RATIO PROPERTIES IN
LOCALLY COMPACT AMENABLE GROUPS

BY

WILLIAM R. EMERSON

0. Introduction and summary. Let Γ be a locally compact (Hausdorff) topological group, and let $|*|$ denote a fixed left Haar measure on the Borel sets of Γ (in the sense of [1, §51]). It is known (see [2]) that the existence of a left invariant mean on $L^\infty(\Gamma)$ is equivalent to the Følner Condition (FC):

To any $\varepsilon > 0$ and any compact set $C \subseteq \Gamma$, there corresponds a Borel set V of finite nonzero measure such that

$$\text{(FC)} \qquad |\gamma V \triangle V|/|V| < \varepsilon$$

for all $\gamma \in C$.

The purpose of the present paper is the investigation of sequences $\{V_n\}$ of Borel sets of Γ which eventually satisfy either condition (FC) *for every* $\varepsilon > 0$ and compact $C \subseteq \Gamma$, or a stronger condition. More precisely, we say a sequence $\{V_n\}$ of Borel sets of Γ has the weak ratio property, written $\{V_n\} \in W_\Gamma$, if and only if for all $\gamma \in \Gamma$

$$\operatorname*{Lim}_{n \to \infty} \frac{|\gamma V_n \triangle V_n|}{|V_n|} = 0.$$

Similarly $\{V_n\}$ is said to have the strong ratio property, written $\{V_n\} \in S_\Gamma$, if and only if for all compact $C \neq \varnothing$ in Γ,

$$\operatorname*{Lim}_{n \to \infty} \frac{|CV_n \triangle V_n|}{|V_n|} = 0.$$

In §I we consider primarily properties of individual sequences of sets with ratio properties, and obtain particularly detailed results in case Γ is Abelian. We also show that $W_\Gamma \neq \varnothing$ if and only if Γ is amenable and σ-compact. In §II properties of the sets W_Γ and S_Γ are investigated, and we show that S_Γ is always "dense" in W_Γ. From this result we conclude that $S_\Gamma \neq \varnothing$ if and only if Γ is amenable and σ-compact, and obtain necessary conditions for a sequence of sets to have the weak ratio property. Finally, in §III a generalization of the weak ratio property is made to product measure spaces $\Gamma \times S$, where S is any positive measure space, and the case of finitely generated discrete Abelian Γ is considered in detail.

The following notational conventions are observed throughout: N is the set of nonnegative integers; N^+ is the set of positive integers. If Γ is a topological group,

e denotes the unit of Γ, $\overline{V}$ denotes the closure of $V \subseteq \Gamma$, $\Delta(\gamma)$ denotes the modular function of Γ evaluated at $\gamma \in \Gamma$, and we frequently use additive notation in case Γ is Abelian. Finally, $A \sim B$ denotes the difference of the sets A and B and I_V denotes the characteristic function of the set V.

I. Sequences of sets satisfying the ratio properties.

I.1. *Definition of the ratio properties and equivalent formulations.* We first make the following basic definitions:

DEFINITION 1. A sequence $\{V_n\}$ of Borel sets of Γ of finite nonzero measure is said to satisfy the weak ratio property with respect to Γ if and only if

$$\operatorname*{Lim}_{n \to \infty} \frac{|\gamma V_n \triangle V_n|}{|V_n|} = 0$$

for all $\gamma \in \Gamma$. Denote the collection of all such sequences by W_Γ. Furthermore, let W_Γ^0 be the subset of W_Γ consisting of those sequences $\{V_n\}$ with only finitely many $\overline{V}_n$ noncompact.

DEFINITION 2. A sequence $\{V_n\}$ of Borel sets of Γ of finite nonzero measure is said to satisfy the strong ratio property with respect to Γ if and only if

$$\operatorname*{Lim}_{n \to \infty} \frac{|CV_n \triangle V_n|}{|V_n|} = 0$$

for all compact subsets $C \neq \varnothing$ of Γ. Denote the collection of all such sequences by S_Γ. Furthermore, let S_Γ^0 be the subset of S_Γ consisting of those sequences $\{V_n\}$ with only finitely many $\overline{V}_n$ noncompact.

The following theorems give some conditions equivalent to the ratio properties:

THEOREM 1. *If $\{V_n\}$ is a sequence of Borel sets of Γ with finite nonzero measure, the following are equivalent:*

(i) $\{V_n\} \in W_\Gamma$.

(ii) $\operatorname{Lim}_{n \to \infty} (|CV_n|/|V_n|) = 1$ *for all finite subsets $C \neq \varnothing$ of Γ.*

(iii) $\operatorname{Lim}_{n \to \infty} (|\bigcap_{\gamma \in C} \gamma V_n|/|V_n|) = 1$ *for all finite subsets $C \neq \varnothing$ of Γ.*

Proof. (ii) implies (i) upon setting $C = \{\gamma, e\}$ and noting that $CV_n = \gamma V_n \cup V_n$ and $\gamma V_n \triangle V_n \subseteq (CV_n \sim V_n) \cup (CV_n \sim \gamma V_n)$ and therefore

$$\operatorname*{Lim\,sup}_{n \to \infty} \frac{|\gamma V_n \triangle V_n|}{|V_n|} \leq \operatorname*{Lim\,sup}_{n \to \infty} \frac{(|CV_n \sim V_n| + |CV_n \sim \gamma V_n|)}{|V_n|}$$

$$= \operatorname*{Lim\,sup}_{n \to \infty} \frac{2(|CV_n| - |V_n|)}{|V_n|} = 0.$$

(iii) also implies (i) upon setting $C = \{\gamma, e\}$ and noting that

$$\gamma V_n \triangle V_n \subseteq (\gamma V_n \sim (\gamma V_n \cap V_n)) \cup (V_n \sim (\gamma V_n \cap V_n))$$

and therefore

$$\operatorname*{Lim\,sup}_{n \to \infty} \frac{|\gamma V_n \triangle V_n|}{|V_n|} \leq \operatorname*{Lim\,sup}_{n \to \infty} \frac{(|\gamma V_n| - |\gamma V_n \cap V_n| + |V_n| - |\gamma V_n \cap V_n|)}{|V_n|} = 0.$$

Now to show (i) implies (ii), let $C = \{\gamma_1, \ldots, \gamma_\kappa\}$ be any nonempty finite subset of Γ. Therefore $\{V_n\} \in W_\Gamma$ implies

$$\operatorname*{Lim}_{n \to \infty} \frac{|\gamma_i V_n \triangle V_n|}{|V_n|} = 0 \quad \text{for } 1 \leq i \leq \kappa.$$

But $\gamma_i V_n \subseteq (\gamma_i V_n \triangle V_n) \cup V_n$, and therefore

$$CV_n = \bigcup_{1 \leq i \leq \kappa} \gamma_i V_n \subseteq \bigcup_{1 \leq i \leq \kappa} (\gamma_i V_n \triangle V_n) \cup V_n$$

which implies

$$1 \leq \operatorname*{Lim\,sup}_{n \to \infty} \frac{|CV_n|}{|V_n|} \leq \operatorname*{Lim\,sup}_{n \to \infty} \frac{\sum\limits_{1 \leq i \leq \kappa} |\gamma_i V_n \triangle V_n| + |V_n|}{|V_n|} = 1.$$

Finally to show that (i) implies (iii) we note that

$$\bigcap_{\gamma \in C} \gamma V_n = \{\{\cdots\{(\gamma_1 V_n) \sim (\gamma_1 V_n \triangle \gamma_2 V_n)\} \sim \cdots\} \sim (\gamma_{\kappa-1} V_n \triangle \gamma_\kappa V_n)\}$$

and consequently

$$1 \geq \operatorname*{Lim\,inf}_{n \to \infty} \frac{\left|\bigcap\limits_{\gamma \in C} \gamma V_n\right|}{|V_n|} \geq \operatorname*{Lim\,inf}_{n \to \infty} \frac{|V_n| - \sum\limits_{1 \leq i < \kappa} |\gamma_i V_n \triangle \gamma_{i+1} V_n|}{|V_n|}$$

$$= 1 - \operatorname*{Lim\,sup}_{n \to \infty} \sum_{1 \leq i < \kappa} \frac{|\gamma_{i+1}^{-1} \gamma_i V_n \triangle V_n|}{|V_n|} = 1.$$

THEOREM 2. *If $\{V_n\}$ is a sequence of Borel sets of Γ with finite nonzero measure, $\{V_n\} \in S_\Gamma$ if and only if $\operatorname{Lim}_{n \to \infty} |CV_n|/|V_n| = 1$ for all compact subsets $C \neq \varnothing$ of Γ.*

Proof. Let $\{V_n\} \in S_\Gamma$, and let $C \neq \varnothing$ be any compact subset of Γ. Then

$$0 \leq \operatorname*{Lim\,sup}_{n \to \infty} \frac{|CV_n| - |V_n|}{|V_n|} \leq \operatorname*{Lim\,sup}_{n \to \infty} \frac{|CV_n \sim V_n|}{|V_n|} \leq \operatorname*{Lim\,sup}_{n \to \infty} \frac{|CV_n \triangle V_n|}{|V_n|} = 0,$$

and consequently $\operatorname{Lim}_{n \to \infty} |CV_n|/|V_n| = 1$.

Conversely if $C \neq \varnothing$ is any compact subset of Γ, let $C' = C \cup \{e\}$ and therefore $\operatorname{Lim}_{n \to \infty} |C'V_n|/|V_n| = 1$. But $C'V_n = V_n \cup CV_n = V_n \cup (CV_n \sim V_n) = CV_n \cup (V_n \sim CV_n)$ and therefore, since $|CV_n| \geq |V_n|$, $\operatorname{Lim}_{n \to \infty} |CV_n \sim V_n|/|V_n| = \operatorname{Lim}_{n \to \infty} |V_n \sim CV_n|/|V_n| = 0$ implying $\operatorname{Lim}_{n \to \infty} |CV_n \triangle V_n|/|V_n| = 0$.

I.2. *Uniformity of convergence.* We now show that the pointwise convergence to 0 in the definition of W_Γ is equivalent to the formally stronger condition of uniform convergence to 0 on compacta, and derive some useful corollaries. The following two lemmas are pivotal in the proof of Theorem 3:

LEMMA 1. *Let K be a compact neighborhood of the unit in Γ, and let $\{f_n\}$ be a sequence of uniformly bounded positive measurable functions on Γ such that*

(1) $\operatorname{Lim}_{n \to \infty} f_n(\gamma) = 0$ *for all $\gamma \in K^2$,*

(2) *There exists an absolute constant $c > 0$ such that $f_n(\alpha\beta) \geqq c|f_n(\alpha) - f_n(\beta)|$ for all $\alpha, \beta \in K$.*

Then $f_n \to 0$ uniformly on K.

Proof. Let $S_n(\varepsilon) = \{\gamma \in K^2 : f_n(\gamma) \geqq \varepsilon\}$ for $n \in N^+$ and $\varepsilon > 0$. Since pointwise convergence implies convergence in measure ($|K^2| < +\infty$) we conclude $\mathrm{Lim}_{n \to \infty} |S_n(\varepsilon)| = 0$ for all $\varepsilon > 0$. Now assume convergence to 0 on K is not uniform. Then without loss of generality (extracting a subsequence if necessary) we may assume that there exists an $\eta > 0$ and a sequence $\{\alpha_n\} \subseteq K$ such that $f_n(\alpha_n) > \eta$. Let $T_n(\varepsilon) = S_n(\varepsilon) \cap K$, and consider $B_n = T_n(\eta/2) \cup \alpha_n(K \sim T_n(\eta/2)) \subseteq K^2$. First $|B_n| \geqq |K|/2 > 0$ by left invariance of Haar measure ($|K| > 0$ since K has nonempty interior). But also $B_n \subseteq S_n(\varepsilon_0)$, where $\varepsilon_0 = \mathrm{Min} \{\eta/2, c\eta/2\}$. For $\gamma \in B_n$ implies either $f_n(\gamma) \geqq \eta/2$ in case $\gamma \in T_n(\eta/2)$, or else $\gamma = \alpha_n\beta$ where $\beta \in K \sim T_n(\eta/2)$ and hence by (2) $f_n(\gamma) = f_n(\alpha_n\beta) \geqq c|f_n(\alpha_n) - f_n(\beta)| \geqq c|\eta - \eta/2| = c\eta/2$. But this clearly violates $\mathrm{Lim}_{n \to \infty} |S_n(\varepsilon_0)| = 0$.

Lemma 2. *Let X be a Borel set of finite nonzero measure in the locally compact group Γ, and let $f_X(\gamma) = |\gamma X \triangle X|/|X|$ for $\gamma \in \Gamma$. Then for all $\alpha, \beta \in \Gamma$, $f_X(\alpha\beta) \geqq |f_X(\alpha) - f_X(\beta)|$.*

Proof. We apply the metric property of symmetric difference and the left invariance of Haar measure to obtain

$$|\beta X \triangle X| \leqq |\beta X \triangle \alpha^{-1} X| + |\alpha^{-1} X \triangle X| = |\alpha\beta X \triangle X| + |X \triangle \alpha X|,$$

$$|\alpha X \triangle X| \leqq |\alpha X \triangle \alpha\beta X| + |\alpha\beta X \triangle X| = |X \triangle \beta X| + |\alpha\beta X \triangle X|.$$

Upon combining these two inequalities we obtain the assertion.

