THE ABSTRACT TIME-DEPENDENT CAUCHY PROBLEM

BY

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Preface. This paper treats existence of elementary solutions $B(t)$ for equations $B'(t) = A(t)B(t)$, where $A(t)$ is an unbounded operator in a Banach space. Two points of view predominate. One is that generalization of the power-series definition of the exponential function is more fruitful than generalization of the product-integral definition ($e^t = \lim_{n \to \infty} (1 + t/n)^n$). The other point of view is that smoothing of unbounded operators exclusively by resolvents (particularly in the time-dependent case) leads to drastic losses in generality with little gain in clarity and simplicity.

Chapter one of the paper is purely introductory. It develops essential measure-theoretic preliminaries which are particularly relevant to the work of the rest of the paper and do not appear to be readily available elsewhere.

Chapter two begins the paper proper. It introduces the information about elementary solutions for the case of bounded operators from which the rest of the paper is developed. In addition to treating existence, uniqueness, and continuity, it includes a special growth estimate for elementary solutions that makes possible certain of the applications of the general theorems for unbounded operators proved later.

Chapter three is in a sense the core of the paper. Its main theorems are decidedly technical in nature, reflecting the unabashedly anatomical outlook taken toward the work of Hille, Kato, Phillips, and some others. These theorems comprise the first part of the chapter. The second part of the chapter is a justification of the first part. It rederives some of the work of Kato from the previous theorems of the chapter, and it exhibits some interesting nontrivial further applications of the theorems to cases not at all covered by the work of Kato inasmuch as projections in Hilbert space take the place of resolvents for smoothing.

Chapter four is inspired by work and suggestions of I. E. Segal. If the operators involved are affiliated with a ring of operators in a Hilbert space, then some ring-theoretic conditions on the operators ensuring adequate smoothing projections for construction of elementary solutions are available. One of the two cases basically considered may be applied to operators which are matrices in constant-coefficient

differential operators. The other case may be applied to exponentiation of operators affiliated with factors of type II.

Chapter five treats briefly a possible direction in which one may proceed if the operators involved are bad boundary-values of well-behaved analytic functions. The basic case it seeks to generalize is mentioned in the last paragraph of the chapter.

1. **Banach space lemmas.** Let X be a separable Banach space.

(a) DEFINITION. If Ω is a measure space, a function $x: \Omega \to X$ is called measurable if $f(x(\cdot))$ is measurable for all $f \in X^*$. A function $A: \Omega \to B(X)$ is called measurable if $A(\cdot)x$ is measurable for all $x \in X$.

The measurable functions from Ω to X and from Ω to $B(X)$, respectively, form vector spaces.

Let $\{x_n\}$ be a sequence dense in X. Let $\{f_n\} \subseteq X^*$ be such that $\|f_n\| = 1$, and $f_n(x_n) = \|x_n\|$. Then if $x: \Omega \to X$ is measurable, $\|x(\omega)\| = \text{lub}_{n=1,\dots,\infty} |f_n(x(\omega))|$, so that $\|x(\cdot)\|$ is measurable. If $A: \Omega \to B(X)$ is measurable, then $\|A(\omega)\| = \text{lub}_{n=1,\dots,\infty} (\|A(\omega)x_n\|)/(\|x_n\|)$ is measurable.

We wish to show that the measurable functions from Ω to $B(X)$ form an algebra.

When X is a Hilbert space this is easy. Let $A(\cdot)$ be measurable. Then $\|A(\cdot)x\|^2 = (A(\cdot)x, A(\cdot)x)$ is measurable for all $x \in X$. Then $2(A(\cdot)x, A(\cdot)y) = \|A(\cdot)(x+y)\|^2 + i\|A(\cdot)(x+iy)\|^2 - (1+i)\|A(\cdot)x\|^2 - (1+i)\|A(\cdot)y\|^2$ is measurable for all x and y in X, so that $A(\cdot)^*A(\cdot)$ is measurable. Since $A(\cdot)^*$ is trivially measurable whenever $A(\cdot)$ is, $A(\cdot) = S(\cdot) + iT(\cdot)$ with $S(\cdot)$ and $T(\cdot)$ self-adjoint and measurable. Thus we need only show that $A(\cdot)B(\cdot)$ is measurable whenever $A(\cdot)$ and $B(\cdot)$ are self-adjoint and measurable. But if $A(\cdot)$ and $B(\cdot)$ are measurable, then $(A(\cdot) + B(\cdot))^*(A(\cdot) + B(\cdot)) = A(\cdot)^2 + B(\cdot)^2 + A(\cdot)B(\cdot) + B(\cdot)A(\cdot)$ is measurable, so that $A(\cdot)B(\cdot) + B(\cdot)A(\cdot)$ is measurable. Similarly, $(A(\cdot) + iB(\cdot))^*(A(\cdot) + iB(\cdot)) = A(\cdot)^2 + B(\cdot)^2 + iA(\cdot)B(\cdot) - iB(\cdot)A(\cdot)$ is measurable, so that $A(\cdot)B(\cdot) - B(\cdot)A(\cdot)$ is measurable. Thus $A(\cdot)B(\cdot)$ is measurable.

More generally, when X is not necessarily a Hilbert space, $A(\cdot)B(\cdot)$ is still measurable when both $A(\cdot)$ and $B(\cdot)$ are. Since $A(\cdot)B(\cdot)$ measurable means that $A(\cdot)B(\cdot)x$ is measurable for all $x \in X$, it suffices to show that $A(\cdot)x(\cdot)$ is measurable whenever $A(\cdot)$ and $x(\cdot)$ both are. Certainly if $x_n(\omega) \to x(\omega)$ in the norm for each ω and if $A(\cdot)x_n(\cdot)$ is measurable for each n, then $A(\cdot)x(\cdot)$ is measurable. If $x_n(\cdot)$ were measurable and countably-valued, then $A(\cdot)x_n(\cdot)$ would be measurable. But by a standard result $x(\cdot)$ can in fact be approximated by a sequence $\{x_n(\cdot)\}$ of countably-valued measurable $x_n(\cdot)$ for which $\|x(\omega) - x_n(\omega)\| \to 0$ uniformly in ω.

(b) We make the following definitions for convenience.

DEFINITION. $\mathscr{M}(X, \Omega)$ denotes the space of measurable functions from Ω into X. $\mathscr{M}(B(X), \Omega)$ denotes the algebra of measurable functions from Ω into $B(X)$.

DEFINITION.

$$L^p(X, \Omega) = \{x(\cdot) \in \mathscr{M}(X, \Omega) : \|x(\cdot)\| \in L^p(\Omega)\},$$

and

$$L^p(B(X), \Omega) = \{A(\cdot) \in \mathcal{M}(B(X), \Omega) : \|A(\cdot)\| \in L^p(\Omega)\}.$$

For $x(\cdot) \in L^p(X, \Omega)$, we set $\|x(\cdot)\|_p = (\int_\Omega \|x(\cdot)\|^p)^{(1/p)}$, and for $A(\cdot) \in L^p(B(X), \Omega)$, we set $\|A(\cdot)\|_p = (\int_\Omega \|A(\cdot)\|^p)^{(1/p)}$.

The spaces $L^p(X, \Omega)$ and $L^p(B(X), \Omega)$ are Banach spaces for $p \geq 1$. $L^1(X, \Omega)$ is precisely the space of Bochner-integrable functions from Ω into X. Unfortunately, only those $A(\cdot) \in L^1(B(X), \Omega)$ which are "almost separably-valued" are Bochner integrable with $B(X)$ considered as a Banach space under the operator norm. (For $A(\cdot) \in \mathcal{M}(B(X), \Omega)$, $A(\cdot)$ is almost separably-valued if there exists a null-set N such that $\{A(\omega) : \omega \in \Omega - N\}$ is contained in a separable subspace of the Banach space $B(X)$.) For example, if Ω happens to be a topological space with a countable basis of open sets, then $A(\cdot)$ is almost separably-valued if it is almost everywhere continuous as a function from Ω to the Banach space $B(X)$.

(c) We specialize to the case that $\Omega = [a, b]$, a bounded interval with Lebesgue measure.

(1) DEFINITION. A function $x(\cdot): [a, b] \to X$ is of bounded variation if $\exists K > 0$ such that $\sum_{i=1}^n \|x(t_i) - x(t_{i-1})\| < K$ for all choices of $a = t_0 < t_1 < \cdots < t_n = b$. A function $A(\cdot): [a, b] \to B(X)$ is of bounded variation if $\exists K > 0$ such that $\sum_{i=1}^n \|A(t_i) - A(t_{i-1})\| < K$ for all choices of $a = t_0 < t_1 < \cdots < t_n = b$.

Since complex-valued functions of bounded variation are measurable, application of continuous linear functionals shows that $x(\cdot)$ is measurable if it is of bounded variation. Directly, $A(\cdot)$ is measurable if it is of bounded variation. Even more, since it is continuous on $[a, b]$ except at a countable number of points, it is almost separably-valued.

For $A(\cdot)$ to be of bounded variation, it suffices to know that $A(\cdot)x$ is of bounded variation $\forall x \in X$. To see this define a continuous pseudo-norm on X by $\rho_S = \sum_{i=1}^n \|A(t_i)x - A(t_{i-1})x\|$, where S represents the subdivision $a = t_0 < \cdots < t_n = b$. Then ρ_S is uniformly bounded on each x, so that there exists c with $\rho_S(x) \leq c\|x\|$ for all $x \in X$ by Bourbaki's version of the Banach-Steinhaus theorem. Thus for all subdivisions of $[a, b]$, $\sum_{i=1}^n \|A(t_i) - A(t_{i-1})\| \leq c$.

(2) DEFINITION. A function $x(\cdot): [a, b] \to X$ is absolutely continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\sum_{i=1}^n \|x(b_i) - x(a_i)\| < \varepsilon$ whenever $\{(a_i, b_i)\}_{i=1}^n$ is a collection of disjoint open subintervals of $[a, b]$ with $\sum_{i=1}^n (b_i - a_i) < \delta$. A function $A(\cdot): [a, b] \to B(X)$ is absolutely continuous if a similar condition holds with $\sum_{i=1}^n \|A(b_i) - A(a_i)\|$ replacing $\sum_{i=1}^n \|x(b_i) - x(a_i)\|$.

Since absolute continuity implies continuity, $x(\cdot)$ and $A(\cdot)$ are uniformly bounded on $[a, b]$ if they are absolutely continuous. Directly, $A(\cdot)x(\cdot)$ is absolutely continuous if $A(\cdot)$ and $x(\cdot)$ are each absolutely continuous, and $A(\cdot)B(\cdot)$ is absolutely continuous if $A(\cdot)$ and $B(\cdot)$ each are. Parenthetically, we note that absolutely continuous functions are automatically of bounded variation.

If $A(\cdot) \in L^1(B(X), [a, b])$ and is, moreover, almost separably-valued, then

$B(t) = \int_a^t A(s)\,ds$ is absolutely continuous on $[a, b]$, and $B(t)$ is uniformly differentiable a.e. on $[a, b]$ with derivative $B'(t) = A(t)$. If $B_1(\cdot)$ and $B_2(\cdot)$ are absolutely continuous and are uniformly differentiable a.e. on $[a, b]$, then so is $B_1(\cdot)B_2(\cdot)$, and $(B_1(t)B_2(t))' = B_1'(t)B_2(t) + B_1(t)B_2'(t)$ a.e. on $[a, b]$.

If, however, $A(\cdot) \in L^1(B(X), [a, b])$ is not necessarily almost separably-valued, we know only that there exists $B(\cdot)$ with $B(t)x = \int_a^t A(s)x\,ds$ $\forall x \in X$. This means that $\|(B(t_2) - B(t_1))x\| \leqq \|x\| \int_{t_1}^{t_2} \|A(s)\|\,ds$ $\forall x \in X$, so that $\|B(t_2) - B(t_1)\| \leqq \int_{t_1}^{t_2} \|A(s)\|\,ds$. Thus $\|B(t)\| \leqq \|A(\cdot)\|_1$ $\forall t \in [a, b]$, $B(\cdot)$ is absolutely continuous on $[a, b]$, and since $\int_a^t \|A(s)\|\,ds$ is differentiable for almost all $t \in [a, b]$, $(1/\delta)\|B(t+\delta) - B(t)\|$ and $(1/\delta)\|B(t) - B(t-\delta)\|$ are bounded uniformly in δ for almost all $t \in [a, b]$.

For $x(\cdot) \in L^1(X, [a, b])$, $\int_a^t x(s)\,ds$ is strongly differentiable for almost all $t \in [a, b]$ with $(\int_a^t x(s)\,ds)' = x(t)$. Let $A_1(\cdot)$ and $A_2(\cdot) \in L^1(B(X), [a, b])$. Setting $B_i(t)x = \int_a^t A_i(s)x\,ds$ $(i = 1, 2)$, we may assert that, for each $x \in X$, $B_i(t)x$ is strongly differentiable for almost all $t \in [a, b]$ with $(B_i(t)x)' = A_i(t)x$. We should like to assert that $B_1(t)B_2(t)x$ is strongly differentiable for almost all $t \in [a, b]$ with $(B_1(t)B_2(t)x)' = A_1(t)B_2(t)x + B_1(t)A_2(t)x$. The following lemma suffices to establish this.

LEMMA. *Let $x(\cdot)$ be strongly differentiable a.e. on $[a, b]$ with derivative $x'(\cdot)$. Let $A(\cdot) \in L^1(B(X), [a, b])$, and let $B(t)x = \int_a^t A(s)x\,ds$ $\forall x \in X$. Then $B(t)x(t)$ is strongly differentiable a.e. on $[a, b]$ with derivative $A(t)x(t) + B(t)x'(t)$.*

Proof. Let N_0 be a null set such that $x(\cdot)$ is differentiable $\forall t \in [a, b] - N_0$ with derivative at t equal to $x'(t)$. Let N_{00} be a null set such that $\int_a^t \|A(s)\|\,ds$ is differentiable $\forall t \in [a, b] - N_{00}$ with derivative at t equal to $\|A(t)\|$. If $\{x_n\}$ is a sequence dense in X, then for each x_n the set of t for which $B(t)x_n$ is differentiable with $(B(t)x_n)'$ equal to $A(t)x_n$ is contained in $[a, b]$ minus a null set N_n.

Set $N = N_{00} \cup \bigcup_{n=0}^{\infty} N_n$, and let $t \in [a, b] - N$. Then

$$\|(A(t)x(t) + B(t)x'(t)) - (1/\delta)(B(t+\delta)x(t+\delta) - B(t)x(t))\|$$
$$\leqq \|(A(t) - (1/\delta)(B(t+\delta) - B(t)))x(t)\|$$
$$+ \|B(t+\delta)\| \cdot \|x'(t) - (1/\delta)(x(t+\delta) - x(t))\|$$
$$+ \|B(t) - B(t+\delta)\| \cdot \|x'(t)\|.$$

Certainly the last two terms go to zero with δ. The first term is majorized by

$$\|(A(t) - (1/\delta)(B(t+\delta) - B(t)))x_n\|$$
$$+ \|A(t)\| \cdot \|x_n - x(t)\| + (1/\delta)\|B(t+\delta) - B(t)\| \cdot \|x_n - x(t)\|.$$

But we may choose x_n to make the last two terms of this uniformly small in δ, and then let δ go to zero to make the first term vanish. ∎

Suppose now $x(\cdot)$ is absolutely continuous on $[a, b]$ and $(f(x(\cdot)))' = 0$ a.e. $\forall f \in X^*$. Then $x(\cdot)$ is constant.

When $x(\cdot)$ is of bounded variation on $[a, b]$ and has a weak derivative $x'(\cdot)$ a.e. on $[a, b]$, $x'(\cdot) \in L^1(X, [a, b])$ by a standard theorem of vector integration. In particular this is true if $x(\cdot)$ is absolutely continuous on $[a, b]$ and has a weak derivative $x'(\cdot)$ a.e. But by the above paragraph $x(t) - \int_a^t x'(s)\, ds$ is then constant, so that $x(t) = x(a) + \int_a^t x'(s)\, ds$, and $x(\cdot)$ is even strongly differentiable a.e. on $[a, b]$.

2. The Cauchy problem for bounded operators in a separable Banach space.

(a) LEMMA 1. *Let* $A(\cdot) \in L^1(B(X), [a, b])$. *Then there exist* $B(\cdot, a)$ *and* $\bar{B}(\cdot, a) \in L^\infty(B(X), [a, b])$ *such that*

(1) $\|B(t, a) - I\| \leq \exp\left(\int_a^t \|A(s)\|\, ds\right) - 1$ *and* $\|\bar{B}(t, a) - I\| \leq \exp\left(\int_a^t \|A(s)\|\, ds\right) - 1$, *and* (2) $(B(t, a) - I)x = \int_a^t A(s)B(s, a)x\, ds$ *and* $(\bar{B}(t, a) - I)x = -\int_a^t \bar{B}(s, a)A(s)x\, ds$, *for all* $x \in X$.

If $A(\cdot)$ *is almost separably-valued, then* $B(\cdot, a)$ *and* $\bar{B}(\cdot, a)$ *are also, and* $(B(t, a) - I) = \int_a^t A(s)B(s, a)\, ds$, *and* $(\bar{B}(t, a) - I) = -\int_a^t \bar{B}(s, a)A(s)\, ds$.

Proof. Set $\alpha(t) = \|A(t)\|$. For any n, $A(t_n)A(t_{n-1})\cdots A(t_1)$ is measurable on $[a, b]^n$, and, since $\|A(t_n)A(t_{n-1})\cdots A(t_1)\| \leq \alpha(t_n)\cdots\alpha(t_1)$,

$$A(t_n)A(t_{n-1})\cdots A(t_1) \in L^1(B(X), [a, b]^n).$$

For $t \in [a, b]$ let $B_n(t) \in B(X)$ be such that $\forall x \in X$,

$$B_n(t)x = \int_{t \geq t_n \geq \cdots \geq t_1 = a} A(t_n)\cdots A(t_1)x\, dt_1\cdots dt_n.$$

Then

$$\|B_n(t)x\| \leq \|x\| \int_{t \geq t_n \geq \cdots \geq t_1 = a} \alpha(t_n)\cdots\alpha(t_1)\, dt_1\cdots dt_n$$

$$= \|x\| \int_a^t \alpha(t_n)\, dt_n \int_a^{t_n} \alpha(t_{n-1})\, dt_{n-1} \int_a^{t_2} \alpha(t_1)\, dt_1.$$

Set $\beta(t) = \int_a^t \alpha(s)\, ds$, so that $\beta'(t) = \alpha(t)$ a.e. on $[a, b]$. Then

$$\|B_n(t)\| \leq \int_a^t \beta'(t_n)\, dt_n \cdots \int_a^{t_2} \beta'(t_1)\, dt_1 = \int_a^t \beta'(t_n)\, dt_n \cdots \int_a^{t_3} \beta(t_2)\beta'(t_2)\, dt_2$$

$$= \int_a^t \beta'(t_n)\, dt_n \cdots \int_a^{t_3} (\beta(t_3)^2/2!)\beta'(t_3)\, dt_3 = \int_a^t (\beta(t_n)^{n-1}/(n-1)!)\beta'(t_n)\, dt_n$$

$$= (\beta(t)^n/n!).$$

If $f(t_1, \ldots, t_n)$ is any nonnegative function in $L^1([a, b]^n)$, then

$$\int_{t \geq t_n \geq \cdots \geq t_1 = a} f(t_1, \ldots, t_n)\, dt_1\cdots dt_n$$

is a monotone increasing function of t and, hence, is measurable. Thus

$$\int_{t \geq t_n \geq \cdots \geq t_1 = a} f(t_1, \ldots, t_n)\, dt_1\cdots dt_n$$

is measurable for any $f(t_1, \ldots, t_n) \in L^1([a, b]^n)$. Thus $\forall f \in X^*$

$$f(B_n(t)x) = \int_{t \geq t_n \geq \cdots \geq t_1 \geq a} f(A(t_n) \cdots A(t_1)x) \, dt_1 \cdots dt_n$$

is measurable, so that $B_n(\cdot) \in L^\infty(B(X), [a, b])$. Since $\|\sum_{k=m}^n B_k(\cdot)\|_\infty$ $\leq \sum_{k=m}^n (\beta(b)/k!)$, $\sum_{k=1}^n B_k(\cdot)$ is Cauchy in $L^\infty(B(X), [a, b])$.

Set $B(\cdot, a) - I = \sum_{k=1}^\infty B_k(\cdot)$. Then

$$\|B(t, a) - I\| \leq \sum_{k=1}^\infty \|B_k(t)\| \leq \sum_{k=1}^\infty (\beta(t)^k/k!) = \exp(\beta(t)) - 1.$$

This proves the first assertion for $B(t, a)$. To prove the second assertion we note that

$$\left\| \sum_{k=1}^n A(t)B_k(t) \right\| \leq \|A(t)\| \sum_{k=1}^n (\beta(t)^k/k!)$$

$$\leq \|A(t)\| \sum_{k=1}^n (\beta(b)^k/k!) \leq \|A(t)\|(\exp \beta(b) - 1),$$

so that for all n the norm of $\sum_{k=1}^n A(t)B_k(t)$ is bounded by the function

$$\|A(t)\|(\exp \beta(b) - 1),$$

which is in $L^1([a, b])$. Since $\sum_{k=1}^n B_k(t)$ is uniformly convergent for each $t \in [a, b]$, for each $x \in X$ we have

$$\int_a^t A(s)B(s, a)x \, ds = \int_a^t A(s)x \, ds + \sum_{k=1}^\infty \int_a^t A(s)B_k(s)x \, ds.$$

But

$$\int_a^t A(s)B_k(s)x \, ds = \int_a^t A(s) \, ds \int_{s \geq s_k \geq \cdots \geq s_1 \geq a} A(s_k) \cdots A(s_1)x \, ds_1 \cdots ds_k$$

$$= \int_{t \geq s \geq s_k \geq \cdots \geq s_1 \geq a} A(s)A(s_k) \cdots A(s_1)x \, ds_1 \cdots ds_k \, ds$$

by the Fubini theorem for Bochner integrals, $= B_{k+1}(t)x$. Thus

$$\int_a^t A(s)B(s, a)x \, ds = \sum_{k=1}^\infty B_k(t)x = (B(t, a) - I)x.$$

All of the assertions for $\bar{B}(\cdot, a)$ are established in exactly the same manner except that $\bar{B}_k(t)$ is defined by

$$\bar{B}_k(t)x = (-1)^n \int_{t \geq t_n \geq \cdots \geq t_1 \geq a} A(t_1) \cdots A(t_n)x \, dt_1 \cdots dt_n, \quad \forall x \in X.$$

When $A(\cdot)$ is almost separably-valued all of the relevant integrals may be taken directly as uniform Bochner integrals. The stronger assertions in this case follow immediately.

COROLLARY. $B(\cdot, a)$ and $\bar{B}(\cdot, a)$ as defined above are absolutely continuous on $[a, b]$ and are strongly differentiable a.e. on $[a, b]$ on each $x \in X$ with derivatives $A(\cdot)B(\cdot, a)$ and $-\bar{B}(\cdot, a)A(\cdot)$, respectively. When $A(\cdot)$ is almost separably-valued, $B(\cdot, a)$ and $\bar{B}(\cdot, a)$ are, in addition, uniformly differentiable a.e. on $[a, b]$.

Proof. The corollary is immediate from the integral formulas and bounds for $(B(\cdot, a) - I)$ and $(\bar{B}(\cdot, a) - I)$. ∎

LEMMA 2. $\bar{B}(t, a)$ is invertible $\forall t \in [a, b]$.

Proof. Let $a = a_0 < a_1 < \cdots < a_n = b$ be any subdivision of $[a, b]$ with

$$\int_{a_i}^{a_{i+1}} \|A(s)\| \, ds < \tfrac{1}{2} \quad \forall i.$$

Then $\forall t \in [a_i, a_{i+1}]$,

$$\|B(t, a_i) - I\| \leq \exp\left(\int_{a_i}^{t} \|A(s)\| \, ds\right) - 1$$

$$\leq \exp\left(\int_{a_i}^{a_{i+1}} \|A(s)\| \, ds\right) - 1 < e^{1/2} - 1 < 2 - 1 = 1.$$

Thus $B(t, a_i)$ is invertible $\forall t \in [a_i, a_{i+1}]$. Set $C(t) = B(t, a_i)B(a_i, a_{i-1}) \cdots B(a_1, a_0)$ for $t \in [a_i, a_{i+1}]$. Then $C(\cdot)$ is absolutely continuous on $[a, b]$, is strongly differentiable a.e. on $[a, b]$ with derivative $A(\cdot)C(\cdot)$, and is invertible everywhere on $[a, b]$.

But $\bar{B}(\cdot, a)C(\cdot)x$ is strongly differentiable a.e. on $[a, b]$ for each $x \in X$ with derivative $-\bar{B}(\cdot, a)A(\cdot)C(\cdot)x + \bar{B}(\cdot, a)A(\cdot)C(\cdot)x = 0$ and is, in addition, absolutely continuous on $[a, b]$. Thus $\bar{B}(t, a)C(t)x = x \quad \forall t \in [a, b]$, so that $\bar{B}(t, a)C(t) = I \quad \forall t \in [a, b]$, so that $\bar{B}(t, a) = C(t)^{-1} \quad \forall t \in a, b$. ∎

COROLLARY 1. If $x(\cdot)$ is absolutely continuous on $[a, b]$ and strongly differentiable a.e. with $x'(\cdot) = A(\cdot)x(\cdot)$, then $x(t) = \bar{B}(t, a)^{-1}x(a) \quad \forall t \in [a, b]$.

Proof. $\bar{B}(\cdot, a)x(\cdot)$ is absolutely continuous on $[a, b]$ and is differentiable a.e. with derivative $-\bar{B}(\cdot, a)A(\cdot)x(\cdot) + \bar{B}(\cdot, a)A(\cdot)x(\cdot) = 0$, so that $\bar{B}(t, a)x(t) = \bar{B}(a, a)x(a) = x(a) \quad \forall t \in [a, b]$, so that $x(t) = \bar{B}(t, a)^{-1}x(a) \quad \forall t \in [a, b]$. ∎

COROLLARY 2. If $C(\cdot)$ is absolutely continuous on $[a, b]$ and if $C(\cdot)x$ is strongly differentiable a.e. $\forall x \in X$ with derivative $A(\cdot)C(\cdot)x$, then $C(\cdot) = \bar{B}(\cdot, a)^{-1}C(a)$.

Proof. Immediate from Corollary 1. ∎

COROLLARY 3. $B(\cdot, a)$ is the unique absolutely continuous function from $[a, b]$ to $B(X)$ such that $B(a, a) = I$ and such that $B(\cdot, a)x$ is strongly differentiable a.e. $\forall x \in X$ with derivative $A(\cdot)B(\cdot, a)x$. $\bar{B}(\cdot, a)^{-1} = B(\cdot, a)$, so that $B(\cdot, a)$ is invertible everywhere on $[a, b]$.

Proof. Immediate from Corollary 2. ∎

COROLLARY 4. $B(t, a) = B(t, s)B(s, a)$ $\forall s \in [a, t]$. *Thus*

$$B(s, a) = B(t, s)^{-1}B(t, a)   \forall t \in [s, b].$$

Proof. For $s \in [a, b]$, let $C(\cdot)$ be defined by $C(t) = B(t, a)$ for $a \leqq t \leqq s$, $C(t) = B(t, s)B(s, a)$ for $s \leqq t \leqq b$. Then $C(\cdot)$ is absolutely continuous on $[a, b]$, and $C(\cdot)x$ is strongly differentiable a.e. with derivative $A(\cdot)C(\cdot)x$. Thus $C(\cdot) = B(\cdot, a)$, or $B(t, s)B(s, a) = B(t, a)$ $\forall t \in [s, b]$, i.e., $\forall s \in [a, t]$. ∎

COROLLARY 5. $B(b, \cdot)x$ *is differentiable a.e. on* $[a, b]$ *for all* $x \in X$ *with derivative* $-B(b, \cdot)A(\cdot)x$. *If* $A(\cdot)$ *is almost separably-valued,* $B(b, \cdot)$ *is uniformly differentiable a.e. on* $[a, b]$.

