=0$, $f_0(x_2)=1$. Let $P\colon C(S)\to C(S_0)$ be defined by

$$Pf = f+(f(x_1)-f(x_2))f_0, \qquad f\in C(S).$$

P is clearly a projection. $u\circ P$ is weakly compact by Lemma 2.4. Let $j\colon C(S_0)\to C(S)$ be the inclusion map. Then $u=(u\circ P)\circ j$ and so u is weakly compact.

The projection P is due to Isbell and Semadeni [7].

3. **Uniform boundedness.** The principal result of this section is the extension of the Nikodym boundedness theorem [9] to ba $(\mathscr{B})$ for $\mathscr{B}$ a Boolean algebra which has the property (I).

DEFINITION. Let $\mathscr{B}$ be a Boolean algebra. ba $(\mathscr{B})$ is the space of bounded, additive, complex-valued functions on $\mathscr{B}$ and is equipped with the total variation norm ($\|\mu\|=\sup\{\sum_{E\in\mathscr{F}}|\mu(E)| : \varnothing\neq\mathscr{F}\subset\mathscr{B}, \mathscr{F}$ finite, $E, F\in\mathscr{F}, E\neq F\Rightarrow E\wedge F=0\}$). For $\mu\in$ ba $(\mathscr{B})$, $|\mu|\in$ ba $(\mathscr{B})$ is defined by

$$|\mu|(E)=\sup\Big\{\sum_{F\in\mathscr{F}}|\mu(E)| : \varnothing\neq\mathscr{F}\subset\mathscr{B}; \;\mathscr{F}\text{ finite}, \bigvee\mathscr{F}\leqq E;\ F, G\in\mathscr{F}, F\neq G\Rightarrow F\wedge G=0\Big\}, \qquad E\in\mathscr{B}.$$

The next two lemmas are simply preparation for Theorem 3.1. The portion of Theorem 3.1 that we use in the sequel is known, and a much shorter proof of that portion can be found in [2, p. 856].

([4]) This proof is due to the referee.

LEMMA 3.1. *Let $\mathscr{B}$ be a Boolean ring, and let $E_1, \ldots, E_n \in \mathscr{B}$ be such that $i>j \Rightarrow E_i \wedge E_j = 0$ or $E_i \leqq E_j$. Then there exist pairwise disjoint $E'_1, \ldots, E'_n \in \mathscr{B}$ such that each E'_i is exceeded by some E_j, and each E_i is the maximum of the E'_j it exceeds.*

Proof. We induct on n. For $n=1$ there is nothing to prove. Suppose the lemma is valid for all $k<n$. Let $E'_1, \ldots, E'_{n-1}$ be pairwise disjoint elements of $\mathscr{B}$ such that each $E_i \sim E_n$ is the maximum of the E'_j it exceeds, and each E'_i is exceeded by some $E_j \sim E_n$. Set $E'_n = E_n$. Let $i<n$. There exist $j_1, \ldots, j_p$ such that $E_i \sim E_n = \bigvee_{k=1}^{p} E'_{j_k}$. If $E_i \wedge E_n = 0$, then $E_i = \bigvee_{k=1}^{p} E'_{j_k}$. If $E_i \geqq E_n$, then $E_i = (E_i \sim E_n) \vee E_n = E'_n \vee \bigvee_{k=1}^{p} E'_{j_k}$. Thus each E_i is the maximum of the E'_j it exceeds. Clearly, each E'_i is exceeded by some E_j.