We now prove the "uniform" analogue of Theorem 1:

Theorem 3. *If $\{V_n\}$ is a sequence of Borel sets of Γ with finite nonzero measure, the following are equivalent:*

(i) $\{V_n\} \in W_\Gamma$.

(ii) $|\gamma V_n \triangle V_n|/|V_n| \to 0$ *uniformly for γ in compacta.*

(iii) *If K is any positive integer and C any compact subset of Γ,*

$$\frac{\left|\bigcap_{1 \leqq i \leqq k} \gamma_i V_n\right|}{|V_n|} \to 1 \quad \text{uniformly for } \{\gamma_1, \ldots, \gamma_k\} \subseteq C.$$

(iv) *If K is any positive integer and C any compact subset of Γ,*

$$\frac{|\{\gamma_1, \ldots, \gamma_k\}V_n|}{|V_n|} \to 1 \quad \text{uniformly for } \{\gamma_1, \ldots, \gamma_k\} \subseteq C.$$

Proof. Condition (ii), (iii), or (iv) trivially implies (i), in light of Theorem 1. Conversely, we prove (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv):

(i) $\Rightarrow$ (ii): Using the notation of Lemma 2, $\mathrm{Lim}_{n \to \infty} f_{V_n}(\gamma) = 0$ for all $\gamma \in \Gamma$. Hence (ii) is an immediate consequence of Lemmas 1 and 2 since each f_{V_n} is a continuous function with values in $[0, 1]$.

(ii) $\Rightarrow$ (iii). Fix any compact set C and $\varepsilon > 0$. By (ii) there exists an n_0 such that $|\gamma V_n \triangle V_n|/|V_n| < \varepsilon$ for all $n \geq n_0$ and γ in the compact set $C_1 = C^{-1} \cdot C$. Hence

$$\left| \bigcap_{1 \leq i \leq k} \gamma_i V_n \right| \geq |\gamma_1 V_n| - \sum_{1 \leq i < k} |\gamma_i V_n \triangle \gamma_{i+1} V_n|$$

$$= |V_n| - \sum_{1 \leq i < k} |\gamma_{i+1}^{-1} \gamma_i V_n \triangle V_n| \geq |V_n| - (k-1)\varepsilon$$

since $\gamma_{i+1}^{-1} \gamma_i \in C^{-1} \cdot C = C_1$ if $\{\gamma_1, \ldots, \gamma_k\} \subseteq C$, and (iii) follows.

(ii) $\Rightarrow$ (iv). $\{\gamma_1, \ldots, \gamma_k\} V_n \subseteq \bigcup_{1 \leq i \leq k} (\gamma_i V_n \triangle V_n) \cup V_n$, implying

$$1 \leq \frac{|\{\gamma_1, \ldots, \gamma_k\} V_n|}{|V_n|} \leq \sum_{1 \leq i \leq k} \frac{|\gamma_i V_n \triangle V_n|}{|V_n|} + 1,$$

and thus (iv) follows readily from (ii).

COROLLARY 1. *If $\{V_n\} \in W_\Gamma$, then $\{V_n\}$ eventually satisfies condition* (FC) *for every $\varepsilon > 0$ and compact $C \subseteq \Gamma$.*

Proof. Clear from (ii) of Theorem 3.

As an immediate consequence of the uniformity of convergence we obtain the following corollaries:

COROLLARY 2. *Let $\{V_n\} \in W_\Gamma$, let C' be a compact subset of Γ, and let $\{\gamma_n'\}$ be any sequence of elements in C'. Then* $\text{Lim}_{n \to \infty} |\gamma_n' V_n \triangle V_n|/|V_n| = 0$.

Proof. Theorem 3, (ii).

COROLLARY 3. *Let $\{V_n\} \in W_\Gamma$, let C be a compact subset of Γ, and let $\{\gamma_n\}$ be any sequence of elements in C. Then $\{\gamma_n V_n\} \in W_\Gamma$.*

Proof. For any $\gamma \in \Gamma$, upon setting $\gamma_n' = \gamma_n^{-1} \gamma \gamma_n$ we obtain

$$\frac{|\gamma \gamma_n V_n \triangle \gamma_n V_n|}{|\gamma_n V_n|} = \frac{|\gamma_n^{-1} \gamma \gamma_n V_n \triangle V_n|}{|V_n|} = \frac{|\gamma_n' V_n \triangle V_n|}{|V_n|},$$

and upon noting that γ_n' is in the compact set $C' = C^{-1} \gamma C$ for all $n \in N^+$ we immediately obtain Corollary 3 from Corollary 2.

REMARK. Corollary 2 will be essential in the proof of Theorem 13.

I.3. Γ *for which* $W_\Gamma \neq \varnothing$. We are now in a position to prove:

THEOREM 4. $W_\Gamma \neq \varnothing$ *if and only if Γ is amenable and σ-compact.*

Proof. Assume $W_\Gamma \neq \varnothing$. Then by Corollary 1 Γ must be amenable. Moreover, let $\{V_n\} \in W_\Gamma$. Since any Borel set of Γ is σ-bounded, $V = \bigcup_n V_n$ is σ-bounded since each V_n, $n \in N^+$, is a Borel set. Consequently $\langle V \rangle$, the subgroup of Γ generated by V, is also σ-bounded. But $\Gamma = \langle V \rangle$; for otherwise there is a $\gamma \in \Gamma \sim \langle V \rangle$, which clearly implies $\gamma V_n \cap V_n = \varnothing$ $(n \in N^+)$, and therefore $|\gamma V_n \triangle V_n| = 2|V_n|$ and $\{V_n\} \notin W_\Gamma$, a contradiction.

Conversely, assume Γ is σ-compact and amenable. Let $\{C_n\}$ be any sequence of

nested compact subsets of Γ such that the union of the interiors of C_n cover Γ. By condition (FC), for each $n \in N^+$ we may find a Borel set V_n of Γ with finite nonzero measure such that $|\gamma V_n \triangle V_n| < 1/n|V_n|$ for all $\gamma \in K_n$. This clearly implies $\{V_n\} \in W_\Gamma \neq \varnothing$.

I.4. *Abelian groups.* We first show that $S_\Gamma^0 \neq \varnothing$ for σ-compact locally compact Abelian groups, and in fact "construct" a rather special sequence $\{V_n\} \in S_\Gamma^0$.

THEOREM 5. *If Γ is any σ-compact locally compact Abelian group, then $S_\Gamma^0 \neq \varnothing$. Moreover there exists $\{V_n\} \in S_\Gamma^0$ satisfying*
  (i) $V_n \subseteq V_{n+1}$ *for all $n \in N^+$,*
  (ii) $\bigcup_{n \in N^+}$ *interior* $(V_n) = \Gamma$,
  (iii) *Each V_n partitions Γ, i.e. Γ is the disjoint union of translates of V_n.*

Proof. It suffices to construct the sequence $\{V_n\}$ of the theorem. It is clear that *any* σ-compact group Γ contains an ascending sequence $\{K_n\}$ of compact neighborhoods of the identity such that the union of the interiors of the K_n is Γ. Consider any such $\{K_n\}$ for the Abelian group we are considering, and let H_n be the subgroup of Γ generated by K_n. Then H_n is a compactly generated open subgroup of Γ, and by the structure theorem for compactly generated Abelian groups there is a topological isomorphism

$$H_n = Z^{a_n} \times R^{b_n} \times C_n$$

where a_n and $b_n \in N$ and C_n is compact. We identify elements of the two groups under this isomorphism. Since H_n is open in Γ we may take Haar measure on H_n to be the restriction of Haar measure on Γ. For all $t \in N^+$, we define

$$V_n^t = \{0, \pm 1, \ldots, \pm t\}^{a_n} \times \{(-t, t]\}^{b_n} \times C_n.$$

Note that V_n^t partitions Γ since the subgroup

$$H_n^t = \{(2t+1)Z\}^{a_n} \times \{2tZ\}^{b_n} \times \{0\} \subseteq H_n$$

has V_n^t as a set of coset representatives. Also for $t_1, t_2 \in N^+$

$$V_n^{t_1} + V_n^{t_2} = V_n^{t_1 + t_2},$$

(1.11)
$$|V_n^{t_2}| = \left(\frac{2t_2 + 1}{2t_1 + 1}\right)^{a_n} \left(\frac{t_2}{t_1}\right)^{b_n} |V_n^{t_1}|.$$

Now since $K_n \subseteq H_n$ is compact there exists $r = r(n) \in N^+$ such that $K_n \subseteq V_n^r$. But by (1.11) there also exists an $s_0 = s_0(n) \in N^+$ such that for $s \geq s_0$, $|V_n^r + V_n^s| < (1 + 1/n)|V_n^s|$. We now construct V_n inductively; let $V_1 = V_1^{s_0(1)}$, and assume V_k has been chosen inductively for $k < n$. We choose $V_n = V_n^{t(n)}$ where $t(n)$ is subject to
  (1) $t(n) \geq s_0(n)$,
  (2) $2V_{n-1} = V_{n-1} + V_{n-1} \subseteq V_n^{t(n)}$.
(2) is true for all sufficiently large t since $2V_{n-1} \subseteq H_{n-1} \subseteq H_n$ and $2V_{n-1}$ has compact closure.

We now verify that $\{V_n\}$ satisfies the conditions of the theorem. (i) and (iii) are clear, and (ii) follows without difficulty from condition (2) of the inductive construction. To verify $\{V_n\} \in S_\Gamma^0$ let $C \neq \varnothing$ be a compact subset of Γ. Since $\{\text{interior } (K_n) : n \in N^+\}$ was constructed to be an open cover for Γ and since the K_n are nested we have $C \subseteq K_n$ for $n \geq n_0$. Hence, since $K_n \subseteq V_n^{r(n)}$

$$1 \leq \operatorname*{Lim\,sup}_{n \to \infty} \frac{|C + V_n|}{|V_n|} \leq \operatorname*{Lim\,sup}_{n \to \infty} \frac{|V_n^{r(n)} + V_n^{t(n)}|}{|V_n|} \leq \operatorname*{Lim\,sup}_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1,$$

by (1). Noting that all V_n have compact closure, we conclude by Theorem 2 that $\{V_n\} \in S_\Gamma^0$.

We now prove that $\{V_n\} \in W_\Gamma$ if the sets V_n are "well-approximated" by disjoint translates of each set which partitions Γ. More precisely:

Theorem 6. *Let Γ be a locally compact Abelian group and $\{V_n\}$ a sequence of Borel sets of Γ with finite nonzero measure. Then $\{V_n\} \in W_\Gamma$ if for every Borel set K of Γ with compact closure which partitions Γ there exist subsets $T_n \subseteq \Gamma$ such that*

$$\operatorname*{Lim}_{n \to \infty} \frac{\left| V_n \bigtriangleup \bigcup_{\tau \in T_n} (K + \tau) \right|}{|V_n|} = 0$$

and all $K + \tau$, $\tau \in T_n$ are mutually disjoint.