Proof. By Corollary 4, $B(b, a)B(\cdot, a)^{-1} = B(b, \cdot)$. Corollary 5 follows at once. ∎

COROLLARY 6. $B(\cdot, a)^{-1}$ *is the unique absolutely continuous function from* $[a, b]$ *to* $B(X)$ *such that* $B(a, a)^{-1} = I$ *and such that* $B(\cdot, a)^{-1}x$ *is strongly differentiable a.e.* $\forall x \in X$ *with derivative* $-B(\cdot, a)A(\cdot)x$.

Proof. Let $C(\cdot)$ be absolutely continuous on $[a, b]$, and let $C(\cdot)x$ be differentiable a.e. $\forall x \in X$ with derivative $-C(\cdot)A(\cdot)x$. Let $C(a) = I$. Then $(C(t) - I)x = -\int_a^t C(s)A(s)x \, ds$ $\forall x \in X$. Then $C(\cdot)B(\cdot, a)x$ is strongly differentiable a.e. with derivative equal to zero, so that $C(t)B(t, a) = I$, so that $C(t) = B(t, a)^{-1}$. ∎

(b) We have obtained a bound $\|B(t, a)\| \leqq \|B(t, a) - I\| + 1 \leqq \exp\left(\int_a^t \|A(s)\| \, ds\right)$ which is by no means the best possible one. If, for example, $A(\cdot) = f(\cdot)I$, where $f(\cdot) \in L^1([a, b])$, $B(t, a) = \exp\left(\int_a^t f(s) \, ds\right)I$, so that $\|B(t, a)\| = \exp\left(\int_a^t \operatorname{Re} f(s) \, ds\right)$, the bound we have obtained is exact only if $\operatorname{Re} f(s) = |f(s)|$ a.e., i.e., only if $f(s) \geqq 0$ a.e.

A bound for $\|B(t, a)\|$ which is exact in this case is, however, obtainable.

DEFINITION. For $A \in B(X)$, $\rho(A) = \inf_{\alpha \geqq 0} (\|A + \alpha\| - \alpha)$. ∎

Since $\|A + \alpha\| \leqq \|A\| + \alpha$ and $\alpha \leqq \|A\| + \|A + \alpha\|$, $-\|A\| \leqq \rho(A) \leqq \|A\|$. If $\alpha_1 \geqq \alpha_2$, then $\|A + \alpha_1\| - \alpha_1 = \|A + \alpha_2 + (\alpha_1 - \alpha_2)\| - \alpha_1 \leqq \|A + \alpha_2\| - \alpha_2$, so that $\|A + \alpha\| - \alpha \downarrow$, so that $\rho(A) = \lim_{n \to \infty} (\|A + n\| - n)$.

Clearly $\rho(0) = 0$. If $\lambda > 0$, $\|\lambda A + \lambda \alpha\| - \lambda \alpha = \lambda(\|A + \alpha\| - \alpha)$, so that $\rho(\lambda A) = \lambda \rho(A)$, and $\rho(\lambda A) = \lambda \rho(A)$ for $\lambda \geqq 0$. Also,

$$\|A + B + \alpha_1 + \alpha_2\| - (\alpha_1 + \alpha_2) \leqq (\|A + \alpha_1\| - \alpha_1) + (\|B + \alpha_2\| - \alpha_2),$$

so that $\rho(A + B) \leqq \rho(A) + \rho(B)$.

Further, if a is any real number, then for α with $\alpha + a > 0$,

$$\|(A + a) + \alpha\| - \alpha = \|A + (a + \alpha)\| - (a + \alpha) + a,$$

so that $\rho(A + aI) = \rho(A) + a$. If a and b are real numbers, then $\rho((a + bi)I) = a + |b|\rho(\pm iI)$. But

$$\|\pm iI + n\| - n = (1 + n^2)^{1/2} - n \leqq (1 + n^2 + (1/4n^2))^{1/2} - n = (n + (1/2n)) - n = 1/2n \to 0$$

as $n \to \infty$, so that $\rho((a + bi)I) = a$.

Finally, if $\lambda > \rho(A)$, then $\exists n > 0$ with $n + \lambda > 0$. Then $\|A + n\| - n < \lambda$ implies $\|A + n\| < (n + \lambda)$, implies $\|A + n\| < (n + \lambda)(1 + (\mu^2/(\lambda + n)^2))^{1/2}$ for any real μ, implies

$$(\|A + n\|)/((n + \lambda)^2 + \mu^2)^{1/2} < 1,$$

implies $(A + n) - (\lambda + i\mu + n) = A - (\lambda + i\mu)$ is invertible (with $\|(A - (\lambda + i\mu))^{-1}\| \leqq (\lambda - \rho(A))^{-1}$), implies $\rho(A) \geqq \sup \operatorname{Re} \sigma(A)$.

Before we state the next proposition, let us note that if $A(\cdot) \in L^1(B(X), [a, b])$, $|\rho(A(\cdot))| \leqq \|A(\cdot)\|$, so that $\rho(A(\cdot)) \in L^1([a, b])$, so that we may consider

$$\int_a^t \rho(A(s)) \, ds.$$

PROPOSITION. $\|B(t, a)\| \leqq \exp\left(\int_a^t \rho(A(s)) \, ds\right)$.

Proof. Let $n > 0$. Then $e^{n(t-a)} B(t, a) = \tilde{B}(t, a)$ is absolutely continuous on $[a, b]$, $\tilde{B}(a, a) = I$, and $\tilde{B}(\cdot, a)x$ is strongly differentiable a.e. $\forall x \in X$ with derivative $(A(\cdot) + n)\tilde{B}(\cdot, a)x$. By the rough bound already obtained,

$$\|\tilde{B}(t, a)\| = e^{n(t-a)}\|B(t, a)\| \leqq \exp\left(\int_a^t \|A(s) + n\| \, ds\right),$$

so that

$$\|B(t, a)\| \leqq \exp\left(\int_a^t (\|A(s) + n\| - n) \, ds\right) \quad \forall t \in [a, b].$$

Since $|\,\|A(\cdot) + n\| - n| \leqq \|A(\cdot)\|$ and $(\|A(s) + n\| - n) \downarrow \rho(A(s))$ $\forall s \in [a, b]$,

$$\int_a^t (\|A(s) + n\| - n) \, ds \downarrow \int_a^t \rho(A(s)) \, ds,$$

so that

$$\|B(t, a)\| \leqq \exp\left(\int_a^t \rho(A(s)) \, ds\right) \quad \forall t \in [a, b]. \quad \blacksquare$$

In general the bound for $\|B(t, a)\|$ given in the above proposition, though perhaps easy to work with, is by no means best possible. For example, if $A \in B(X)$ is nilpotent with $A^{n+1} = 0$, then $\exp(tA) = 1 + tA + \cdots + (t^n/n!)A^n$, so that

$$\|\exp(tA)\| \leqq 1 + t\|A\| + \cdots + (t^n/n!)\|A\|^n = p(t),$$

a polynomial. Since $\sigma(A) = 0$, $\rho(A) \geqq 0$. The estimate given by the proposition in this case is $\|\exp(tA)\| \leqq e^{t\rho(A)}$. If $\rho(A) > 0$, this is certainly not very good.

At least one amelioration of the estimate of the proposition is possible. Suppose $A \in B(X)$ is invertible. Then $A^{-1}B(\cdot, a)A$ is absolutely continuous on $[a, b]$, is the identity at a, and $A^{-1}B(\cdot, a)Ax$ is strongly differentiable a.e. $\forall x \in X$ with derivative

$A^{-1}A(\cdot)B(\cdot, a)Ax = (A^{-1}A(\cdot)A)(A^{-1}B(\cdot, a)A)x$. By the estimate of the proposition $\|A^{-1}B(t, a)A\| \leq \exp\left(\int_a^t \rho(A^{-1}A(s)A)\,ds\right)$. Thus

$$\|B(t, a)\| = \|AA^{-1}B(t, a)AA^{-1}\| \leq (\|A\| \cdot \|A^{-1}\|)\|A^{-1}B(t, a)A\|$$

$$\leq (\|A\| \cdot \|A^{-1}\|) \exp\left(\int_a^t \rho(A^{-1}A(s)A)\,ds\right),$$

so that

$$\|B(t, a)\| \leq \inf_{\text{invertible } A \in B(X)} (\|A\| \cdot \|A^{-1}\|) \exp\left(\int_a^t \rho(A^{-1}A(s)A)\,ds\right).$$

We shall give an example shortly of a nilpotent operator A in Hilbert space for which this gives a fairly reasonable estimate of polynomial type on $\|\exp(tA)\|$. An estimate of this sort would seem to be of most value when

$$A(\cdot) \in L^1(B(X), [a, b]) \quad \forall b > a$$

and we wish to consider a bound on $\|B(t, a)\|$ as $t \to \infty$.

(c) Consider the function θ from $L^1(B(X), [a, b])$ into $L^\infty(B(X), [a, b])$ defined by $\theta: A(\cdot) \to B(\cdot, a)$. We know the range of θ consists of functions which are absolutely continuous on $[a, b]$, which are invertible everywhere on $[a, b]$, which are equal to I at a, and which have $B(\cdot, a)x$ strongly differentiable $\forall x \in X$ with $B'(\cdot, a) \in L^1(B(X), [a, b])$. In fact, the range of θ consists just of all such functions since $\theta(B'(\cdot, a)B(\cdot, a)^{-1}) = B(\cdot, a)$ by uniqueness. Moreover, θ is 1-1 since $\theta^{-1}(B(\cdot, a)) = B'(\cdot, a)B(\cdot, a)^{-1}$.

Let us investigate the continuity of θ. We state two propositions.

PROPOSITION. *Let $A_n(\cdot)$ and $A(\cdot) \in L^1(B(X), [a, b])$, and let $A_n(\cdot) \to A(\cdot)$ in $L^1(B(X), [a, b])$. Then $B_n(\cdot, a_0) \to B(\cdot, a_0)$ in $L^\infty(B(X), [a_0, b])$, uniformly over all $a_0 \in [a, b]$.*

Proof. Since $A_n(\cdot) \to A(\cdot)$ in $L^1(B(X), [a, b])$, $\exists k$ such that $\|A_n(\cdot)\|_1 + \|A(\cdot)\|_1 \leq k$. Since $\|B_n(t_2, t_1)\| \leq \exp\left(\int_{t_1}^{t_2} \|A_n(s)\|\,ds\right)$ for $a \leq t_1 \leq t_2 \leq b$, it follows that $\|B_n(t_2, t_1)\| \leq e^k = K$, and, similarly, $\|B(t_2, t_1)\| \leq K$.

We know that $\forall x \in X$, $B_n(t, s)B(s, a_0)x$ is absolutely continuous and differentiable a.e. in s with derivative $B_n(t, s)(A(s) - A_n(s))B(s, a_0)x$, so that

$$(B(t, a_0) - B_n(t, a_0))x = \int_{a_0}^t B_n(t, s)(A(s) - A_n(s))B(s, a_0)x\,ds,$$

implying

$$\|(B(t, a_0) - B_n(t, a_0))x\| \leq \|x\|K^2 \int_{a_0}^t \|A(s) - A_n(s)\|\,ds$$

$$\leq \|x\|K^2 \int_a^b \|A(s) - A_n(s)\|\,ds = \|x\|K^2\|A(\cdot) - A_n(\cdot)\|_1,$$

so that

$$\|B(\cdot, a_0) - B_n(\cdot, a_0)\|_\infty \leq K^2 \|A(\cdot) - A_n(\cdot)\|_1,$$

and $B_n(\cdot, a_0) \to B(\cdot, a_0)$ in $L^\infty(B(X), [a_0, b])$, uniformly over $a_0 \in [a, b]$.

PROPOSITION. *Let* $A_n(\cdot)$ *and* $A(\cdot) \in L^1(B(X), [a, b])$, *and let* $A_n(\cdot)x \to A(\cdot)x$ *in* $L^1(X, [a, b])$ $\forall x \in X$. *Suppose further that* $\exists k > 0$ *such that* $\|A_n(\cdot)\|_1 + \|A(\cdot)\|_1 \leq k$. *Then* $B_n(\cdot, a_0)x \to B(\cdot, a_0)x$ *in* $L^\infty(X, [a_0, b])$ $\forall x \in X$, *uniformly in* a_0.

Proof. As before, $\|B_n(t_2, t_1)\| \leq K = e^k$, and $\|B(t_2, t_1)\| \leq K$ for $a \leq t_1 \leq t_2 \leq b$.
If $x(\cdot)$ is any finite Riemann sum $\in L^\infty(X, [a, b])$, i.e., if

$$x(\cdot) = \sum_{i=1}^n x_i \chi_{[t_{i-1} t_i]}(\cdot) \quad (a = t_0 \leq \cdots \leq t_n = b),$$

then $A_n(\cdot)x(\cdot) \to A(\cdot)x(\cdot)$ in $L^1(X, [a, b])$.
Let $x \in X$, and let $\{x_m(\cdot)\}$ be a sequence of finite Riemann sums in $L^\infty(X, [a, b])$ such that $\|B(\cdot, a)x - x_m(\cdot)\| \leq 1/m$. Then as in the previous proposition, for $t \in [a, b]$,

$$(B(t, a_0) - B_n(t, a_0))x = \int_{a_0}^t B_n(t, s)(A(s) - A_n(s))B(s, a_0)x \, ds,$$

so that

$$\|(B(t, a_0) - B_n(t, a_0))x\| \leq K \int_a^t \|A(s) - A_n(s)\| \cdot \|B(s, a)x - x_m(s)\| \, ds$$

$$+ K \int_a^t \|(A(s) - A_n(s))x_m(s)\| \, ds \leq (kK)/m + K\|(A(\cdot) - A_n(\cdot))x_m(\cdot)\|_1,$$

so that $\|(B(\cdot, a_0) - B_n(\cdot, a_0))x\| \leq (kK)/m + K\|(A(\cdot) - A_n(\cdot))x_m(\cdot)\|_1$, and

$$B_n(\cdot, a_0)x \to B(\cdot, a_0)x$$

in $L^\infty(X, [a_0, b])$ by first choosing m large and then letting $n \to \infty$, convergence being uniform over $a_0 \in [a, b]$. ∎

(d) When X is a Hilbert space, some further statements can be made.
Note that if $A(\cdot) \in L^1(B(X), [a, b])$,

$$\left(\int_a^t A(s)x \, ds, y\right) = \int_a^t (A(s)x, y) \, ds = \int_a^t (x, A(s)^*y) \, ds = \left(x, \int_a^t A(s)^*y \, ds\right),$$

so that the operators $C(t)$ and $D(t)$ defined by $C(t)x = \int_a^t A(s)x \, ds$, and $D(t)x = \int_a^t A(s)^*x \, ds$, are adjoints.
Thus if we set $B_1(\cdot, a) = \theta(-A(\cdot)^*)$, $B_1(t, a)^*x = -\int_a^t B_1(s, a)^*A(s)x \, ds$, so that $B_1(t, a)^* = B(\cdot, a)^{-1}$. For example, if $A(t)$ is skew-adjoint $\forall t \in [a, b]$, $B_1(\cdot, a) = \theta(A(\cdot)) = B(\cdot, a)$, so that $B(\cdot, a)^* = B(\cdot, a)^{-1}$ and $B(t, a)$ is unitary $\forall t \in [a, b]$.
Let $T = A + iB$, where A and B are self-adjoint. Suppose $M > \rho(T)$. Then $\exists n_0 > 0$

such that $\|A+iB+n\| - n < M$, or $\|A+iB+n\| < M+n$, for $n > n_0$. If x is any unit vector,

$$((A+iB+n)x, (A+iB+n)x) < M^2+n^2+2Mn,$$

so that

$$(1/2n)((\|Ax\|^2 + \|Bx\|^2 + 2\text{Im }(Ax, Bx) - M^2)) + (Ax, x) < M.$$

The expression in the double parenthesis is majorized in absolute value by some $K > 0$ independent of x, so that $(Ax, x) < M + (K/2n)$ $\forall n > n_0$, and $(Ax, x) \leqq M$ for all unit vectors x. Thus sup $\sigma(A) \leqq M$, and sup $\sigma(A) \leqq \rho(T)$.

If $M \leqq \rho(T)$, then the same argument carries through with $>$ replaced everywhere by $\leqq$ and with $K/2n$ appearing on the left. Thus sup $\sigma(A) \geqq \rho(T)$, and $\rho(T) = \text{sup } \sigma(A)$.

[We digress briefly to state some pleasant consequences of the fact just proved.

(1) If T has pure imaginary spectrum, then $(T+T^*)$ can be neither strictly positive nor strictly negative.

To show this we need only show that $(T+T^*)$ cannot be strictly negative, since $-T$ also has pure imaginary spectrum. But $\rho(T) \geqq \text{sup Re } \sigma(T) = 0$, so that sup $\sigma((T+T^*)/2) \geqq 0$.

(2) If A is self-adjoint and S is invertible, sup $\sigma(\text{Re } SAS^{-1})$ is minimal for $S = I$.

To see this we need only notice that SAS^{-1} has the same spectrum as A, so that $\rho(SAS^{-1}) \geqq \rho(A) = \text{sup } \sigma(A)$.

If A is skew-adjoint, $\sigma(SAS^{-1})$ is pure imaginary so that $\rho(SAS^{-1}) \geqq 0 = \rho(A)$.]

From the digression it is clear that the amelioration of the estimate $\|B(t, a)\| \leqq \exp\left(\int_a^t \rho(A(s))\, ds\right)$ is no amelioration at all if $A(t)$ is either self-adjoint or skew-adjoint for each t. Consider, however, the following example.

Let $X = L^2([0, 1])$, and let Af for $f \in X$ be defined by $(Af)(x) = f(x - \frac{1}{2})$ for $x > \frac{1}{2}$, $= 0$ for $x < \frac{1}{2}$. Then $A^2 = 0$. Since $(A^*f)(x) = f(x + \frac{1}{2})$ for $x < \frac{1}{2}$, $= 0$ for $x > \frac{1}{2}$, $(A+A^*)$ simply permutes the sections of a function on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, so that $\|A+A^*\| = 1$. But $(A+A^*)(1) = 1$ ($1 = $ the function identically equal to 1 on $[0, 1]$), so that sup $\sigma(A+A^*) = 1$, and $\rho(A) = \frac{1}{2}$. Then the estimate gives $\|\exp(tA)\| \leqq e^{t/2}$.

We may, however, compute $\|\exp(tA)\|$ directly. Certainly $\exp(tA) = 1 + tA$, and $t - 1 \leqq \|\exp(tA)\| \leqq t + 1$. More precisely, let $h = h\chi_{[0,1/2]} + h\chi_{[1/2,1]} = f + g$ be a function of unit norm in X. Then

$$(\exp(tA)h, \exp(tA)h) = \int_0^1 (h(x) + tf(x - \tfrac{1}{2})\chi_{[1/2,1]}(x))(\bar{h}(x) + t\bar{f}(x - \tfrac{1}{2})\chi_{[1/2,1]}(x))\, dx$$

$$= 1 + t^2\|f\|^2 + 2t\, \text{Re} \int_{1/2}^1 f(x - \tfrac{1}{2})\bar{g}(x)\, dx.$$

This can only increase if we assume that f and g are $\geqq 0$. If we fix f, then $\|g\| = (1 - \|f\|^2)^{1/2}$. Thus by the Schwartz inequality we can choose g to maximize

(exp $(tA)h$, exp $(tA)h$) to $1 + 2t \|f\|(1 - \|f\|^2)^{1/2} + t^2\|f\|^2$. By calculus we may maximize this to $1 + (t/2)((t^2 + 4)^{1/2} + t)$.

Let us now consider the estimate

$$\|\exp (tA)\| \leq \inf_{\substack{\text{invertible } S \in B(X)}} \left(\|S\| \cdot \|S^{-1}\| \exp \int_0^t \rho(S^{-1}AS) \, ds \right).$$

We consider only invertible S of the form $M_a =$ multiplication by e^{ax} for real a. For $f \in L^2([0, 1])$, $(AM_a f) = e^{-a/2}M_a Af$, so that $M_a^{-1}AM_a = e^{-a/2}A$, and $\rho(M_a^{-1}AM_a)$ $= e^{-a/2}\rho(A) = \frac{1}{2}e^{-a/2}$. We need consider only M_a for $a \geq 0$ if we wish to obtain an improvement of the original estimate. Clearly $\|M_a\| = e^a$, and $\|M_a^{-1}\| = 1$. We must therefore minimize

$$\|M_a\| \cdot \|M_a^{-1}\| \exp \int_0^t \rho(M_a^{-1}AM_a) \, ds = e^a e^{t(e^{-a/2}/2)} = \exp (a + t(e^{-a/2}/2))$$

for $a \geq 0$. I.e., we must minimize $a + t(e^{-a/2}/2)$ for $a \geq 0$. We have

$$(d/da)(a + t(e^{-a/2}/2)) = 1 - (te^{-a/2}/2)/4.$$

Since this is always nonnegative for $t \leq 4$ ($a \geq 0$), the expression is minimized by choosing $a = 0$ for $t \leq 4$. For $t > 4$, the derivative is negative at $a = 0$. We therefore set $1 - (t/4)e^{-a/2} = 0$ to get $e^a = t^2/16$ and $e^{-a/2} = 4/t$ at this point, so that the minimum of the expression over $a \geq 0$ for $t > 4$ is $e^2 t^2/16$. Hence the improved general estimate is at least as good as $\|\exp (tA)\| \leq e^{t/2}$ for $t \leq 4$, $\leq e^2 t^2/16$ for $t > 4$. Though not ideal, it is certainly better than $\|\exp (tA)\| \leq e^{t/2}$ for all t.

3. The Cauchy problem for unbounded operators in a separable Banach space.

(a) DEFINITION. $CD(X) =$ the set of closed, densely-defined operators in X. $L(X) =$ the set of operators in $B(X)$ which are 1-1 with dense range. ∎

Let $A \in L(X)$. Then $A^{-1} \in CD(X)$. Under the norm $\|x\|_{R(A)} = \|A^{-1}x\|$, the range of $A = R(A)$ becomes a Banach space since $\{x_n\}$ Cauchy in $R(A)$ means that $\{A^{-1}x_n\}$ is Cauchy in X, so that $\exists y \in X$ with $A^{-1}x_n \to y$ in X, so that $x_n \to Ay$ in $R(A)$.

For $T \in B(X)$ with $T(R(A)) \subseteq R(A)$, $A^{-1}TA$ is bounded by the closed graph theorem because TA is bounded and has range in the domain of A^{-1}. For such T,

$$\|T\|_{R(A)} = \sup_{\|x\|_{R(A)} = 1} \|Tx\|_{R(A)} = \sup_{\|x\|_{R(A)} = 1} \|A^{-1}Tx\|$$

$$= \sup_{\|x\| = 1} \|A^{-1}TAx\| = \|A^{-1}TA\|.$$

If we set $\rho_{R(A)}(T) = \sup_{\alpha \geq 0} (\|T + \alpha\|_{R(A)} - \alpha)$, then

$$\rho_{R(A)}(T) = \sup_{\alpha \geq 0} (\|A^{-1}(T + \alpha)A\| - \alpha) = \sup_{\alpha \geq 0} (\|A^{-1}TA + \alpha\| - \alpha) = \rho(A^{-1}TA).$$

Finally, if X is reflexive and $x_n \to x$ with $\|A^{-1}x_n\| \leq K$ $\forall n$, then $\{A^{-1}x_n\}$ is Cauchy on all elements of X^* of the form $f \circ A$, so that $\{A^{-1}x_n\}$ is Cauchy on all $f \in X^*$, being uniformly bounded. By weak completeness of the closed sphere

$\exists y \in X$ such that $A^{-1}x_n \to y$ weakly. Since the graph of A^{-1} is weakly closed, x is in the domain of A^{-1}, and $A^{-1}x = y$. Also $\|y\| \leq K$ since the norm-closed sphere is weakly closed.

(b) Although reflexivity of X seems to be required to prove much in the way of existence for solutions of the Cauchy problem when unbounded operators are involved, we may derive the essential uniqueness lemma without it.

LEMMA. *Let $A(\cdot)$ be a function from $[a, b]$ to $CD(X)$. Suppose $A \in L(X)$ has range in the domain of $A(t)$ $\forall t \in [a, b]$, and $A(\cdot)A \in L^1(B(X), [a, b])$. Suppose further that there exists a sequence $\{A_n(\cdot)\} \subset L^1(B(X), [a, b])$ for which the associated $\{f \circ B_n(t, s)\}$ are uniformly bounded by a constant $K(f)$ for each f in a w^*-dense subspace Y^* of X^* as $n = 1, 2, \ldots$ and $a \leq s \leq t \leq b$, and for which the functions $\|A_n(\cdot)A\|$ are all dominated by some fixed function in $L^1([a, b])$. Suppose $A_n(t)A \to A(t)A$ strongly for almost all $t \in [a, b]$.*

Then if $x_i(\cdot)$, $i = 1, 2$, is a function from $[a, b]$ into the range of A which is absolutely continuous and strongly differentiable a.e. with derivative $A(\cdot)x_i(\cdot)$, and which has $A^{-1}x_i(\cdot) \in L^\infty(X, [a, b])$, $x_1(a) = x_2(a)$ entails $x_1(t) = x_2(t)$ $\forall t \in [a, b]$.

Proof. Let $x(\cdot) = x_1(\cdot) - x_2(\cdot)$. Then $x(\cdot)$ satisfies the same hypotheses as $x_1(\cdot)$ and $x_2(\cdot)$, and, in addition, $x(a) = 0$.

For any n, $B_n(t, \cdot)x(\cdot)$ is absolutely continuous on $[a, b]$ and strongly differentiable a.e. with derivative $B_n(t, \cdot)(-A_n(\cdot) + A(\cdot))x(\cdot)$, so that for any $f \in Y^*$,

$$f(x(t) - B_n(t, a)x(a)) = f(x(t)) = \int_a^t f(B_n(t, s)(-A_n(s)A + A(s)A)A^{-1}x(s)) \, ds,$$

so that $|f(x(t))| \leq K(f) \int_a^t \|(A(s)A - A_n(s)A)A^{-1}x(s)\| \, ds \to 0$ as $n \to \infty$ because the integrand converges to zero for each s and is dominated by a fixed function in $L^1([a, b])$. Thus $x(t) = 0$ $\forall t \in [a, b]$, and $x_1(\cdot) = x_2(\cdot)$.

(c) From now on unless otherwise stated we shall assume that X is reflexive.

Note that if $x(\cdot) \in \mathcal{M}(X, [a, b])$ and if all $x(t)$ for $t \in [a, b]$ are in the range of $A \in L(X)$, $A^{-1}x(\cdot) \in \mathcal{M}(X, [a, b])$. For if $f \in X^*$, there exists a sequence $\{f_n \circ A\}$ with $f_n \circ A \to f$ in X^*, so that $(f_n \cdot A)(A^{-1}x(\cdot)) = f_n(x(\cdot)) \to f(A^{-1}x(\cdot))$ pointwise, implying that $f(A^{-1}x(\cdot))$ is measurable on $[a, b]$.

Our fundamental existence theorems follow.