LEMMA 3.2. *Let $\mathscr{B}$ be a Boolean ring, and let $\{E_n\}_{n \in \omega}$ be a sequence in $\mathscr{B}$ such that, $n, m \in \omega$, $n>m \Rightarrow E_n \leqq E_m$ or $E_n \wedge E_m = 0$. Then there is a strictly increasing sequence $\{n_k\}_{n \in \omega}$ in ω such that $\{E_{n_k}\}_{n \in \omega}$ is either decreasing or disjoint.*

Proof. A sequence $\{F_n\}_{n \in \omega}$ in $\mathscr{B}$ has the property (P) iff $m_0 < \cdots < m_p$, $F_{m_i} \wedge F_{m_j} = 0$ for $i \neq j \Rightarrow$ there exists $m > m_p$ such that $F_m \wedge \bigvee_{i=0}^{p} F_{m_i} = 0$. Clearly, a sequence with the property (P) has a disjoint subsequence. Consequently, if some subsequence of $\{E_n\}_{n \in \omega}$ has the property (P) then $\{E_n\}_{n \in \omega}$ has a disjoint-subsequence. Suppose, then, that no subsequence of $\{E_n\}_{n \in \omega}$ has the property (P). There exist $m_0 < \cdots < m_p$ such that $E_{m_i} \wedge E_{m_j} = 0$ for $i \neq j$ and such that $j > m_p \Rightarrow E_j \wedge \bigvee_{i=0}^{p} E_{m_i} \neq 0$. By the condition on $\{E_n\}_{n \in \omega}$ $j > m_p \Rightarrow$ there exists uniquely i such that $E_j \leqq E_{m_i}$. It follows that one of the E_{m_i}, which we take to be E_{n_0}, exceeds E_j for infinitely many j. Suppose $n_0 < \cdots < n_k$ have been selected such that $i>j \Rightarrow E_{n_i} \leqq E_{n_j}$, and such that $E_{n_k} \geqq E_j$ for infinitely many j. Let $\{p_i\}_{i \in \omega}$ be a strictly increasing sequence in ω such that $\{p_i : i \in \omega\} = \{i \in \omega : i > n_k,\ E_i \leqq E_{n_k}\}$. $\{E_{p_i}\}_{i \in \omega}$ does not have the property (P) so that there exist $i_0 < \cdots < i_q$ such that $E_{p_{i_j}} \wedge E_{p_{i_h}} = 0$ for $h \neq j$, and such that $h > i_q \Rightarrow E_{p_h} \wedge \bigvee_{j=0}^{q} E_{p_{i_j}} \neq 0$. Thus one of the E_{p_j}, which we take to be $E_{n_{k+1}}$, exceeds E_{p_i} for infinitely many i.

THEOREM 3.1. *Let $\mathscr{B}$ be a Boolean ring, and let ρ be a real-valued, nonnegative, subadditive*[5] *function on $\mathscr{B}$ such that $\rho[\mathscr{B}]$ is unbounded. Then*

(i) *there exists a sequence $\{E_n\}_{n \in \omega}$ in $\mathscr{B}$ which is either decreasing or disjoint and is such that $\rho(E_n) \to \infty$,*

(ii) *if ρ satisfies the additional condition that $E, F \in \mathscr{B}$, $E \geqq F \Rightarrow \rho(E \sim F) \geqq |\rho(E) - \rho(F)|$, then the sequence $\{E_n\}_{n \in \omega}$ can be taken to be disjoint.*

Proof. To prove (i) it is enough to show the existence of a sequence $\{F_n\}_{n \in \omega}$ in $\mathscr{B}$ such that $n, m \in \omega$, $n>m \Rightarrow \rho(F_m) > m$, and either $F_n \leqq F_m$ or $F_n \wedge F_m = 0$. We construct such a sequence inductively. Let $F_0 \in \mathscr{B}$ be such that $\rho(F_0) > 0$. Suppose $F_0, \ldots, F_k$ have been chosen such that $\rho(F_i) > i$, $i = 0, \ldots, k$, and such that $k \geqq i > j \geqq 0 \Rightarrow F_i \leqq F_j$ or $F_i \wedge F_j = 0$. Let $F \in \mathscr{B}$ be such that $\rho(F) > (k+1)(k+2)$. By

[5] By subadditive we mean: $E \wedge F = 0 \Rightarrow \rho(E \vee F) \leqq \rho(E) + \rho(F)$.