Proof. Fix any $\gamma \in \Gamma$, let U be any compact neighborhood of γ, and let H be the compactly generated open subgroup of Γ generated by U. By the structure theorem there exist $a, b \in N$ and a compact group C such that $H \cong Z^a \times R^b \times C$. We identify corresponding elements of these two groups. Now fix any $\varepsilon > 0$. For sufficiently large r the set

$$K = K(r) = \{0, \pm 1, \ldots, \pm r\}^a \times ((-r, r])^b \times C$$

clearly satisfies $|(K + \gamma) \bigtriangleup K| < \varepsilon |K|$, where Haar measure on H is taken to be the restriction of Haar measure on Γ. Moreover K is a Borel set of Γ with (finite) nonzero measure and compact closure which partitions Γ. Hence, by hypothesis there exist subsets $T_n \subseteq \Gamma$ such that

$$\operatorname*{Lim}_{n \to \infty} \frac{\left| V_n \bigtriangleup \bigcup_{\tau \in T_n} (K + \tau) \right|}{|V_n|} = 0,$$

where the union is disjoint. Using the translation invariance of Haar measure we also infer

$$\operatorname*{Lim}_{n \to \infty} \frac{\left| (V_n + \gamma) \bigtriangleup \bigcup_{\tau \in T_n} (K + \tau + \gamma) \right|}{|V_n|} = 0.$$

Thus, by the metric property of $\rho(A, B) = |A \triangle B|$,

$$\frac{|V_n \triangle (V_n + \gamma)|}{|V_n|} \leqq \frac{\left|V_n \triangle \bigcup_{\tau \in T_n} (K + \tau)\right|}{|V_n|} + \frac{\left|\left(\bigcup_{\tau \in T_n} (K + \tau)\right) \triangle \left(\bigcup_{\tau \in T_n} (K + \tau + \gamma)\right)\right|}{|V_n|}$$

$$+ \frac{\left|\bigcup_{\tau \in T_n} (K + \tau + \gamma) \triangle (V_n + \gamma)\right|}{|V_n|}$$

and consequently

$$\underset{n \to \infty}{\text{Lim sup}} \frac{|V_n \triangle (V_n + \gamma)|}{|V_n|} \leqq \underset{n \to \infty}{\text{Lim sup}} \frac{\left|\bigcup_{\tau \in T_n} (K + \tau) \triangle \bigcup_{\tau \in T_n} (K + \tau + \gamma)\right|}{|V_n|}.$$

Now in general

$$\left|\bigcup_{i \in I} A_i \triangle \bigcup_{i \in I} B_i\right| \leqq \sum_{i \in I} |A_i \triangle B_i|,$$

and therefore

$$\frac{\left|\bigcup_{\tau \in T_n} (K + \tau) \triangle \bigcup_{\tau \in T_n} (K + \tau + \gamma)\right|}{|V_n|} \leqq \frac{\sum_{\tau \in T_n} |(K + \tau) \triangle (K + \tau + \gamma)|}{|V_n|}$$

$$= \sum_{\tau \in T_n} \frac{|K \triangle (K + \gamma)|}{|V_n|} \leqq \frac{\varepsilon}{|V_n|} \sum_{\tau \in T_n} |K| = \varepsilon \frac{\left|\bigcup_{\tau \in T_n} (K + \tau)\right|}{|V_n|},$$

by the construction of K, translation invariance, and the disjointness of $K + \tau$, $\tau \in T_n$. But as $n \to +\infty$

$$\frac{\left|\bigcup_{\tau \in T_n} (K + \tau)\right|}{|V_n|} \to 1$$

since

$$\frac{\left|V_n \triangle \bigcup_{\tau \in T_n} (K + \tau)\right|}{|V_n|} \to 0,$$

and we finally obtain Lim $\sup_{n \to \infty} |V_n \triangle (V_n + \gamma)|/|V_n| \leqq \varepsilon$. Since ε may be arbitrarily small the proof is complete.

II. **Properties of** W_Γ, W_Γ^0, S_Γ, **and** S_Γ^0. By Theorem 4 the theory is vacuous unless Γ is σ-compact and amenable. Hence unless otherwise stated we always assume Γ is a group of this category.

II.1. *Inclusions among the basic sets.* We first prove a useful proposition:

PROPOSITION 1. *If* Γ *is any locally compact group,* (i) $S_\Gamma = \varnothing \Leftrightarrow S_\Gamma^0 = \varnothing$, (ii) $W_\Gamma = \varnothing \Leftrightarrow W_\Gamma^0 = \varnothing$.

Proof. The forward implications are trivial. Conversely, assume $\{V_n\} \in S_\Gamma$ or $\{V_n\} \in W_\Gamma$. Since Haar measure is regular ([1, Theorem B, p. 254]), for each $n \in N^+$ we may find a compact set $V_n' \subseteq V_n$ such that $|V_n \sim V_n'| < 1/n |V_n|$. It is now easily verified that $\{V_n'\} \in S_\Gamma^0$ or $\{V_n'\} \in W_\Gamma^0$, respectively.

In the following results we study the questions of relations among W_Γ, W_Γ^0, S_Γ, and S_Γ^0.

THEOREM 7. *In any locally compact group* Γ, (i) $W_\Gamma^0 \subseteq W_\Gamma$, (ii) $S_\Gamma^0 \subseteq S_\Gamma$, (iii) $S_\Gamma^0 \subseteq W_\Gamma^0$, (iv) $S_\Gamma \subseteq W_\Gamma$.

Proof. (i) and (ii) are immediate from the definitions. (iv) follows since single points are compact sets, and (iii) follows from (iv) since $S_\Gamma^0 \subseteq S_\Gamma \subseteq W_\Gamma$.

THEOREM 8. *Suppose* $S_\Gamma^0 \neq \varnothing$. *Then* $S_\Gamma = S_\Gamma^0 \Leftrightarrow \Gamma$ *is a unimodular group.*

Proof. $S_\Gamma^0 \subseteq S_\Gamma$ always. Hence it suffices to prove $S_\Gamma \subseteq S_\Gamma^0 \Leftrightarrow \Gamma$ is unimodular, assuming $S_\Gamma^0 \neq \varnothing$.

First assume Γ is not unimodular and let $\{V_n\} \in S_\Gamma^0$. Let $\{\gamma_m\}$ be a sequence of elements in Γ such that,

(i) $\sum_{m \geq n} \Delta(\gamma_m) < 1/n |V_n|$ for all $n \in N^+$,

(ii) $\Delta(\gamma_n) < 1/n$ for all $n \in N^+$,

where Δ is the modular function. Such a sequence exists because $|V_n| > 0$ and Δ assumes arbitrarily small values on Γ. Let $E_n = \{\gamma_m : m \geq n\}$, and set $V_n' = V_n \cup E_n$. We shall show that $\{V_n'\} \in S_\Gamma \sim S_\Gamma^0$. The set $\overline{V}_n'$ is not compact since the continuous nonvanishing function Δ takes arbitrarily small values on $E_n \subseteq \overline{V}_n'$. Therefore $\{V_n'\} \notin S_\Gamma^0$. But $\{V_n'\} \in S_\Gamma$. For, let $C \neq \varnothing$ be a compact subset of Γ. Since Γ is not unimodular, it is certainly not discrete. Thus $|E_n| = 0$ and hence $|V_n'| = |V_n|$. Also

$$CV_n' = CV_n \cup CE_n = CV_n \cup \bigcup_{m \geq n} C\gamma_m,$$

implying

$$\frac{|CV_n' \triangle CV_n|}{|V_n|} \leq \sum_{m \geq n} \frac{|C\gamma_m|}{|V_n|} = \frac{|C|}{|V_n|} \sum_{m \geq n} \Delta(\gamma_m) < \frac{|C|}{n}.$$

Moreover,

$$\frac{|CV_n' \triangle V_n'|}{|V_n'|} = \frac{|CV_n' \triangle V_n|}{|V_n|} \leq \frac{|CV_n' \triangle CV_n|}{|V_n|} + \frac{|CV_n \triangle V_n|}{|V_n|}.$$

But

$$\operatorname*{Lim}_{n \to \infty} \frac{|CV_n' \triangle CV_n|}{|V_n|} = \operatorname*{Lim}_{n \to \infty} \frac{|CV_n \triangle V_n|}{|V_n|} = 0,$$

and therefore $\{V_n'\} \in S_\Gamma$.

To prove the converse, assume Γ is unimodular and $\{V_n\} \in S_\Gamma \sim S_\Gamma^0$. For notational simplicity assume *all* $\overline{V}_n$ noncompact (if $\{V_n\} \in S_\Gamma$, any subsequence $\{V_{n_i}\}$ $\in S_\Gamma$). Let C be any compact subset of Γ of positive measure. We shall show $|CV_n| = \infty$, contradicting $\{V_n\} \in S_\Gamma$. It suffices to construct a sequence $\{v_i\} \subseteq V_n$ such that

(*) $\qquad Cv_{t+1} \cap \bigcup_{1 \leq i \leq t} Cv_i = \varnothing$ for all $t \in N^+$;

for this would imply

$$|CV_n| \geq \left| \bigcup_{1 \leq i \leq t} Cv_i \right| = \sum_{1 \leq i \leq t} |Cv_i| = t|C|$$

since Γ is unimodular. Therefore $|CV_n| = +\infty$ since $|C| > 0$ and we may choose t arbitrarily large. We now inductively construct $\{v_i\}$. Choose $v_1 \in V_n$ arbitrarily, and assume v_i for $1 \leq i \leq t$ have already been chosen. Let $C_t = \bigcup_{1 \leq i \leq t} Cv_i$. Then $C^{-1}C_t$ is compact. Since $\overline{V}_n$ is *not* compact $V_n \not\subseteq C^{-1}C_t$. Then (∗) holds if v_{t+1} is chosen to be any point in $V_n \sim C^{-1}C_t \neq \varnothing$, and the proof is completed.

The following lemma, which is of independent interest, is useful in the proof of the next theorem.

LEMMA 3. *Let Γ be a noncompact locally compact group with modular function Δ and let R be a relatively compact subset of Γ. Then there exists an infinite subset $\mathscr{D} = \{\gamma_i\}$ in Γ such that the $R\gamma_i$ are pairwise disjoint and $\Delta(\gamma_i) \geq 1$ for all i.*

Proof. Obviously we may assume that R is compact. Let $\Gamma_+ = \{\gamma \in \Gamma \mid \Delta(\gamma) \geq 1\}$ and let $\mathscr{D}$ be maximal in Γ_+ such that the sets $R\gamma$, $\gamma \in \mathscr{D}$, are pairwise disjoint. This implies $R\mu \cap R\mathscr{D} \neq \varnothing$ for every $\mu \in \Gamma_+$. Hence $\Gamma_+ \subset R^{-1}R\mathscr{D}$ and so $\mathscr{D}$ cannot be finite (or relatively compact) because Γ_+ is not compact (observe $\Gamma = \Gamma_+ \cup \Gamma_+^{-1}$).

REMARK. If $\{\gamma_i\} \subset \Gamma$ is such that $\{\Delta(\gamma_i)\}$ is a discrete subset of the positive real numbers, in particular if $\lim \Delta(\gamma_i) = \infty$, then $\{\gamma_i\}$ is closed and discrete because Δ is continuous. If R in Lemma 3 is chosen with nonvoid interior, the set $\mathscr{D}$ necessarily is closed and discrete in Γ_+.

THEOREM 9. *Let $W_\Gamma \neq \varnothing$. The following properties are equivalent*:
   (i) Γ *is compact or discrete.*
   (ii) $W_\Gamma = W_\Gamma^0$.
   (iii) $S_\Gamma = W_\Gamma$.
   (iv) $S_\Gamma^0 = W_\Gamma^0$.
Moreover, if Γ is neither compact nor discrete, there exists $\{V_n'\} \in W_\Gamma$ such that $|CV_n'| = +\infty$ for all $n \in N^+$ and any Borel set C of positive measure.

Proof. If Γ is discrete $W_\Gamma = W_\Gamma^0$ and $S_\Gamma = S_\Gamma^0$ immediately from the definition. Also by Theorem 2 and Theorem 1 (ii) we conclude $S_\Gamma = W_\Gamma$. If Γ is compact, trivially $S_\Gamma = S_\Gamma^0$ and $W_\Gamma = W_\Gamma^0$. To show that $S_\Gamma = W_\Gamma$ we first assume Haar measure on Γ is normalized so that $|\Gamma| = 1$. Let $\{V_n\} \in W_\Gamma$. We shall show that this implies $\text{Lim}_{n \to \infty} |V_n| = 1$, from which we clearly conclude $\{V_n\} \in S_\Gamma$. For $\{V_n\} \in W_\Gamma$ implies

$$\underset{n \to \infty}{\text{Lim}} \frac{|\gamma V_n \cap V_n|}{|V_n|} = \underset{n \to \infty}{\text{Lim}} \frac{|V_n| - \frac{1}{2}|\gamma V_n \triangle V_n|}{|V_n|} = 1 \quad \text{for all } \gamma \in \Gamma.$$

Hence upon recalling the formula $\int_\Gamma |\gamma A \cap B| \, d\gamma = |A| \cdot |B^{-1}|$, by dominated convergence we obtain

$$1 = \underset{n \to \infty}{\text{Lim}} \int_\Gamma \frac{|\gamma V_n \cap V_n| \, d\gamma}{|V_n|} = \underset{n \to \infty}{\text{Lim}} \frac{|V_n| \cdot |V_n^{-1}|}{|V_n|} = \underset{n \to \infty}{\text{Lim}} |V_n|.$$

Conversely, if Γ is noncompact and nondiscrete let R be any compact neighborhood of the unit in Γ and let $\mathscr{D} = \{\gamma_1, \gamma_2, \ldots\} \subset \Gamma_+$ be a sequence as guaranteed by Lemma 3. Let $\{V_n\} \in W_\Gamma$ and define $V_n' = V_n \cup \mathscr{D}$.

(i) Clearly $\gamma V_n' \bigtriangleup V_n' \subseteq (\gamma V_n \bigtriangleup V_n) \cup (\mathscr{D} \cup \gamma\mathscr{D})$, and consequently

$$\frac{|\gamma V_n' \bigtriangleup V_n'|}{|V_n'|} \leqq \frac{|\gamma V_n \bigtriangleup V_n|}{|V_n|}$$

since countable sets have zero measure in nondiscrete groups, and therefore $\{V_n'\} \in W_\Gamma$. Since $\mathscr{D} \subset \overline{V}_n'$, we have $\{V_n'\} \notin W_\Gamma^0$ and thus $W_\Gamma \neq W_\Gamma^0$.