THEOREM 1. *Let $A(\cdot)$ be a function from $[a, b]$ into $CD(X)$. Let $A \in L(X)$ have range in the domain of $A(t)$ $\forall t \in [a, b]$, and let $A(\cdot)A \in L^1(B(X), [a, b])$. Let C_1 and $C_2 \in B(X)$, C_2 with dense range. Suppose $\exists$ a sequence $\{A_n(\cdot)\} \subset L^1(B(X), [a, b])$ (with associated $\{B_n(t, s)\}_{a \leq s \leq t \leq b}$) such that*

(1) *the range of $B_n(t, s)C_1C_2$ is in the domain of A^{-1} $\forall n$, $a \leq s \leq t \leq b$,*

(2) *$\exists K > 0$ such that $\|B_n(t, s)C_1\| + \|A^{-1}B_n(t, s)C_1C_2\| \leq K$ $\forall n$, $a \leq s \leq t \leq b$,*

(3) *$\exists k(\cdot) \in L^1([a, b])$ such that $\|A_n(t)A\| + \|A(t)A\| \leq k(t)$ $\forall t \in [a, b]$, and*

(4) *$(A_n(\cdot)A)^*f \to (A(\cdot)A)^*f$ strongly a.e. on $[a, b]$ $\forall f \in X^*$.*

Then for $a \leq s \leq t \leq b$, there exists $B_{C_1C_2}(t, s) \in B(X)$ bounded by K and strongly continuous in t for each s such that

(1) $B_{C_1C_2}(t, t) = C_1$ $\forall t \in [a, b]$, *and* $B_{C_1C_2}(t, s)x = B_{C_1C_2}(t, s)y$ *if* $C_1x = C_1y$,

(2) $B_{C_1C_2}(t, s)$ *carries the range of C_2 into the range of A, and*

$$\|A^{-1}B_{C_1C_2}(t, s)C_2\| \leq K,$$

(3) $B_{C_1C_2}(\cdot, s)C_2$ *is absolutely continuous on $[s, b]$, and $B_{C_1C_2}(\cdot, s)C_2x$ is strongly differentiable a.e. on $[s, b]$ $\forall x \in X$ with derivative $A(\cdot)B_{C_1C_2}(\cdot, s)C_2x$, so that*

$$B_{C_1C_2}(t, s)C_2x = C_1C_2x + \int_s^t A(r)B_{C_1C_2}(r, s)C_2x\, dr \quad \forall x \in X.$$

Proof. Let $\{x_j\}$ and $\{f_j\}$ be sequences in X and X^*, respectively, such that $\forall x \in X, f \in X^*, \varepsilon > 0$, $\exists j$ with $\|x - x_j\| < \varepsilon$ and $\|f - f_j\| < \varepsilon$. Fix s and t, $a \leq s \leq t \leq b$. We have $\forall x \in X$, $(B_n(t, s) - I)C_1C_2x = \int_s^t A_n(r)B_n(r, s)C_1C_2x\, dr$, so that

$$|f((B_n(t, s) - I))C_1C_2x| \leq \int_s^t |f(A_n(r)AA^{-1}B_n(r, s)C_1C_2x)|\, dr$$

$$\leq K\|f\| \cdot \|x\| \int_s^t k(r)\, dr.$$

Choose a subsequence N_1 of the integers such that $f_1((B_{N_1(n)}(t, s) - I)C_1C_2x_1)$ converges to some value $B(t, s, f_1, x_1)$. We have

$$|B(t, s, f_1, x_1)| \leq K\|f_1\| \cdot \|x_1\| \int_s^t k(r)\, dr.$$

Choose a subsequence N_2 of N_1 such that $f_2((B_{N_2(N_1(n))}(t, s) - I)C_1C_2x_2)$ converges to some value $B(t, s, f_2, x_2)$, $|B(t, s, f_2, x_2)| \leq K\|f_2\| \cdot \|x_2\| \int_s^t k(r)\, dr$. Continue, choosing a subsequence N_j of $N_{j-1} \circ \cdots \circ N_1$ for which $f_j((B_{N_j \circ \cdots \circ N_1(n)}(t, s) - I)C_1C_2x_j)$ converges to some value $B(t, s, f_j, x_j)$, $|B(t, s, f_j, x_j)| \leq K\|f_j\| \cdot \|x_j\| \int_s^t k(r)\, dr$. Relabel, calling the diagonal sequence $= \{N_n \circ \cdots \circ N_1(n)\}$ simply $\{n\}$.

Then for each j, $f_j((B_n(t, s) - I)C_1C_2x_j) \to B(t, s, f_j, x_j)$. Let $x \in X, f \in X^*$. Then $f((B_n(t, s) - I)C_1C_2x)$ is Cauchy. For given $\varepsilon > 0$, choose $x_j \in X$ and $f_j \in X^*$ such that $\|f - f_j\| < \min\{\|f\|, (6K\|C_2\| \cdot \|x\|)^{-1}\varepsilon\}$, and $\|x - x_j\| < \varepsilon(12K\|C_2\| \cdot \|f\|)^{-1}$ Then

$$|f((B_n(t, s) - B_m(t, s))C_1C_2x)| \leq |(f - f_j)((B_n(t, s) - B_m(t, s))C_1C_2x)|$$
$$+ |f_j((B_n(t, s) - B_m(t, s))C_1C_2(x - x_j))|$$
$$+ |f_j((B_n(t, s) - B_m(t, s))C_1C_2x_j)|$$
$$< (2/3)\varepsilon + |f_j((B_n(t, s) - B_m(t, s))C_1C_2x_j)|.$$

We have only to choose n_0 such that $m, n > n_0$ makes the last term $< \varepsilon/3$.

Call the limit of $f((B_n(t, s) - I)C_1C_2x)$, $B(t, s, f, x)$.

Now let $\{t_j\}$ be a sequence dense in $[s, b]$. Again by using the diagonal process and ultimately relabeling the final subsequence, we may assume that $f((B_n(t_j, s) - I)C_1C_2x)$ converges to some $B(t_j, s, f, x)$ $\forall x \in X, f \in X^*$. Then $f((B_n(t, s) - I)C_1C_2x)$ converges to some $B(t, s, f, x)$ $\forall x \in X, f \in X^*, t \in [s, b]$. For, taking $s \leqq t_j \leqq t$, we have

$$|f((B_n(t, s) - I)C_1C_2x) - f((B_m(t, s) - I)C_1C_2x)|$$
$$\leqq |f((B_n(t, s) - B_n(t_j, s))C_1C_2x)| + |f((B_m(t, s) - B_m(t_j, s))C_1C_2x)|$$
$$+ |f((B_m(t_j, s) - B_n(t_j, s))C_1C_2x)|.$$

The last term can be made arbitrarily small for fixed j by taking m and n suitably large. Since

$$|f((B_n(t, s) - B_n(t_j, s))C_1C_2x)| \leqq \int_{t_j}^{t} |f(A_n(r)AA^{-1}B_n(r, s)C_1C_2x)| \, dr$$

$$\leqq K\|f\| \cdot \|x\| \int_{t_j}^{t} k(r) \, dr,$$

the first two terms can be made arbitrarily small by choosing t_j sufficiently close to t.

Consider now $f((B_n(t, s) - I)C_1x)$. This is Cauchy $\forall x \in X, f \in X^*, t \in [s, b]$, because $f \circ (B_n(t, s) - I)C_1$ is a bounded sequence of linear functionals (bounded by $\|f\|(K + \|C_1\|)$) converging pointwise on a dense subset of X, namely, the range of C_2.

Thus $f(B_n(t, s)C_1x) \to$ some $f(B_{C_1C_2}(t, s)x)$. This expression is clearly linear in f and x since each of the convergents is, and since each $f(B_n(t, s)C_1x)$ is bounded by $K\|f\| \cdot \|x\|$, $|f(B_{C_1C_2}(t, s)x)| \leqq K\|f\| \cdot \|x\|$. By reflexivity of X, $B_{C_1C_2}(t, s) \in B(X)$ with $\|B_{C_1C_2}(t, s)\| \leqq K$. Since each $B_n(s, s) = I$, $B_{C_1C_2}(s, s) = C_1$. If we anticipate the remainder of this proof by assuming that $B_{C_1C_2}(\cdot, s)C_2$ is absolutely continuous on $[s, b]$ (or even only continuous), then $B_{C_1C_2}(\cdot, s)$ is strongly continuous on $[s, b]$ since bounded strong convergence on the dense subset consisting of the range of C_2 implies strong convergence on all of X. Clearly $B_{C_1C_2}(t, s)x = 0$ if $C_1x = 0$.

We have now that each $f(B_{C_1C_2}(\cdot, s)x)$ is a pointwise limit on $[s, b]$ of the measurable (indeed, continuous) functions $f(B_n(\cdot, s)C_1x)$. Hence $B_{C_1C_2}(\cdot, s) \in L^\infty(B(X), [s, b])$ with $\|B_{C_1C_2}(\cdot, s)\|_\infty \leqq K$, and $B_{C_1C_2}(\cdot, s)C_2 \in L^\infty(B(X), [s, b])$ with $\|B_{C_1C_2}(\cdot, s)C_2\| \leqq K\|C_2\|$. By an earlier remark, $B_{C_1C_2}(t, s)C_2x$ is in the domain of A^{-1} $\forall x \in X, t \in [s, b]$, and $\|A^{-1}B_{C_1C_2}(t, s)C_2x\| \leqq K\|x\|$. I.e., $A^{-1}B_{C_1C_2}(\cdot, s)C_2$ belongs to $L^\infty(B(X), [s, b])$ with $\|A^{-1}B_{C_1C_2}(\cdot, s)C_2\|_\infty \leqq K$ ($A^{-1}B_{C_1C_2}(\cdot, s)C_2$ is measurable by the remark immediately preceding this theorem).

Let us show that $\forall x \in X, f \in X^*, t \in [s, b]$,

$$f(B_{C_1C_2}(t, s)C_2x) = f(C_1C_2x) + \int_s^t f((A(r)A)(A^{-1}B_{C_1C_2}(r, s)C_2)x) \, dr.$$

We have

$$\left| f(C_1C_2x) + \int_s^t f((A(r)A)(A^{-1}B_{C_1C_2}(r,s)C_2x)\,dr - f(B_n(t,s)C_1C_2x) \right|$$

$$\leq \int_s^t |f((A(r)A - A_n(r)A)(A^{-1}B_n(r,s)C_1C_2)x)|\,dr$$

$$+ \int_s^t |f((A(r)A)((A^{-1}B_n(r,s)C_1C_2) - (A^{-1}B_{C_1C_2}(r,s)C_2))x)|\,dr.$$

The integrand of the first term is dominated by $K\|f\| \cdot \|x\| k(r)$, and it goes to zero pointwise a.e. on $[s, t]$ since $(A_n(r)A)^*f \to (A(r)A)^*f$ strongly for almost all $r \in [s, b]$, and $\|A^{-1}B_n(r, s)C_1C_2x\| \leq K\|x\|$ $\forall r \in [s, b]$. Thus the first term goes to zero as $n \to \infty$. The integrand of the second term is similarly dominated, and

$$f \circ ((A(r)A)(A^{-1}B_n(r,s)C_1C_2 - A^{-1}B_{C_1C_2}(r,s)C_2)x) \to 0$$

as $n \to \infty$ since for fixed r and s, $A^{-1}B_n(r, s)C_1C_2 \to A^{-1}B_{C_1C_2}(r, s)C_2$ weakly. Thus the second term goes to zero. I.e., $f(B_n(t, s)C_1C_2x)$ converges to the right side of the desired equality. But it also converges to $f(B_{C_1C_2}(t, s)C_2x)$, so that

$$B_{C_1C_2}(t, s)C_2x = C_1C_2x + \int_s^t A(r)B_{C_1C_2}(r, s)C_2x\,dr \quad \forall t \in [s, b],$$

the integrand being in $L^1(B(X), [s, b])$.

All remaining conclusions of the theorem now follow directly. ∎

THEOREM 2. *Suppose in addition to the hypothesis of Theorem 1 that $A_n(\cdot)Ax \to A(\cdot)Ax$ strongly a.e. on $[a, b]$ for each $x \in X$, and that $\exists C_3 \in B(X)$ such that C_3 is 1-1 and $\exists K' > 0$ with $\|C_3B_n(t, s)\| \leq K'$ $\forall n, a \leq s \leq t \leq b$. Then if $x(\cdot) \in L^\infty(X, [s, b])$ is absolutely continuous with $x(t)$ in the domain of A^{-1} $\forall t \in [s, b]$ in such wise that $A^{-1}x(\cdot) \in L^\infty(X, [s, b])$, and if $x(\cdot)$ is strongly differentiable a.e. on $[s, b]$ with derivative $A(\cdot)x(\cdot)$, then $x(s) = C_1C_2x$ entails $x(t) = B_{C_1C_2}(t, s)C_2x$ $\forall t \in [s, b]$.*

Proof. Apply the uniqueness lemma to $x(\cdot)$ and to $B_{C_1C_2}(\cdot, s)C_2x$. ∎

COROLLARY 1. *If $C(\cdot) \in L^\infty(B(X), [s, b])$ is such that $C(s) = C_1$, $C(\cdot)C_2$ is absolutely continuous on $[s, b]$ with range in the domain of A^{-1} in such wise that $A^{-1}C(\cdot)C_2 \in L^\infty(B(X), [s, b])$, and if $C(\cdot)C_2x$ is strongly differentiable a.e. on $[s, b]$ with derivative $A(\cdot)C(\cdot)C_2x$ $\forall x \in X$, then $C(\cdot) = B_{C_1C_2}(\cdot, s)$.*

Proof. By the theorem $C(\cdot)C_2x = B_{C_1C_2}(\cdot, s)C_2x$ $\forall x \in X$. Since the range of C_2 is dense in X, $C(\cdot) = B_{C_1C_2}(\cdot, s)$. ∎

COROLLARY 2. *Let $s_0 \in [s, b]$. Let $x \in X$. If $B_{C_1C_2}(s_0, s)C_2x = C_1y$ for some $y \in X$, then $B_{C_1C_2}(t, s)C_2x = B_{C_1C_2}(t, s_0)C_1^{-1}B_{C_1C_2}(s_0, s)C_2x$ for all $t \in [s_0, b]$. If the range of $B_{C_1C_2}(s_0, s)$ lies in the range of C_1, then*

$$B_{C_1C_2}(t, s) = B_{C_1C_2}(t, s_0)C_1^{-1}B_{C_1C_2}(s_0, s) \quad \forall t \in [s_0, b].$$

Proof. Let $x(t) = B_{C_1C_2}(t, s)C_2x$ for $t \in [s, s_0]$, $= B_{C_1C_2}(t, s_0)C_1^{-1}B_{C_1C_2}(s_0, s)C_2x$ for

$t \in [s_0, b]$. Then $x(\cdot)$ satisfies the hypotheses of the theorem, so that $B_{C_1C_2}(t, s)C_2x = x(t) \quad \forall t \in [s, b]$, in particular, for $t \in [s_0, b]$.

If the range of $B_{C_1C_2}(s_0, s)$ lies in the range of C_1, $B_{C_1C_2}(t, s_0)C_1^{-1}B_{C_1C_2}(s_0, s)$ is well-defined, and $B_{C_1C_2}(t, s)C_2x = B_{C_1C_2}(t, s_0)C_1^{-1}B_{C_1C_2}(s_0, s)C_2x$ by the first part of the corollary $\forall x \in X$, so that $B_{C_1C_2}(t, s) = B_{C_1C_2}(t, s_0)C_1^{-1}B_{C_1C_2}(s_0, s)$ for $t \in [s_0, b]$ by density of the range of C_2.

THEOREM 3. *Suppose in addition to the hypotheses of Theorems 1 and 2 that $C_1 \in L(X)$. Then $C_3B_{C_1C_2}(t, s)C_1^{-1} \in B(X)$, $a \leq s \leq t \leq b$, and $\|C_3B_{C_1C_2}(t, s)C_1^{-1}\| \leq K'$. Further, if $B_n(t, s)C_1 \to B_{C_1C_2}(t, s)$ weakly, $C_3B_n(t, s) \to C_3B_{C_1C_2}(t, s)C_1^{-1}$ weakly. Also, $C_3B_{C_1C_2}(t, \cdot)C_1^{-1}Ax$ is absolutely continuous on $[a, t]$, and $C_3B_{C_1C_2}(t, \cdot)C_1^{-1}Ax$ is strongly differentiable a.e. on $[a, t]$ with derivative $-C_3B_{C_1C_2}(t, \cdot)C_1^{-1}A(\cdot)Ax$.*

Proof. Let us assume that $B_n(t, s)C_1 \to B_{C_1C_2}(t, s)$ weakly. Then $C_3B_n(t, s)C_1 \to C_3B_{C_1C_2}(t, s)$ weakly. Let $y = C_1x$, $\|y\| \leq 1$. We have

$$C_3B_n(t, s)C_1x = C_3B_n(t, s)y \to C_3B_{C_1C_2}(t, s)x = C_3B_{C_1C_2}(t, s)C_1^{-1}y$$

weakly. Then $\|C_3B_{C_1C_2}(t, s)C_1^{-1}y\| \leq \liminf \|C_3B_n(t, s)y\| \leq K'$, so that

$$\|C_3B_{C_1C_2}(t, s)C_1^{-1}\| \leq K'.$$

Since $C_3B_n(t, s)$ is bounded by $K' \ \forall n$ and since $C_3B_n(t, s)$ converges weakly to $C_3B_{C_1C_2}(t, s)C_1^{-1}$ on the range of C_1, which is dense, $C_3B_n(t, s) \to C_3B_{C_1C_2}(t, s)C_1^{-1}$ weakly.

Let us show that

$$f(C_3B_{C_1C_2}(t, s)C_1^{-1}Ax) = f(C_3Ax) + \int_s^t f(C_3B_{C_1C_2}(t, r)C_1^{-1}A(r)Ax) \, dr.$$

We have $B_n(t, s)x = x + \int_s^t B_n(t, r)A_n(r)x \, dr$, so that

$$\left| f(C_3Ax) + \int_s^t f(C_3B_{C_1C_2}(t, r)C_1^{-1}A(r)Ax) \, dr - f(C_3B_n(t, s)Ax) \right|$$

$$\leq \int_s^t |f(C_3B_n(t, r)(A(r)A - A_n(r)Ax))| \, dr$$

$$+ \int_s^t |f((C_3B_n(t, r) - C_3B_{C_1C_2}(t, r)C_1^{-1})A(r)Ax)| \, dr.$$

The integrand of the first integral is dominated by $K'\|f\| \cdot \|x\|k(r)$, and since $A_n(r)A \to A(r)A$ strongly a.e., the integrand goes to zero pointwise as $n \to \infty$. Thus the integral goes to zero as $n \to \infty$. The integrand of the second integral is dominated by $2K'\|f\| \cdot \|x\|k(r)$, and since $C_3B_n(t, r) \to C_3B_{C_1C_2}(t, r)C_1^{-1}$ weakly for fixed t and r, the integrand goes pointwise to zero as $n \to \infty$, and hence this integral, too, goes to zero as $n \to \infty$. But $f(C_3B_n(t, s)Ax) \to f(C_3B_{C_1C_2}(t, s)C_1^{-1}Ax)$, so that $C_3B_{C_1C_2}(t, s)C_1^{-1}Ax = C_3Ax + \int_s^t C_3B_{C_1C_2}(t, r)C_1^{-1}A(r)Ax \, dr$, the integrand of the integral being in $L^1(B(X), [s, t])$.

The remaining conclusions of the theorem are now immediate.

At least one generalization of these theorems is possible. Suppose $U(\cdot) \in L^\infty(B(X), [a, b])$ is absolutely continuous, and suppose that $U(\cdot)x$ is strongly differentiable a.e. $\forall x \in X$ with derivative $U'(\cdot)x$, where $U'(\cdot) \in L^1(B(X), [a, b])$. If, in addition, each $U(t)$ is invertible, we may write $U'(\cdot) = (U'(\cdot)U(\cdot)^{-1})U(\cdot)$, where $U'(\cdot)U(\cdot)^{-1} \in L^1(B(X), [a, b])$ by continuity, and, hence, boundedness of $U(\cdot)^{-1}$. Then, since $U(\cdot)U(a)^{-1}$ is the identity at a, $U(t)U(a)^{-1}x = x + \int_a^t (U'(s)U(s)^{-1})(U(s)U(a)^{-1})x \, ds$, and, consequently,

$$U(a)U(t)^{-1}x = x - \int_a^t (U(a)U(s)^{-1})(U'(s)U(s)^{-1})x \, ds$$

$\forall x \in X$ by the fundamental propositions for bounded operators. Thus $U(\cdot)^{-1}$ is absolutely continuous, and $U(\cdot)^{-1}x$ is strongly differentiable a.e. $\forall x \in X$ with derivative $-U(\cdot)^{-1}U'(\cdot)U(\cdot)^{-1}x$.

Having made these remarks, we may now generalize the three above theorems as follows.

THEOREM 1′. *Let* $U(\cdot) \in L^\infty(B(X), [a, b])$ *be absolutely continuous, and let* $U(t)$ *be invertible* $\forall t \in [a, b]$. *Set* K_0 *equal to* $\|U(\cdot)\|_\infty \|U(\cdot)^{-1}\|_\infty$. *Let* $U(\cdot)x$ *be strongly differentiable a.e.* $\forall x \in X$ *with derivative* $U'(\cdot)x$, *where*

$$U'(\cdot) \in L^1(B(X), [a, b]).$$

Let $\bar{A}(\cdot)$ *be a function from* $[a, b]$ *to* $CD(X)$ *such that* $A(\cdot) = -U(\cdot)^{-1}U'(\cdot) + U(\cdot)^{-1}\bar{A}(\cdot)U(\cdot)$ *satisfies the hypotheses of Theorem 1 with* A, C_1, C_2, *and* K *as given in that theorem. Set* $C_1(t) = U(t)C_1U(t)^{-1}$, $C_2(t) = U(t)C_2U(t)^{-1}$, *and* $P(t) = U(t)AU(t)^{-1}$ $\forall t \in [a, b]$. (*Clearly all of* $C_1(\cdot)$, $C_2(\cdot)$, *and* $P(\cdot)$ *are absolutely continuous on* $[a, b]$ *and strongly differentiable a.e. with the obvious derivatives. The range of* $C_2(t)$ *is dense in* X $\forall t \in [a, b]$, *and* $P(t) \in L(X)$ $\forall t \in [a, b]$.)

Then $\bar{A}(\cdot)P(\cdot) \in L^1(B(X), [a, b])$, *and for* $a \le s \le t \le b$, $\exists \bar{B}_{C_1C_2}(t, s) \in B(X)$ *bounded by* KK_0 *and strongly continuous in* t *for each* s *such that*

(1) $\bar{B}_{C_1C_2}(t, t) = C_1(t)$ $\forall t \in [a, b]$, *and* $\bar{B}_{C_1C_2}(t, s)x = \bar{B}_{C_1C_2}(t, s)y$ *if* $C_1(s)x = C_1(s)y$,

(2) $\bar{B}_{C_1C_2}(t, s)$ *carries the range of* $C_2(s)$ *into the domain of* $P(t)^{-1}$, *and*

$$\|P(t)^{-1}\bar{B}_{C_1C_2}(t, s)C_2(s)\| \le KK_0,$$

(3) $\bar{B}_{C_1C_2}(\cdot, s)C_2(s)$ *is absolutely continuous on* $[s, b]$ *and* $\bar{B}_{C_1C_2}(\cdot, s)C_2(s)x$ *is strongly differentiable a.e. on* $[s, b]$ $\forall x \in X$ *with derivative* $\bar{A}(\cdot)\bar{B}_{C_1C_2}(\cdot, s)C_2(s)x$, *so that* $\bar{B}_{C_1C_2}(t, s)C_2(s)x = C_1(s)C_2(s)x + \int_s^t \bar{A}(r)\bar{B}_{C_1C_2}(r, s)C_2(s)x \, dr$ $\forall x \in X$.

Proof. Since $A(\cdot)A \in L^1(B(X), [a, b])$, $A(\cdot)A + U(\cdot)^{-1}U'(\cdot)A$ belongs to $L^1(B(X), [a, b])$, and, consequently, $\bar{A}(\cdot)U(\cdot)AU(\cdot)^{-1} = \bar{A}(\cdot)P(\cdot)$ belongs to $L^1(B(X), [a, b])$. Let $B_{C_1C_2}(t, s)$ be as in Theorem 1, and set $\bar{B}_{C_1C_2}(t, s) = U(t)B_{C_1C_2}(t, s)U(s)^{-1}$. Then $\|\bar{B}_{C_1C_2}(t, s)\| \le KK_0$. By strong continuity and boundedness of $B_{C_1C_2}(t, s)$ in t, and by uniform continuity of $U(t)$, $\bar{B}_{C_1C_2}(t, s)$ is strongly continuous in t.

We have $\bar{B}_{C_1C_2}(t, t) = U(t)B_{C_1C_2}(t, t)U(t)^{-1} = C_1(t)$ $\forall t \in [a, b]$. If $C_1(s)x = C_1(s)y$, then $C_1U(s)^{-1}x = C_1U(s)^{-1}y$, so that

$$B_{C_1C_2}(t, s)U(s)^{-1}x = B_{C_1C_2}(t, s)U(s)^{-1}y,$$

and $\bar{B}_{C_1C_2}(t, s)x = \bar{B}_{C_1C_2}(t, s)y$.

The range of $\bar{B}_{C_1C_2}(t, s)C_2(s)$ is the range of $U(t)B_{C_1C_2}(t, s)U(s)^{-1}U(s)C_2U(s)^{-1}$, which has the same range as $U(t)B_{C_1C_2}(t, s)C_2$, which has range in the range of $U(t)A$, which lies in the domain of $P(t)^{-1} = U(t)A^{-1}U(t)^{-1}$. We have

$$\|P(t)^{-1}\bar{B}_{C_1C_2}(t, s)C_2(s)\| = \|U(t)A^{-1}B_{C_1C_2}(t, s)C_2U(s)^{-1}\| \leq KK_0.$$

Since $\bar{B}_{C_1C_2}(\cdot, s)C_2(s) = U(\cdot)(B_{C_1C_2}(\cdot, s)C_2)U(s)^{-1}$, it is absolutely continuous on $[a, b]$, and it is strongly differentiable a.e. on all $x \in X$ with derivative

$$U'(\cdot)(B_{C_1C_2}(\cdot, s)C_2)U(s)^{-1} + U(\cdot)(-U(\cdot)^{-1}U'(\cdot) + U(\cdot)^{-1}\bar{A}(\cdot)U(\cdot))$$
$$\cdot B_{C_1C_2}(\cdot, s)C_2U(s)^{-1} = \bar{A}(\cdot)\bar{B}_{C_1C_2}(\cdot, s)C_2(s). \quad \blacksquare$$

THEOREM 2′. *Suppose in addition to the hypotheses of Theorem 1′ the hypotheses of Theorem 2 hold for $A(\cdot)$. Then if $x(\cdot) \in L^\infty(X, [s, b])$ is absolutely continuous with $x(t)$ in the domain of $P(t)^{-1}$ $\forall t \in [s, b]$ in such wise that $P(t)^{-1}x(t) \in L^\infty(X, [s, b])$, and if $x(\cdot)$ is strongly differentiable a.e. on $[s, b]$ with derivative $\bar{A}(\cdot)x(\cdot)$, then $x(s) = C_1(s)C_2(s)x$ entails $x(t) = \bar{B}_{C_1C_2}(t, s)C_2(s)x$ $\forall t \in [s, b]$.*

Proof. It suffices to show that $U(t)^{-1}x(t) = B_{C_1C_2}(t, s)C_2U(s)^{-1}x$ $\forall t \in [s, b]$. But $U(\cdot)^{-1}x(\cdot)$ is absolutely continuous, $U(t)^{-1}x(t)$ is in the domain of A^{-1} because $x(t)$ is in the domain of $P(t)^{-1} = U(t)A^{-1}U(t)^{-1}$, and $A^{-1}U(\cdot)^{-1}x(\cdot) = U(\cdot)^{-1}P(\cdot)^{-1}x(\cdot) \in L^\infty(X, [s, b])$. Further, $U(\cdot)^{-1}x(\cdot)$ is strongly differentiable a.e. on $[s, b]$ with derivative

$$U(\cdot)^{-1'}x(\cdot) + U(\cdot)^{-1}x'(\cdot) = -U(\cdot)^{-1}U'(\cdot)U(\cdot)^{-1}x(\cdot) + U(\cdot)^{-1}\bar{A}(\cdot)x(\cdot)$$
$$= (-U(\cdot)^{-1}U'(\cdot) + U(\cdot)^{-1}\bar{A}(\cdot)U(\cdot))U(\cdot)^{-1}x(\cdot) = A(\cdot)(U(\cdot)^{-1}x(\cdot)).$$

Finally, $U(s)^{-1}x(s) = U(s)^{-1}C_1(s)C_2(s)x = C_1C_2U(s)^{-1}x$.