Lemma 3.1 there exist pairwise disjoint members $F'_0, \ldots, F'_k$ of $\mathscr{B}$ such that each $F \wedge F_i$ is the maximum of the F'_j it exceeds, and each F'_i is exceeded by some $F \wedge F_j$. Set $F'_{k+1} = F \sim \bigvee_{i=0}^{k} F_i$. $F = \bigvee_{i=0}^{k+1} F'_i$, and the F'_i are pairwise disjoint so that by the subadditivity of ρ, $(k+1)(k+2) < \rho(F) \leqq \sum_{i=0}^{k+1} \rho(F'_i)$. Let F_{k+1} be one of the F'_i for which $\rho(F'_i) > k+1$. $F'_i \wedge F_j \neq 0 \Rightarrow F'_i \leqq F_j$. Thus $k+1 \geqq i > j \geqq 0 \Rightarrow F_i \wedge F_j = 0$ or $F_i \leqq F_j$.

Now suppose ρ satisfies the condition of (ii). Let $\{F_n\}_{n \in \omega}$ be a decreasing sequence in $\mathscr{B}$ such that $\rho(F_n) \to \infty$. Let $n_0 = 0$. Suppose $n_0 < \cdots < n_k$ have been chosen such that $\rho(F_{n_i} \sim F_{n_{i+1}}) > i$, $i = 0, \ldots, k-1$. Let $n_{k+1} > n_k$ be such that $\rho(F_{n_{k+1}}) > k + \rho(F_{n_k})$. $\rho(F_{n_k} \sim F_{n_{k+1}}) \geqq |\rho(F_{n_k}) - \rho(F_{n_{k+1}})| \geqq \rho(F_{n_{k+1}}) - \rho(F_{n_k}) > k$. Set $E_k = F_{n_k} \sim F_{n_{k+1}}$, $k \in \omega$. Then $\{E_k\}_{k \in \omega}$ is disjoint and $\rho(E_k) \to \infty$.

(ii) of Theorem 3.1 is due to Badé and Curtis [2, p. 856]. We are indebted to Professor Badé for the observation that our proof of the Badé-Curtis result yields the stronger theorem stated here. A simple application of Theorem 3.1 is: if $\mathscr{B}$ is a σ-complete Boolean ring, and $\mu : \mathscr{B} \to \boldsymbol{R} \cup \{-\infty\}$ is countably additive, then $\sup \mu[\mathscr{B}] < \infty$. [Proof: take $\rho(E) = \max(\mu(E), 0)$. ρ is bounded on both decreasing and disjoint sequences.]

The following lemma is the Boolean algebra analogue of Theorem 2.2. It can be obtained as a corollary to Theorem 2.2 by an appeal to the Stone representation theorem.

LEMMA 3.3. *Let $\mathscr{B}$ be a Boolean algebra which has the property* (I), $\mu \in \mathrm{ba}(\mathscr{B})$, *and let $\mathscr{N} = \{E \in \mathscr{B} : |\mu|(E) = 0\}$. Then $\mathscr{B}/\mathscr{N}$ is complete.*

Proof. First suppose $\mathscr{N} = \{0\}$. Let $\{E_n\}_{n \in \omega}$ be an increasing sequence in $\mathscr{B}$. Let $\{F_n\}_{n \in \omega}$ be a decreasing sequence in the set $\mathscr{U}$ of upper bounds of $\{E_n : n \in \omega\}$ such that $\inf_n |\mu|(F_n) = \inf_{F \in \mathscr{U}} |\mu|(F)$. Let $E \in \mathscr{B}$ be such that $E_n \leqq E \leqq F_n$ for all $n \in \omega$. We assert that $E = \bigvee_{n \in \omega} E_n$. Let $F \in \mathscr{U}$. $E \wedge F \in \mathscr{U}$ so that $|\mu|(E \wedge F) \geqq \inf_n |\mu|(F_n) \geqq |\mu|(E)$. $0 = |\mu|(E \sim E \wedge F) \Rightarrow E \sim E \wedge F = 0$. Therefore $E \leqq F$. The proof that $\mathscr{B}$ is complete is just about the same as the analogous part of the proof of Theorem 2.2.