(ii) Continuing from (i) we shall also show that $\{V_n'\} \notin S_\Gamma$, and in fact $|CV_n'| = \infty$ for any Borel set C in Γ of positive measure. First, there exists $\nu \in \Gamma$ with $|\nu C \cap R| > 0$. It follows that

$$|CV_n'| = |\nu CV_n'| \geqq |(\nu C \cap R)\mathscr{D}| = \sum_1^\infty |(\nu C \cap R)\gamma_i|$$

$$= \sum_1^\infty \Delta(\gamma_i)|\nu C \cap R| = \infty$$

since $\Delta(\gamma_i) \geqq 1$.

(iii) If $W_\Gamma^0 = \varnothing$, then $W_\Gamma = \varnothing$ by Proposition 1. Hence assume $\{V_n\} \in W_\Gamma^0$. If $\{t_n\}$ is any sequence of natural numbers and we set $V_n' = V_n \cup \{\gamma_i \in \mathscr{D} \mid 1 \leqq i \leqq t_n\}$ we clearly have $\{V_n'\} \in W_\Gamma^0$. But as in the above paragraph, if C is any Borel set of Γ of positive measure $|C\mathscr{D}| = \infty$ and therefore if we choose t_n increasing sufficiently rapidly (depending on our choice of V_n and C) we may guarantee $|CV_n'|/|V_n'| \to \infty$ as $n \to \infty$. We conclude $\{V_n'\} \notin S_\Gamma^0$ by Theorem 2.

II.2. *Closure properties of the basic sets.* We generically let R_Γ denote any of the four sets W_Γ, W_Γ^0, S_Γ, S_Γ^0. Trivially, if $\{V_n\} \in R_\Gamma$ any subsequence $\{V_{n_i}\}$ is also in R_Γ as well as any sequence $\{U_n\}$ of Borel sets of Γ with finite nonzero measure with only finitely many $U_n \neq V_n$. Also, if $\{V_n^{(i)}\} \in R_\Gamma$ for $1 \leqq i \leqq t$ and $\{V_m^*\}$ is an arbitrary sequence such that for all $m \in N^+$, $V_m^* = V_n^{(i)}$ for some $n \in N^+$, $1 \leqq i \leqq t$, and every $V_n^{(i)}$ occurs only finitely often as a V_m^*, then $\{V_m^*\} \in R_\Gamma$. The following theorem lists further closure properties of R_Γ.

THEOREM 10. (i) *If $\{V_n\} \in R_\Gamma$ and $\{\gamma_n\}$ is any sequence of elements of Γ, then* $\{V_n\gamma_n\} \in R_\Gamma$.

(ii) *If $\{U_n\}$ and $\{V_n\} \in R_\Gamma$, then $\{U_n \cup V_n\} \in R_\Gamma$.*

(iii) *If $\{V_n\}$ is a sequence of Borel sets of Γ with finite nonzero measure such that for every $\varepsilon > 0$ there exists a $\{V_n^\varepsilon\} \in W_\Gamma$ satisfying* $\mathrm{Lim\ sup}_{n\to\infty} |V_n \bigtriangleup V_n^\varepsilon|/|V_n| < \varepsilon$, *then $\{V_n\} \in W_\Gamma$.*

(iv) *If $\{V_n\}$ is a sequence of Borel sets of Γ with finite nonzero measure such that for every $\varepsilon > 0$ there exists a $\{V_n^\varepsilon\} \in S_\Gamma$ satisfying $V_n \subseteq V_n^\varepsilon$ and* $\mathrm{Lim\ sup}_{n\to\infty} |V_n^\varepsilon \sim V_n|/|V_n| < \varepsilon$, *then $\{V_n\} \in S_\Gamma$.*

Proof. (i) A Borel set V of Γ has finite nonzero measure or compact closure if and only if $V\gamma$ does, where γ is any element of Γ. Also, if $C \neq \varnothing$ is a compact set,

$|CV \triangle V|/|V| = |C(V\gamma) \triangle (V\gamma)|/|V\gamma|$ and (i) clearly follows. (ii) follows since the union of two Borel sets with compact closure has compact closure and

$$|C(U_n \cup V_n) \triangle (U_n \cup V_n)| \leq |CU_n \triangle U_n| + |CV_n \triangle V_n|.$$

To prove (iii), we apply the metric property of $\rho(A, B) = |A \triangle B|$ and infer

$$|\gamma V_n \triangle V_n| \leq |\gamma V_n \triangle \gamma V_n^\varepsilon| + |\gamma V_n^\varepsilon \triangle V_n^\varepsilon| + |V_n^\varepsilon \triangle V_n|.$$

Consequently

$$\underset{n \to \infty}{\text{Lim sup}} \frac{|\gamma V_n \triangle V_n|}{|V_n|} \leq \underset{n \to \infty}{\text{Lim sup}} \frac{(|\gamma V_n \triangle \gamma V_n^\varepsilon| + |\gamma V_n^\varepsilon \triangle V_n^\varepsilon| + |V_n^\varepsilon \triangle V_n|)}{|V_n|} < 2\varepsilon,$$

and since $\varepsilon > 0$ is arbitrary we have $\{V_n\} \in W_\Gamma$.

Finally, to prove (iv) we note that $V_n \subseteq V_n^\varepsilon$ implies

$$CV_n \triangle V_n \subseteq (CV_n \triangle CV_n^\varepsilon) \cup (CV_n^\varepsilon \triangle V_n^\varepsilon) \cup (V_n^\varepsilon \triangle V_n)$$
$$= (CV_n^\varepsilon \sim CV_n) \cup (CV_n^\varepsilon \triangle V_n^\varepsilon) \cup (V_n^\varepsilon \sim V_n).$$

But

$$|CV_n^\varepsilon \sim CV_n| = |CV_n^\varepsilon| - |CV_n| \leq |CV_n^\varepsilon| - |V_n|$$
$$\leq |CV_n^\varepsilon \triangle V_n^\varepsilon| + |V_n^\varepsilon| - |V_n| = |CV_n^\varepsilon \triangle V_n^\varepsilon| + |V_n^\varepsilon \sim V_n|$$

and therefore

$$\underset{n \to \infty}{\text{Lim sup}} \frac{|CV_n \triangle V_n|}{|V_n|} \leq \underset{n \to \infty}{\text{Lim sup}} \frac{2(|CV_n^\varepsilon \triangle V_n^\varepsilon| + |V_n^\varepsilon \sim V_n|)}{|V_n|} < 2\varepsilon.$$

Hence $\{V_n\} \in S_\Gamma$.

By Theorem 10 (i) we see that any sequence $\{V_n\} \in R_\Gamma$ can be arbitrarily translated on the *right* and the resulting sequence $\{V_n\gamma_n\}$ remains in R_Γ. We now consider the more delicate problem of *left* translation, and prove the following theorem:

THEOREM 11. $\{V_n\} \in W_\Gamma$ *implies* $\{\gamma_n V_n\} \in W_\Gamma$ *for any sequence of elements* $\{\gamma_n\} \subseteq \Gamma$ *if and only if every conjugacy class* $\alpha^\Gamma = \{\gamma^{-1}\alpha\gamma : \gamma \in \Gamma\}$ *in* Γ *has compact closure* cl (α^Γ).

Proof. Assume cl (α^Γ) is compact, and let $\{\gamma_n\} \subseteq \Gamma$. Then

$$\underset{n \to \infty}{\text{Lim}} \frac{|\alpha(\gamma_n V_n) \triangle (\gamma_n V_n)|}{|\gamma_n V_n|} = \underset{n \to \infty}{\text{Lim}} \frac{|(\gamma_n^{-1}\alpha\gamma_n) V_n \triangle V_n|}{|V_n|} = 0$$

by Corollary 2, since $\gamma_n^{-1}\alpha\gamma_n = \gamma_n' \in \alpha^\Gamma$ for all $n \in N^+$. Conversely, assume cl (α^Γ) noncompact for some $\alpha \in \Gamma$ and let $\{V_n\} \in W_\Gamma^0 \ (\subseteq W_\Gamma)$. Construct $\{\gamma_n\}$ subject to the condition $\gamma_n^{-1}\alpha\gamma_n \notin V_n \cdot V_n^{-1}$, which is feasible since we may assume all cl (V_n) compact, hence all cl $(V_n \cdot V_n^{-1})$ compact, and cl (α^Γ) is noncompact. Then for this sequence $\{\gamma_n\}$ we have

$$\alpha\gamma_n V_n \triangle \gamma_n V_n = \gamma_n\{(\gamma_n^{-1}\alpha\gamma_n) V_n \triangle V_n\} = \alpha\gamma_n V_n \cup \gamma_n V_n$$

and hence $|\alpha\gamma_n V_n \triangle \gamma_n V_n|/|V_n| = 2$ and $\{\gamma_n V_n\} \notin W_\Gamma$. We now prove a result of independent interest which will also be used in the proof of Theorem 13.

THEOREM 12. *Let $\{V_n\}$ be a sequence of Borel sets of finite nonzero measure in any locally compact group Γ satisfying*

(*) $$\underset{n\to\infty}{\text{Lim}} |\gamma V_n \triangle V_n|/|V_n| = 0 \quad \text{for all } \gamma \in U,$$

where U is a measurable subset of Γ. Then $\text{Lim inf}_{n\to\infty} |V_n^{-1}| \geqq |U|$.

Proof. First assume $|U| < +\infty$. Then by (*) $\text{Lim}_{n\to\infty} |\gamma V_n \cap V_n|/|V_n| = 1$ for all $\gamma \in U$, and hence by dominated convergence

$$\underset{n\to\infty}{\text{Lim}} \int_U \frac{|\gamma V_n \cap V_n|}{|V_n|} \, d\gamma = |U|.$$

But by Fubini's Theorem

$$\int_U \frac{|\gamma V_n \cap V_n|}{|V_n|} \, d\gamma = \frac{1}{|V_n|} \int_U \left[\int_{V_n} I_{V_n}(\gamma^{-1}\tau) \, d\tau \right] d\gamma$$

$$= \frac{1}{|V_n|} \int_{V_n} \left[\int_U I_{V_n}(\gamma^{-1}\tau) \, d\gamma \right] d\tau = \frac{1}{|V_n|} \int_{V_n} |U \cap \tau V_n^{-1}| \, d\tau$$

$$\leqq \frac{1}{|V_n|} \int_{V_n} |\tau V_n^{-1}| \, d\tau = |V_n^{-1}|,$$

and therefore $\text{Lim inf}_{n\to\infty} |V_n^{-1}| \geqq |U|$.

If $|U| = +\infty$ we may find measurable subsets of U with arbitrarily large finite measure, and since (*) trivially is satisfied on any subset of U, we conclude $\text{Lim}_{n\to\infty} |V_n^{-1}| = +\infty$ by the estimate above.

COROLLARY 4. *If Γ is not compact, then*

(i) $\{V_n\} \in W_\Gamma$ *implies* $\text{Lim}_{n\to\infty} |V_n^{-1}| = +\infty$.

(ii) *If $W_\Gamma \neq \varnothing$, $\text{Lim}_{n\to\infty} |V_n| = +\infty$ for all $\{V_n\} \in W_\Gamma$ if and only if Γ is unimodular. In fact, if Γ is not unimodular and $\varepsilon > 0$ there exists a sequence $\{V_n^\varepsilon\} \in W_\Gamma$ such that $\sum_n |V_n^\varepsilon| < \varepsilon$.*

Proof. (i) is immediate since $\{V_n\} \in W_\Gamma$ implies (*) is satisfied for $U = \Gamma$ and $|\Gamma| = +\infty$ since Γ is assumed noncompact.

(ii): If Γ is unimodular $|V_n^{-1}| = |V_n|$ and therefore $\{V_n\} \in W_\Gamma$ implies $\text{Lim}_{n\to\infty} |V_n| = +\infty$ by (i). On the other hand, if Γ is not unimodular and $\{V_n\}$ is any sequence in W_Γ then $\{V_n\gamma_n\} \in W_\Gamma$ for any sequence $\{\gamma_n\} \subseteq \Gamma$ by Theorem 10 (i). Now since Γ is not unimodular, for each $n \in N^+$ we may choose $\gamma_n^\varepsilon \in \Gamma$ such that $\Delta(\gamma_n^\varepsilon) < \varepsilon/(2^n |V_n|)$. Thus if we set $V_n^\varepsilon = V_n\gamma_n^\varepsilon$, $\{V_n^\varepsilon\} \in W_\Gamma$ and

$$\sum_n |V_n^\varepsilon| = \sum_n |V_n\gamma_n^\varepsilon| = \sum_n |V_n|\Delta(\gamma_n^\varepsilon) < \sum_n \varepsilon/2^n = \varepsilon$$

as asserted.

THEOREM 13. *If Γ is a nonunimodular σ-compact amenable group,*

$$C(\Gamma) = \{\alpha \in \Gamma : \text{cl } (\alpha^\Gamma) \text{ is compact}\}$$

has measure zero. Moreover, if $\{\gamma_n\}$ is any sequence of elements of Γ for which $\{\Delta(\gamma_n)\}$ is unbounded

$$C(\{\gamma_n\}) = \{\alpha \in \Gamma : \text{cl } (\{\gamma_n^{-1}\alpha\gamma_n\}) \text{ is compact}\}$$

has measure zero.