By Theorem 2, $U(t)^{-1}x(t) = B_{C_1C_2}(t, s)C_2U(s)^{-1}x$ $\forall t \in [a, b]$. $\blacksquare$

COROLLARY 1′. *If $C(\cdot) \in L^\infty(B(X), [s, b])$ is such that $C(s) = C_1(s)$, $C(\cdot)C_2(s)$ is absolutely continuous on $[s, b]$ with range in the domain of $P(t)^{-1}$ in such wise that $P(t)^{-1}C(\cdot)C_2(s)$ belongs to $L^\infty(B(X), [s, b])$, and if $C(\cdot)C_2(s)x$ is strongly differentiable a.e. on $[s, b]$ with derivative $\bar{A}(\cdot)C(\cdot)C_2(s)x$ $\forall x \in X$, then $C(\cdot) = \bar{B}_{C_1C_2}(\cdot, s)$.*

Proof. By Theorem 2′, $C(\cdot)C_2(s)x = \bar{B}_{C_1C_2}(\cdot, s)C_2(s)x$ $\forall x \in X$. Since $C_2(s)$ has dense range, $C(\cdot) = \bar{B}_{C_1C_2}(\cdot, s)$. $\blacksquare$

COROLLARY 2′. *Let $s_0 \in [s, b]$, and let $x \in X$. If $\bar{B}_{C_1C_2}(s_0, s)C_2(s)x = C_1(s_0)y$ for some $y \in X$, then*

$$\bar{B}_{C_1C_2}(t, s)C_2(s)x = \bar{B}_{C_1C_2}(t, s_0)C_1(s_0)^{-1}\bar{B}_{C_1C_2}(s_0, s)C_2(s)x \quad \forall t \in [s_0, b].$$

If the range of $\bar{B}_{c_1 c_2}(s_0, s)$ lies in the range of $C_1(s_0)$, then

$$\bar{B}_{c_1 c_2}(t, s) = \bar{B}_{c_1 c_2}(t, s_0)C_1(s_0)^{-1}\bar{B}_{c_1 c_2}(s_0, s) \quad \forall t \in [s_0, b].$$

Proof. Exactly as for Corollary 2. ∎

THEOREM 3'. *Suppose in addition to the hypotheses of Theorems 1' and 2', $C_1(t) \in L(X)$ $\forall t \in [a, b]$, i.e., $C_1 \in L(X)$. Let C_3 and K' be as implicitly contained in the statement of Theorem 2' (recall that C_3 is 1-1). Set $C_3(t) = U(t)C_3 U(t)^{-1}$. Then $C_3(t)\bar{B}_{c_1 c_2}(t, s)C_1(s)^{-1} \in B(X)$ for $a \leqq s \leqq t \leqq b$, and*

$$\| C_3(t)\bar{B}_{c_1 c_2}(t, s)C_1(s)^{-1}\| \leqq K'K_0.$$

Also, $C_3(t)\bar{B}_{c_1 c_2}(t, \cdot)C_1(\cdot)^{-1}P(\cdot)x$ is strongly differentiable a.e. on $[a, t]$ with derivative

$$C_3\bar{B}_{c_1 c_2}(t, \cdot)C_1(\cdot)^{-1}(-\bar{A}(\cdot)P(\cdot)+P'(\cdot))x \quad \forall x \in X.$$

Proof. Since $C_3(t)\bar{B}_{c_1 c_2}(t, s)C_1(s)^{-1} = U(t)(C_3 B_{c_1 c_2}(t, s)C_1^{-1})U(s)^{-1}$, it is bounded by $K'K_0$ by Theorem 3. Also,

$$C_3(t)\bar{B}_{c_1 c_2}(t, \cdot)C_1(\cdot)^{-1}P(\cdot) = U(t)(C_3 B_{c_1 c_2}(t, \cdot)C_1^{-1}A)U(\cdot)^{-1},$$

which is absolutely continuous on $[a, t]$ by Theorem 3, and which is almost everywhere strongly differentiable $\forall x \in X$ with derivative

$$U(t)(C_3 B_{c_1 c_2}(t, \cdot)C_1^{-1}A)'U(\cdot)^{-1} + U(t)(C_3 B_{c_1 c_2}(t, \cdot)C_1^{-1}A)U(\cdot)^{-1'}$$

$$= U(t)(C_3 B_{c_1 c_2}(t, \cdot)C_1^{-1})((-A(\cdot)AU(\cdot)^{-1} - AU(\cdot)^{-1}U'(\cdot)U(\cdot)^{-1}))$$

$$= C_3(t)\bar{B}_{c_1 c_2}(t, \cdot)C_1(\cdot)^{-1}((-U(\cdot)A(\cdot)U(\cdot)^{-1}P(\cdot) - P(\cdot)U'(\cdot)U(\cdot)^{-1})).$$

But $A(\cdot) = -U(\cdot)^{-1}U'(\cdot) + U(\cdot)^{-1}\bar{A}(\cdot)U(\cdot)$, so that $-U(\cdot)A(\cdot)U(\cdot)^{-1} = U'(\cdot)U(\cdot)^{-1} - \bar{A}(\cdot)$, and the expression in double parentheses is $-\bar{A}(\cdot)P(\cdot) + (U'(\cdot)U(\cdot)^{-1}P(\cdot) - P(\cdot)U'(\cdot)U(\cdot)^{-1})$. But since $P(\cdot) = U(\cdot)AU(\cdot)^{-1}$,

$$P'(\cdot) = U'(\cdot)AU(\cdot)^{-1} - U(\cdot)AU(\cdot)^{-1}U'(\cdot)U(\cdot)^{-1}$$

$$= U'(\cdot)U(\cdot)^{-1}P(\cdot) - P(\cdot)U'(\cdot)U(\cdot)^{-1}. ∎$$

(d) We shall immediately take up some specific cases of the above theorems. We deal first with Kato's basic theorem. Though we do not prove precisely the same result as he does, the change seems in the direction of improvement and generalization.

Consider first a function $N(\cdot) \in L^\infty(B(X), [a, b])$ and an associated operator $A \in L(X)$. We assume each $N(t)$ to be in $L(X)$ and to have the same range as A. Furthermore, we assume that $\exists M \geqq 1$ such that $\|A^{-1}N(\cdot)\|_\infty \leqq M$ and $\|N(\cdot)^{-1}A\|_\infty \leqq M$ (each $N(t)^{-1}A$ and $A^{-1}N(t)$ is clearly bounded). Finally, we assume that $N(\cdot)$ is of bounded variation productwise over every subinterval of $[a, b]$, with

variation bounded by M. I.e., we assume that for any subdivision $a \leq t_0 \leq t_1 \leq \cdots \leq t_n \leq b$, $\prod_{i=1}^n \|N(t_i)^{-1}N(t_{i-1})\| \leq M$.

[Digression: Kato assumes that $N(\cdot)^{-1}A$ is of ordinary bounded variation. This implies $N(\cdot)$ is of bounded variation productwise. For (essentially reproducing his proof), if we assume the variation of $N(\cdot)^{-1}A$ to be bounded by M_0, we have

$$
\begin{aligned}
\|N(t_i)^{-1}N(t_{i-1})\| &= \|(N(t_i)^{-1}-N(t_{i-1})^{-1})N(t_{i-1})+1\| \\
&\leq 1+\|(N(t_i)^{-1}-N(t_{i-1})^{-1})A(A^{-1}N(t_{i-1}))\| \\
&\leq 1+\|(N(t_i)^{-1}-N(t_{i-1})^{-1})A\| \cdot M,
\end{aligned}
$$

so that

$$
\prod_{i=1}^n \|N(t_i)^{-1}N(t_{i-1})\| \leq \prod_{i=1}^n (1+M\|(N(t_i)^{-1}-N(t_{i-1})^{-1})A\|)
$$

$$
\leq \exp\left(M\sum_{i=1}^n \|(N(t_i)^{-1}-N(t_{i-1})^{-1})A\|\right) \leq e^{MM_0}
$$

since $\ln(x+1) \leq x$ for $x \geq 0$, and so the product $\prod_{i=1}^n (1+x_i) \leq \exp\left(\sum_{i=1}^n x_i\right)$.

However, suppose f is a function on $[a, b]$ with values in $[0, \pi/2]$ (where $\sin x \geq x/2$) which is not of bounded variation, and we set $N(t)=e^{if(t)}A$. Since $N(t_i)^{-1}N(t_{i-1})=\exp(i(f(t_{i-1})-f(t_i)))$, $\|N(t_i)^{-1}N(t_{i-1})\|=1$, so that $N(\cdot)$ is of bounded variation productwise over every subinterval of $[a, b]$ with bound 1. But

$$
\begin{aligned}
\|(N(t_i)^{-1}-N(t_{i-1})^{-1})A\| &= \|e^{-if(t_i)}-e^{-if(t_{i-1})}\| = \|1-e^{i(f(t_i)-f(t_{i-1}))}\| \\
&= ((1-\cos(f(t_i)-f(t_{i-1})))^2+\sin^2(f(t_i)-f(t_{i-1})))^{1/2} \\
&\geq \sin|f(t_i)-f(t_{i-1})| \geq \tfrac{1}{2}|f(t_i)-f(t_{i-1})|,
\end{aligned}
$$

so that $N(\cdot)^{-1}A$ is not of bounded variation.]

Suppose now that $A(\cdot) \in L^1(B(X), [a, b])$, and that for each t, $A(t)$ preserves the range of A in such a way that $\tilde{A}(\cdot)=A^{-1}A(\cdot)A$ belongs to $L^1(B(X), [a, b])$. Then for the associated $\tilde{B}(t, s)$, $a \leq s \leq t \leq b$, $B(t, s)$ preserves the range of A in such a way that $\tilde{B}(t, s)=A^{-1}B(t, s)A$. For if we consider $A(t)$ as an operator $A_0(t)$ in the Banach space $R(A)$, $A_0(\cdot) \in L^1(R(A), [a, b])$, and there exists an associated $B_0(t, s)$ for $a \leq s \leq t \leq b$. For $x \in R(A)$, $B_0(\cdot, s)x$ is absolutely continuous in the norm of $R(A)$, and, hence, in the norm of X. Since it is strongly differentiable a.e. on $[s, b]$ in the norm of $R(A)$ with derivative $A_0(\cdot)B_0(\cdot, s)x=A(\cdot)B_0(\cdot, s)x$, the same holds in the norm of X, so that $B_0(\cdot, s)x=B(\cdot, s)x$. Thus $B_0(t, s)=B(t, s)|R(A)$. $B_0(\cdot, s)$ is absolutely continuous on $[s, b]$ in the norm of $R(A)$, i.e., $A^{-1}B_0(\cdot, s)A =A^{-1}B(\cdot, s)A$ is absolutely continuous in the norm of X on $[s, b]$. Similarly, $B_0(\cdot, s)x$ is a.e. strongly differentiable in the norm of $R(A)$ $\forall x \in R(A)$ with derivative $A_0(\cdot)B_0(\cdot, s)x=A(\cdot)B(\cdot, s)x$, so that $A^{-1}B(\cdot, s)Ax$ is a.e. strongly differentiable in the norm of X $\forall x \in X$ with derivative $(A^{-1}A(\cdot)A)(A^{-1}B(\cdot, s)A)x$.

Thus $\tilde{B}(t, s) = A^{-1}B(t, s)A$ by uniqueness. Let us consider $\|A^{-1}B(t, s)A\|$ $= \|\tilde{B}(t, s)\|$. Recall that $\tilde{B}(\cdot, s) = I + \sum_{k=1}^{\infty} B_k(\cdot)$, where $B_k(t)$ is defined by

$$B_k(t)x = \int_{t \geq t_n \geq \ldots \geq t_1 \geq s} \tilde{A}(t_n) \cdots \tilde{A}(t_1)x \, dt_1 \cdots dt_n$$

$$= \int_{t \geq t_n \geq \ldots \geq t_1 \geq s} ((A^{-1}N(t_n)) \cdot (N(t_n)^{-1}A(t_n)N(t_n)) \cdot (N(t_n)^{-1}N(t_{n-1}))$$

$$\cdot (N(t_{n-1})^{-1}A(t_{n-1})N(t_{n-1})) \cdots (N(t_2)^{-1}N(t_1))$$

$$\cdot (N(t_1)^{-1}A(t_1)N(t_1)) \cdot (N(t_1)^{-1}A)x) \, dt_1 \cdots dt_n.$$

Thus

$$\|\tilde{B}_k(t)x\| \leq M^3\|x\| \int_s^t \|N(t_n)^{-1}A(t_n)N(t_n)\| \, dt_n$$

$$\cdot \int_s^{t_n} \|N(t_{n-1})^{-1}A(t_{n-1})N(t_{n-1})\| \, dt_{n-1} \cdots \int_s^{t_2} \|N(t_1)^{-1}A(t_1)N(t_1)\| \, dt_1$$

by boundedness of product variation of $N(\cdot)$ by M and by $\|A^{-1}N(\cdot)\|_\infty$ and $\|N(\cdot)^{-1}A\|_\infty \leq M$. Thus exactly as in the first lemma for the Cauchy problem for bounded operators,

$$\|\tilde{B}_k(t)\| \leq (M^3/n!)\left(\int_s^t \|N(r)^{-1}A(r)N(r)\| \, dr\right)^n,$$

so that

$$\|\tilde{B}(t, s)\| \leq M^3 \exp\left(\int_s^t \|N(r)^{-1}A(r)N(r)\| \, dr\right)$$

since $M \geq 1$. Exactly as in the boundedness proposition we get

$$\|e^{n(t-s)}\tilde{B}(t, s)\| \leq M^3 \exp\left(\int_s^t \|N(r)^{-1}(A(r)+n)N(r)\| \, dr\right),$$

so that

$$\|\tilde{B}(t, s)\| \leq M^3 \exp\left(\int_s^t (\|N(r)^{-1}A(r)N(r)+n\| - n) \, dr\right),$$

and

$$\|\tilde{B}(t, s)\| \leq M^3 \exp\left(\int_s^t \rho(N(r)^{-1}A(r)N(r)) \, dr\right).$$

We may now prove the following theorem.

THEOREM. *Let $A \in L(X)$. Let $U(\cdot) \in L^\infty(B(X), [a, b])$ be absolutely continuous with each $U(t)$ invertible, and let $U(\cdot)x$ be a.e. strongly differentiable $\forall x \in X$ with derivative $U'(\cdot)x$, where $U'(\cdot) \in L^1(B(X), [a, b])$. If $P(t) = U(t)AU(t)^{-1}$, assume that $U'(t)$ takes the range of A into the domain of $P(t)^{-1}$ in such a way that*

$$P(\cdot)^{-1}U'(\cdot)A \in L^1(B(X), [a, b]).$$

Suppose now that $\overline{A}(\cdot)$ is a function from $[a, b]$ to $CD(X)$ with the following properties:

(1) *the range of $P(t)$ is the domain of $\overline{A}(t)$, and $\overline{A}(\cdot)P(\cdot)$ belongs to $L^1(B(X), [a, b])$,*

(2) *the spectrum of $\overline{A}(t)$ lies in a left half-plane,*

(3) $\exists m(\cdot) \in L^1([a, b])$ *and strictly positive $\varepsilon(\cdot)$ with $\varepsilon(\cdot)^{-1} \in L^1([a, b])$ such that for almost all t, $0 \leq \varepsilon \leq \varepsilon(t)$ entails $\|U(t)^{-1}(1 - \varepsilon\overline{A}(t))^{-1}U(t)\| \leq 1 + m(t)\varepsilon$,*

(4) *there exists a measurable function $n(\cdot)$ on $[a, b]$ such that*

$$N(\cdot) = U(\cdot)^{-1}(1 - n(\cdot)\overline{A}(\cdot))^{-1}U(\cdot)$$

(which has the same range as A) has $N(\cdot)^{-1}A$ and $A^{-1}N(\cdot)$ belonging to $L^\infty(B(X), [a, b])$, and has product variation over every subinterval of $[a, b]$ bounded by a constant M.

Then for $a \leq s \leq t \leq b$ $\exists \overline{B}(t, s) \in B(X)$ simultaneously strongly continuous in s and t with $\|\overline{B}(t, s)\| \leq \|U(t)\| \exp\left(\int_s^t (m(r) + \|U(r)^{-1}U'(r)\| \, dr)\right)$ such that

(1) $\overline{B}(t, t) = I \quad \forall t \in [a, b]$,

(2) $\overline{B}(t, s)P(s)$ *has range in the domain of $P(t)^{-1}$, and $P(\cdot)^{-1}\overline{B}(\cdot, s)P(s) \in L^\infty(B(X), [s, b])$,*

(3) $\overline{B}(\cdot, s)P(s)$ *is absolutely continuous on $[s, b]$, and $\overline{B}(\cdot, s)P(s)x$ is strongly differentiable a.e. on $[s, b]$ $\forall x \in X$ with derivative $\overline{A}(\cdot)\overline{B}(\cdot, s)P(s)x$,*

(4) $\overline{B}(t, \cdot)P(\cdot)$ *is absolutely continuous on $[a, t]$, and $\overline{B}(t, \cdot)P(\cdot)x$ is differentiable a.e. $\forall x \in X$ with derivative $\overline{B}(t, \cdot)(P'(\cdot) - \overline{A}(\cdot)P(\cdot))x$,*

(5) $\overline{B}(t, s_0)\overline{B}(s_0, s) = \overline{B}(t, s)$ *for $a \leq s \leq s_0 \leq t \leq b$.*

$\overline{B}(\cdot, s) \in L^\infty(B(X), [s, b])$ *is uniquely characterized by properties (1), (2), and (3).*

Proof. Set $A(\cdot) = -U(\cdot)^{-1}U'(\cdot) + U(\cdot)^{-1}\overline{A}(\cdot)U(\cdot)$. We shall show that $A(\cdot)$ satisfies the hypotheses of Theorem 1 with $C_1 = I$ and $C_2 = A$.

Let $\varepsilon_0(t) = \min\{\varepsilon(t), |2m(t)|^{-1}\}$, and set $\varepsilon_n(t) = \varepsilon_0(t)/n$. Set $A_n(t) = -U(t)^{-1}U'(t) + U(t)^{-1}\overline{A}(t)(1 - \varepsilon_n(t)\overline{A}(t))^{-1}U(t)$. We have

$$\|A_n(t)A\| \leq \|U(t)^{-1}\| \cdot \|U'(t)\| \cdot \|A\|$$
$$+ \|U(t)^{-1}(1 - \varepsilon_n(t)\overline{A}(t))^{-1}U(t)\| \cdot \|U(t)^{-1}\| \cdot \|\overline{A}(t)P(t)\| \cdot \|U(t)\|,$$

so that

$$\|A_n(t)A\| \leq \|U(\cdot)^{-1}\|_\infty\|A\| \, \|U'(t)\| + (1 + 1/2n)\|U(\cdot)^{-1}\|_\infty\|U(\cdot)\|_\infty\|\overline{A}(t)P(t)\|,$$

and, similarly,

$$\|A(\cdot)A\| \leq \|U(\cdot)^{-1}\|_\infty\|A\| \, \|U'(t)\| + \|U(\cdot)^{-1}\|_\infty\|U(\cdot)\|_\infty\|\overline{A}(t)P(t)\|.$$

Thus $\exists k(\cdot) \in L^1([a, b])$ with $\|A_n(\cdot)A\| + \|A(\cdot)A\| \leq k(\cdot)$. Since

$$\overline{A}(t)(1 - \varepsilon_n(t)\overline{A}(t))^{-1} = \varepsilon_n(t)^{-1}((1 - \varepsilon_n(t)\overline{A}(t))^{-1} - I),$$

we have

$$\|A_n(t)\| \leq \|U(\cdot)^{-1}\|_\infty\|U'(t)\| + 3\varepsilon_n(t)^{-1}$$
$$\leq \|U(\cdot)^{-1}\|_\infty\|U'(t)\| + 3n(2m(t) + \varepsilon(t)^{-1}),$$

and each $A_n(\cdot) \in L^1(B(X), [a, b])$. We may take $A_n(\cdot)$ to be as in Theorem 1, then, and we have already verified hypothesis (3) of that theorem.

Let us verify hypothesis (1) of Theorem 1, that the range of $B_n(t, s)A$ is in the domain of A^{-1} $\forall n$ and $a \leq s \leq t \leq b$. By the remarks preceding this theorem it suffices to show that $A_n(t)$ preserves the range of A in such a way that $A^{-1}A_n(\cdot)A \in L^1(B(X), [a, b])$. Since $P(\cdot)^{-1} = U(\cdot)A^{-1}U(\cdot)^{-1}$, $P(\cdot)^{-1}U'(\cdot)A \in L^1(B(X), [a, b])$, so that $U(\cdot)A^{-1}U(\cdot)^{-1}U'(\cdot)A \in L^1(B(X), [a, b])$, $-A^{-1}U(\cdot)^{-1}U'(\cdot)A \in L^1(B(X), [a, b])$. For the other term,

$$\|A^{-1}U(t)^{-1}(1 - \varepsilon_n(t)\overline{A}(t))^{-1}\overline{A}(t)U(t)A\|$$
$$\leq \|A^{-1}N(t)\| \cdot \|U(t)^{-1}(1 - \varepsilon_n(t)\overline{A}(t))^{-1}\overline{A}(t)U(t)\| \cdot \|N(t)^{-1}A\|$$
$$\leq \|A^{-1}N(\cdot)\|_\infty \|N(\cdot)^{-1}A\|_\infty 3n(2m(t) + \varepsilon(t)^{-1}).$$

Let us now verify hypothesis (2) of Theorem 1 for $A(\cdot)$. We have, first, for uniform boundedness of $B_n(t, s)$, only to show that all of the $\rho(A_n(\cdot))$ are bounded from above by a fixed function in $L^1([a, b])$. We have

$$\rho(A_n(t)) \leq \rho(U(t)^{-1}U'(t)) + \rho(U(t)^{-1}\overline{A}(t)(1 - \varepsilon_n(t)\overline{A}(t))^{-1}U(t))$$
$$\leq \|U(t)^{-1}\| \|U'(t)\|$$
$$\quad + (\|U(t)^{-1}\overline{A}(t)(1 - \varepsilon_n(t)\overline{A}(t))^{-1}U(t) + \varepsilon_n(t)^{-1}\| - \varepsilon_n(t)^{-1})$$
$$= \|U(t)^{-1}\| \|U'(t)\| + (\varepsilon_n(t)^{-1}\|U(t)^{-1}(1 - \varepsilon_n(t)\overline{A}(t))^{-1}U(t)\| - \varepsilon_n(t)^{-1})$$
$$\leq \|U(t)^{-1}\| \|U'(t)\| + \varepsilon_n(t)^{-1}(1 + \varepsilon_n(t)m(t)) - \varepsilon_n(t)^{-1}$$
$$= \|U(t)^{-1}\| \|U'(t)\| + m(t),$$

so that

$$\|B_n(t, s)\| \leq \exp\left(\int_s^t (m(r) + \|U(r)^{-1}\| \|U'(r)\|)\, dr\right).$$

To show uniform boundedness of $A^{-1}B_n(t, s)A$, we have, by the remarks directly preceding the theorem, only to show that $\rho(N(t)^{-1}A_n(t)N(t))$ is bounded from above by a fixed function in $L^1([a, b])$. But

$$\rho(N(t)^{-1}A_n(t)N(t)) \leq \rho(N(t)^{-1}U(t)^{-1}U'(t)N(t))$$
$$\quad + \rho(N(t)^{-1}U(t)^{-1}\overline{A}(t)(1 - \varepsilon_n(t)\overline{A}(t))^{-1}U(t)N(t)).$$

Since $N(t) = U(t)^{-1}(1 - n(t)\overline{A}(t))^{-1}U(t)$, $N(t)$ commutes through the argument of the last term, so that

$$\rho(N(t)^{-1}A_n(t)N(t))$$
$$\leq \|N(t)^{-1}A\| \|A^{-1}U(t)^{-1}U'(t)A\| \|A^{-1}N(t)\|$$
$$\quad + \rho(U(t)^{-1}\overline{A}(t)(1 - \varepsilon_n(t)\overline{A}(t))^{-1}U(t))$$
$$\leq \|N(\cdot)^{-1}A\|_\infty \|U(\cdot)^{-1}\|_\infty \|P(t)^{-1}U'(t)A\| \|A^{-1}N(\cdot)\|_\infty + m(t),$$

as before.

Let us now verify hypothesis (4) of Theorem 1, that $(A_n(t)A)^* \to (A(t)A)^*$ strongly for almost all $t \in [a, b]$. First, $\|(A_n(t)A)^*\| = \|A_n(t)A\|$ is uniformly bounded in n for fixed t. For

$$\|A_n(t)A\| \leqq \|U(t)^{-1}U'(t)A\| + \|U(t)^{-1}(1 - \varepsilon_n(t)\bar{A}_n(t))^{-1}\bar{A}(t)U(t)A\|$$
$$\leqq \|U(t)^{-1}U'(t)A\| + (1 + \varepsilon_n(t)m(t))\|U(t)^{-1}\| \; \|\bar{A}(t)P(t)\| \; \|U(t)\|,$$

and $(1 + \varepsilon_n(t)m(t)) \leqq 1 + 1/2n < 2$. Thus it suffices to show strong convergence on a dense subspace of X^*, for example, the set of $f \circ N(t)$. Clearly, it suffices to show that

$$(f \circ N(t)) \circ (U(t)^{-1}\bar{A}(t)(1 - \varepsilon_n(t)\bar{A}(t))^{-1}U(t))A \to (f \circ N(t)) \circ (U(t)^{-1}\bar{A}(t)U(t))A.$$

I.e., we must show that $f \circ (N(t)U(t)^{-1}\bar{A}(t)) \circ \varepsilon_n(t)\bar{A}(t)(1 - \varepsilon_n(t)\bar{A}(t))^{-1} \circ U(t)A$ goes to zero. Since $\varepsilon_n(t) \to 0$, it suffices to show that

$$f \circ (N(t)U(t)^{-1}\bar{A}(t)) \circ \bar{A}(t)(1 - \varepsilon_n(t)\bar{A}(t))^{-1} \circ U(t)A$$

is uniformly bounded. But

$$N(t)U(t)^{-1}\bar{A}(t) = U(t)^{-1}(1 - n(t)\bar{A}(t))^{-1}\bar{A}(t) = U(t)^{-1}\bar{A}(t)(1 - n(t)\bar{A}(t))^{-1},$$

so that this is bounded, $\bar{A}(t)U(t)A = \bar{A}(t)P(t)U(t)$, so that this is bounded, and

$$\|(1 - \varepsilon_n(t)\bar{A}(t))^{-1}\| \leqq \|U(t)\| \; \|U(t)^{-1}\|(1 + 1/2n)$$
$$< 2\|U(t)\| \; \|U(t)^{-1}\| \quad \forall n.$$

Thus by Theorem 1', (1), (2), and (3) of this theorem hold. To verify the hypotheses of Theorem 2', and, *a fortiori*, of Theorem 3', we need only show that $A_n(t)A \to A(t)A$ strongly for fixed t. Since we have already shown that $\|A_n(t)A\|$ is uniformly bounded in n, it suffices to show strong convergence on elements of the form $A^{-1}N(t)Ax$, i.e., it suffices to show that $A_n(t)N(t)Ax$ converges strongly to $A(t)N(t)Ax \quad \forall x \in X$. As before,

$$(A_n(t)N(t) - A(t)N(t))Ax$$
$$= (U(t)^{-1})(\bar{A}(t)U(t)N(t))U(t)^{-1}\varepsilon_n(t)(1 - \varepsilon_n(t)\bar{A}(t))^{-1}\bar{A}(t)U(t)Ax$$
$$= \varepsilon_n(t)(U(t)^{-1}(\bar{A}(t)U(t)A)(A^{-1}N(t))U(t)^{-1}(1 - \varepsilon_n(t)\bar{A}(t))^{-1}(\bar{A}(t)U(t)A)x),$$

and $\varepsilon_n(t) \to 0$, while the other term remains bounded by

$$\|U(t)^{-1}\|^2 \|\bar{A}(t)P(t)\| \; \|U(t)\| \; \|A^{-1}N(t)\|2\|U(t)\| \; \|U(t)^{-1}\| \; \|\bar{A}(t)P(t)\| \; \|U(t)\| \; \|x\|.$$

Thus Theorems 2' and 3' and Corollary 2' hold, proving (4), (5), and the uniqueness assertion of this theorem. The bounds mentioned have been proved in passing. Only joint strong continuity of $\bar{B}(t, s)$ in s and t remains to be shown. Assuming for convenience that δ and δ' are small and $\geqq 0$, we have for $x \in X$,

$$\|(\bar{B}(t + \delta, s + \delta') - \bar{B}(t, s))x\| \leqq \|(\bar{B}(t + \delta, s) - \bar{B}(t, s))x\|$$
$$+ \|\bar{B}(t + \delta, s + \delta')\| \; \|(\bar{B}(s + \delta', s) - I)x\|.$$

(e) We shall now take up some specific cases of the existence theorems in the case that X is a Hilbert space. In this case approximation of unbounded operators by use of projections on which they are bounded seems most natural and seems, at least, to be possible in most of the cases where approximations using resolvents can be used.