Now let $\mathscr{B}$ and μ be as in the lemma. Let $q : \mathscr{B} \to \mathscr{B}_0 = \mathscr{B}/\mathscr{N}$ be the projection. By Theorem 2.1 $\mathscr{B}_0$ has the property (I). Let $\mu_0 \in \mathrm{ba}(\mathscr{B}_0)$ be defined by $\mu_0(q(E)) = \mu(E)$, $E \in \mathscr{B}_0$. $|\mu_0|(q(E)) = 0 \Rightarrow \mu_0(q(F)) = 0$ whenever $F \leqq E \Rightarrow \mu(F) = 0$ whenever $F \leqq E \Rightarrow |\mu|(E) = 0 \Rightarrow q(E) = 0$. By the above paragraph $\mathscr{B}_0$ is complete.

THEOREM 3.2. *Let $\mathscr{B}$ be a Boolean algebra which has the property* (I). *Then $K \subset \mathrm{ba}(\mathscr{B})$ is bounded iff $\sup_{\mu \in K} |\mu(E)| < \infty$ for all $E \in \mathscr{B}$.*

Proof. The 'only if' part is obvious. Suppose that K is unbounded, but that $\sup_{\mu \in K} |\mu(E)| < \infty$ for all $E \in \mathscr{B}$. Let $\{\mu_n\}_{n \in \omega}$ be an unbounded sequence in K. Set $\mu = \sum_{n \in \omega} 2^{-n} |\mu_n| \, \|\mu_n\|^{-1}$, and set $\mathscr{N} = \{E \in \mathscr{B} : \mu(E) = 0\}$. Let $q : \mathscr{B} \to \mathscr{B}' = \mathscr{B}/\mathscr{N}$ be the projection, and let $\mu'_n \in \mathrm{ba}(\mathscr{B}')$ be defined by $\mu'_n(q(E)) = \mu_n(E)$, $E \in \mathscr{B}$. $\mathscr{B}'$ is a complete Boolean algebra, $\{\mu'_n : n \in \omega\}$ is an unbounded subset of $\mathrm{ba}(\mathscr{B}')$ such that $\sup_n |\mu'_n(E)| < \infty$ for all $E \in \mathscr{B}'$. Thus it is enough to prove the theorem under the additional hypothesis that $\mathscr{B}$ is complete. For $E \in \mathscr{B}$, set $\rho(E) = \sup_{\mu \in K} |\mu(E)|$.

ρ satisfies all the conditions of Theorem 3.1 so that there exists a disjoint sequence $\{E_n\}_{n\in\omega}$ in $\mathscr{B}$ such that $\sup_n \rho(E_n)=\infty$. Let $\{\nu_n\}_{n\in\omega}$ be a sequence in K such that $\lim_n |\nu_n(E_n)|=\infty$. For $A\subset\omega$, let $\lambda_n(A)=\nu_n(\bigvee_{k\in A} E_k)\nu_n(E_n)^{-1}$. Clearly, $\lambda_n\in \mathrm{ba}\,(2^\omega)$ for all n, and $\lambda_n(A)\to 0$ for all $A\subset\omega$. By Lemma 2.3 $\lim_n \lambda_n(\{n\})=0$. But $\lambda_n(\{n\}) =\nu_n(E_n)\nu_n(E_n)^{-1}=1$ for all n.

Corollary. *Let S be a totally disconnected F-space, and let $K\subset M(S)$. Then K is bounded iff $\sup_{\mu\in K} |\mu(E)|<\infty$ for all open closed $E\subset S$.*

Proof. Let $\mathscr{B}$ be the Boolean algebra of open closed subsets of S, and for $\mu\in M(S)$, let $\hat\mu\in \mathrm{ba}\,(\mathscr{B})$ be the restriction of μ to $\mathscr{B}$. Then K is bounded iff $\{\hat\mu : \mu\in K\}$ is bounded.