Proof. Since $C(\Gamma) \subseteq C(\{\gamma_n\})$ it suffices to prove the latter statement. First note that $C(\{\gamma_n\})$ is measurable: for if $\{K_i\}$ is any nested sequence of compacta in Γ whose interiors cover Γ (such sequences clearly exist in arbitrary σ-compact groups), we have $C(\{\gamma_n\}) = \bigcup_i \{\bigcap_n \gamma_n K_i \gamma_n^{-1}\}$ because cl $(\{\gamma_n^{-1}\alpha\gamma_n\})$ is compact if and only if $\{\gamma_n^{-1}\alpha\gamma_n\} \subseteq K_i$ for some i if and only if $\alpha \in \bigcap_n \gamma_n K_i \gamma_n^{-1}$ for some i. Now let $\{V_n\} \in W_\Gamma^0$ and set $V_n' = \gamma_n V_n$. Let $\alpha \in C(\{\gamma_n\})$, and consider $|\alpha V_n' \bigtriangleup V_n'|/|V_n'|$ $= |(\gamma_n^{-1}\alpha\gamma_n)V_n \bigtriangleup V_n|/|V_n|$: once more by Corollary 2 we see that this quotient tends to zero as $n \to +\infty$. Hence by Theorem 12,

$$|C(\{\gamma_n\})| \leqq \operatorname*{Lim\,inf}_{n \to +\infty} |V_n'^{-1}| = \operatorname*{Lim\,inf}_{n \to \infty} |V_n^{-1}\gamma_n^{-1}| = \operatorname*{Lim\,inf}_{n \to \infty} |V_n^{-1}| \bigtriangleup (\gamma_n^{-1}).$$

Now the left hand side of this inequality is independent of our choice of $\{V_n\}$, and we may clearly assume $\Delta(\gamma_n^{-1}) \to 0$. Hence for an appropriate choice of $\{V_n\}$ (e.g., "repeating" terms of *any* $\{V_n\}$ with appropriate increasing "multiplicites") the right hand side is zero, and the proof is complete.

COROLLARY 5. *If Γ is a nonunimodular amenable σ-compact group and $\{V_n\} \in W_\Gamma$ then $\{\gamma_n V_n\} \notin W_\Gamma$ for some sequence $\{\gamma_n\} \subseteq \Gamma$.*

Proof. Theorem 13 and Theorem 11.

II.3. *The approximation theorem.* In Theorem 9 we proved that if $W_\Gamma \neq \varnothing$ then $W_\Gamma^0 \neq S_\Gamma^0$ unless Γ is compact or discrete. In this section we show that nevertheless S_Γ^0 is in a sense dense in W_Γ, and apply this result to obtain a characterization of W_Γ. We first make a definition:

DEFINITION 3. Let Γ be any group and let $K \subseteq \Gamma$ be any subset of Γ. For any subset $T \subseteq \Gamma$ we define $C(K, T)$ to be the maximum number of $K\tau$, $\tau \in T$ which intersect, possibly $+\infty$. $I_R(\Gamma, K)$, the right index of K in Γ, is defined by

$$I_R(\Gamma, K) = \text{Inf} \left\{ C(K, T) : T \subseteq \Gamma, \Gamma = \bigcup_{\tau \in T} K\tau \right\}.$$

We need the following result of Greenleaf and the author [2]:

THEOREM 14. *If Γ is any locally compact group, there exists a Borel set $K \subseteq \Gamma$ with compact closure and nonvoid interior such that $I_R(\Gamma, K) < +\infty$.*

We next prove:

PROPOSITION 2. *If Γ is any topological group and there exists a subset $K \subseteq \Gamma$ with compact closure, nonvoid interior, and finite right index in Γ, then every subset $C \subseteq \Gamma$ with compact closure and nonvoid interior has finite right index.*

Proof. Let $\Gamma = \bigcup_{\tau \in T} K\tau$ be such that at most n right translates $K\tau$, $\tau \in T$ intersect. Since K has compact closure and C has nonvoid interior, we may cover K by finitely many right translates of C, say

$$K \subseteq C\alpha_1 \cup C\alpha_2 \cup \cdots \cup C\alpha_r = C'.$$

But C' has compact closure since C does, and K has nonvoid interior and therefore we may cover C' by finitely many left translates of K, say

$$C' \subseteq \beta_1 K \cup \beta_2 K \cup \cdots \cup \beta_s K = K'.$$

We shall prove that this implies $I_R(\Gamma, C) \leq nrs$. For consider the decomposition

$$\Gamma = \bigcup_{\tau \in T} K\tau = \bigcup_{\tau \in T} ((C\alpha_1\tau) \cup (C\alpha_2\tau) \cup \cdots \cup (C\alpha_r\tau)),$$

and assume more than nrs of the $C\alpha_i\tau$, $1 \leq i \leq r$, $\tau \in T$, intersect. This implies more than ns different τ must occur. Hence more than ns of the $C'\tau$, $\tau \in T$ intersect, and consequently $m > ns$ of the $K'\tau$, $\tau \in T$ intersect, say $\bigcap_{1 \leq i \leq m} K'\tau_i \neq \varnothing$. Let γ_0 be any element of this intersection, and for each i, $1 \leq i \leq m$, let $u = u(i)$ be some integer $1 \leq u \leq s$ for which $\gamma_0 \in \beta_u K\tau_i$. Hence $u(i)$ must take some value u_0 more than n times, say $u(i_1) = u(i_2) = \cdots = u(i_{n'}) = u_0$, where $n' > n$. But this implies

$$\gamma_0 \in \bigcap_{1 \leq j \leq n'} \beta_{u_0} K\tau_{i_j}, \quad \beta_{u_0}^{-1}\gamma_0 \in \bigcap_{1 \leq j \leq n'} K\tau_{i_j} \neq \varnothing.$$

Combining the two preceding results we obtain

PROPOSITION 3. *If Γ is any locally compact group, every Borel set with compact closure and nonvoid interior has finite right index in Γ.*

We now prove a lemma which is critical in the proof of The Approximation Theorem:

LEMMA 4. *Let Γ be a locally compact group and let K be a Borel set of Γ containing e with compact closure and nonvoid interior. Then if $\{V_n\} \in W_\Gamma^0$ and $0 < \varepsilon < 1$, there exists a sequence $\{V_n'\}$ of Borel sets of Γ such that*
  (i) $V_n' \subseteq V_n$, $|V_n \sim V_n'| \leq \varepsilon |V_n|$ *for all* $n \in N^+$.
  (ii) $\text{Lim sup}_{n \to \infty} |KV_n'|/|V_n'| \leq 1/(1-\varepsilon)$.

Proof. $|K| > 0$ since K has nonvoid interior. Also $K' = K \cdot K \cdot K^{-1}$ is a set with compact closure. For any $n \in N^+$ and $\delta > 0$ we define the Borel set $E_n(\delta)$ by

$$E_n(\delta) = \{\gamma \in V_n : |K' \sim V_n\gamma^{-1}| > \delta\},$$

and let

$$\delta_n = \text{Inf}\{\delta : |E_n(\delta)| \leq \varepsilon |V_n|\}.$$

Therefore $E_n(\delta) > \varepsilon |V_n|$ if $\delta < \delta_n$, and $|E_n(\delta_n)| \leqq \varepsilon |V_n|$ since Haar measure is continuous.

We shall now prove that $V_n' = V_n \sim E_n(\delta_n)$ satisfies conditions (i) and (ii) of the lemma. Clearly (i) is true. As a first step in proving (ii) we shall first show $\text{Lim}_{n \to \infty} \delta_n = 0$. For assume $\text{Lim sup}_{n \to \infty} \delta_n > 0$; then without loss of generality we may assume there exists an $\eta > 0$ and an n_0 such that for $n \geqq n_0$ we have $\eta < \delta_n$ and therefore

$$(2.1) \qquad\qquad\qquad |E_n(\eta)| > \varepsilon |V_n|.$$

This will result in a contradiction. For $\gamma \in E_n(\eta)$, $K_\gamma = K_\gamma(n) = K' \sim V_n \gamma^{-1} \subseteq K'$ clearly satisfies

$$(2.2) \qquad\qquad |K_\gamma| > \eta, \qquad K_\gamma \gamma \cap V_n = \varnothing .$$

Also, for each $\alpha \in K'$, let

$$S_\alpha = S_\alpha(n) = \{\gamma \in E_n(\eta) : \alpha \in K_\gamma\}.$$

We then have

$$(2.3) \qquad\qquad \int_{E_n(\eta)} |K_\gamma| \, d\gamma = \int_{K'} |S_\alpha| \, d\alpha$$

since upon application of Fubini's Theorem

$$\int_{E_n(\eta)} |K_\gamma| \, d\gamma = \int_{E_n(\eta)} \left[\iint_{K'} I_{K_\gamma}(\alpha) \, d\alpha \right] d\gamma = \int_{K'} \left[\iint_{E_n(\eta)} I_{K_\gamma}(\alpha) \, d\gamma \right] d\alpha$$

$$= \int_{K'} |\{\gamma \in E_n(\eta) : \alpha \in K_\gamma\}| \, d\alpha = \int_{K'} |S_\alpha| \, d\alpha.$$

Now $\gamma \in S_\alpha$ implies $\alpha \in K_\gamma = K' \sim V_n \gamma^{-1}$ and consequently $\alpha\gamma \notin V_n$ and thus $\alpha S_\alpha \cap V_n = \varnothing$. But $S_\alpha \subseteq V_n$ for $\alpha \in K'$ and therefore $\alpha V_n \cap V_n = \alpha(V_n \sim S_\alpha) \cap V_n$. But $\{V_n\} \in W_\Gamma$ implies

$$1 = \underset{n \to \infty}{\text{Lim}} \frac{|\alpha V_n \cap V_n|}{|V_n|} = \underset{n \to \infty}{\text{Lim}} \frac{|\alpha(V_n \sim S_\alpha) \cap V_n|}{|V_n|} \leqq \underset{n \to \infty}{\text{Lim inf}} \frac{|\alpha(V_n \sim S_\alpha)|}{|V_n|}$$

$$= \underset{n \to \infty}{\text{Lim inf}} \frac{|V_n| - |S_\alpha|}{|V_n|} = 1 - \underset{n \to \infty}{\text{Lim sup}} \frac{|S_\alpha|}{|V_n|},$$

and hence we may conclude $\text{Lim}_{n \to \infty} |S_\alpha(n)|/|V_n| = 0$ for $\alpha \in K'$. Therefore by dominated convergence

$$(2.4) \qquad\qquad \underset{n \to \infty}{\text{Lim}} \int_{K'} \frac{|S_\alpha|}{|V_n|} \, d\alpha = 0.$$

By (2.1) to (2.4), $|E_n(\eta)| > \varepsilon |V_n|$, $|K_\gamma(n)| > \eta$ on $E_n(\eta)$, and

$$0 = \underset{n \to \infty}{\text{Lim}} \int_{K'} \frac{|S_\alpha|}{|V_n|} \, d\alpha = \underset{n \to \infty}{\text{Lim}} \frac{1}{|V_n|} \int_{E_n(\eta)} |K_\gamma(n)| \, d\gamma \geqq \underset{n \to \infty}{\text{Lim inf}} \frac{1}{|V_n|} (\varepsilon |V_n| \eta) = \varepsilon\eta,$$

which is impossible.