Because the additional hypothesis that X is a Hilbert space makes possible a number of existence theorems, we shall treat only two relatively simple ones by way of example.

THEOREM. *Let $A \in L(X)$ be self-adjoint. Let $U(\cdot) \in L^\infty(B(X), [a, b])$ have each $U(t)$ unitary, and let $U(\cdot)x$ be a.e. strongly differentiable $\forall x \in X$ with derivative $U'(\cdot)x$, where $U'(\cdot) \in L^1(B(X), [a, b])$. If $P(\cdot) = U(\cdot)AU(\cdot)^{-1}$, assume that $U'(t)$ takes the range of A into the domain of $P(\cdot)$ in such fashion that $P(\cdot)^{-1}U'(\cdot)A \in L^1(B(X), [a, b])$.*

Suppose now that $\overline{A}(\cdot)$ is a function from $[a, b]$ to $CD(X)$ with the following properties:

(1) each $\overline{A}(t)$ is skew-adjoint,

(2) the domain of $\overline{A}(t)$ contains the range of $P(t)$, and $\overline{A}(\cdot)P(\cdot) \in L^1(B(X), [a, b])$,

(3) $\exists m(\cdot) \in L^1([a, b])$ such that $[\overline{A}(\cdot), P(\cdot)^2] \leq m(\cdot)P(\cdot)^2$.

Then for $a \leq s \leq t \leq b$ $\exists \overline{B}(t, s) \in B(X)$ jointly strongly continuous in t and s such that $\overline{B}(t, s)$ is isometric, and

(1) $\overline{B}(t, t) = I$ $\forall t \in [a, b]$,

(2) $\overline{B}(t, s)$ takes the range of $P(s)$ into the domain of $P(t)^{-1}$, and

$$P(\cdot)^{-1}\overline{B}(\cdot, s)P(s) \in L^\infty(B(X), [s, b]),$$

(3) $\overline{B}(\cdot, s)P(s)$ is absolutely continuous on $[s, b]$, and $\overline{B}(\cdot, s)P(s)x$ is strongly differentiable a.e. on $[s, b]$ $\forall x \in X$ with derivative $\overline{A}(\cdot)\overline{B}(\cdot, s)P(s)x$,

(4) $\overline{B}(t, \cdot)P(\cdot)$ is absolutely continuous on $[a, t]$, and $\overline{B}(t, \cdot)P(\cdot)x$ is strongly differentiable a.e. $\forall x \in X$ with derivative $\overline{B}(t, \cdot)(P'(\cdot) - \overline{A}(\cdot)P(\cdot))x$,

(5) $\overline{B}(t, s_0)\overline{B}(s_0, s) = \overline{B}(t, s)$ for $a \leq s \leq s_0 \leq t \leq b$.

$\overline{B}(\cdot, s) \in L^\infty(B(X), [s, b])$ is uniquely characterized by properties (1), (2), and (3).

Proof. Note first that since $U(t)x = U(s)x + \int_s^t U'(r)x\,dr$ $\forall x \in X$ entails $U(t)^*x = U(s)^*x + \int_s^t (U'(r))^*x\,dr$ $\forall x \in X$ by the discussion for bounded operators in Hilbert space, $U'(\cdot)^*$ is the strong derivative of $U(\cdot)^{-1}$, so that $U'(\cdot)^* = -U(\cdot)^{-1}U'(\cdot)U(\cdot)^{-1}$. Thus $(U(t)^{-1}U'(t))^* = U'(t)^*U(t) = -U(t)^{-1}U'(t)$, so that each $U(t)^{-1}U'(t)$ is skew-adjoint.

Let us show that the hypotheses of Theorem 1 hold for $A(\cdot) = -U(\cdot)^{-1}U'(\cdot) + U(\cdot)^{-1}\overline{A}(\cdot)U(\cdot)$ with $C_1 = I$ and $C_2 = A$.

Since

$$\|A(t)A\| \leq \|U(t)^{-1}U'(t)A\| + \|U(t)^{-1}\overline{A}(t)P(t)U(t)\|$$
$$\leq \|U'(t)\|\,\|A\| + \|\overline{A}(t)P(t)\|,$$

we have $A(\cdot)A \in L^1(B(X), [a, b])$.

If $\{E_\lambda\}$ is the spectral family for A, set $P_n = (1 - E_{1/n}) + E_{-1/n}$. Then $A^{-1}P_n \in B(X)$, and so $\|A(t)P_n\| \leq \|A(t)A(A^{-1}P_n)\| \leq n\|A(t)A\|$. Set $A_n(\cdot) = P_n A(\cdot)P_n$. Then each $A_n(t)$ is skew-adjoint because each $A(\cdot)$ is, and $A_n(\cdot) \in L^1(B(X), [a, b])$ $\forall n$. Since, by the discussion for bounded operators in Hilbert space, each $B_n(t, s)$ is thus unitary, $\|B_n(t, s)\| \leq 1$ $\forall n, a \leq s \leq t \leq b$. Thus the first part of hypothesis (2) of Theorem 1 is satisfied.

Since the range of $A_n(t)$ lies in the range of P_n, which lies in the domain of A^{-1}, we may consider $A^{-1}A_n(\cdot)A$. By an earlier discussion, $A^{-1}A_n(\cdot)A \in L^1(B(X), [a, b])$ will imply that the range of $B_n(t, s)A$ lies in the domain of A^{-1}, thus proving hypothesis (1) of Theorem 1. But $\|(A^{-1}P_n)A(\cdot)(P_nA)\| \leq \|A^{-1}P_n\| \|A(\cdot)A\| \|P_n\| \leq n\|A(\cdot)A\|$.

Let us now verify the remainder of hypothesis (2) of Theorem 1, that $A^{-1}B_n(t, s)A$ is uniformly bounded. We compute $\rho(A^{-1}A_n(\cdot)A)$. We have

$$
\begin{aligned}
\rho(A^{-1}A_n(t)A) &= \rho(A^{-1}P_nA(t)AP_n) \\
&\leq \rho(-P_nA^{-1}U(t)^{-1}U'(t)AP_n) + \rho((A^{-1}P_n)U(t)^{-1}\overline{A}(t)U(t)(AP_n)) \\
&\leq \|P(t)^{-1}U'(t)A\| \\
&\quad + \tfrac{1}{2}\sup \sigma((A^{-1}P_n)U(t)^{-1}\overline{A}(t)U(t)(AP_n) - (AP_n)U(t)^{-1}\overline{A}(t)U(t)(A^{-1}P_n)).
\end{aligned}
$$

Since

$$\overline{A}(t)P(t)^2 - P(t)^2\overline{A}(t) \leq m(t)P(t)^2,$$

or

$$\overline{A}(t)U(t)A^2U(t)^{-1} - U(t)A^2U(t)^{-1}\overline{A}(t) \leq m(t)U(t)A^2U(t)^{-1},$$

or

$$U(t)^{-1}\overline{A}(t)U(t)A^2 - A^2U(t)^{-1}\overline{A}(t)U(t) \leq m(t)A^2,$$

it follows that

$$P_nU(t)^{-1}\overline{A}(t)U(t)A^2P_n - P_nA^2U(t)^{-1}\overline{A}(t)U(t)P_n \leq m(t)A^2P_n,$$

and, therefore,

$$(A^{-1}P_n)U(t)^{-1}\overline{A}(t)U(t)(AP_n) - (AP_n)U(t)^{-1}\overline{A}(t)U(t)(A^{-1}P_n) \leq m(t)P_n.$$

Thus $\rho(A^{-1}A_n(\cdot)A) \leq \|P(\cdot)^{-1}U'(\cdot)A\| + \tfrac{1}{2}m(\cdot)$, and

$$\|A^{-1}B_n(t, s)A\| \leq \exp\left(\int_s^t (\tfrac{1}{2}m(r) + \|P(r)^{-1}U'(r)A\|)\, dr\right)$$

$$\leq \exp(\tfrac{1}{2}\|m(\cdot)\|_1 + \|P(\cdot)^{-1}U'(\cdot)A\|_1).$$

For hypothesis (3) of Theorem 1, we have $\|A_n(t)A\| = \|P_nA(t)AP_n\| \leq \|A(t)A\|$, so that $k(\cdot) = 2\|A(\cdot)A\| \in L^1([a, b])$ satisfies $\|A_n(\cdot)A\| + \|A(\cdot)A\| \leq k(\cdot)$.

For hypothesis (4) of Theorem 1 we must show that $(A_n(t)A)^*$ converges strongly

to $(A(t)A)^*$ for almost all t. But $\|(A_n(t)A)^*\| = \|A_n(t)A\| \leq \|A(t)A\|$, so that $\|(A_n(t)A)^*\|$ is uniformly bounded for fixed t. Thus we need only show that $AA_n(t) \to \overline{AA(t)}$ on a dense subset of X. For x in the range of P_k and for $n \geq k$, $AA_n(t)x = (AP_n)A(t)x$, and $AP_n \to A$ strongly.

Thus conclusions (1), (2), and (3) of this theorem together with strong continuity of $\bar{B}(\cdot, s)$ follow from Theorem 1'. To verify the hypotheses of Theorem 2, and, *a fortiori*, of Theorem 3, for $A(\cdot)$, we need only show that $A_n(t)A \to A(t)A$ strongly for almost all t. Again by uniform boundedness of $\|A_n(t)A\|$ for fixed t, it suffices to show strong convergence on x in the range of P_k for some k. For $n \geq k$, $A_n(t)Ax \to P_nA(t)Ax$, and $P_n \to I$ strongly.

Thus the uniqueness assertion holds by Corollary 1', conclusion (4) holds by Theorem 3', and conclusion (5) holds by Corollary 2'. As before, joint strong continuity of $\bar{B}(t, s)$ in t and s follows from conclusion (5), boundedness of $\bar{B}(t, s)$, and strong continuity in t.

It remains only to show that $\bar{B}(t, s)$ is isometric. But

$$(\bar{B}(\cdot, s)P(s)x, \bar{B}(\cdot, s)P(s)x)'$$
$$= (\bar{A}(\cdot)\bar{B}(\cdot, s)P(s)x, \bar{B}(\cdot, s)P(s)x) + (\bar{B}(\cdot, s)P(s)x, \bar{A}(\cdot)\bar{B}(\cdot, s)P(s)x) = 0$$

$\forall x \in X$, so that $\|\bar{B}(t, s)P(s)x\|^2 = \|P(s)x\|^2$ $\forall x \in X$. ∎

None of the examples up to now has made use of the auxiliary operator C_1 of Theorem 1. In the following example it is used to control the bounds of the $B_n(t, s)$ which appear. This time we do not attempt a formulation in terms of the generalized theorems.

THEOREM. *Let $\mathscr{A}$ be a weakly-closed ring of operators in X with commutor $\tilde{\mathscr{A}}$. Let $A(\cdot)$ be a function from $[a, b]$ to $CD(X)$ such that each $A(t)$ is affiliated with $\mathscr{A}$. Suppose $\exists A \in \tilde{\mathscr{A}} \cap L(X)$ (which by polar decomposition we may take to be positive self-adjoint) such that the range of A lies in the domain of each $A(t)$, and $A(\cdot)A \in L^1(B(X), [a, b])$.*

Then $\exists C \in \tilde{\mathscr{A}}^+ \cap L(X)$ such that $C = g(A)$ for some nonnegative Borel function g, and for $a \leq s \leq t \leq b$ $\exists B(t, s)$ in the weakly-closed ring $\mathscr{A}'$ generated by $\mathscr{A}$ and A, simultaneously strongly continuous in t and s, such that

(1) $B(t, t) = C$ $\forall t \in [a, b]$, and $\|B(t, s)\| \leq K\|A\|$, $a \leq s \leq t \leq b$, (where K may be chosen equal to 2),

(2) the range of $B(t, s)$ lies in the domain of A^{-1} for $a \leq s \leq t \leq b$, and $A^{-1}B(\cdot, s) \in L^\infty(B(X), [s, b])$,

(3) $B(\cdot, s)$ is absolutely continuous on $[s, b]$, and $B(\cdot, s)x$ is a.e. strongly differentiable $\forall x \in X$ with derivative $A(\cdot)B(\cdot, s)x$,

(4) $B(t, \cdot)$ is absolutely continuous on $[a, t]$, and $B(t, \cdot)x$ is a.e. strongly differentiable $\forall x \in X$ with derivative

$$-\overline{B(t, \cdot)A(\cdot)}x \quad (-\overline{B(t, s)A(s)} = -A^{-1}A\overline{B(t, s)A(s)} = -A^{-1}B(t, s)A(s)A),$$

(5) *the range of* $B(t, s)C$ *lies in the domain of* C^{-1}, *and* $B(t, s)C = B(t, s_0)C^{-1}B(s_0, s)C$ *for* $s \leq s_0 \leq t \leq b$.

$B(\cdot, s) \in L^\infty(B(X), [s, b])$ *is uniquely characterized by properties* (1), (2), *and* (3).

Proof. Let us construct C_1. Let $\{E_\lambda\}$ be the spectral family of A. Set $P_n = E_{1/n}^\perp$. Since the range of A lies in the domain of every $A(t)$, the range of $A(A^{-1}P_n) = P_n$ lies in the domain of every $A(t)$, and

$$\|A(\cdot)P_n\| = \|A(\cdot)A(A^{-1}P_n)\| \leq \|A(\cdot)A\| \|A^{-1}P_n\| \leq n\|A(\cdot)A\|,$$

so that $A(\cdot)P_n \in L^1(B(X), [a, b])$ $\forall n$. Letting $a_n = \|A(\cdot)P_n\|_1$, so that $a_n \uparrow$, set $C_1 = \frac{1}{2}e^{-a_1}P_1 + \sum_{n=2}^\infty 2^{-n}e^{-a_n}(P_n - P_{n-1})$, set $C_2 = A$, and set $C = C_1C_2 = C_1A$.

For the $A_n(\cdot)$ of Theorem 1 let us take $A(\cdot)P_n$. Since $B_n(t, s)$ is a uniform limit of sums of strong integrals of operators in $\mathscr{A}'$, $B_n(t, s) \in \mathscr{A}'$, so that $B_n(t, s)C_1C_2 = B_n(t, s)C_1A = AB_n(t, s)C_1$ has range in the domain of A^{-1}, and $A^{-1}B_n(t, s)C_1A = B_n(t, s)C_1$. We have shown hypothesis (1) of Theorem 1 to hold and will have shown hypothesis (2) also to hold as soon as we show that $\|B_n(t, s)C_1\|$ is uniformly bounded for $a \leq s \leq t \leq b$, $\forall n$.

Consider $(B_n(t, s) - I)P_m + I$ for $m \leq n$. We have

$$(B_n(t, s) - I)P_m x = \int_s^t A(r)P_nB_n(r, s)P_m x \, dr = \int_s^t A(r)P_mB_n(r, s)P_m x \, dr,$$

so that $(B_n(t, s) - I)P_m + I = B_m(t, s)$ by uniqueness. Thus

$$\|B_n(t, s)C_1\| \leq \|(B_n(t, s) - I)C_1\| + \|C_1\| \leq \|(B_n(t, s) - I)P_nC_1\| + \frac{1}{2}$$

$$\leq \|(B_n(t, s) - I)P_1C_1\| + \sum_{k=2}^n \|(B_n(t, s) - I)(P_k - P_{k-1})C_1\| + 1$$

$$\leq \|(B_1(t, s) - I)\| \|P_1C_1\| + \sum_{k=2}^n \|(B_n(t, s) - I)P_k\| \|(P_k - P_{k-1})C_1\| + 1$$

$$\leq \sum_{k=1}^n 2^{-k}e^{-a_k}\|B_k(t, s) - I\| + 1 \leq \sum_{k=1}^n 2^{-k}e^{-a_k}(e^{a_k} - 1) + 1 \leq 2.$$

For hypothesis (3) of Theorem 1 we have $\|A_n(\cdot)A\| \leq \|A(\cdot)P_nA\| = \|A(\cdot)AP_n\| \leq \|A(\cdot)A\|$, and we need only choose $k(\cdot) \in L^1([a, b]) = 2\|A(\cdot)A\|$.

Finally, $(A_n(t)A)^* \to (A(t)A)^*$ strongly for each t. As usual, since we have just shown that $\|(A_n(\cdot)A)^*\| = \|A_n(\cdot)A\| \leq \|A(\cdot)A\|$, we need only show strong convergence for x in the range of some P_k. But then for $n \geq k$, $AA_n(t)^*x = AA(t)^*P_nx = AA(t)^*x$.

To verify the hypotheses of Theorems 2 and 3 with $C_3 = C_1$, we need only show that $A_n(t)A \to A(t)A$ strongly, and, again, for x in the range of P_k, when $n \geq k$ we have $A_n(t)Ax = A(t)P_nAx = A(t)AP_nx = A(t)Ax$.

Now if $\tilde{B}(t, s)$ is the relevant solution according to Theorem 1, it is in $\mathscr{A}'$ since it is a weak limit of operators $B_n(t, s)C_1$. Setting $B(t, s) = \tilde{B}(t, s)A$, we may immediately verify all the assertions of the present theorem except joint strong

continuity of $B(t, s)$ in t and s. But this is immediate in the usual way: assuming δ and $\delta' > 0$ and small,

$$\|(B(t+\delta, s+\delta') - B(t, s))Cx\|$$

$$\leq \|(B(t+\delta, s) - B(t, s))Cx\| + \|B(t+\delta, s+\delta')C^{-1}(B(s+\delta', s) - I)Cx\|$$

$$\leq \|(B(t+\delta, s) - B(t, s))Cx\| + \|B(t+\delta, s+\delta')\| \, \|(B(s+\delta', s) - I)x\|. \quad \blacksquare$$

COROLLARY. $C^{-1}B(t, s)$ is a closed, densely-defined operator $\forall a \leq s \leq t \leq b$. For elements x in the range of C, $C^{-1}B(\cdot, s)x$ and $C^{-1}B(t, \cdot)x$ are absolutely continuous on $[s, b]$ and $[a, t]$, respectively, and they are a.e. strongly differentiable with derivatives $A(\cdot)C^{-1}B(\cdot, s)x$ and $-C^{-1}B(t, \cdot)(A(\cdot)x)$. If $B_1(t, s)$ is related to $-A(\cdot)^*$ in the same way that $B(t, s)$ is related to $A(\cdot)$, C may be chosen the same for B and B_1. $B_1(\cdot, s)^*$ and $B_1(t, \cdot)^*$ are absolutely continuous on $[s, b]$ and $[a, t]$, respectively, and $B_1(\cdot, s)^*x$ and $B_1(t, \cdot)^*x$ are a.e. strongly differentiable $\forall x \in X$ with derivatives $[-B_1(\cdot, s)^*A(\cdot)]^-x$ and $A(\cdot)B_1(t, \cdot)^*x$. $C^{-1}B_1(t, s)^*$ is a closed, densely-defined operator, and

$$\overline{(C^{-1}B_1(t, s)^*)(C^{-1}B(t, s))} = I,$$

and

$$\overline{(C^{-1}B(t, s))(C^{-1}B_1(t, s)^*)} = I.$$

Proof. Only the last statement is not immediate. But $C^{-1}B_1(t, s)^*C^{-1}B(t, s)C^2x = B_1(t, s)^*B(t, s)x$. By taking derivatives in t, $B_1(t, s)^*B(t, s)x = B_1(s, s)^*B(s, s)x = C^2x$ $\forall t \in [s, b]$, so that $(C^{-1}B_1(t, s)^*)(C^{-1}B(t, s))$ is the identity on the range of C^2.

A similar computation taking derivatives in s for

$$B(t, s)B_1(t, s)^*x = C^{-1}B(t, s)C^{-1}B_1(t, s)^*C^2x$$

shows that $(C^{-1}B(t, s))(C^{-1}B_1(t, s)^*)$ is the identity on the range of C^2 as well. $\quad \blacksquare$

4. The Cauchy problem for measurable operators affiliated with a weakly-closed ring of operators in a separable Hilbert space. (a) Some observations on measurable functions from a measure space Ω to $B(X)$ when X is a separable Hilbert space are in order here.

For $T \in B(X)$, let us define $|T|$ by the polar decomposition $T = U|T|$. Unless otherwise indicated, let us assume all spectral families of projections $\{E_\lambda\}$ to be normalized from above from now on, so that $E_{\lambda_0} = \bigcap_{\lambda > \lambda_0} E_\lambda$.

Suppose now that $A(\cdot) \in \mathcal{M}(B(X), \Omega)$, where each $A(t)$ is positive self-adjoint with spectral family $\{E_\lambda(t)\}$. Suppose first that $\|A(\cdot)\| < M$ a.e. If f is any continuous function on $[0, M]$, there exists a sequence $\{p_n\}$ of polynomials converging uniformly to f. Thus $(p_n(A(\cdot))x, y) \to (f(A(\cdot))x, y)$ pointwise, so that $f(A(\cdot)) \in \mathcal{M}(B(X), \Omega)$. If $\lambda < M$, set $f_n(x) = 1$ for $0 \leq x \leq \lambda$, $= n(1/n + \lambda - x)$ for $\lambda \leq x \leq \lambda + 1/n$,

$=0$ for $\lambda+1/n \le x \le M$. Then $(f_n(A(\cdot))x, y) \to (E_\lambda(\cdot)x, y)$ pointwise, so that $E_\lambda(\cdot) \in \mathscr{M}(B(X), \Omega)$. Since $\|A(\cdot)\|$ is in general measurable, $E_\lambda(\cdot) \in \mathscr{M}(B(X), \Omega)$ generally by splitting up Ω into a countable number of pieces on which $\|A(\cdot)\|$ is uniformly bounded.

Let $A(\cdot) \in \mathscr{M}(B(X), \Omega)$ be arbitrary, $= U(\cdot)|A(\cdot)|$. Then $A(\cdot)^*A(\cdot) \in \mathscr{M}(B(X), \Omega)$, so that $(A(\cdot)^*A(\cdot))^{1/2} = |A(\cdot)| \in \mathscr{M}(B(X), \Omega)$. If $\{E_\lambda(t)\}$ is the spectral family for $|A(t)|$, $\int_{1/n}^n (1/\lambda)\, dE_\lambda(\cdot)$ is measurable, so that

$$U(\cdot)|A(\cdot)| \int_{1/n}^n (1/\lambda)\, dE_\lambda(\cdot) = U(\cdot)(E_n^-(\cdot) - E_{1/n}(\cdot)),$$

and, ultimately, $U(\cdot) \in \mathscr{M}(B(X), \Omega)$.

In general let us make the following definition.

DEFINITION. Let $A(\cdot)$ be a function from the measure space Ω to $CD(X)$, where X is a separable Hilbert space. Let $A(\cdot) = U(\cdot)|A(\cdot)|$ be the polar decomposition for $A(\cdot)$, and let $\{E_\lambda(\cdot)\}$ be the spectral family of $|A(\cdot)|$. Then $A(\cdot)$ will be called measurable on Ω if $A(\cdot)E_\lambda(\cdot)$ is measurable on Ω $\forall \lambda \ge 0$. ∎

Note that $A(\cdot)$ measurable implies that $A(\cdot)^*$ is measurable since $A(\cdot)^* = U(\cdot)^*(U(\cdot)|A(\cdot)|U(\cdot)^*)$, and the spectral family for $U(\cdot)|A(\cdot)|U(\cdot)^*$ is $U(\cdot)E_\lambda(\cdot)U(\cdot)^*$, so that

$$A(\cdot)^*(U(\cdot)E_\lambda(\cdot)U(\cdot)^*) = U(\cdot)^*(U(\cdot)E_\lambda(\cdot)U(\cdot)^*)(U(\cdot)|A(\cdot)|U(\cdot)^*)$$

$$= E_\lambda(\cdot)|A(\cdot)|U(\cdot)^*$$

$$= (U(\cdot)|A(\cdot)|E_\lambda(\cdot))^* \in \mathscr{M}(B(X), \Omega).$$

Note also that $U(\cdot)E_n(\cdot) \in \mathscr{M}(B(X), \Omega)$ $\forall n \ge 0$, so that $U(\cdot) \in \mathscr{M}(B(X), \Omega)$.

Consider now $P(\cdot) \in \mathscr{M}(B(X), \Omega)$, where each $P(t)$ is a projection. Since $((P_1(\cdot)P_2(\cdot))^n x, y) \to (P_1(\cdot) \cap P_2(\cdot)x, y)$, $P_1(\cdot) \cap P_2(\cdot) \in \mathscr{M}(B(X), \Omega)$. Also,

$$P_1(\cdot) \cup P_2(\cdot) = 1 - (1 - P_1(\cdot)) \cap (1 - P_2(\cdot)),$$

so that $P_1(\cdot) \cup P_2(\cdot)$ is measurable. Thus $\bigcup_{i=1}^\infty P_i(\cdot) \in \mathscr{M}(B(X), \Omega)$.

[If $\mathscr{A}$ is a weakly-closed ring of operators in X, and if P is a projection in $\mathscr{A}$, then P', the central support of P, $= \bigcup_{i=1}^\infty U_i P U_i^{-1}$, where $\{U_i\}$ ranges over a countable set of unitaries dense in the separable metric space of all unitaries of $\mathscr{A}$ under the strong topology. For if x is in the range of P and if U is any unitary in $\mathscr{A}$, there exists a sequence $\{U_{i_k}\}$ with $U_{i_k} x \to Ux$, which means that $\bigcup_{i=1}^\infty U_i P U_i^{-1}$ contains all images of elements in the range of P under unitaries in $\mathscr{A}$, which means that it is stable under all unitaries of $\mathscr{A}$, which means that it is in $\tilde{\mathscr{A}}$, which means that it is in the center of $\mathscr{A}$.