For the moment define an N-space as a totally disconnected, compact Hausdorff space for which the conclusion of the above corollary holds. Thus the corollary states that totally disconnected F-spaces are N-spaces. We wish now to define N-spaces in such a way that all F-spaces are N-spaces. To this end we make first the

Definition. Let S be a compact Hausdorff space. A subset N of $C(S)$ is *normal* iff for any pair A, B of disjoint closed subsets of S, there exists $f\in N$ such that $f\mid A=0, f\mid B=1$.

Definition. An *N-space* is a compact Hausdorff space S such that for any $K\subset M(S)$, K is bounded if $\{f\in C(S) : \sup_{\mu\in K} |\mu(f)|<\infty\}$ is normal.

Lemma 3.4. *A closed subspace of an N-space is an N-space.*

Proof. Let S be an N-space, and let S_0 be a closed subspace of S. Let $K\subset M(S_0)$ be such that $N=\{f\in C(S_0) : \sup_{\mu\in K} |\mu(f)|<\infty\}$ is normal. Let $\kappa: M(S_0)\to M(S)$ be the map of Lemma 2.1. Let A and B be disjoint closed subsets of S. There exists $f_0\in N$ such that f_0 is 0 on $A\cap S_0$ and 1 on $B\cap S_0$. Let $f_1: A\cup B\cup S_0\to \boldsymbol{R}$ be 0 on A, 1 on B, and equal f_0 on S_0. f_1 is continuous and so has a continuous extension, say f, to all of S. $\sup_{\mu\in K} |\kappa(\mu)(f)|=\sup_{\mu\in\kappa[K]} |\mu(f_0)|<\infty$. Thus

$$\Big\{g\in C(S) : \sup_{\mu\in\kappa[K]} |\mu(g)| < \infty\Big\}$$

is normal. Since S is an N-space, $\kappa[K]$ is bounded. Therefore K is bounded.

Lemma 3.5. *Let S be a compact Hausdorff space, E a Banach space, and $u: E\to C(S)$ a continuous linear map. Then u is onto iff for any $\mu\in M(S)$, $\{u(x)|\mathrm{car}\,(\mu) : x\in E\}=C(\mathrm{car}\,(\mu))$.*

Proof. The 'only if' part is obvious. We first show that u^* has closed range which will imply that u has closed range [3, p. 488]. Let $\{\mu_n\}_{n\in\omega}$ be a sequence in $M(S)$ such that $\{u^*(\mu_n)\}_{n\in\omega}$ converges. Let $\nu=\sum_{n\in\omega} 2^{-n}\|\mu_n\|^{-1} |\mu_n|$, and let $S_0 =\mathrm{car}\,(\nu)$. For $x\in E$, let $u_0(x)=u(x)|S_0$. u_0 is onto so that there exists $r>0$ such that for $\mu\in M(S_0)$, $\|\mu\|\leqq r\|u_0^*(\mu)\|$. $\mathrm{car}\,(\mu_n)\subset S_0$ so that there exist $\mu_n'\in M(S_0)$ such that $\kappa(\mu_n')=\mu_n$ (κ as in Lemma 2.1). $u_0^*(\mu_n')=u^*(\mu_n)$. Thus for $n, m\in\omega$, $\|\mu_n-\mu_m\|=$

$\|\mu_n'-\mu_m'\| \leqq r\,\|u_0^*(\mu_n'-\mu_m')\| = r\,\|u^*(\mu_n-\mu_m)\|$. Thus $\{\mu_n\}_{n\in\omega}$ is Cauchy. Whence $\{u^*(\mu_n)\}_{n\in\omega}$ converges to a member of $u^*[M(S)]$.