By Proposition 3, K has finite right index N in Γ, and we may write $\Gamma = \bigcup_{\tau \in T} K\tau$ where at most N $K\tau$, $\tau \in T$ have common intersection. Let

$$T_0 = T_0(n) = \{\tau \in T : K\tau \cap V_n' \neq \varnothing\}.$$

Hence

$$V_n' \subseteq \bigcup_{\tau \in T_0} K\tau.$$

Let $\gamma_\tau \in K\tau \cap V_n'$ for $\tau \in T_0$, implying $\tau \in K^{-1}\gamma_\tau$ and $K\tau \subseteq K \cdot K^{-1}\gamma_\tau$, and therefore

$$KV_n' \subseteq \bigcup_{\tau \in T_0} K'\gamma_\tau$$

where each $\gamma_\tau \in V_n''$. Let $M = \mathrm{Sup}_{\gamma \in K}\, \Delta(\gamma)$. Since $\gamma_\tau \in K\tau$ for $\tau \in T_0$ we conclude

$$(2.5) \qquad\qquad \Delta(\gamma_\tau) \leqq M\,\Delta(\tau).$$

Now assume n_0 is chosen so large that for $n \geqq n_0$, $\delta_n < |K|/M$. By the construction, $\gamma \in V_n'$ implies $|K' \sim V_n\gamma^{-1}| \leqq \delta_n$ and therefore $|K'\gamma \sim V_n| \leqq \delta_n \Delta(\gamma)$. Also, if V is a Borel set of Γ,

$$(2.6) \qquad\qquad \sum_{\tau \in T} |K\tau \cap V| \leqq N|V|$$

by Fubini's Theorem, since any point of Γ lies in at most N different $K\tau$, $\tau \in T$. Since $e \in K$, for $\tau \in T_0$

$$(2.7) \qquad |K\tau \sim V_n| \leqq |K'\gamma_\tau \sim V_n| \leqq \delta_n \Delta(\gamma_\tau) \leqq \delta_n M\,\Delta(\tau),$$

and consequently

$$(2.8) \qquad |K\tau \cap V_n| \geqq |K\tau| - \delta_n M\,\Delta(\tau) = (|K| - \delta_n M)\,\Delta(\tau).$$

Hence by (2.6) and (2.8)

$$N|V_n| \geqq \sum_{\tau \in T} |K\tau \cap V_n| \geqq \sum_{\tau \in T_0} |K\tau \cap V_n| \geqq (|K| - \delta_n M) \sum_{\tau \in T_0} \Delta(\tau),$$

and therefore for $n \geqq n_0$

$$(2.9) \qquad\qquad \sum_{\tau \in T_0} \Delta(\tau) \leqq \frac{N|V_n|}{|K| - \delta_n M}.$$

But also

$$KV_n' \subseteq \bigcup_{\tau \in T_0} K'\gamma_\tau \subseteq \bigcup_{\tau \in T_0} (K'\gamma_\tau \sim V_n) \cup V_n,$$

and therefore by (2.7) and (2.9)

$$|KV_n'| \leqq \sum_{\tau \in T_0} |K'\gamma_\tau \sim V_n| + |V_n| \leqq \delta_n M \sum_{\tau \in T_0} \Delta(\tau) + |V_n|$$

$$\leqq \left(1 + \frac{\delta_n MN}{|K| - \delta_n M}\right) |V_n|.$$

Finally, since $\delta_n \to 0$ as $n \to \infty$ and $|V_n'| \geq (1-\varepsilon)|V_n|$ for $n \in N^+$,

$$\text{Lim} \sup_{n \to \infty} \frac{|KV_n'|}{|V_n'|} \leq \text{Lim} \sup_{n \to \infty} \left(1 + \frac{\delta_n MN}{|K| - \delta_n M}\right) \frac{|V_n|}{|V_n'|} = \text{Lim} \sup_{n \to \infty} \frac{|V_n|}{|V_n'|} \leq \frac{1}{1-\varepsilon}.$$

THEOREM 15 (THE APPROXIMATION THEOREM). *Let Γ be any locally compact group, and let $\{V_n\} \in W_\Gamma$. Then there exists $\{V_n^*\} \in S_\Gamma^0$ such that $V_n^* \subseteq V_n$ for all $n \in N^+$ and $\text{Lim}_{n \to \infty} |V_n \sim V_n^*|/|V_n| = 0$.*

A sequence $\{V_n\}$ of Borel sets of Γ with finite nonzero measure is in W_Γ if and only if there exists $V_n^* \in S_\Gamma^0$ such that $\text{Lim}_{n \to \infty} |V_n \triangle V_n^*|/|V_n| = 0$.

Proof. By the regularity of Haar measure we may find Borel sets $V_n^0 \subseteq V_n$ for all $n \in N^+$ such that $\{V_n^0\} \in W_\Gamma^0$ and $\text{Lim}_{n \to \infty} |V_n \sim V_n^0|/|V_n| = 0$. Hence we may assume $\{V_n\} \in W_\Gamma^0$ initially. Furthermore, Γ must be σ-compact if $W_\Gamma \neq \varnothing$ by Theorem 4, say $\Gamma = \bigcup C_i$ where all the countably many C_i are compact. But this readily leads to the construction of a sequence $\{K_i\}$ of Borel sets of Γ with compact closure, nonvoid interior, and all containing the identity such that

(1) $K_i \subseteq K_{i+1}$, $i \in N^+$.

(2) $\bigcup_i$ interior $(K_i) = \Gamma$.

(3) K_i has finite right index in Γ.

For set $C_i' = \bigcup_{j \leq i} C_j \cup \{e\}$ and let U be any open neighborhood of e with compact closure. Then it is easily verified that the above conditions are satisfied by $K_i = UC_i'$. Note that (3) follows directly from Proposition 3.

We now apply Lemma 4 to each $K = K_i$, taking $\varepsilon = \varepsilon_i = 1/3i$. Consequently there exists a sequence $\{V_{n,i}'\}$ of Borel sets of Γ satisfying

(i) $V_{n,i}' \subseteq V_n$ for all $n \in N^+$.

(ii) $|V_n \sim V_{n,i}'| \leq (1/3i)|V_n|$ for all $n \in N^+$.

(iii) $\text{Lim} \sup_{n \to \infty} |K_i V_{n,i}'|/|V_{n,i}'| \leq 1/(1 - 1/3i)$.

From (iii) we infer the existence of $n_0 = n_0(i)$ such that for all $n \geq n_0$ we have

(iii)' $|K_i V_{n,i}'|/|V_{n,i}'| < 1/(1 - 1/2i)$.

Note also that all $V_{n,i}'$ have compact closure for sufficiently large n by virtue of (i), since $\{V_n\} \in W_\Gamma^0$. We now construct $\{V_n^*\}$. Without loss of generality assume $n_0(i)$ is chosen such that $n_0(i) < n_0(i+1)$ for $i \in N^+$. For $1 \leq n < n_0(1)$ let $V_n^* = V_n$, and for $i \in N^+$ and $n_0(i) \leq n < n_0(i+1)$ let $V_n^* = V_{n,i}'$. Then each $V_n^* \subseteq V_n$ by (i), each V_n^* is a Borel set of finite nonzero measure, and almost all V_n^* have compact closure. Also by (ii), for $n_0(i) \leq n < n_0(i+1)$, $|V_n \sim V_n^*|/|V_n| \leq 1/3i$ and therefore $\text{Lim}_{n \to \infty} |V_n \sim V_n^*|/|V_n| = 0$.

Finally to show that $\{V_n^*\} \in S_\Gamma^0$, let $C \neq \varnothing$ be a compact subset of Γ. Since $\bigcup$ interior $(K_i) = \Gamma$ is certainly an open cover of C and the K_i are nested, there exists an integer i_0 such that $C \subseteq K_i$ for $i \geq i_0$. Therefore if $i \geq i_0$ and $n_0(i) \leq n < n_0(i+1)$, we have by (iii)'

$$1 \leq \frac{|CV_n^*|}{|V_n^*|} \leq \frac{|K_i V_n^*|}{|V_n^*|} = \frac{|K_i V_{n,i}'|}{|V_{n,i}'|} < \frac{1}{1 - 1/2i},$$

and consequently $\text{Lim}_{n \to \infty} |CV_n^*|/|V_n^*| = 1$.

We have just shown that if $\{V_n\} \in W_\Gamma$ there exists $\{V_n^*\} \in S_\Gamma^0$ such that

$$(2.10) \qquad \lim_{n \to \infty} |V_n \triangle V_n^*|/|V_n| = 0,$$

and in fact V_n^* could be taken to be a subset of V_n. Conversely, assume $\{V_n^*\} \in S_\Gamma^0$ and $\{V_n\}$ is a sequence of Borel sets of Γ with finite nonzero measure satisfying (2.10).

Fix any $\gamma \in \Gamma$. Since $\{V_n^*\} \in S_\Gamma^0 \subseteq W_\Gamma^0$, we have $\lim_{n \to \infty} |\gamma V_n^* \triangle V_n^*|/|V_n^*| = 0$. But

$$\gamma V_n \triangle V_n \subseteq (\gamma V_n \triangle \gamma V_n^*) \cup (\gamma V_n^* \triangle V_n^*) \cup (V_n^* \triangle V_n),$$

and consequently,

$$\limsup_{n \to \infty} \frac{|\gamma V_n \triangle V_n|}{|V_n|} \leqq \limsup_{n \to \infty} \frac{|\gamma V_n \triangle \gamma V_n^*| + |\gamma V_n^* \triangle V_n^*| + |V_n^* \triangle V_n|}{|V_n|}$$

$$= \limsup_{n \to \infty} \frac{2|V_n \triangle V_n^*| + |\gamma V_n^* \triangle V_n^*|)}{|V_n|} = 0$$

since $\lim_{n \to \infty} |V_n|/|V_n^*| = 1$; and therefore $\{V_n\} \in W_\Gamma$.

As an immediate corollary to The Approximation Theorem we have:

COROLLARY 6. $W_\Gamma \neq \varnothing \Leftrightarrow S_\Gamma^0 \neq \varnothing$.

II.4. *A necessary condition for* $\{V_n\} \in W_\Gamma$. In this section we apply The Approximation Theorem to obtain a necessary condition for $\{V_n\} \in W_\Gamma$. It should be noted that the condition is also sufficient if Γ is Abelian by Theorem 6. It is reasonable to expect that the condition stated in Theorem 16 is in fact sufficient in general also, but the author has been unable to prove this.

THEOREM 16. *Let* Γ *be a locally compact group, and let* $\{V_n\} \in W_\Gamma$. *Then for any Borel set* K *of* Γ *with compact closure and finite right index* N *there exist subsets* $T_n \subseteq \Gamma$ *such that*

$$\lim_{n \to \infty} \frac{\left| V_n \triangle \bigcup_{\tau \in T_n} K\tau \right|}{|V_n|} = 0$$

and at most N *of the* $K\tau$, $\tau \in T_n$ *intersect.*

Proof. Since $\{V_n\} \in W_\Gamma$, by Theorem 15 there exists $\{V_n^*\} \in S_\Gamma^0$ such that $\lim_{n \to \infty} |V_n^* \triangle V_n|/|V_n| = 0$. Fix K of right index N, say $\Gamma = \bigcup_{\tau \in T} K\tau$, where at most N of the $K\tau$, $\tau \in T$ intersect. Let $K' = K \cdot K^{-1}$ a Borel set of Γ with compact closure. Since $\{V_n^*\} \in S_\Gamma^0$, $\lim_{n \to \infty} |K'V_n^*|/|V_n^*| = 1$. Let $T_n = \{\tau \in T : K\tau \cap V_n^* \neq \varnothing\}$, implying

$$V_n^* \subseteq \bigcup_{\tau \in T_n} K\tau.$$

Let $\alpha_\tau \in K\tau \cap V_n^*$ for $\tau \in T_n$, and hence $\tau \in K^{-1}\alpha_\tau$ and consequently $K\tau \subseteq K \cdot K^{-1}\alpha_\tau$ $= K'\alpha_\tau \subseteq K'V_n^*$. Therefore

$$\bigcup_{\tau \in T_n} K\tau \subseteq K'V_n^*.$$

Finally,

$$\underset{n \to \infty}{\text{Lim sup}} \frac{\left|V_n \triangle \bigcup_{\tau \in T_n} K\tau\right|}{|V_n|} \leq \underset{n \to \infty}{\text{Lim sup}} \left(\frac{|V_n \triangle V_n^*| + \left|\bigcup_{\tau \in T_n} K\tau \triangle V_n^*\right|}{|V_n|}\right)$$

$$\leq \underset{n \to \infty}{\text{Lim sup}} \left(\frac{|V_n \triangle V_n^*| + |K'V_n^* \triangle V_n^*|}{|V_n|}\right) = 0,$$

and by construction at most N of the $K\tau$, $\tau \in T_n \subseteq T$ intersect.

III. **A generalization and some special considerations.** In this section we use combinatorial arguments to study a generalization of the (weak) ratio property in the case where the locally compact group involved is a finitely generated discrete Abelian group. We first make a definition:

DEFINITION 4. Let $\Lambda = \Gamma \times S$ be the product measure space of the locally compact topological group $(\Gamma, B, |\cdot|)$, where B is taken to be the Borel sets of Γ and $|\cdot|$ is a left invariant Haar measure on B, and the arbitrary measure space $(S, \Sigma, |\cdot|)$. A sequence $\{\Lambda_n\}$ of measurable subsets of Λ with finite nonzero measure is said to satisfy the ratio property with respect to Γ in Λ, written $\{\Lambda_n\} \in W_{\Gamma, \Lambda}$, if and only if $\text{Lim}_{n \to} |\gamma\Lambda_n \triangle \Lambda_n|/|\Lambda_n| = 0$ for all $\gamma \in \Gamma$, where $\gamma\Lambda_0 = \{(\gamma\tau, s) : (\tau, s) \in \Lambda_0 \subseteq \Gamma \times S\}$ for any $\Lambda_0 \subseteq \Lambda = \Gamma \times S$. Note that if S is a singleton of unit measure this definition collapses to the weak ratio property of Definition 1.

If a and b are real numbers, we shall use the notation $a \wedge b = \text{Min}\{a, b\}$, $a \vee b = \text{Max}\{a, b\}$.