Then if $P(\cdot) \in \mathscr{M}(B(X), \Omega)$, with each $P(t)$ a projection in $\mathscr{A}$, $P'(\cdot) \in \mathscr{M}(B(X), \Omega)$.

Suppose $A \in \mathscr{A}^+$, with $A = \int_0^M \lambda\, dE_\lambda$. Set $\tilde{E}_\lambda^\perp$ equal to the central support of $(1 - E_\lambda)$. Then $\{\tilde{E}_\lambda\}$ is a spectral family continuous from above. For $\tilde{E}_{\lambda_0} = \bigcap_{\lambda > \lambda_0} E_\lambda$

is equivalent to $\tilde{E}^\perp_{\lambda_0} = \bigcup_{\lambda > \lambda_0} \tilde{E}^\perp_\lambda$. Set $\tilde{A} = \int_0^M \lambda \, d\tilde{E}_\lambda$. Since $\tilde{E}^\perp_\lambda$ is the central support of $E^\perp_\lambda$, $\tilde{E}_\lambda \leqq E_\lambda$. Then $\forall x \in X$, $(\tilde{E}_\lambda x, x) \leqq (E_\lambda x, x)$. But then

$$(\tilde{A}x, x) = \int_0^M \lambda \, d(\tilde{E}_\lambda x, x) = M(x, x) - \int_0^M (\tilde{E}_\lambda x, x) \, d\lambda$$

$$\geqq M(x, x) - \int_0^M (E_\lambda x, x) \, d\lambda = \int_0^M \lambda d(E_\lambda x, x) = (Ax, x).$$

If A' is any central operator in $\mathscr{A}$ with $A' \geqq A$,

$$A'^n - A^n = ((A' - A)^{1/2}(A'^{n-1} + A'^{n-2}A + \cdots + A'A^{n-2} + A^{n-1})^{1/2})^2 \geqq 0,$$

so that $A'^n \geqq A^n$ $\forall n \geqq 0$. Then if $A' = \int_0^M \lambda \, dE'_\lambda$ and if x is a unit vector in the range of E'_{λ_0}, $(A^n x, x) \leqq (A'^n x, x) \leqq \lambda_0^n$ $\forall n \geqq 0$, so that $\int_0^M (\lambda/\lambda_0)^n \, d(E_\lambda x, x) \leqq 1$ $\forall n \geqq 0$, which means $(E_{\lambda_0} x, x) = 1$, which means x is in the range of E_{λ_0}, which means $E'_{\lambda_0} \leqq E_{\lambda_0}$, which means $E'^\perp_{\lambda_0} \geqq E^\perp_{\lambda_0}$, which means $E'^\perp_{\lambda_0} \geqq \tilde{E}^\perp_{\lambda_0}$, which means $E'_{\lambda_0} \leqq \tilde{E}_{\lambda_0}$, which means $(A'x, x) \geqq (\tilde{A}x, x)$, just as above, which means $A' \geqq \tilde{A}$, so that $\tilde{A}$ is the least central operator in $\mathscr{A}$ majorizing A.

If $A(\cdot) \in \mathscr{M}(B(X), \Omega)$ with each $A(t) \in \mathscr{A}^+$, then $\tilde{A}(\cdot) \in \mathscr{M}(B(X), \Omega)$ since the associated $\tilde{E}_\lambda(\cdot) \in \mathscr{M}(B(X), \Omega)$ $\forall \lambda \geqq 0$.]

(b) Let $\mathscr{M}(\mathscr{A})$ be the *-algebra of measurable operators affiliated with $\mathscr{A}$. We shall determine some ring-theoretic conditions for solution of the Cauchy problem for measurable functions on $[a, b]$ with values in $\mathscr{M}(\mathscr{A})$.

If tr is any trace on the ring $\mathscr{A}$, there exists a sequence of vectors $\{x_n\}$ such that tr $A = \sum_{i=1}^\infty (Ax_i, x_i)$ $\forall A \in \mathscr{A}^+$. Thus if $A(\cdot)$ is a measurable function on a measure space Ω with values in $\mathscr{M}(\mathscr{A})$, we have

$$\text{tr } |A(\cdot)| = \lim_{k \uparrow \infty} \text{tr } (|A(\cdot)| E_k(\cdot)) = \lim_{k \uparrow \infty} \sum_{i=1}^\infty (|A(\cdot)| E_k(\cdot) x_i, x_i),$$

so that tr $|A(\cdot)|$ is measurable.

We may now give some ring-theoretic conditions for solution of the Cauchy problem. The essential details of the construction amount to a direct generalization of the basic propositions for bounded operators. We first prove a lemma.

LEMMA. *Let Ω be a finite measure space for which $L^2(\Omega)$ is separable. Let $\mathscr{A}$ be the multiplication ring of $L^2(\Omega)$. Then if $A(\cdot)$ is a measurable function from a finite measure space Ω' to operators affiliated with $\mathscr{A}$, there exists a measurable function $k(\omega, \omega')$ on $\Omega \times \Omega'$ such that for almost all $\omega' \in \Omega'$, $A(\omega')$ is multiplication by $k(\omega, \omega')$.*

Proof. Let $\{E_\lambda(\cdot)\}$ be the spectral family associated with $|A(\cdot)|$. Set $A_n(\cdot) = A(\cdot) E_n(\cdot)$. Then if $v \in L^2(\Omega)$ is the function constantly equal to one, $A_n(\cdot)v \in L^\infty(L^2(\Omega), \Omega') \subset L^2(L^2(\Omega), \Omega')$.

Thus there exists a sequence $\{v_n^j(\cdot)\}$ of countably-valued measurable functions

from Ω' to $L^2(\Omega)$ such that $\|A_n(\cdot)v - v_n^j(\cdot)\| \leqq 1/j$ a.e. on Ω' (the norm being the norm in $L^2(\Omega)$). Certainly $v_n^j(\omega')(\omega) = k_n^j(\omega, \omega')$ is measurable on $\Omega \times \Omega'$. Also,

$$\int_{\Omega \times \Omega'} |k_n^i(\omega, \omega') - k_n^j(\omega, \omega')|^2 \, d\omega \, d\omega'$$

$$\leqq \int_{\Omega'} 2(1/i^2 + 1/j^2) \, d\omega' = 2(1/i^2 + 1/j^2) \int_{\Omega'} d\omega',$$

so that $\{k_n^j(\omega, \omega')\}_{j=1}^\infty$ is Cauchy in $L^2(\Omega \times \Omega')$. Let $k_n(\omega, \omega') \in L^2(\Omega \times \Omega')$ be a limit function. Then $k_n(\omega, \omega') = A_n(\omega')v$ (in $L^2(\Omega)$), and, hence, in $L^\infty(\Omega)$) for almost all $\omega' \in \Omega'$ since

$$\int_{\Omega'} d\omega' \left(\int_\Omega |k_n(\omega, \omega') - k_n^j(\omega, \omega')|^2 \, d\omega \right) \to 0,$$

so that

$$\int_\Omega |k_n(\omega, \omega') - k_n^{j_m}(\omega, \omega')|^2 \, d\omega \to 0$$

for almost all ω' for an appropriate subsequence $\{j_m\}$. Thus for almost all $\omega' \in \Omega'$, $A_n(\omega')$ is multiplication by $k_n(\omega, \omega')$.

Application of this to $E_n(\cdot)$ gives a function $p_n(\omega, \omega')$, which, since it is equal to 0 or 1 a.e. on $\Omega \times \Omega'$, must be the characteristic function of a measurable set G_n. Clearly $k_m \chi_{G_n} = k_n$ almost everywhere $\forall m \geqq n$, so that, except on a null set of G_n, k_m remains constant when $m \geqq n$. Then $\{k_n\}$ converges to a measurable function k a.e. on $\Omega \times \Omega'$. Since $k \chi_{G_n} = k_n$, $A(\omega')$ is multiplication by $k(\omega, \omega')$ for almost all $\omega' \in \Omega'$. ∎

THEOREM. *Let $\mathscr{A}$ be a discrete ring, and let $A_\alpha(\cdot)$ be a measurable function from $[a, b]$ to $\mathscr{M}(\mathscr{A})$, $\alpha = 1, 2$. Let $A_1(\cdot) \in L^1(B(X), [a, b])$, and let $[\mathrm{tr}\, A_2(\cdot)^* A_2(\cdot)]^{1/2} \in L^1([a, b])$, where tr is an essential trace on $\mathscr{A}$. Set $A_0(\cdot) = A_1(\cdot) + A_2(\cdot)$. Then $\exists B(t, s) \in \mathscr{M}(\mathscr{A})$ $\forall a \leqq s \leqq t \leqq b$ with the following properties:*

(1) *$B(t, t) = I$ $\forall t \in [a, b]$,*

(2) *there exists a central $A \in \mathscr{A}^+ \cap L(X)$ whose range is in the domain of every $B(t, s)$, and which has $B(\cdot, s)A$ in $L^\infty(B(X), [s, b])$ $\forall s \in [a, b]$ (indeed, $B(t, s)A$ is uniformly bounded over all $a \leqq s \leqq t \leqq b$),*

(3) *if $x \in X$ has $(Tx, x) \leqq k(x) \mathrm{tr}\, T$ $\forall T \in \mathscr{A}^+$, $B(\cdot, s)Ax$ is absolutely continuous on $[s, b]$ and a.e. strongly differentiable on $[s, b]$ with derivative $A_0(\cdot)B(\cdot, s)Ax$ (such x are dense in X),*

(4) *for x as above, $B(t, \cdot)Ax$ is absolutely continuous on $[a, t]$ and a.e. strongly differentiable on $[a, t]$ with derivative $-B(t, \cdot)AA_0(\cdot)x$,*

(5) *if $\bar{B}(t, s)$ is related to $-A_0(\cdot)^*$ as $B(t, s)$ is related to $A_0(\cdot)$, $\bar{B}(t, s)$ has all of the above properties with respect to $-A_0(\cdot)^*$ and the same A, and $\bar{B}(t, s)^* = B(t, s)^{-1}$,*

(6) *$\forall s_0 \in [s, t]$, $B(t, s_0)B(s_0, s) = B(t, s)$,*

(7) *$B(t, s)A$ is jointly strongly continuous in t and s.*

Properties (1), (2), and (3) uniquely characterize $B(t, s)$. Further, $\forall a \leqq s \leqq t \leqq b$, $\int_s^t |A_0(r)|^\sim E_n(r) \, dr$ is a monotone increasing sequence of central operators in $\mathscr{A}$

converging a.e. to an unbounded positive central operator $\int_s^t |A_0(\tilde{r})|\ dr$ affiliated with $\mathscr{A}$, and $|B(t, s) - I| + I \leq \exp\left(\int_s^t |A_0(\tilde{r})|\ dr\right)$. ($\{E_\lambda(r)\}$ is the spectral family for $|A_0(r)|^\sim$.)

Proof. Suppose $\{P_i\}$ is a sequence of mutually orthogonal central projections in $\mathscr{A}$ with $\sum_{i=1}^\infty P_i = I$. Set $X_i = P_i X$, set $\mathscr{A}_i = \mathscr{A} P_i$, and set $\mathrm{tr}_i\ T = \mathrm{tr}\ T\ \ \forall T \in \mathscr{A}_i^+$. Then $[\mathrm{tr}_i\ A_2(\cdot)^* A_2(\cdot) P_i]^{1/2} \in L^1([a, b])$.

Suppose that all of the assertions of the theorem hold for $A_1(\cdot)P_i + A_2(\cdot)P_i$. Let $B_i(t, s)$ and A_i be the relevant "$B(t, s)$" and "A". Set

$$a_i = \sup_{a \leq s \leq t \leq b} \{\|\bar{B}_i(t, s)A\|,\ \|B_i(t, s)A\|\},$$

and let $A = \sum_{i=1}^\infty a_i^{-1} A_i$. Let $B(t, s) = \sum_{i=1}^\infty B_i(t, s)$.

Then assertions (1) and (2) certainly hold. For assertion (3), let $x \in X$ be such that $(Tx, x) \leq k(x)\ \mathrm{tr}\ T\ \ \forall T \in \mathscr{A}^+$. We have $\forall t \in [a, b]$,

$$\mathrm{tr}\ (AB(t, s)^* A_2(t)^* A_2(t)B(t, s)A) \leq K^2\ \mathrm{tr}\ A_2(t)^* A_2(t)$$

(K^2 is independent of s and t), so that $\|A_2(t)B(t, s)Ax\|^2 \leq k(x)K^2\ \mathrm{tr}\ A_2(t)^* A_2(t)$, which means $\|A_2(\cdot)B(\cdot, s)Ax\| \in L^1([s, b])$, and, therefore, $(A_1(\cdot) + A_2(\cdot))B(\cdot, s)Ax \in L^1(B(X), [s, b])$. Thus $\int_s^t A_0(r)B(r, s)Ax\ dr$ exists, and we have only to show that it is equal to $B(t, s)Ax - Ax$. But this is immediate upon application of P_i.

Assertion (4) follows in exactly the same manner. For assertion (5) we need only take $\bar{B}(t, s) = \sum_{i=1}^\infty \bar{B}_i(t, s)$. Assertion (6) is immediate. Assertion (7) follows from joint strong continuity on elements in the range of $\sum_{i=1}^n P_i$ and uniform boundedness of $B(t, s)A$.

Uniqueness follows by application of P_i and the observation that properties (1), (2), and (3) are inherited when everything is reduced to $\mathscr{A}_i$. The operator bound is immediate.

Since $\mathscr{A}$ is discrete and X is separable, there exists a sequence of mutually orthogonal central projections $\{P_n\}$ such that each P_n is the sum of n orthogonal equivalent abelian projections, $n = 1, 2, \ldots, \infty$. By the above discussion we may assume that some $P_n = I$.

We assume then that there exists a sequence $\{P_i\}_{i=1}^n$ of mutually orthogonal equivalent abelian projections with $\sum_{i=1}^n P_i = I$. Let $U_{ij} \in \mathscr{A}$ be a partial isometry with $U_{ij}^* = U_{ji}$, and $U_{ij}U_{ji} = P_i$. Since $P_1 \mathscr{A} P_1$ is abelian, it is isomorphic to the multiplication algebra $\mathscr{M}$ of $L^\infty(\Omega)$ over $L^2(\Omega)$, where Ω is a finite measure space with $L^2(\Omega)$ separable. The integral of a function in $L^\infty(\Omega)$ gives a trace tr_1 on $P_1 \mathscr{A} P_1$. If we set $\mathrm{tr}'\ T = \sum_{j=1}^n \mathrm{tr}_1\ (U_{1j}TU_{j1})\ \ \forall T \in \mathscr{A}^+$, tr' is clearly additive, homogeneous with respect to nonnegative real numbers, and normal. For all $T \in \mathscr{A}$,

$$\mathrm{tr}'\ T^*T = \sum_{j=1}^n \mathrm{tr}_1\ (U_{1j}T^*TU_{j1}) = \sum_{j,k=1}^n \mathrm{tr}_1\ ((U_{1j}T^*U_{k1})(U_{1k}TU_{j1}))$$

$$= \sum_{j,k=1}^n \mathrm{tr}_1\ ((U_{1k}TU_{j1})(U_{1j}T^*U_{k1})) = \sum_{k=1}^n \mathrm{tr}_1\ (U_{1k}TT^*U_{k1}) = \mathrm{tr}'\ TT^*,$$

so that tr' is a trace. Since for $T \in \mathscr{A}^+$, tr' $T = 0$ if and only if $P_i T P_i = 0$ $\forall i$, if and only if $\|T^{1/2}P_i x\|^2 = 0$ $\forall x \in X$, if and only if $T = 0$, tr' is essential.

Let S be an unbounded 1-1 positive operator affiliated with the center of $\mathscr{A}$ such that tr $T = $ tr' ST $\forall T \in \mathscr{A}^+$. Let S have spectral resolution $\{E_\lambda\}$, and set $Q_k = E_{2^k} - E_{2^{k-1}}$. Then $\{Q_k\}$ is a sequence of mutually orthogonal central projections with $\sum_{-\infty}^{\infty} Q_k = I$, and tr $TQ_k = $ tr' $SQ_k T \leq 2^k$ tr' TQ_k, tr' $TQ_k = $ tr $TS^{-1}Q_k$ $\leq 2^{1-k}$ tr TQ_k. If we assume, as we may, that some $Q_k = I$, then we may assume the isomorphism of $P_1 \mathscr{A} P_1$ with the multiplication algebra $\mathscr{M}$ of $L^\infty(\Omega)$ over $L^2(\Omega)$ to be such that tr gives the same measure as tr_1 on Ω since tr and tr_1 give equivalent measures inducing a unitary transformation of $L^2(\Omega)$. Call the isomorphism θ.

Consider now $\theta(U_{1i}A_\alpha(\cdot)U_{j1})$. This is a measurable function from $[a, b]$ to unbounded operators affiliated with $\mathscr{M}$. Thus by the lemma there exists a function $k_{ij}^\alpha(\omega, t)$ measurable on $\Omega \times [a, b]$ with $\theta(U_{1i}A_\alpha(t)U_{j1})$ multiplication by $k_{ij}^\alpha(\cdot, t)$ for almost all $t \in [a, b]$. Further, for almost all t, $k_{ij}^\alpha(\omega, t)$ defines a bounded matrix operator on $l^2(\{j\}_{j=1}^n)$ with norm $\|k_{ij}^\alpha(\omega, t)\|_{l^2}$ $\forall \omega \in \Omega$ if we set $k_{ij}^\alpha(\omega, t)$ equal to zero on a suitable null set of $\Omega \times [a, b]$, an adjustment which changes $k_{ij}^\alpha(\omega, t)$ only on a null set of Ω for almost all t since a null set of $\Omega \times [a, b]$ has null t-sections for almost all t by the Fubini theorem. Indeed, the function $\check{k}_{ij}^\alpha(\omega, t)$ for $|A_\alpha(\cdot)|^\sim$ has $\check{k}_{ij}^\alpha(\omega, t) = 0$ for $i \neq j$, and $\check{k}_{11}^\alpha(\omega, t) = \check{k}_{ii}^\alpha(\omega, t) \geq 0$ $\forall i$. If F is the set of all finite sequences of complex rationals with l^2-norm ≤ 1, the function

$$\sup_{v \in F} \|(k_{ij}^\alpha(\omega, t))(v)\|_{l^2}$$

is measurable and ≥ 0 on $\Omega \times [a, b]$, and must by its properties relative to $A_\alpha(\cdot)$ be equal a.e. on $\Omega \times [a, b]$ to $\check{k}_{11}^\alpha(\cdot, t)$.

We have

$$\text{tr } A_2(t)^* A_2(t) = \sum_{j=1}^n \text{tr } (U_{1j}A_2(t)^* A_2(t)U_{j1})$$

$$= \sum_{i,j=1}^n \text{tr } (U_{1j}A_2(t)^* U_{i1}U_{1i}A_2(t)U_{j1})$$

$$= \int_\Omega \sum_{i,j=1}^n k_{ij}^2(\omega, t)\check{k}_{ij}^2(\omega, t)\, d\omega \geq \int_\Omega \check{k}_{11}^2(\omega, t)^2\, d\omega$$

since the Hilbert-Schmidt norm majorizes the operator norm, so that

$$\int_\Omega \check{k}_{11}^2(\omega, \cdot)\, d\omega \in L^1([a, b]),$$

and, consequently, $\int_a^b \check{k}_{11}^2(\cdot, t)\, dt \in L^1(\Omega)$ by the Fubini theorem. Set $\Omega_m = \{\omega \in \Omega : m - 1 \leq \int_a^b \check{k}_{11}(\omega, t)^2\, dt < m\}$, so that the Ω_m are disjoint and $\Omega = \bigcup_{m=1}^\infty \Omega_m$ up to a null set. Let $Q_m = \sum_{j=1}^n U_{j1}\theta^{-1}(\chi_{\Omega_m})U_{1j}$. Then $\forall T \in \mathscr{A}$,

$$Q_m T = \sum_{j=1}^n U_{j1}\theta^{-1}(\chi_{\Omega_m})U_{1j}T = \sum_{j,k=1}^n U_{j1}\theta^{-1}(\chi_{\Omega_m})U_{1j}TU_{kj}U_{jk}$$

$$= \sum_{j,k=1}^n P_j TU_{kj}U_{j1}\theta^{-1}(\chi_{\Omega_m})U_{1j}U_{jk} = TQ_m,$$

so that Q_m is a central projection. Since $\sum_{m=1}^{\infty} Q_m = I$, we may, as twice before, assume that $Q_m = I$ for some m, so that $\int_a^b \tilde{k}_{11}^0(\cdot, t) \, dt \leqq M$ a.e. on Ω.

Let $K(\omega, t)$ be the linear transformation on $l^2(\{j\}_{j=1}^n)$ defined by the matrix $k_{ij}^0(\omega, t)$. By the propositions for bounded operators $\exists B(\omega, t, s)$ for almost all ω and $\forall a \leqq s \leqq t \leqq b$ with $\|B(\omega, t, s) - I\|_{l^2} + 1 \leqq \exp\left(\int_s^t \tilde{k}_{11}^0(\omega, r) \, dr\right) \leqq e^M$, and with $B(\omega, \cdot, s)$ absolutely continuous on $[s, b]$ and a.e. strongly differentiable with derivative $K(\omega, \cdot)B(\omega, \cdot, s)$.

Since $K(\omega, t)$ is measurable from $\Omega \times [a, b]$ to $B(l^2(\{j\}_{j=1}^n)) = B(H)$, $K(\omega, t_n)$ $\cdots K(\omega, t_1)$ is measurable from $\Omega \times [a, b]^n$ to $B(H)$. We have by the Fubini theorem, defining $B_n(\omega, t, s)$ by

$$B_n(\omega, t, s)h = \int_{t \geqq t_n \geqq \ldots \geqq t_1 \geqq s} K(\omega, t_n) \cdots K(\omega, t_1)h \, dt_1 \cdots dt_n$$

for $h \in H$, that

$$(B_n(\omega, t, s)h, h') = \int_{t \geqq t_n \geqq \ldots \geqq t_1 \geqq s} (K(\omega, t_n) \cdots K(\omega, t_1)h, h') \, dt_1 \cdots dt_n$$

is measurable in ω, t, and s. Thus $B(\omega, t, s) = I + \sum_{n=1}^{\infty} B_n(\omega, t, s)$ is measurable in ω, t, and s. Set $B(t, s) = \sum_{i,j=1}^n U_{i1} \theta^{-1}(B(\cdot, t, s)_{ij}) U_{1j}$, and set A of assertion (2) of the theorem equal to I. Certainly $\|B(t, s)\| \leqq e^M$ $\forall a \leqq s \leqq t \leqq b$, and $B(t, t) = I$ $\forall t \in [a, b]$. Measurability of $B(t, s)$ in t and s is also clear (since it may be expressed in the form "$\rho(B(t, s))$ is measurable in t and s for any positive normal form ρ on $\mathscr{A}$").

Suppose $(Tx, x) \leqq k(x) \operatorname{tr} T$ $\forall T \in \mathscr{A}^+$ for some $x \in X$. Then

$$(A_2(\cdot)B(\cdot, s)x, A_2(\cdot)B(\cdot, s)x) = (B(\cdot, s)^* A_2(\cdot)^* A_2(\cdot)B(\cdot, s)x, x)$$
$$\leqq k(x) \operatorname{tr} (B(\cdot, s)^* A_2(\cdot)^* A_2(\cdot)B(\cdot, s))$$
$$\leqq k(x)e^{2M} \operatorname{tr} A_2(\cdot)^* A_2(\cdot),$$

so that $\|A_2(\cdot)B(\cdot, s)x\| \in L^1([s, b])$, and so

$$A_0(\cdot)B(\cdot, s)x \in L^1(B(X), [s, b]).$$

We wish to show that

$$B(t, s)x = x + \int_s^t A_0(r)B(r, s)x \, dr.$$

Note that the set of $x \in X$ with $(Tx, x) \leqq k(x) \operatorname{tr} T$, or $\|T^{1/2}x\|^2 \leqq k(x) \operatorname{tr} T$, forms a linear space $(T \in \mathscr{A}^+)$. For

$$\|T^{1/2}\lambda x\|^2 \leqq |\lambda|^2 k(x) \operatorname{tr} T,$$

and

$$\|T^{1/2}(x+y)\|^2 \leqq (\|T^{1/2}x\| + \|T^{1/2}y\|)^2 \leqq 2(\|T^{1/2}x\|^2 + \|T^{1/2}y\|^2)$$
$$\leqq 2(k(x) + k(y)) \operatorname{tr} T.$$

Since tr $T = \sum_{i=1}^{\infty} (Tx_i, x_i)$ for some separating sequence of vectors $\{x_i\}$, the space of $x \in X$ with $\|T^{1/2}x\|^2 \leq k(x)$ tr T is a separating set for $\mathscr{A}$. For any unitary $U \in \tilde{\mathscr{A}}$, $\|T^{1/2}(Ux)\|^2 = \|UT^{1/2}x\|^2 = \|T^{1/2}x\|^2 \leq k(x)$ tr T, so that the space of $x \in X$ with $\|T^{1/2}x\|^2 \leq k(x)$ tr T is dense in X.

Thus it suffices to show that $(B(t, s)x, y) = (x, y) + \int_s^t (A_0(r)B(r, s)x, y)\, dr$ $\forall y$ with $\|T^{1/2}y\|^2 \leq k(y)$ tr T. By decomposing each term in the form

$$(x, y) = \tfrac{1}{4}((x+y, x+y) - (x-y, x-y) + i(x+iy, x+iy) - i(x-iy, x-iy)),$$

we see that it suffices to show that $(B(t, s)x, x) = (x, x) + \int_s^t (A_0(r)B(r, s)x, x)\, dr$ $\forall x \in X$ with $\|T^{1/2}x\|^2 \leq k(x)$ tr T for $T \in \mathscr{A}^+$. Since $T \to (Tx, x)$ is a positive normal form on $\mathscr{A}$, it defines a positive normal form on $B(H) \otimes \mathscr{M}$, given by $T(\cdot) \to \sum_{m=1}^{\infty} \int_{\Omega} (T(\omega)f_m(\omega), f_m(\omega))\, d\omega$, where $f_m(\omega) \in H$ for each $\omega \in \Omega$ with $\sum_{m=1}^{\infty} \int_{\Omega} \|f_m(\omega)\|^2\, d\omega < \infty$, and where $T(\omega) \in B(H)$ for each ω is defined by the matrix $T_{ij}(\cdot) = \theta(U_{1i}TU_{j1})$. Since

$$(B(\omega, t, s)f_m(\omega), f_m(\omega)) = (f_m(\omega), f_m(\omega)) + \int_s^t (K(\omega, r)B(\omega, r, s)f_m(\omega), f_m(\omega))\, dr$$

for almost all $\omega \in \Omega$, we have

$$\int_{\Omega} (B(\omega, t, s)f_m(\omega), f_m(\omega))\, d\omega = \int_{\Omega} \|f_m(\omega)\|^2\, d\omega$$
$$+ \int_s^t dr \int_{\Omega} (K(\omega, r)B(\omega, r, s)f_m(\omega), f_m(\omega))\, d\omega$$

by the Fubini theorem because

$$\int_s^t dr \int_{\Omega} |(K(\omega, r)B(\omega, r, s)f_m(\omega), f_m(\omega))|\, d\omega$$

$$\leq e^M \int_s^t dr \int_{\Omega} \|f_m(\omega)\|\, \|K(\omega, r)^* f_m(\omega)\|\, dr$$

$$\leq e^M \int_s^t dr \left(\int_{\Omega} \|f_m(\omega)\|^2\, d\omega \right)^{1/2} \left(\int_{\Omega} \|K(\omega, r)^* f_m(\omega)\|^2\, d\omega \right)^{1/2}$$

$$\leq e^M \|f_m\|_2 \int_s^t (A_0(r)A_0(r)^*x, x)^{1/2}\, dr$$

$$\leq e^M \|f_m\|_2 \left(\|x\| \int_s^t \|A_1(r)\|\, dr + k(x) \int_s^t (\operatorname{tr} A_2(r)^* A_2(r))^{1/2}\, dr \right),$$

which is finite. By the Lebesgue dominated convergence theorem

$$\sum_{m=1}^{\infty} \int_{\Omega} (B(\omega, t, s)f_m(\omega), f_m(\omega))\, d\omega = \sum_{m=1}^{\infty} \int_{\Omega} \|f_m(\omega)\|^2\, d\omega$$

$$+ \int_s^t dr \sum_{m=1}^{\infty} \int_{\Omega} (K(\omega, r)B(\omega, r, s)f_m(\omega), f_m(\omega))\, d\omega$$

since

$$\sum_{m=1}^{N} \left| \int_{\Omega} (K(\omega, r)B(\omega, r, s)f_m(\omega), f_m(\omega))\, d\omega \right|$$

$$\leq e^M \sum_{m=1}^{N} \int_{\Omega} \|f_m(\omega)\| \, \|K(\omega, r)^* f_m(\omega)\| \, d\omega$$

$$\leq e^M \sum_{m=1}^{N} \left(\int_{\Omega} \|f_m(\omega)\|^2 \right)^{1/2} \left(\int_{\Omega} \|K(\omega, r)^* f_m(\omega)\|^2 \right)^{1/2}$$

$$\leq e^M \left(\sum_{m=1}^{N} \int_{\Omega} \|f_m(\omega)\|^2 \, d\omega \right)^{1/2} \left(\sum_{m=1}^{N} \int_{\Omega} \|K(\omega, r)^* f_m(\omega)\|^2 \right)^{1/2}$$

$$\leq e^M \left(\sum_{m=1}^{\infty} \int_{\Omega} \|f_m(\omega)\|^2 \right)^{1/2} (A_0(r)A_0(r)^* x, x)^{1/2}$$

$$\leq e^M (x, x)(\|x\| \, \|A_1(r)\| + k(x)(\operatorname{tr} A_2(r)^* A_2(r))^{1/2}) \in L^1([a, b]).$$

But then $(B(t, s)x, x) = (x, x) + \int_s^t (A_0(r)B(r, s)x, x)\, dr$.