We complete the proof by showing that $u[E]$ is dense, and this is the same as showing that u^* is one-to-one. Let $u^*(\mu)=0$, and let $f\in C(S)$. Let $x\in E$ be such that $u(x)|\text{car}(\mu)=f|\text{car}(\mu)$. Then $\mu(f)=\mu(u(x))=u^*(\mu)(x)=0$. Thus $\mu=0$.

LEMMA 3.6. *If S is an N-space, then each normal subspace of $C(S)$ is dense.*

Proof. Let N be a normal subspace of $C(S)$. Let $\mu\in M(S)$ vanish on N. $N\subset\{f\in C(S) : \sup_n|(n\mu)(f)|<\infty\}$ so that the latter set is normal. Since S is an N-space, $\{n\mu : n\in\omega\}$ is thus bounded. Therefore $\mu=0$. It follows that N is $\sigma(C(S), M(S))$-dense and so is norm dense [3, p. 422].

THEOREM 3.3. *Let S be a compact Hausdorff space. Then the following conditions are equivalent.*

(a) *S is an N-space.*

(b) *A sequence $\{\mu_n\}_{n\in\omega}$ in $M(S)$ is $\sigma(M(S), C(S))$-convergent iff*

$$\left\{f\in C(S) : \lim_n \mu_n(f) \text{ exists}\right\}$$

is normal.

(c) *If E is a Banach space and $u\colon E\to C(S)$ is a bounded linear map with normal range, then u is onto.*

(d) *For each $\mu\in M(S)$,* car (μ) *is an N-space.*

Proof. (a) $\Rightarrow$ (b). $\{f\in C(S) : \sup_n|\mu_n(f)|<\infty\}$ is normal so that $\{\mu_n : n\in\omega\}$ is bounded. Alaoglu's theorem [3, p. 424] insures the existence of a $\sigma(M(S), C(S))$-cluster point of $\{\mu_n\}_{n\in\omega}$, and the density of $\{f\in C(S) : \lim_n \mu_n(f) \text{ exists}\}$ insures the unicity of this cluster point, i.e. $\{\mu_n\}_{n\in\omega}$ $\sigma(M(S), C(S))$-converges.

(b) $\Rightarrow$ (c). It is enough to prove that u^* has closed range. Let $\{u^*(\mu_n)\}_{n\in\omega}$ converge to x^*. $\{\mu_n(f)\}_{n\in\omega}$ converges for all $f\in u[E]$ and so $\{\mu_n\}_{n\in\omega}$ $\sigma(M(S), C(S))$-converges, say to μ. Then $u^*(\mu)=x^*$.

(c) $\Rightarrow$ (a). Let $K\subset M(S)$ be such that $E_0=\{f\in C(S) : \sup_{\mu\in K}|\mu(f)|<\infty\}$ is normal. We may assume that K contains the unit ball of $M(S)$. For $f\in E_0$, let $\|f\|_0=\sup_{\mu\in K}|\mu(f)|$. $(E_0, \|\cdot\|_0)$ is a normed space and (since K contains the unit ball of $M(S)$) the inclusion map $u_0\colon E_0\to C(S)$ has norm at most 1. Let $(E, \|\cdot\|)$ be the completion of $(E_0, \|\cdot\|_0)$, and let $u\colon E\to C(S)$ be the continuous extension of u_0. u has normal range and so is onto. Thus there exists $r>0$ such that for $\mu\in M(S)$, $\|\mu\|\leqq r\,\|u^*(\mu)\|$. Let $\mu\in K$. $\|u^*(\mu)\|=\sup\{|\mu(f)| : f\in E_0, \|f\|_0\leqq 1\}$ and for $f\in E_0$, $|\mu(f)|\leqq\|f\|_0$. Thus $\|\mu\|\leqq r$.

(a) $\Rightarrow$ (d). For $\mu\in M(S)$, car (μ) is a closed subspace of S.