LEMMA 4. *Let $b_1, \ldots, b_r$ be $r \geq 2$ real numbers, and set*

$$S = \sum_{1 \leq i \leq r} b_i, \qquad S_0 = \frac{b_1 + b_r}{2} + \sum_{1 < i < r} b_i, \qquad M = \sum_{1 \leq i \leq r} b_i \wedge b_{i+1}.$$

Then

$$S - M \geq \frac{1}{r} S \geq \frac{1}{r-1} M, \qquad S_0 - M \geq \frac{1}{2} \underset{1 \leq i < j \leq r}{\text{Max}} |b_i - b_j|.$$

Proof. From the identity $a \wedge b = (a+b)/2 - |a-b|/2$ we derive

$$M = \sum_{1 \leq i < r} \left(\frac{b_i + b_{i+1}}{2} - \frac{|b_i - b_{i+1}|}{2}\right) = S_0 - \frac{1}{2}\sum_{1 \leq i < r} |b_i - b_{i+1}|.$$

And hence by the triangle inequality

$$S_0 - M \geq \frac{1}{2} \underset{1 \leq i < j \leq r}{\text{Max}} |b_i - b_j|.$$

Also if we set $b = b_m = \text{Max}_{1 \leq i \leq r} b_i$,

$$M \leq S_0 - \frac{1}{2}(|b_1 - b_m| + |b_m - b_r|) = S - b_m \leq S - S/r,$$

and consequently $S - M \geq S/r$, $S/r \geq M/(r-1)$.

THEOREM 17. *Let* $\Lambda = Z \times S$ *as in Definition* 4 *where* Z *is the discrete group of integers with the counting measure. Let* Λ_0 *be any measurable subset of* Λ *with finite nonzero measure and assume*

$$|(1 + \Lambda_0) \cap \Lambda_0| \geqq (1 - \varepsilon)|\Lambda_0|.$$

Then if $c \in N^+$, *we may partition* Λ_0 *as*

$$\Lambda_0 = \left(\bigcup_n A_n \times S_n \right) \cup R$$

where n *ranges over an appropriate index set* $I = I(c)$, *each* A_n *consists of exactly* c *consecutive integers, the* A_n *are mutually disjoint, each* $S_n \in \Sigma$, *and* $|R| \leqq 6c\sqrt{\varepsilon}|\Lambda_0|$.

Proof. The measurable sets of Λ are precisely the sets of the form $\bigcup_{n \in Z} \{n\} \times B_n$ where $B_n \in \Sigma$ for $n \in Z$. Hence assume $\Lambda_0 = \bigcup_{n \in Z} \{n\} \times B_n$ and let $b_n = |B_n|$. Then $|\Lambda_0| = \sum_{n \in Z} b_n$ and

$$1 + \Lambda_0 = \bigcup_{n \in Z} \{n\} \times B_{n-1}, \qquad (1 + \Lambda_0) \cap \Lambda_0 = \bigcup_{n \in Z} (\{n\} \times B_n \cap B_{n-1}).$$

By hypothesis,

$$\varepsilon|\Lambda_0| \geqq |\Lambda_0 \sim ((1 + \Lambda_0) \cap \Lambda_0)| = \sum_{n \in Z} (b_n - |B_n \cap B_{n-1}|)$$

(3.1)

$$= \sum_{n \in Z} (b_{n-1} - |B_n \cap B_{n-1}|),$$

and consequently

(3.2) $$2\varepsilon|\Lambda_0| \geqq \sum_{n \in Z} (b_n \vee b_{n-1} - |B_n \cap B_{n-1}|).$$

Fix any $c \in N^+$ and let $V_n = \{n\} \times B_n \cap B_{n-1}$, $A_n = \{cn, cn+1, \ldots, cn+c-1\}$, and $\Delta_n = \bigcup_{K \in A_n} V_K$. Therefore $(1 + \Lambda_0) \cap \Lambda_0 = \bigcup_{n \in Z} \Delta_n$ and

$$\varepsilon|\Lambda_0| \geqq \sum_{n \in Z} \left(\sum_{K \in A_n} \{b_K - |B_K \cap B_{K-1}|\} \right)$$

$$\geqq \sum_{n \in Z} \left(b_{cn} + \cdots + b_{cn+c-1} - \sum_{K \in A_n} (b_K \wedge b_{K-1}) \right)$$

(3.3)

$$= \sum_{n \in Z} \left(\frac{b_{cn-1}}{2} + b_{cn} + \cdots + \frac{b_{cn+c-1}}{2} - \sum_{K \in A_n} (b_K \wedge b_{K-1}) \right)$$

$$\geqq \frac{1}{2} \sum_{n \in Z} \text{Max} \{|b_u - b_v| : u, v \in A_n \cup \{cn-1\}\}$$

by Lemma 4. Now fix any δ, $0 < \delta < 1$, and let

$$E = E_\delta = \{m \in Z : |B_m \cap B_{m-1}| \leqq (1 - \delta)(b_m \vee b_{m-1})\}.$$

Then by (3.2)

$$2\varepsilon|\Lambda_0| \geqq \sum_{n \in Z} (b_n \vee b_{n-1} - |B_n \cap B_{n-1}|) \geqq \sum_{m \in E} (b_m \vee b_{m-1} - |B_m \cap B_{m-1}|)$$

(3.4)

$$\geqq \delta \sum_{m \in E} b_m \vee b_{m-1} \geqq \delta \sum_{m \in E} b_m.$$

Now let $E^* = \{n \in Z : A_n \cap E \neq \varnothing\}$. We shall show

(3.5)
$$\sum_{n \in E^*} |\Delta_n| \leq \left(\frac{2c\varepsilon}{\delta} + 2c\varepsilon\right)|\Lambda_0|.$$

For, in general by Lemma 4,

$$|\Delta_n| \leq b_{cn-1}/2 + b_{cn} + \cdots + b_{cn+c-1}/2$$

and consequently

$$\sum_{n \in E^*} |\Delta_n| \leq \sum_{n \in E^*} \{c b_m + (\tfrac{1}{2}(b_{cn-1} - b_m) + (b_{cn} - b_m) + \cdots + \tfrac{1}{2}(b_{cn+c-1} - b_m))\}$$

where since $A_n \cap E \neq \varnothing$ we have taken any $m \in A_n \cap E$ and the corresponding term b_m. Hence,

$$\sum_{n \in E^*} |\Delta_n| \leq c \sum_{n \in E^*} b_m + \sum_{n \in E^*} \{\tfrac{1}{2}|b_{cn-1} - b_m| + \cdots + \tfrac{1}{2}|b_{cn+c-1} - b_m|\}$$

$$\leq c \sum_{m \in E} b_m + c \sum_{n \in E^*} \text{Max}\, \{|b_u - b_v| : u, v \in \{cn-1, \ldots, cn+c-1\}\}$$

$$\leq \frac{2c\varepsilon}{\delta} |\Lambda_0| + 2c\varepsilon |\Lambda_0|$$

by (3.4) and (3.3).

We now set $I = Z \sim E^*$, and therefore $n \in I$ implies $A_n \cap E = \varnothing$, i.e. $|V_K| > (1 - \delta)(b_K \vee b_{K-1})$ for all $K \in A_n$ where $V_K = \{K\} \times (B_K \cap B_{K-1})$. For any $n \in I$, set

$$S_n = \bigcap_{K \in \{cn-1, \ldots, cn+c-1\}} B_K \in \Sigma.$$

We next prove for $n \in I$,

(3.6)
$$|\Lambda_n \sim (A_n \times S_n)| \leq c\delta \sum_{T \in A_n} b_T.$$

For $k \in A_n$, $|B_k \cap B_{k-1}| > (1-\delta)(b_k \vee b_{k-1})$, and therefore $|B_k \sim B_{k-1}| < \delta b_k$, $|B_{k-1} \sim B_k| < \delta b_{k-1}$. But for any $k \in A_n$,

$$S_n = \bigcap_{T \in \{cn-1, \ldots, cn+c-1\}} B_T$$
$$\supseteq (B_k \cap B_{k-1}) \sim \{(B_{k-1} \sim B_{k-2}) \vee \cdots \vee (B_{cn} \sim B_{cn-1})$$
$$\cup (B_k \sim B_{k+1}) \vee \cdots \vee (B_{cn+c-2} \sim B_{cn+c-1})\},$$

and therefore

$$|S_n| \geq |B_k \cap B_{k-1}| - \delta \sum_{T \in A_n} b_T.$$

Thus

$$|\Lambda_n \sim (A_n \times S_n)| = \sum_{k \in A_n} (|B_k \cap B_{k-1}| - |S_n|) \leq \sum_{k \in A_n} \delta \sum_{T \in A_n} b_T = c\delta \sum_{T \in A_n} b_T.$$

Consequently,

(3.7)
$$\sum_{n \in I} |\Lambda_n \sim (A_n \times S_n)| \leq c\delta \sum_{n \in I} \sum_{T \in A_n} b_T \leq c\delta |\Lambda_0|,$$

and finally by (3.1), (3.5) and (3.7),

$$\left| \Lambda_0 \sim \bigcup_{n \in I} (A_n \times S_n) \right| = \left| \Lambda_0 \sim (\Lambda_0 \cap (1 + \Lambda_0)) \right| + \left| (\Lambda_0 \cap (1 + \Lambda_0)) \sim \bigcup_{n \in I} (A_n \times S_n) \right|$$

$$= \left| \Lambda_0 \sim (\Lambda_0 \cap (1 + \Lambda_0)) \right| + \sum_{n \in E^*} |\Delta_n| + \sum_{n \in I} |\Delta_n \sim (A_n \times S_n)|$$

$$\leq \varepsilon |\Lambda_0| + \left(\frac{2c\varepsilon}{\delta} |\Lambda_0| + 2c\varepsilon |\Lambda_0| \right) + c\delta |\Lambda_0|.$$

Upon setting $\delta = \sqrt{\varepsilon}$, we obtain the theorem.

COROLLARY 7. *There exist two functions* ω_1 *and* δ_1 *defined on* (0, 1) *satisfying* $\mathrm{Lim}_{\varepsilon \to 0^+} \delta_1(\varepsilon) = 0$, $\mathrm{Lim}_{\varepsilon \to 0^+} \omega_1(\varepsilon) = +\infty$ *such that* $|\Lambda_0 \cap (1 + \Lambda_0)| \geq (1 - \varepsilon)|\Lambda_0|$ *implies*

$$\Lambda_0 = \left(\bigcup_n A_n \times S_n \right) \cup R$$

where each A_n *consists of exactly* $\omega_1(\varepsilon)$ *consecutive integers, the* A_n *are pairwise disjoint,* $S_n \in \Sigma$, *and* $|R| \leq \delta_1(\varepsilon)|\Lambda_0|$. *In fact, we may take* $\omega_1(\varepsilon) = [\varepsilon^{-1/4}]$, $\delta_1(\varepsilon) = 6\varepsilon^{1/4}$.

Proof. Set $c = [\varepsilon^{-1/4}]$ in Theorem 17, and note that we may take $\delta_1(\varepsilon) = 6\sqrt{\varepsilon}[\varepsilon^{-1/4}] \leq 6\varepsilon^{1/4}$.

We now generalize Corollary 7:

THEOREM 18. *Let* $\Lambda = Z^k \times S$ *be the product measure space of* Z^k ($k \in N^+$) *with the counting measure and the measure space* S. *Let* $e_j = (\delta_{1j}, \ldots, \delta_{kj}) \in Z^k$ *for* $1 \leq j \leq k$. *Then there exist two functions* ω_k *and* δ_k *defined on* (0, 1) *satisfying* $\mathrm{Lim}_{\varepsilon \to 0^+} \delta_k(\varepsilon) = 0$, $\mathrm{Lim}_{\varepsilon \to 0^+} \omega_k(\varepsilon) = +\infty$ *such that*

$$|\Lambda_0 \cap (e_j + \Lambda_0)| \geq (1 - \varepsilon)|\Lambda_0| \quad \text{for all } j, \ 1 \leq j \leq k,$$

implies

$$\Lambda_0 = \left(\bigcup_n A_1^n \times \cdots \times A_k^n \times S_n \right) \cup R'$$

where $|R'| \leq \delta_k(\varepsilon)|\Lambda_0|$, *each* A_i^n *is a set of at least* $\omega_k(\varepsilon)$ *consecutive integers,* $A^n = A_1^n \times \cdots \times A_k^n$ *are pairwise disjoint, and* $S_n \in \Sigma$. *In fact, we may take*

$$\omega_k(\varepsilon) = [\varepsilon^{-\lambda_k}], \qquad \delta_k(\varepsilon) = (7k^2 + 7k - 2)\,\varepsilon^{\lambda_k}/2,$$

where

$$\lambda_k = \left\{ \left(\frac{2 + \sqrt{2}}{4} \right) (2 + 2\sqrt{2})^k + \left(\frac{2 - \sqrt{2}}{4} \right) (2 - 2\sqrt{2})^k \right\}^{-1}.$$

Proof. We proceed by induction, noting that Corollary 7 covers the case $k = 1$. Assume the theorem true for $j < k$. Let $\Lambda = Z^k \times S = Z \times S^*$ where $S^* = Z^{k-1} \times S$. We shall find it convenient to consider Λ decomposed as a product in both these ways. Assume $\Lambda_0 \subseteq \Lambda$ is such that $|\Lambda_0 \cap (e_j + \Lambda_0)| \geq (1 - \varepsilon^\rho)|\Lambda_0|$ for $1 \leq j \leq k$, where $\rho > 4$ is a parameter to be optimally chosen later. Then in particular considering