Assertion (4) is proved in exactly the same manner. Assertions (5) and (6) follow from the previous assertions and the isomorphism of $\mathscr{A}$ with $B(H) \otimes \mathscr{M}$. Assertion (7) follows from boundedness of $B(t, s)$ and absolute continuity of $B(t, s)x$ in t and s separately for x with $(Tx, x) \leq k(x) \operatorname{tr} T$ for $T \in \mathscr{A}^+$, together with (6). The operator bound has already been established in the course of computation.

Only uniqueness of $B(t, s)$ remains to be shown. Suppose $C(\cdot)$ is a function from $[s, b]$ to $\mathscr{M}(\mathscr{A})$ with $C(s) = I$, with an associated central $A \in \mathscr{A}^+ \cap L(X)$ whose range is in the domain of every $C(t)$, and which has $C(\cdot)A \in L^\infty(B(X), [s, b])$, and $C(\cdot)Ax$ absolutely continuous and a.e. strongly differentiable on $[s, b]$ with derivative $A_0(\cdot)C(\cdot)Ax$ $\forall x$ with $(Tx, x) \leq k(x) \operatorname{tr} T$ $\forall T \in \mathscr{A}^+$. (Note that for $S \in \mathscr{A}$,

$$(TSx, Sx) = (S^*TSx, x) \leq k(x) \operatorname{tr} S^*TS \leq \|S\|^2 k(x) \operatorname{tr} T,$$

so that

$$\|A_2(\cdot)C(\cdot)Ax\|^2 = (A_2(\cdot)^* A_2(\cdot)C(\cdot)Ax, C(\cdot)Ax)$$
$$\leq \|C(\cdot)A\|^2 k(x) \operatorname{tr} A_2(\cdot)^* A_2(\cdot)$$

and thus $\|A_2(\cdot)C(\cdot)Ax\|$ is in $L^1[(s, b])$.) Let x and y have (Tx, x) and (Ty, y) dominated by $\operatorname{tr} T$ $\forall T \in \mathscr{A}^+$. Then $(C(\cdot)Ax, \bar{B}(\cdot, s)y)$ is differentiable with derivative $(A_0(\cdot)C(\cdot)Ax, \bar{B}(\cdot, s)y) + (C(\cdot)Ax, -A_0(\cdot)^* \bar{B}(\cdot, s)y) = 0$, so that

$$(\bar{B}(t, s)^* C(t)Ax, y) = (Ax, y) \quad \forall t \in [s, b],$$

which means that $\bar{B}(t, s)^* C(t)A = A$ $\forall t \in [s, b]$, which means that $C(t)A = B(t, s)A$ $\forall t \in [s, b]$, which means that $C(t) = B(t, s)$ $\forall t \in [s, b]$. ∎

(c) If the ring $\mathscr{A}$ is not discrete, no theorem like the above one seems to be available. However, some kind of control over the traces of the operators involved can help in keeping domains under control. Let us give an example of an operator A affiliated with a finite ring $\mathscr{A}$ for which $\sum_{k=0}^{\infty} (1/k!)A^k x$ converges weakly for no nonzero $x \in X$.

We take the ring $\mathscr{A}$ to be one of the standard examples of a finite factor of type II_1. Let C be the group of reals mod one. Let $H_n \cong L^2(C)$, and set $X = \oplus_\infty^\infty H_n$. Let $a \in (0, 1)$ be a fixed irrational number. For $f = \{f_n\} \in X$, let $(Uf)_n(x) = f_{n-1}(x-a)$, so that U is unitary on X. Let $(Sf)_n = xf_n(x)$. Then the ring $\mathscr{A}$ generated by U and S is a factor of type II_1. If $m(\cdot)$ is any measurable function on C, and if $(Tf)_n(x) = m(x)f_n(x)$ for suitable $f \in X$, then $UT = A$ is affiliated with $\mathscr{A}$. If we take $m(x) = \exp(x^{-4})$, A will furnish the example desired.

To see this, let us first note that there exist arbitrarily large integers m such that the set of iterates $\{ja\}_{j=1}^{(m'+2)^2}$ intersects any open subinterval of $[0, 1)$ of length $\geq 1/m$. For by the box argument some ra and sa for $1 \leq r < s \leq (m'+1)$ must fall in an interval $(k/m', (k+1)/m')$, so that $0 < (s-r)a < 1/m'$ with $(s-r) \leq m'$, i.e., $0 < ra < 1/m'$ for some integer r between 1 and m'. Let m be such that $1/(m+1) < ra < 1/m \leq 1/m'$. Then $\{sra\}_{s=1}^{m+1}$ intersects every open subinterval of $[0, 1)$ of length $\geq 1/m$, and $sr \leq (m+1)m' < (m+1)^2$.

Suppose now that for some nonzero $f = \{f_n\} \in X$, $\sum_{n=0}^N (1/k!)A^k f$ were weakly convergent as $N \to \infty$. Then $(1/k!)A^k f \to 0$ weakly as $k \to \infty$, so that $(1/k!)^2 \|A^k f\|^2 \leq$ some constant K $\forall k$. Let N be such that $f_N \neq 0$. We have $(1/k!)^2 \|(A^k f)_{N+k}\|^2 \leq K$ $\forall k$, or

$$(1/k!)^2 \int_0^1 \left(\prod_{j=1}^k \exp(x-ja)^{-4} \right)^2 |f_N(x-ka)|^2 \, dx \leq K \quad \forall k,$$

so that

$$(1/k!) \inf \prod_{j=1}^k \exp(x-ja)^{-4} \leq K' \quad \forall k \qquad (K' = K^{1/2}/\|f_N\|).$$

Consider $k = (m+1)^2$, where m is as described above. For any $x \in [0, 1)$, some $ja \in (x-1/m, x)$ for j between 1 and k, so that $\inf \prod_{j=1}^k \exp(x-ja)^{-4} \geq e^{m^4} = \exp(k^{1/2}-1)^4 \geq e^{k^2/2}$ for $k \geq 50$. Thus $(1/k!)e^{k^2/2} \leq K'$ for arbitrarily large k, so that $k^{-k}e^{k^2/2} \leq K'$ for arbitrarily large k, or $e^{k(k/2 - \ln k)} \leq K'$ for arbitrarily large k, which is impossible.

Note that $T = |A|$ has very badly divergent trace ($\operatorname{tr} T = \int_0^1 e^{x^{-4}} \, dx$). If we consider only operators A which are sums of bounded operators and operators whose absolute values have finite trace, exponentiation is always possible.

We preface the statement of the theorem by remarking that if $A \in \mathscr{M}(\mathscr{A})$, and if $A = UT = R + VS$, where UT and VS are polar decompositions, R is bounded with $\|R\| = r$, and $\operatorname{tr} S < \infty$, then $\operatorname{tr}(TE_{r+2}^\perp) \leq (r+1) \operatorname{tr}(S)$ (where $\{E_\lambda\}$ is the spectral family for T). For if $E_{r+2}^\perp = 0$, this is immediate. Otherwise, if v is any unit vector in the range of $E_{r+2}^\perp$, $(r+2) \leq \|Rv\| + \|VSv\| \leq r + \|Sv\|$, so that $2 \leq \|Sv\|$, $E_{r+2}^\perp \cap F_1 = 0$ in consequence (writing $\{F_\lambda\}$ for the spectral family of S), and thus $E_{r+2}^\perp \prec F_1^\perp$, giving $\operatorname{tr}(E_{r+2}^\perp |R|^2 E_{r+2}^\perp)^{1/2} \leq r(\operatorname{tr} S)$, and, hence, $\operatorname{tr}(TE_{r+2}^\perp) \leq (r+1) \operatorname{tr}(S)$.

THEOREM. *Let $A \in \mathscr{M}(\mathscr{A})$ be the sum of a bounded operator and an operator whose absolute value has finite trace for an essential trace tr on $\mathscr{A}$.*

Then $\forall R > 0$ $\exists C_R \in \mathscr{A}^+ \cap L(X)$ with measurable inverse such that the range of C_R lies in the domain of A^k $\forall k \geqq 0$, and $\|(z^k/k!)A^k C_R\| < 2^{-k}$ for $|z| < R$, and $\sum_{k=0}^{\infty} (1/k!)z^k A^k C_R$ is analytic on $|z| < R$ with values in $\mathscr{A}$. It is equal to $e^{zA}C_R$, where e^{zA} is a function on the complex numbers to $\mathscr{M}(\mathscr{A})$ with $Ae^{zA} = e^{zA}A$, $e^{(z_1 + z_2)A} = e^{z_1 A}e^{z_2 A}$, and $(e^{zA})^* = e^{\bar{z}A^*}$.

The function $F(z) = e^{zA}$ from the complex plane to $\mathscr{M}(\mathscr{A})$ is uniquely characterized by the property that $F(0) = I$, and that $\forall R > 0$, $\exists C_R \in \mathscr{A}^+ \cap L(X)$ with measurable inverse such that the range of C_R lies in the domains of $F(z)$ and $AF(z)$ $\forall z$ with $|z| < R$, and $F(z)C_R$ is analytic in $|z| < R$ with derivative $AF(z)C_R$.

Proof. Let $A = UT$ be the polar decomposition of A, and let $\{E_\lambda\}$ be the spectral family of T. Since A is measurable, $\exists \lambda_0 > 0$ with $E_{\lambda_0}^\perp$ finite. By the remark we may assume that $\operatorname{tr}(TE_{\lambda_0}^\perp) < \infty$.

Choose $R > 0$, and set $r = (4R^2)^{-1}$. Let Q_n^k be the null space of $E_{\lambda_0 + (n+k)r}^\perp A^n$, so that $Q_n^{k\perp} \prec E_{\lambda_0 + (n+k)r}^\perp$ and the range of $A^n Q_n^k$ lies in the range of $E_{\lambda_0 + (n+k)r}$. Set $P_k = \bigcap_{n=0}^\infty Q_n^k$. Since $Q_n^{k+1} \geqq Q_n^k$, $P_k \uparrow$, $k = 1, 2, \ldots$.

We have

$$\operatorname{tr} P_k^\perp = \operatorname{tr}\left(\bigcup_{n=0}^\infty Q_n^{k\perp}\right) \leqq \sum_{n=0}^\infty \operatorname{tr} Q_n^{k\perp} \leqq \sum_{n=0}^\infty \operatorname{tr}(E_{\lambda_0 + (n+k)r}^\perp)$$

$$= \sum_{n=0}^\infty (1+n) \operatorname{tr}(E_{\lambda_0 + (n+1+k)r} - E_{\lambda_0 + (n+k)r})$$

$$\leqq r^{-1} \sum_{n=0}^\infty (\lambda_0 + (n+k)r) \operatorname{tr}(E_{\lambda_0 + (n+1+k)r} - E_{\lambda_0 + (n+k)r})$$

$$\leqq r^{-1} \operatorname{tr}(TE_{\lambda_0 + kr}^\perp) \downarrow 0 \quad \text{as } k \to \infty,$$

so that $P_k \uparrow I$ and P_k is cofinite.

For fixed k,

$$A^n P_k = (AE_{\lambda_0 + (n-1+k)r})(AE_{\lambda_0 + (n-2+k)r}) \cdots (AE_{\lambda_0 + kr})P_k,$$

so that $\|A^n P_k\| \leqq \prod_{j=1}^n (\lambda_0 + (j+(k-1))r)$, so that

$$(1/n!)\|A^n P_k\| \leqq r^n \prod_{j=1}^n ((\lambda_0/(j+(k-1))r) + 1)(((k-1)/j) + 1)$$

$$\leqq r^n \prod_{j=1}^n ((K/j) + 1)^2 \leqq r^n \exp\left(2K \sum_{j=1}^n (1/j)\right)$$

$$\leqq r^n \exp(2K(\ln n + 1)) \leqq (e^{2K})r^{n/2} \exp(2K \ln n + (n/2)\ln r)$$

$$\leqq (e^{2K})(2R)^{-n} \exp(2K \ln n - (n/2)\ln 4R^2) \leqq K'(R,k)(2R)^{-n}$$

(where K depends on λ_0, r, and k; and λ_0 is considered forever fixed). Thus

$$\|(1/n!)z^n A^n P_k\| \leqq K'(R,k)2^{-n} \quad \text{for } |z| < R.$$

Set $C_R = \sum_{k=1}^\infty 2^{-k/2}(K'(R,k)+1)^{-1}(P_k - P_{k-1})$, where $P_0 = 0$. Then $C_R \in \mathscr{A}^+ \cap L(X)$, and C_R^{-1} is measurable because P_k is cofinite for $k \geqq 1$.

Let $x \in X$ have norm one, and set $x_k = (P_k - P_{k-1})x$. Then

$$C_R x = \sum_{k=1}^{\infty} 2^{-k/2}(K'(R, k) + 1)^{-1} x_k,$$

and

$$\left\| (1/n!)z^n A^n \sum_{k=1}^{M} 2^{-k/2}(K'(R, k) + 1)^{-1} x_k \right\|$$

$$\leq \sum_{k=1}^{M} 2^{-k/2}(K'(R, k) + 1)^{-1} \| (1/n!)z^n A^n x_k \|$$

$$< \sum_{k=1}^{M} 2^{-k/2} \| x_k \| 2^{-n}$$

$$\leq 2^{-n} \left(\sum_{k=1}^{\infty} 2^{-k} \right)^{1/2} \left(\sum_{k=1}^{\infty} \| x_k \|^2 \right)^{1/2} = 2^{-n},$$

so that $\|(1/n!)z^n A^n C_R\| < 2^{-n} \quad \forall n \geq 0, \forall z$ with $|z| < R$.

Thus $F(z, A, C_R) = \sum_{n=0}^{\infty} (1/n!)z^n A^n C_R$ is an analytic function from $|z| < R$ into $\mathscr{A}$ with derivative $AF(z, A, C_R)$, and with $F(0, A, C_R) = C_R$.

Suppose $F(z)$ is a function on $|z| < R$ to $\mathscr{M}(\mathscr{A})$ with $F(0) = I$, and suppose that $\exists C \in \mathscr{A}^+ \cap L(X)$ with measurable inverse such that the range of C lies in the domains of $F(z)$ and $AF(z)$, and $F(z)C$ is analytic on $|z| < R$ with derivative $AF(z)C$.

Set $D = ((C^{-1})^2 + (C_R^{-1})^2)^{-1}$. Then $C^{-1}D$ and $C_R^{-1}D$ are bounded. Consider $F(z, A, C_R)C_R^{-1}D - F(z)D$. This is analytic on $|z| < R$ with derivative

$$A(F(z, A, C_R)C_R^{-1}D - F(z)D).$$

But it is equal to zero at $z = 0$, so that all derivatives are also equal to zero at $z = 0$, so that $F(z, A, C_R)C_R^{-1}D - F(z)D = 0$ in $|z| < R$, and $F(z) = F(z, A, C_R)C_R^{-1}$ in $|z| < R$.

Thus $F(z, A, C_R)C_R^{-1}$ is independent of R. Denote it by e^{zA}. For any $x \in X$ in the range of C_R which also has Ax in the range of C_R $(Ae^{zA} - e^{zA}A)x$ is an analytic function on $|z| < R$ which is equal to zero at $z = 0$. Since its nth derivative is $A^n(Ae^{zA} - e^{zA}A)x$, it is zero at $z = 0$, so that $(Ae^{zA} - e^{zA}A)x = 0$ for all x in the range of C_R which have Ax also in the range of C_R, which means that $Ae^{zA} - e^{zA}A = 0$ since such x are dense in X.

Since $(e^{zA}C_R)^* = C_R(e^{zA})^*$ has derivatives with respect to z at zero equal to $C_R A^{*n}$, a uniqueness result exactly like the first one above, but with auxiliary operators applied on the left, gives $(e^{zA})^* = e^{zA^*}$, or $(e^{zA})^* = e^{zA^*}$.

Finally, consider $e^{zA}e^{z_0 A}(|z_0| + |z| < R)$. For x in the range of C_R with $e^{z_0 A}x$ in the range of C_R, $(e^{zA}e^{z_0 A} - e^{(z+z_0)A})x$ is an analytic function in $|z| < R - |z_0|$ equal to zero at $z = 0$ and with all derivatives thus equal to zero at $z = 0$, so that $e^{zA}e^{z_0 A} = e^{(z+z_0)A}$ by density of the x's. ∎

Some weak generalization of the above theorem is possible. Let $A(\cdot)$ be a measurable function from a measure space Ω to unbounded operators affiliated with a ring $\mathscr{A}$, and let $P \in \mathscr{A}$ be a projection. Then the family of projections $Q \in \mathscr{A}$ with

$P^{\perp}A(\cdot)Q=0$ a.e. forms an ascending net. Since the unit ball of $\mathscr{A}$ is metrizable in the strong topology (X is separable), there exists a sequence $\{Q_n\}$ of such projections converging strongly to the limit projection Q_0, so that $P^{\perp}A(\cdot)Q_0=0$ a.e. Set $\theta_{A(\cdot)}(P)=Q_0$.

THEOREM. *Let $A \in \mathscr{A}^+ \cap L(X)$ have measurable inverse, and let tr be an essential trace on $\mathscr{A}$. Let μ be a measure on a right half line $[\alpha, \infty)$, $\alpha > 0$, for which $\int_\alpha^\infty \lambda\, d\mu(\lambda) < \infty$.*

Then if $A(\cdot)$ is a measurable function from $[a, b]$ to $\mathscr{M}(\mathscr{A})$ such that

(1) *$A(\cdot)A \in L^1(B(X), [a, b])$, and*

(2) *$\exists \varepsilon > 0$ such that $\mathrm{tr}\,(\theta_{A(\cdot)}^m(E_\lambda^\perp)^\perp) \leqq \mu([1/\lambda, \infty))$ for $\varepsilon > \lambda > 0$ and $\forall m \geqq 0$, where $\{E_\lambda\}$ is the spectral family for A, then $\forall a \leqq s \leqq t \leqq b$ $\exists B(t, s) \in \mathscr{M}(\mathscr{A})$ with properties*

(1) *$B(t, t)=I$ $\forall t \in [a, b]$, and*

(2) *$\exists C \in \mathscr{A}^+ \cap L(X)$ with measurable inverse such that $A^{-1}B(\cdot, s)C$ belongs to $L^\infty(B(X), [s, b])$, and such that $B(\cdot, s)C$ is absolutely continuous on $[s, b]$ and a.e. strongly differentiable with derivative $A(\cdot)B(\cdot, s)C$.*

If, in addition to the other hypotheses on $A(\cdot)$, $\mathrm{tr}\,(\theta_{A(\cdot)}^m(E_{\frac{1}{\lambda}}^\perp)^\perp) \leqq \mu([1/\lambda, \infty))$ for $\varepsilon > \lambda > 0$ and $\forall m \geqq 0$, and $AA(\cdot)$ belongs to $L^1(B(X), [a, b])$, then $B(t, s)$ is uniquely characterized by the above two properties. $B(t, s)$ then has the additional properties

(3) *$B(t, s_0)B(s_0, s)=B(t, s)$ $\forall a \leqq s \leqq s_0 \leqq t \leqq b$,*

(4) *$\exists C' \in \mathscr{A}^+ \cap L(X)$ with measurable inverse such that $C'B(t, \cdot)A(\cdot)$ belongs to $L^1(B(X), [a, t])$, and such that $C'B(t, \cdot)$ is absolutely continuous on $[a, t]$ and a.e. strongly differentiable with derivative $-C'B(t, \cdot)A(\cdot)$, and*

(5) *if $B_1(t, s)$ bears the same relation to $-A(\cdot)^*$ as $B(t, s)$ does to $A(\cdot)$, then $B_1(t, s)^* = B(t, s)^{-1}$.*

Proof. We shall show that $A(\cdot)$ satisfies the hypotheses of Theorem 1 on the Cauchy problem for unbounded operators. We will set C_2 of that theorem equal to I.

Let $\lambda_0 > 1/\varepsilon$ be such that $\int_{\lambda_0}^\infty \lambda\, d\mu(\lambda) < \infty$, where $\bar{E}_\lambda = E_{1/\lambda}^\perp$ defines the spectral family of A^{-1}, set $Q_n^k = \theta_{A(\cdot)}^n(\bar{E}_{\lambda_0 + (n+k)r})$, and let $r = (2\|A(\cdot)A\|_1)^{-2}$. Let $P_k = \bigcap_{n=0}^\infty Q_n^k$. Since $Q_n^{k+1} \geqq Q_n^k$ (trivial for $n=0$, and then clear by induction), $P_k \uparrow$, $k=1, 2, \ldots$.

We have

$$\mathrm{tr}\, P_k^\perp = \mathrm{tr}\left(\bigcup_{n=0}^\infty Q_n^{k\perp}\right) \leqq \sum_{n=0}^\infty \mathrm{tr}\, Q_n^{k\perp} \leqq \sum_{n=0}^\infty \mu([\lambda_0 + (n+k)r, \infty))$$

$$= \sum_{n=0}^\infty (1+n)\mu([\lambda_0 + (n+k)r, \lambda_0 + (n+k+1)r))$$

$$\leqq r^{-1} \sum_{n=0}^\infty (\lambda_0 + (n+k)r)\mu([\lambda_0 + (n+k)r, \lambda_0 + (n+k+1)r))$$

$$\leqq r^{-1} \int_{\lambda_0 + kr}^\infty \lambda\, d\mu(\lambda) \downarrow 0$$

as $k \to \infty$, so that $P_k \uparrow I$, and P_k is eventually cofinite.

For almost all $t \in [a, b]$ the range of $A(t)Q_{n+1}^k$ lies in the range of Q_n^{k+1}, so that the range of $A(t)P_k$ lies in the range of P_{k+1} for almost all $t \in [a, b]$. Thus for almost all $(t_1, \ldots, t_m) \in [a, b]^m$,

$$P_i A(t_m) P_i A(t_{m-1}) P_i \cdots P_i A(t_1) P_i P_k$$
$$= (P_i P_{k+m}) A(t_m)(P_i P_{k+m-1}) A(t_{m-1}) \cdots (P_i P_{k+1}) A(t_1) P_i P_k,$$

so that the range of $(P_i A(t_m) P_i)(P_i A(t_{m-1}) P_i) \cdots (P_i A(t_1) P_i) P_k$ lies in the range of P_{k+m}, and

$$\|(P_i A(t_m) P_i) \cdots (P_i A(t_1) P_i) P_k\|$$
$$\leq \|A(t_m) P_{k+m-1}\| \cdots \|A(t_1) P_k\|$$
$$\leq \|A(t_m) A\| \, \|A^{-1} \bar{E}_{\lambda_0 + (k+m-1)r}\| \cdots \|A(t_1) A\| \, \|A^{-1} \bar{E}_{\lambda_0 + kr}\|$$
$$\leq \left(\prod_{j=1}^{m} (\lambda_0 + (k-1+j)r) \right) \|A(t_m) A\| \cdots \|A(t_1) A\|$$
$$\leq m! \, r^{m/2} c(k) \|A(t_m) A\| \cdots \|A(t_1) A\|$$
$$= c(k) m! (2\|A(\cdot) A\|_1)^{-m} \|A(t_m) A\| \cdots \|A(t_1) A\|$$

as in the previous theorem.

Set $A_n(\cdot) = P_n A(\cdot) P_n$. Then $\|A_n(t)\| \leq \|A(t) \bar{E}_{\lambda_0 + kr}\| \leq \|A(t) A\| \, \|A^{-1} \bar{E}_{\lambda_0 + kr}\| \leq (\lambda_0 + kr) \|A(t) A\|$, so that $A_n(\cdot) \in L^1(B(X), [a, b])$. Set

$$C = \sum_{k=1}^{\infty} 2^{-k/2} (c(k) + 1)^{-1} (\lambda_0 + kr + 1)^{-1} (P_k - P_{k-1}),$$

and let $C = C_1$ of Theorem 1. Let us verify hypotheses (1) and (2) of Theorem 1.

We have $B_n(t, s) = I + \sum_{m=1}^{\infty} B_n^m(t, s)$, where $B_n^m(t, s)$ is defined by

$$B_n^m(t, s) x = \int_{t \geq t_m \geq \ldots \geq t_1 \geq s} A_n(t_m) \cdots A_n(t_1) x \, dt_1 \cdots dt_m \quad \forall x \in X.$$

Thus $B_n^m(t, s) P_k$ has range contained in the range of P_{k+m}, and

$$\|B_n^m(t, s) P_k\| \leq \int_{t \geq t_m \geq \ldots \geq t_1 \geq s} \|A_n(t_m) \cdots A_n(t_1) P_k\| \, dt_1 \cdots dt_m$$
$$\leq c(k) m! (2\|A(\cdot) A\|_1)^{-m} \int_s^t \|A(t_m) A\| \, dt_m \cdots \int_s^{t_2} \|A(t_1) A\| \, dt_1$$
$$\leq c(k) 2^{-m},$$

so that, letting $\|x\| = 1$ and $x_k = (P_k - P_{k-1}) x$, we get

$$\|A^{-1} B_n^m(t, s) C x\| = \left\| A^{-1} \sum_{k=1}^{\infty} 2^{-k/2} (c(k) + 1)^{-1} (\lambda_0 + kr + 1)^{-1} B_n^m(t, s) P_k x_k \right\|$$
$$\leq \sum_{k=1}^{\infty} 2^{-k/2} (c(k) + 1)^{-1} (\lambda_0 + kr + 1)^{-1} \|A^{-1} \bar{E}_{\lambda_0 + kr}\| \, \|B_n^m(t, s) P_k\| \, \|x_k\|$$
$$\leq 2^{-m} \sum_{k=1}^{\infty} 2^{-k/2} \|x_k\| \leq 2^{-m},$$

so that $\|A^{-1}B_n^m(t,s)C\| \leqq 2^{-m}$ $\forall n$. It follows easily that $\|A^{-1}B_n(t,s)C\| \leqq 2$ $\forall n$, $\forall a \leqq s \leqq t \leqq b$.