(d) $\Rightarrow$ (c). Let $\mu\in M(S)$. Let $S_0=\text{car}(\mu)$. $\{u(x)|S_0 : x\in E\}$ is obviously normal so that $x\to u(x)|S_0$ has normal range. By Lemma 3.5 u is onto.

COROLLARY 1. *An F-space is an N-space.*

Proof. Let S be an F-space. Let $\mu \in M(S)$. car (μ) is Stonian and so is totally disconnected. By the corollary to Theorem 3.2 car (μ) is an N-space.

COROLLARY 2. *Let A be a commutative Banach algebra whose spectrum is a totally disconnected N-space. Then the Gelfand homomorphism of A is onto.*

Proof. Let S be the spectrum of A, and let $\gamma: A \to C(S)$ be the Gelfand homomorphism. By the Šilov idempotent theorem [11, p. 168] $\gamma[A] \supset \{\chi_E : E \subset S, E \text{ open and closed}\}$ and so is normal. Thus $\gamma[A] = C(S)$.

Corollary 2 was proved by Badé and Curtis [2, p. 858] for totally disconnected F-spaces. I do not know if the hypothesis of total disconnectedness can be dropped. The following theorem shows that Corollary 2 is indeed a generalization of the Badé-Curtis result.

THEOREM 3.4 (BADÉ). *Let S be an infinite totally disconnected F-space, and let S_0 be constructed from S as in Theorem* 2.6. *Then S_0 is an N-space but not an F-space.*

Proof. We have already seen that S_0 is not an F-space. Let $\{\mu_n\}_{n \in \omega}$ be a sequence in $M(S)$ such that $\{\mu_n(E)\}_{n \in \omega}$ converges for all open closed $E \subset S_0$. Let E_1 and E_2 be disjoint open closed subsets of S such that $E_1 \cup E_2 = S$ and $x_i \in E_i$, $i = 1, 2$ (we use the notation of Theorem 2.6). Let $f_0 = \chi_{E_2}$ and now define P as in Theorem 2.6. Let E be an open closed subset of E_i. Then

$$\begin{aligned} P(\chi_E) &= \chi_F && \text{if } x_i \notin E, \\ &= \tfrac{1}{2} - \chi_{E_i \sim E} && \text{if } x_i \in E. \end{aligned}$$

Thus $\{(P^*\mu_n)(\chi_E)\}_{n \in \omega}$ converges whenever E is an open closed subset of E_i. It follows that $\{(P^*\mu_n)(\chi_E)\}_{n \in \omega}$ converges whenever E is an open closed subset of S. Thus $\{P^*\mu_n\}_{n \in \omega}$ $\sigma(M(S), C(S))$-converges, say to ν. Let $\mu = \nu | C(S_0)$. Let $f \in C(S_0)$. $\mu_n(f) = \mu_n(Pf) = (P^*\mu_n)(f) \to \nu(f) = \mu(f)$.

The hypothesis of total disconnectedness can be dropped in Theorem 3.4. The proof of the more general theorem has more length than the theorem has interest and so is not given.

I do not know if there are G-spaces which are not N-spaces or if there are N-spaces which are not G-spaces.

4. **The Vitali-Hahn-Saks theorem.** In this section we combine Theorems 2.4 and 3.3 to obtain a generalization (Theorem 4.1) of the Vitali-Hahn-Saks theorem (for whose statement see the remarks following the proof of Theorem 4.1).