$\Lambda = Z \times S^*$, in which case translation by e_1 corresponds to translating Z by 1, we obtain from Corollary 7

$$\Lambda_0 = \left(\bigcup A_n \times S_n^*\right) \cup R,$$

where $|R| \leqq 6\varepsilon^{\rho/4}|\Lambda_0|$, $|A_n| = [\varepsilon^{-\rho/4}]$, the A_n are pairwise disjoint, $S_n^* \subseteq S^*$ and measurable. For $2 \leqq j \leqq k$, $|\Lambda_0 \cap (e_j + \Lambda_0)| \geqq (1 - \varepsilon^\rho)|\Lambda_0|$; let I_j index all n in the above representation for which $|S_n^* \cap (e_j + S_n^*)| \leqq (1 - \varepsilon)|S_n^*|$ $(2 \leqq j \leqq k)$, where we interpret $e_j + S_n^*$ by projecting e_j on its last $(k - 1)$ factors and consider $S_n^* \subseteq Z^{k-1} \times S$ in the natural way. We then have for $2 \leqq j \leqq k$

$$\varepsilon^\rho|\Lambda_0| \geqq |\Lambda_0 \sim \{(e_j + \Lambda_0) \cap \Lambda_0\}| \geqq \left|\bigcup_n A_n \times \{S_n^* \sim (S_n^* \cap (e_j + S_n^*))\}\right| - 6\varepsilon^{\rho/4}|\Lambda_0|,$$

implying

$$7\varepsilon^{\rho/4}|\Lambda_0| \geqq \varepsilon \sum_{n \in I_j} |A_n \times S_n^*|.$$

Hence

$$\Lambda_0 = \left({\bigcup_n}' A_n \times S_n^*\right) \cup R^*$$

where

$$R^* = \left(\bigcup_{n \in I_2 \cup \cdots \cup I_k} A_n \times S_n^*\right) \cup R, \quad |R^*| \leqq (6\varepsilon^{\rho/4} + 7\varepsilon^{\rho/4 - 1}(k - 1))|\Lambda_0| \leqq 7k\varepsilon^{\rho/4 - 1}|\Lambda_0|,$$

and the restricted union $\bigcup'$ is over $n \notin I_2 \cup \cdots \cup I_k$. By construction, for every such n,

$$|S_n^* \cap (e_j + S_n^*)| \geqq (1 - \varepsilon)|S_n^*| \quad \text{for } 2 \leqq j \leqq k,$$

and by applying the induction hypothesis to S_n^*,

$$S_n^* = \left(\bigcup_i A_{2,i}^n \times \cdots \times A_{k,i}^n \times S_{n,i}\right) \cup R_n$$

where $|R_n| \leqq \delta_{k-1}(\varepsilon)|S_n^*|$, each $A_{j,i}^n$ is a set of at least $\omega_{k-1}(\varepsilon)$ consecutive integers, and the $A_{2,i}^n \times \cdots \times A_{k,i}^n$ are disjoint. Therefore for such n

$$A_n \times S_n^* = \left(\bigcup_i A_n \times A_{2,i}^n \times \cdots \times A_{k,i}^n \times S_{n,i}\right) \cup \bar{R}_n$$

where

$$\bar{R}_n = A_n \times R_n, \quad |\bar{R}_n| \leqq \delta_{k-1}(\varepsilon)|A_n \times S_n^*|.$$

Thus we finally obtain

$$\Lambda_0 = \left\{{\bigcup_n}' \bigcup_i (A_n \times A_{2,i}^n \times \cdots \times A_{k,i}^n \times S_{n,i})\right\} \cup R'$$

where $R' = \bigcup_n' \bar{R}_n \cup R^*$ and $|R'| \leqq (\delta_{k-1}(\varepsilon) + 7\varepsilon^{\rho/4 - 1}k)|\Lambda_0|$. The double union is of terms of the desired type where each factor from a component of Z^k consists of at least Min $([\varepsilon^{-\rho/4}], \omega_{k-1}(\varepsilon))$ consecutive integers. Hence we may take

$$\delta_k(\varepsilon) = \delta_{k-1}(\varepsilon^{1/\rho}) + 7\varepsilon^{1/4 - 1/\rho}k, \quad \omega_k(\varepsilon) = \text{Min } ([\varepsilon^{-1/4}], \omega_{k-1}(\varepsilon^{1/\rho})).$$

We now wish to choose $\rho_0 = \rho_0(k)$ so as to optimize the estimate of δ_k inductively. Assume δ_{k-1} is of the form: $\delta_{k-1}(\varepsilon) = C_{k-1}\varepsilon^{\lambda_{k-1}}$. Then $\delta_k(\varepsilon) = C_{k-1}\varepsilon^{\lambda_{k-1}/\rho} + 7k\varepsilon^{1/4 - 1/\rho}$ is admissible for any $\rho > 4$. We choose ρ_0 such that $\lambda_{k-1}/\rho_0 = 1/4 - 1/\rho_0$ and set $\lambda_k = \lambda_{k-1}/\rho_0$. In this way, if we set $\mu_k = 1/\lambda_k$ we obtain the recursions:

$$\mu_k = 4\mu_{k-1} + 4, \quad \mu_1 = 4,$$
$$C_k = 7k + C_{k-1}, \quad C_1 = 6.$$

Consequently, $C_k = (7k^2 + 7k - 2)/2$ for $k \geq 1$ and

$$\mu_k = \left(\frac{2+\sqrt{2}}{4}\right)(2+2\sqrt{2})^k + \left(\frac{2-\sqrt{2}}{4}\right)(2-2\sqrt{2})^k \quad \text{for } k \geq 1,$$

by standard methods of linear difference equations.

The following simple lemma is used in proving the next theorem:

LEMMA 5. $\{\Lambda_n\} \in W_{\Gamma,\Lambda}$ if and only if $\mathrm{Lim}_{n \to \infty} |g\Lambda_n \triangle \Lambda_n|/|\Lambda_n| = 0$ for all g in a generating set G of Γ.

Proof. Let $\gamma \in \Gamma$, say $\gamma = g_r g_{r-1} \cdots g_2 g_1$, $g_i \in G$, $1 \leq i \leq r$, and choose $n_0(\varepsilon)$ such that for $n \geq n_0$, $|g_i\Lambda_n \triangle \Lambda_n| \leq \varepsilon|\Lambda_n|$ for $1 \leq i \leq r$. Then for $n \geq n_0$

$$|\gamma\Lambda_n \triangle \Lambda_n| = |g_r \cdots g_2 g_1 \Lambda_n \triangle \Lambda_n| \leq |g_r \cdots g_1 \Lambda_n \triangle g_r \cdots g_2 \Lambda_n|$$
$$+ |g_r \cdots g_2 \Lambda_n \triangle g_r \cdots g_3 \Lambda_n| + \cdots + |g_r \Lambda_n \triangle \Lambda_n|$$
$$= |g_1 \Lambda_n \triangle \Lambda_n| + |g_2 \Lambda_n \triangle \Lambda_n| + \cdots + |g_r \Lambda_n \triangle \Lambda_n| \leq r\varepsilon|\Lambda_n|,$$

and therefore, since ε may be arbitrarily small, $\mathrm{Lim}_{n \to \infty} |\gamma\Lambda_n \triangle \Lambda_n|/|\Lambda_n| = 0$.

THEOREM 19. Let $\Gamma = C \times Z^k$, $|C| < +\infty$, be a finitely generated discrete Abelian group, $\Lambda = \Gamma \times S$ as in Definition 4, and $\{\Lambda_n\}$ a sequence of measurable subsets of Λ with finite nonzero measure. Then $\{\Lambda_n\} \in W_{\Gamma,\Lambda}$ if and only if there exist two sequences $\{\omega_n\}$, $\{\gamma_n\}$, $\mathrm{Lim}_{n \to \infty} \omega_n = +\infty$, $\mathrm{Lim}_{n \to \infty} \gamma_n = 0$, such that

$$\Lambda_n = \left(\bigcup_m C \times A_1^m \times \cdots \times A_k^m \times S_m\right) \cup R_n$$

where $|R_n| \leq \delta_n|\Lambda_n|$, each A_i^m is a set of at least ω_n consecutive integers, the $A_1^m \times \cdots \times A_k^m$ are pairwise disjoint and each $S_m \in \Sigma$.

Proof. By Lemma 5 we need only verify the ratio property for a generating set for Γ; we take $G = C \cup \{e_1, \ldots, e_k\}$. For $c \in C$, $c + \Lambda_n = (\bigcup_m (c+C) \times A_1^m \times \cdots \times A_k^m \times S_m) \cup (c + R_n) = (\bigcup_m C \times A_1^m \times \cdots \times A_k^m \times S_m) \cup R_n'$ where $|R_n'| = |R_n| \leq \delta_n|\Lambda_n|$ and therefore $|(c + \Lambda_n) \cap \Lambda_n| \geq (1 - \delta_n)|\Lambda_n|$ implying $\mathrm{Lim}_{n \to \infty} |(c + \Lambda_n) \triangle \Lambda_n|/|\Lambda_n| = 0$. Similarly for $1 \leq i \leq k$,

$$e_i + \Lambda_n = \left\{\bigcup_m C \times A_1^m \times \cdots \times (A_i^m + 1) \times \cdots \times A_k^m \times S_m\right\} \cup R_n'$$

where $|R_n'| = |R_n + e_i| \leq \delta_n |\Lambda_n|$. Since A_i^m consists of at least ω_n consecutive integers,

$$|(e_i + \Lambda_n) \cap \Lambda_n| \geq \left| \bigcup_m C \times A_1^m \times \cdots \times (A_i^m \cap (A_i^m + 1)) \times \cdots \times A_k^m \times S_m \right|$$

$$\geq \left(1 - \frac{1}{\omega_n}\right) \left| \bigcup_m C \times A_1^m \times \cdots \times A_k^m \times S_m \right| \geq \left(1 - \frac{1}{\omega_n}\right)(1 - \delta_n)|\Lambda_n|,$$

and hence $\mathrm{Lim}_{n \to \infty} |(e_i + \Lambda_n) \triangle \Lambda_n|/|\Lambda_n| = 0$ since $\omega_n \to \infty$, $\delta_n \to 0$.

Conversely, let $C = \{c_1, \ldots, c_T\}$ and write

$$\Lambda_n = \bigcup_{1 \leq i \leq T} \{c_i\} \times A_n^i$$

where $A_n^i \subseteq Z^k \times S$. Choose $n_0 = n_0(\varepsilon)$ such that for $n \geq n_0$, $1 \leq i \leq T$,

$$|(C_i + \Lambda_n) \cap \Lambda_n| \geq (1 - \varepsilon)|\Lambda_n|.$$

Therefore, for $1 \leq i, j \leq T$, $|A_n^i \sim A_n^j| \leq \varepsilon |\Lambda_n|$. Let $|A_n^{i_0}| = \mathrm{Max}_{1 \leq i \leq T} |A_n^i|$, and

$$A_n = \bigcap_{1 \leq i \leq T} A_n^i = \{\{\cdots\{A_n^{i_0} \sim (A_n^{i_0} \sim A_n^{i_1})\} \sim \cdots\} \sim (A_n^{i_{T-2}} \sim A_n^{i_{T-1}})\},$$

where $i_0, i_1, \ldots, i_{T-1}$ is an ordering of $1, 2, \ldots, T$. We then have

$$|A_n| \geq |A^{i_0}| - T\varepsilon |\Lambda_n| \geq \left(\frac{1}{T} - T\varepsilon\right)|\Lambda_n|,$$

and consequently $\Lambda_n = (C \times A_n) \cup R_n$ where $|R_n| \leq T^2 \varepsilon |\Lambda_n|$. Choose $n_0' = n_0'(\varepsilon)$ such that $n \geq n_0'$ implies $|(e_i + \Lambda_n) \cap \Lambda_n| \geq (1 - \varepsilon)|\Lambda_n|$ for $1 \leq i \leq k$. Then

$$|(e_i + \Lambda_n) \cap \Lambda_n| = |\{(C \times (e_i + A_n)) \cup (e_i + R_n)\} \cap \{(C \times (A_n \cup R_n))\}|$$

$$\leq |C \times (A_n \cap (e_i + A_n))| + 2|R_n| = T|A_n \cap (e_i + A_n)| + 2|R_n|.$$

Therefore for $1 \leq i \leq k$

$$|(e_i + A_n) \cap \Lambda_n| \geq \frac{(1 - \varepsilon)|\Lambda_n| - 2|R_n|}{T} \geq (1 - \varepsilon - 2T^2\varepsilon)\frac{|\Lambda_n|}{T}$$

$$\geq (1 - (2T + 1)\varepsilon)|A_n|.$$

The proof is now completed by applying Theorem 18 to $\Lambda_0 = A_n$.

BIBLIOGRAPHY

1. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N. J., 1961.

2. W. Emerson and F. P. Greenleaf, *Covering properties and Følner conditions for locally compact groups*, Math. Z. **102** (1967), 370–384.

UNIVERSITY OF CALIFORNIA,
     BERKELEY, CALIFORNIA