For hypothesis (3) of Theorem 1 we have $\|A_n(t)A\| \leqq \|A(t)P_nA\| \leqq \|A(t)A\|$. For hypothesis (4) we must show that $(A_n(t)A)^* \to (A(t)A)^*$ strongly for each $t \in [a, b]$. Since $\|(A_n(t)A)^*\| = \|A_n(t)A\| \leqq \|A(t)A\|$, it suffices to show strong convergence on a dense subspace of X. If x is in the range of P_m, then for $n \geqq m$, $(A_n(t)A)^*x = P_n(A(t)A)^*x$, which converges to $(A(t)A)^*x$.

We have only to set $B_{C_1C_2}(t,s)C^{-1} = B(t,s)$, where $B_{C_1C_2}(t,s)$ is as in Theorem 1, and properties (1) and (2) of the present theorem are immediate.

Suppose now that tr $(\theta_{A(\cdot)^*}^m(E_\lambda^\perp)^\perp) \leqq \mu([1/\lambda, \infty))$ for $\varepsilon > \lambda > 0$ and $\forall m \geqq 0$, and $AA(\cdot) \in L^1(B(X), [a, b])$. Let $\bar{A}(t) = A(t)$ for $a \leqq t \leqq b$, $= A(2b-t)^*$ for $b < t \leqq 2b-a$. Then $\bar{A}(\cdot)A \in L^1(B(X), [a, 2b-a])$, and tr $(\theta_{\bar{A}(\cdot)}^m(E_\lambda^\perp)^\perp) \leqq 2\mu([1/\lambda, \infty))$ for $\varepsilon > \lambda > 0$ and $\forall m \geqq 0$. By following through the proof of the first part of this theorem we get that the $\bar{B}_n(t,s) = \bar{B}_n(2b-s, 2b-t)^*$ for $a \leqq s \leqq t \leqq b$ since $\bar{B}_n(t,s) = I + \sum_{m=1}^\infty \bar{B}_n^m(t,s)$, and $\bar{B}_n^m(t,s)$ is defined by

$$\bar{B}_n^m(t,s)x = \int_{t \geqq t_m \geqq \ldots \geqq t_1 \geqq s} \bar{A}_n(t_m) \cdots \bar{A}_n(t_1)x \, dt_1 \cdots dt_m,$$

so that

$$\bar{B}_n^m(2b-s, 2b-t)^*x = \int_{2b-s \geqq t_m \geqq \ldots \geqq t_1 \geqq 2b-t} \bar{A}_n(t_1)^* \cdots \bar{A}_n(t_m)^*x \, dt_1 \cdots dt_m$$

$$= \int_{t \geqq t_m \geqq \ldots \geqq t_1 \geqq s} \bar{A}_n(t_m) \cdots \bar{A}_n(t_1)x \, dt_1 \cdots dt_m \quad \forall x \in X.$$

Thus $\|A^{-1}\bar{B}_n(t,s)\bar{C}\| \leqq 2$ $\forall a \leqq s \leqq t \leqq 2b-a$, entails $\|\bar{C}\bar{B}_n(t,s)\| \leqq 2\|A\|$ $\forall a \leqq s \leqq t \leqq b$.

To apply Theorem 2 on the Cauchy problem for unbounded operators and its corollaries to $\bar{B}(t,s)$, $a \leqq s \leqq t \leqq b$, to demonstrate uniqueness and property (3) of this theorem, we need only set C_3 of Theorem 2 equal to $\bar{C}$ and show that $\bar{A}_n(t)A \to \bar{A}(t)A$ strongly for fixed t, $a \leqq t \leqq b$. By uniform boundedness it suffices to show strong convergence on the range of each $\bar{P}_k$. But for x with $\bar{P}_kx = x$, $\forall n \geqq k$, $\bar{A}_n(t)Ax = \bar{P}_n\bar{A}(t)\bar{P}_nAx = \bar{P}_n\bar{A}(t)Ax \to \bar{A}(t)Ax$.

Theorem 3 applies immediately to yield property (4) for $B(t,s) = \bar{B}(t,s)$, $a \leqq s \leqq t \leqq b$. Only property (5) remains to be verified. If we let C_1 and C_1' bear the same relation to $B_1(t,s)$ and $-A(\cdot)^*$ as C and C' do to $B(t,s)$, we have that $C_1B_1(\cdot, s)^*B(\cdot, s)Cx$ is absolutely continuous on $[s, b]$ and strongly differentiable with derivative $= 0$ $\forall x \in X$, so that $C_1B_1(t,s)^*B(t,s)Cx = C_1Cx$ $\forall t \in [s, b]$, and thus $B_1(t,s)^*B(t,s) = I$ $\forall a \leqq s \leqq t \leqq b$. Similarly, $C'B(t, \cdot)B_1(t, \cdot)^*C_1'x$ is absolutely continuous on $[a, t]$ and strongly differentiable with derivative $= 0$ $\forall x \in X$, so that $C'B(t,s)B_1(t,s)^*C_1'x = C'C_1'x$ $\forall s \in [a, t]$, giving $B(t,s)B_1(t,s)^* = I$ $\forall a \leqq s \leqq t \leqq b$, and thus $B_1(t,s)^* = B(t,s)^{-1}$. ∎

Embellishment of the above theorem by allowing variable domains in the style

of Theorems $1'$, $2'$, and $3'$ on the Cauchy problem for unbounded operators is very straightforward.

5. **The Cauchy problem in simply-connected regions of the complex plane for operators in a Banach space.** (a) We shall present what appear to be some basic theorems for bounded operators, but we make no attempt at a general treatment for unbounded operators. Indeed, we shall treat only one particular case for unbounded operators, having chosen it on grounds of simplicity and interest.

THEOREM. *Let R be a simply-connected region of the complex plane, and let $A(\cdot)$ be an analytic function from R to $B(X)$. Then for all z_1 and z_2 in R, $\exists B(z_1, z_2) \in B(X)$ analytic in z_1 and such that $B(z, z) = I$ and $(\partial/\partial z_1)B(z_1, z_2) = A(z_1)B(z_1, z_2)$. $B(z_1, z_2)$ is uniquely characterized by these properties. It is also simultaneously analytic in z_1 and z_2, and has $(\partial/\partial z_2)B(z_1, z_2) = -B(z_1, z_2)A(z_2)$. It is invertible $\forall z_1, z_2 \in R$, and $B(z_1, z_2)B(z_2, z_3) = B(z_1, z_3)$ $\forall z_1, z_2$, and $z_3 \in R$. Further, if the disc $|z - z_0| \leqq r$ is in R, then $\|B(z_1, z_2)\| \leqq e^{Mr}$ for $|z_1 - z_0| < r$ and $|z_2 - z_0| < r$, where $M = 2 \sup_{|z - z_0| < r} \|A(z)\|$.*

Proof. Consider first a disc $|z - z_0| \leqq r$ contained in R. We have

$$A(z) = \sum_{k=0}^{\infty} A_k(z - z_0)^k$$

in $|z - z_0| < r$, $A_k \in B(X)$. Certainly

$$B_1(z, z_0) = \sum_{k=0}^{\infty} (k+1)^{-1} A_k(z - z_0)^{k+1}$$

is simultaneously analytic in z and z_0 in $|z - z_0| < r$, $B_1(z_0, z_0) = 0$, and $(\partial/\partial z)B_1(z, z_0) = A(z)$. Similarly, we may start with $A(z)B_1(z, z_0)$ and find $B_2(z, z_0)$ simultaneously analytic in z and z_0 in $|z - z_0| < r$ with $B_2(z_0, z_0) = 0$ and with $(\partial/\partial z)B_2(z, z_0) = A(z)B_1(z, z_0)$, and, in general, we may find $B_{n+1}(z, z_0)$ simultaneously analytic in z and z_0 in $|z - z_0| < r$ such that $(\partial/\partial z)B_{n+1}(z, z_0) = A(z)B_n(z, z_0)$, and such that $B_{n+1}(z_0, z_0) = 0$.

Consider $\|B_n(z_1, z_2)\|$ for $|z_1 - z_0| < r$ and $|z_2 - z_0| < r$. Let $M = 2 \sup_{|z - z_0| < r} \|A(z)\|$. Then

$$B_{n+1}(z_1 t + z_2(1 - t), z_2) = (z_1 - z_2) \int_0^t A(z_1 s + z_2(1 - s))B_n(z_1 s + z_2(1 - s), z_2)\, ds,$$

so that

$$\|B_{n+1}(z_1 t + z_2(1 - t), z_2)\| \leqq (Mr) \int_0^t \|B_n(z_1 s + z_2(1 - s), z_2)\|\, ds,$$

and thus

$$\|B_1(z_1 t + z_2(1 - t), z_2)\| \leqq (Mr)t,$$

and

$$\|B_n(z_1 t + z_2(1 - t), z_2)\| \leqq (Mrt)^n(1/n!) \leqq (Mr)^n(1/n!).$$

Thus $\|B_n(z_1, z_2)\| \leqq (1/n!)(Mr)^n$, and $B(z_1, z_2) = I + \sum_{k=1}^{\infty} B_k(z_1, z_2)$ is analytic in z_1 and z_2 for $|z_1 - z_0| < r$ and $|z_2 - z_0| < r$, with $\|B(z_1, z_2) - I\| \leqq e^{Mr} - 1$, $B(z_1, z_1) = I$, and $(\partial/\partial z_1)B(z_1, z_2) = A(z_1)B(z_1, z_2)$ for such z_1 and z_2.

Now let z_1 and z_2 be arbitrary in R, and let $z(t)$ be a piecewise-smooth path in R with $z(0) = z_2$ and $z(1) = z_1$. Set

$$B(z_1, z_2) = B(z(t_n), z(t_{n-1})) \cdots B(z(t_1), z(t_0)),$$

where $0 = t_0 \leqq t_1 \leqq \cdots \leqq t_n = 1$, and the $B(z(t_i), z(t_{i-1}))$ are as determined above. By the general results on bounded operators applied to $z'(\cdot)A(z(\cdot))$, $B(z_1, z_2)$ is independent of the subdivision taken. Further, by the usual deformation arguments, it is independent of the path $z(t)$ chosen. Uniqueness follows from the general results on bounded operators, and analyticity in z_1 and z_2 are direct from the expression of $B(z_1, z_2)$. The other statements follow from $B(z_1, z_2)B(z_2, z_3) = B(z_1, z_3)$. ∎

Behavior toward the boundary of R is of some interest. Perhaps the simplest theorem of this sort is the one that follows. Note first that if R is a convex region with closure $\bar{R}$, then any piecewise-smooth path $\gamma(t), 0 \leqq t \leqq 1$, in $\bar{R}$ may be approximated by polygonal paths $\{\gamma_n(t)\}$ in R in such fashion that $\|\gamma(\cdot) - \gamma_n(\cdot)\|_\infty \to 0$, and $\|\gamma'(\cdot) - \gamma_n'(\cdot)\|_1 \to 0$.

THEOREM. *Let R be a convex region of the complex plane, and let $A(\cdot)$ be analytic from R to $B(X)$. Let $B(z_1, z_2)$ be as in the above theorem.*

Suppose $\bar{A}(\cdot)$ is a function on the closure $\bar{R}$ of R to $B(X)$ with $\bar{A}(z) = A(z)$ $\forall z \in R$ and with $\bar{A}(\cdot)$ continuous from $\bar{R}$ to the Banach space $B(X)$.

Then $\exists \bar{B}(z_1, z_2)$ continuous from $\bar{R} \times \bar{R}$ to $B(X)$ with $\bar{B}(z_1, z_2) = B(z_1, z_2)$ $\forall z_1, z_2 \in R$. $\bar{B}(z, z) = I$ $\forall z \in \bar{R}$, and $\bar{B}(z_1, z_2)\bar{B}(z_2, z_3) = \bar{B}(z_1, z_3)$ $\forall z_1, z_2,$ and $z_3 \in \bar{R}$. If $\gamma(t), 0 \leqq t \leqq 1$, is any piecewise-smooth path in $\bar{R}$, then $\bar{B}(\gamma(\cdot), \gamma(s))$ is absolutely continuous on $[s, 1]$, and uniformly differentiable with derivative $\gamma'(\cdot)\bar{A}(\cdot)\bar{B}(\gamma(\cdot), \gamma(s))$ a.e. on $[s, 1]$. Similarly, $\bar{B}(\gamma(t), \gamma(\cdot))$ is absolutely continuous on $[0, t]$, and uniformly differentiable with derivative $-\bar{B}(\gamma(t), \gamma(\cdot))\bar{A}(\cdot)\gamma'(\cdot)$ a.e.

Proof. Let z_1 and z_2 be points of $\bar{R}$, and let $\gamma(t), 0 \leqq t \leqq 1$, be a simple piecewise-smooth path in R for $0 < t < 1$ with $\gamma(0) = z_2$ and $\gamma(1) = z_1$. Define $\bar{B}(\gamma(t), \gamma(s))$ for $0 \leqq s \leqq t \leqq 1$ by $\bar{B}(\gamma(t), \gamma(t)) = I$, and $\bar{B}(\gamma(\cdot), \gamma(s))$ is absolutely continuous on $[s, 1]$ and a.e. uniformly differentiable with derivative $\gamma'(\cdot)\bar{A}(\cdot)\bar{B}(\gamma(\cdot), \gamma(s))$. Then $\bar{B}(\gamma(t), \gamma(s))$ is jointly uniformly continuous in t and s, and $\bar{B}(\gamma(t), \gamma(s)) = B(\gamma(t), \gamma(s))$ for $0 < s \leqq t < 1$, so that $B(\gamma(t), \gamma(s))$ converges uniformly to $\bar{B}(\gamma(1), \gamma(0))$ as $t \uparrow 1$ and $s \downarrow 0$. Set $\bar{B}(z_1, z_2) = \bar{B}(\gamma(1), \gamma(0))$.

It is easy to see that this definition is independent of the path chosen, and that $\bar{B}(z_1, z_2)$ is continuous on $\bar{R} \times \bar{R}$. The other properties are straightforward. ∎

(b) In at least one special case we may prove analogous theorems for unbounded operators. We let $\mathscr{A}$ be a weakly-closed ring of operators over a Hilbert space X with center $\mathscr{Z}$. An operator in $CD(X)$ affiliated with $\mathscr{A}$ is measurable with respect

to $\mathcal{Z}$ if there exists a sequence of central projections $P_n \uparrow I$ in the domain of the operator. We call the *-algebra of such operators $\mathcal{M}(\mathcal{A}, \mathcal{Z})$ (clearly such an operator is the closure of its restriction to the ranges of any sequence of central projections in its domain converging monotonely upwards to the identity).

THEOREM. *Let R be a simply-connected region of the complex plane, and let $A(\cdot)$ be a function from R into $\mathcal{M}(\mathcal{A}, \mathcal{Z})$.*

Suppose that $\forall z_0 \in R$, $\exists \varepsilon > 0$ and $A \in \mathcal{Z}^+ \cap L(X)$ such that $A(\cdot)A$ is analytic in $|z - z_0| < \varepsilon$. Then $\forall z_1, z_2 \in R$ $\exists B(z_1, z_2) \in \mathcal{M}(\mathcal{A}, \mathcal{Z})$ with the following properties:

(1) if R_0 is a subregion of R with compact closure $\bar{R}_0 \subset R$, $\exists C \in \mathcal{Z}^+ \cap L(X)$ such that $B(z_1, z_2)C \in \mathcal{A}$ $\forall z_1, z_2 \in R_0$,

(2) $B(z_1, z_2)C$ is analytic in z_1 for $z_1, z_2 \in R_0$, and

$$(\partial/\partial z_1)(B(z_1, z_2)C) = A(z_1)B(z_1, z_2)C,$$

(3) $B(z, z) = I$ $\forall z \in R$.

These properties uniquely characterize $B(z_1, z_2)$. Moreover, $B(z_1, z_2)C$ is simultaneously analytic in z_1 and z_2 over R_0 with $(\partial/\partial z_2)(B(z_1, z_2)C) = -B(z_1, z_2)A(z_2)C$, and $B(z_1, z_2)B(z_2, z_3) = B(z_1, z_3)$ $\forall z_1, z_2,$ and $z_3 \in R$.

Proof. Let R_0 be a simply-connected subregion of R with compact closure $\bar{R}_0 \subset R$, and let $A_1, \ldots, A_n \in \mathcal{Z}^+ \cap L(X)$ be such that $A(\cdot)A_i$ is analytic in $|z - z_i| < \varepsilon_i$, where the discs $|z - z_i| < \varepsilon_i/2$ cover $\bar{R}_0$. Set $A = A_1 \cdots A_N \in \mathcal{Z}^+ \cap L(X)$. Then $A(\cdot)A$ is analytic in a neighborhood of $\bar{R}_0$.

If $\{E_\lambda\}$ is the spectral family for A, set $P_n = E_{1/n}^\perp$. Letting

$$a_n = \sup\{\|A(z)P_n\| : |z - z_i| \leqq \varepsilon_i/2 \text{ for some } i\} \quad \text{and} \quad M = \sum_{i=1}^{N} \varepsilon_i,$$

set

$$C_1 = \tfrac{1}{2}e^{-Ma_1}P_1 + \sum_{n=2}^{\infty} 2^{-n}e^{-Ma_n}(P_n - P_{n-1}), \quad \text{and} \quad C = C_1A.$$

Since $A_n(\cdot) = A(\cdot)P_n$ is analytic in R_0, $\exists B_n(z_1, z_2) \in \mathcal{A}$ as in the preceding theorem jointly analytic in z_1 and z_2, and with $(\partial/\partial z_1)B_n(z_1, z_2) = A(z_1)P_nB_n(z_1, z_2)$, $(\partial/\partial z_2)B_n(z_1, z_2) = -B_n(z_1, z_2)A(z_2)P_n$, and $B_n(z_1, z_2)B_n(z_2, z_3) = B_n(z_1, z_3)$. By following paths through the discs $|z - z_i| < \varepsilon_i/2$, we have $\|B_n(z_1, z_2)\| \leqq \exp(a_n \sum_{i=1}^{N} \varepsilon_i) = e^{Ma_n}$. By uniqueness $B_n(z_1, z_2) = (I - P_n) + P_nB_n(z_1, z_2)$, and $P_nB_m(z_1, z_2) = P_nB_n(z_1, z_2)$ for $m \geqq n$. Then for $m > n > n_0$,

$$\|A^{-1}(B_m(z_1, z_2) - B_n(z_1, z_2))C\| = \|(B_m(z_1, z_2) - B_n(z_1, z_2))C_1\|$$

$$\leqq \|(P_m - P_n)C_1\| + \|(P_mB_m(z_1, z_2) - P_nB_n(z_1, z_2))C_1\|$$

$$\leqq 2\sum_{j=n_0}^{\infty} 2^{-j},$$

so that $\{A^{-1}B_n(z_1, z_2)C\}$ is uniformly Cauchy over $R_0 \times R_0$, and therefore converges to an analytic $f(z_1, z_2)$.

Set $B(z_1, z_2) = C^{-1}Af(z_1, z_2)$. Then $B(z, z) = I$, and $B(z_1, z_2)C \in \mathscr{A}$ is simultaneously analytic in z_1 and z_2 over R_0. By evaluating derivatives on projections P_n, we have that $(\partial/\partial z_1)B(z_1, z_2)C = A(z_1)B(z_1, z_2)C$, and $(\partial/\partial z_2)B(z_1, z_2)C = -B(z_1, z_2)A(z_2)C$. Also by considering behavior on projections P_n,

$$(B(z_1, z_2)C)(B(z_2, z_3)C) = B(z_1, z_3)C^2,$$

so that $B(z_1, z_2)B(z_2, z_3) = B(z_1, z_3)$.

If $\bar{B}(z_1, z_2)$ defined on R_0 with values in $\mathcal{M}(\mathscr{A}, \mathscr{Z})$ has $\bar{B}(z, z) = I$, and $\bar{B}(z_1, z_2)\bar{C}$ analytic in z_1 with derivative $A(z_1)\bar{B}(z_1, z_2)\bar{C}$ for some $\bar{C} \in \mathscr{Z}^+ \cap L(X)$, then $CB(z_2, z_1)\bar{B}(z_1, z_2)\bar{C}$ is analytic in z_1 with derivative equal to zero, so that $CB(z_2, z_1)\bar{B}(z_1, z_2)\bar{C} = C\bar{C}$ $\forall z_1, z_2 \in R_0$, so that $B(z_2, z_1)\bar{B}(z_1, z_2) = I$ $\forall z_1, z_2 \in R_0$, and, hence, $B(z_1, z_2)B(z_2, z_1)\bar{B}(z_1, z_2) = \bar{B}(z_1, z_2) = B(z_1, z_2)$ $\forall z_1, z_2 \in R_0$. Thus $B(z_1, z_2)$ is uniquely defined independently of the construction used. ∎

Using the same notations, we may prove a theorem on values at the boundary of R.

THEOREM. *Let R be a convex region of the complex plane with closure $\bar{R}$, and let $A(\cdot)$ and $B(z_1, z_2)$ be as in the previous theorem.*

Suppose there exists a function $\bar{A}(\cdot)$ on $\bar{R}$ with values in $\mathcal{M}(\mathscr{A}, \mathscr{Z})$ such that $\bar{A}(z) = A(z)$ $\forall z \in R$, and such that $\bar{A}(\cdot)$ is continuous on $\bar{R}$ in the following sense: $\forall z_0 \in \bar{R}$, there exists a sequence $\{P_n\}$ of projections in $\mathscr{Z}$ with $P_n \uparrow I$, $\bar{A}(z_0)P_n \in \mathscr{A}$ $\forall n$, and with $\bar{A}(z_j)P_n \to \bar{A}(z_0)P_n$ uniformly for each n and each sequence $\{z_j\}$ in $\bar{R}$ with $z_j \to z_0$.

Then there exists a function $\bar{B}(z_1, z_2)$ on $\bar{R} \times \bar{R}$ with values in $\mathcal{M}(\mathscr{A}, \mathscr{Z})$ equal to $B(z_1, z_2)$ on $R \times R$ and continuous on $\bar{R} \times \bar{R}$ in the sense just described for $\bar{A}(\cdot)$. $\bar{B}(z, z) = I$ $\forall z \in \bar{R}$, $\bar{B}(z_1, z_2)\bar{B}(z_2, z_3) = \bar{B}(z_1, z_3)$ $\forall z_1, z_2$, and $z_3 \in \bar{R}$, and if $\gamma(t)$ is any piecewise-smooth path in $\bar{R}$, $0 \leq t \leq 1$, there exists a sequence of projections $\{P_n\}$ in $\mathscr{Z}$ with $P_n \uparrow I$ for each $0 \leq s_0 \leq t_0 \leq 1$ such that $\bar{B}(\gamma(\cdot), \gamma(s_0))P_n$ and $\bar{B}(\gamma(t_0), \gamma(\cdot))P_n$ are absolutely continuous in neighborhoods of t_0 and s_0, respectively, where the neighborhoods depend on n, and they are a.e. uniformly differentiable in these neighborhoods with

$$(\partial/\partial t)\bar{B}(\gamma(t), \gamma(s_0))P_n = \gamma'(t)\bar{A}(t)\bar{B}(\gamma(t), \gamma(s_0))P_n,$$

and

$$(\partial/\partial s)\bar{B}(\gamma(t_0), \gamma(s))P_n = -\bar{B}(\gamma(t_0), \gamma(s))\bar{A}(\gamma(s))\gamma'(s)P_n.$$

Proof. Let z_1 and $z_2 \in \bar{R}$, and choose a sequence $\{P_n\}$ of projections in $\mathscr{Z}$ with $P_n \uparrow I$ satisfying the needs of the definition of continuity of $\bar{A}(\cdot)$ at z_1 and z_2. Let $\gamma(t)$, $0 \leq t \leq 1$, be a piecewise-smooth path in $\bar{R}$ with $\gamma(t) \in R$ for $0 < t < 1$, and $\gamma(0) = z_2$, $\gamma(1) = z_1$.

Fix n, and choose ε such that $\bar{A}(\cdot)P_n$ is uniformly continuous on neighborhoods $|z - z_1| < \varepsilon$ and $|z - z_2| < \varepsilon$ in $\bar{R}$. Choose $\delta > 0$ such that $|\gamma(t) - z_1| < \varepsilon$ for $1 - \delta \leq t \leq 1$, and $|\gamma(t) - z_2| < \varepsilon$ for $0 \leq t \leq \delta$. Define $f_1(\cdot)$ by $f_1(1 - \delta) = I$, $f_1(\cdot)$ is absolutely

continuous on $[1-\delta, 1]$, and $f_1(\cdot)$ is a.e. uniformly differentiable on $[1-\delta, 1]$ with derivative $\gamma'(\cdot)\bar{A}(\cdot)P_n f_1(\cdot)$. Define $f_0(\cdot)$ by $f_0(0)=I$, $f_0(\cdot)$ is absolutely continuous on $[0, \delta]$, and $f_0(\cdot)$ is a.e. uniformly differentiable on $[0, \delta]$ with derivative $\gamma'(\cdot)\bar{A}(\cdot)P_n f_0(\cdot)$.

Set $B_n(z_1, z_2)=f_1(1)B(\gamma(1-\delta), \gamma(\delta))f_0(\delta)P_n$. It is not hard to see that $B_n(z_1, z_2)$ depends only on the projection P_n, and that $P_n B_m(z_1, z_2)=B_n(z_1, z_2)$ for $m \geqq n$.

Define $\bar{B}(z_1, z_2)$ by $\bar{B}(z_1, z_2)(P_{n+1}-P_n)=B_{n+1}(z_1, z_2)-B_n(z_1, z_2)$. Then $\bar{B}(z_1, z_2) \in \mathscr{M}(\mathscr{A}, \mathscr{Z})$. By the usual uniqueness properties and taking refinements of sequences of projections, we see that the definition of $\bar{B}(z_1, z_2)$ is really independent of the choice of $\{P_n\}$. The properties claimed for $\bar{B}(z_1, z_2)$ follow by application of suitable projections. ∎

For an example of the theorem just proved let us take $X=L^2([0, 1])$, $\mathscr{A}=$ the ring of multiplications by bounded measurable functions, $R=$ the square $(0, 1)\times(0, 1)$, and $A(z)$ the operator defined by $(A(z)f)(x)=(x-z)^{-1}f(x)$ for $f \in L^2([0, 1])$. Then $(B(z_1, z_2)f)(x)=((x-z_2)/(x-z_1))f(x)$. The extensions $\bar{A}(\cdot)$ and $\bar{B}(z_1, z_2)$ to $[0, 1]\times[0, 1]$ are defined by the same formulas.

BIBLIOGRAPHY

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1957.

2. E. Hille and R. Phillips, *Functional analysis and semi-groups*, rev. ed., Amer. Math. Soc. Colloq. Publ. Vol. 31, Amer. Math. Soc., Providence, R. I., 1957.

3. T. Kato, *Integration of the equation of evolution in a Banach space*, J. Math. Soc. Japan 5 (1954), 208–234.

4. ———, *On linear differential equations in Banach spaces*, Comm. Pure Appl. Math. 9 (1956), 479–486.

5. F. Riesz and B. von Sz.-Nagy, *Functional analysis*, Ungar, New York, 1955.

6. I. E. Segal, *A non-commutative extension of abstract integration*, Ann. of Math. 57 (1953), 401–457.

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