LEMMA 4.1. *Let S be a compact Hausdorff space, $\mu \in M(S)$, and let $\{\mu_n\}_{n \in \omega}$ be a $\sigma(M(S), M(S)^*)$-convergent sequence in $M(S)$ such that each μ_n is absolutely continuous with respect to μ. Then $\{\mu_n : n \in \omega\}$ is uniformly absolutely continuous with respect to μ.*

Proof. Let $\{\mu_n\}_{n \in \omega}$ $\sigma(M(S), M(S)^*)$-converge to ν. The set of measures in $M(S)$ which are absolutely continuous with respect to μ is a norm-closed subspace of

$M(S)$ and hence is $\sigma(M(S), M(S)^*)$-closed. Thus ν is absolutely continuous with respect to μ. Thus we may assume $\nu=0$. Now suppose the lemma is false. Then there is a sequence $\{E_n\}_{n\in\omega}$ of Borel sets such that $|\mu|(E_n)\to 0$ and $\{\mu_k(E_n)\}_{n\in\omega}$ does not converge to 0 uniformly for $k\in\omega$. By passing to subsequences if necessary we may assume the existence of an $\varepsilon>0$ such that $|\mu_n(E_n)|>\varepsilon$ for all n. One easily constructs inductively a strictly increasing sequence $\{n_k\}_{k\in\omega}$ in ω such that for each k,

$$|\mu_{n_k}(E_{n_k})|-\varepsilon > |\mu_{n_k}|\Big(\bigcup_{i\geqq n_{k+1}} E_i\Big).$$

Let $E_k'=E_{n_k}\sim\bigcup_{i\geqq n_{k+1}} E_i$, $\mu_k'=\mu_{n_k}$. Then $\{\mu_k'\}_{k\in\omega}$ $\sigma(M(S), M(S)^*)$-converges to 0, each μ_k' is absolutely continuous with respect to μ, $\{E_k'\}_{k\in\omega}$ is disjoint, and $|\mu_k'(E_k')|>\varepsilon$ for all k. For $A\subset\omega$, let $\nu_k(A)=\mu_k'(\bigcup_{i\in A} E_i')$. $\{\nu_k\}_{k\in\omega}$ is a sequence in ba (2^ω) such that $\nu_k(A)\to 0$ for all $A\in 2^\omega$. By Lemma 2.3 $\nu_k(\{k\})\to 0$. But $\nu_k(\{k\}) =\mu_k'(E_k')$ and $|\mu_k'(E_k')|>\varepsilon$ for all k.

THEOREM 4.1. *Let $\mathscr{B}$ be a Boolean algebra which has the property* (I), $\lambda\in$ ba $(\mathscr{B})$, *and let $\{\mu_n\}_{n\in\omega}$ be a sequence in* ba $(\mathscr{B})$ *such that*

(1) *each μ_n is absolutely continuous with respect to λ*,

(2) *for $E\in\mathscr{B}$, $\{\mu_n(E)\}_{n\in\omega}$ converges.*

Then $\{\mu_n : n\in\omega\}$ is uniformly absolutely continuous with respect to λ.

Proof. Let S be the Stone space of $\mathscr{B}$ and η the Stone isomorphism of $\mathscr{B}$ to the algebra $\mathscr{O}$ of open closed subsets of S. For $\nu\in$ ba $(\mathscr{B})$, let $H(\nu)$ be the unique member of $M(S)$ which satisfies

$$H(\nu)(\eta(E)) = \nu(E), \qquad E\in\mathscr{B}.$$

H is a linear isometry of ba $(\mathscr{B})$ onto $M(S)$. S is an N-space, and $\{H(\mu_n)(E)\}_{n\in\omega}$ converges for all open closed $E\subset S$ so that $\{H(\mu_n)\}_{n\in\omega}$ $\sigma(M(S), C(S))$-converges. S is also a G-space so that $\{H(\mu_n)\}_{n\in\omega}$ $\sigma(M(S), M(S)^*)$-converges. By Lemma 4.1 $\{H(\mu_n)\}_{n\in\omega}$ is uniformly absolutely continuous with respect to $H(\lambda)$. The theorem now follows.

The statement of the Vitali-Hahn-Saks Theorem is obtained by replacing in the statement of Theorem 4.1 'has the property (I)' by 'is σ-complete' and 'ba $(\mathscr{B})$' by 'ca $(\mathscr{B})$'. Andô proved Theorem 4.1 for the case of $\mathscr{B}$ σ-complete.

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