

ON DISCONJUGATE DIFFERENTIAL EQUATIONS

BY
PHILIP HARTMAN⁽¹⁾

Introduction. In Wintner's [16] terminology (when $n=2$), a linear differential equation

$$(0.1) \quad u^{(n)} + p_{n-1}^{(t)} u^{(n-1)} + \cdots + p_0(t) u = 0$$

with continuous coefficients on a t -interval I is said to be *disconjugate* on I if no solution ($\neq 0$) has n zeros on I . For $n > 2$, most known results giving conditions which assure that (0.1) is disconjugate concern perturbations of $u^{(n)} = 0$ either for a fixed finite interval [as, e.g., in the theorem of de la Vallée Poussin (cf. [2, Exercise 5.3(d), p. 346])] or for large t . This paper deals with equations (1.1) which are perturbations of equations with constant coefficients, disconjugate on $-\infty < t < \infty$. As corollaries, we obtain theorems which are refinements of known results concerning perturbations of $u^{(n)} = 0$, but we do not obtain the "best" constants occurring in some of these results ($n=2$).

The proofs depend on the technique introduced in [5] for discussing asymptotic behavior of solutions of perturbed linear systems with constant coefficients (cf. [2, Chapter X]). This technique is based on suitable changes of variables and arguments which have been subsumed by general theorems of Ważewski [15], (cf. [2, pp. 278–283]). §§1 and 2 use the simple Lemma 4.2, [2, p. 285]; §3 requires a generalization given as Theorem (*) in an Appendix below.

In addition to arguments from the theory of asymptotic integration, the proofs use a theorem of Pólya [14] characterizing equations (0.1) disconjugate on an open interval I in terms of Wronskians of subsets of solutions of (0.1); for a generalization, see [2, pp. 51–54] (also obtained in [6]). Theorems I** and IV of Pólya [14] show that no solution ($\neq 0$) of (0.1) on an open interval I has n distinct zeros if and only if no solution ($\neq 0$) has n zeros counting multiplicities; cf. also [13]. (A generalization of this last fact, when the linear family of solutions of (0.1) is replaced by an arbitrary (not necessarily linear) interpolating family of functions, is given in [1].)

In §4, it is observed that the results of the previous sections, together with theorems and methods of Lasota and Opial [8], give criteria for the existence of solutions of certain nonlinear boundary value problems.

Received by the editors June 27, 1967.

⁽¹⁾ Research partially sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Contract No. AF 49(638)–1382.

1. **Distinct real roots.** We shall first consider linear differential equations

$$(1.1) \quad u^{(n)} + a_{n-1}u^{(n-1)} + \dots + a_0u = q_{n-1}(t)u^{(n-1)} + \dots + q_0(t)u,$$

which are perturbations of equations

$$u^{(n)} + a_{n-1}u^{(n-1)} + \dots + a_0u = 0$$

with constant coefficients, having real distinct characteristic values.

ASSUMPTION (A₁). Let a_0, \dots, a_{n-1} be n real numbers such that the polynomial

$$(1.2) \quad \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = 0$$

has real distinct roots $\lambda_1 < \lambda_2 < \dots < \lambda_n$ and let

$$(1.3) \quad c = \min (\lambda_2 - \lambda_1, \dots, \lambda_n - \lambda_{n-1}) > 0.$$

THEOREM 1.1. *Let Assumption (A₁) hold. Then there exists a positive number η_0 , depending on $\lambda_1, \dots, \lambda_n$ but independent of T , with the following properties: If $q_0(t), \dots, q_{n-1}(t)$ are continuous functions on $0 \leq t < T$ ($\leq \infty$) [or on $0 \leq t \leq T$ ($< \infty$)] such that*

$$(1.4) \quad q(t) = \left(\sum_{j=0}^{n-1} |q_j(t)|^2 \right)^{1/2} \geq 0$$

satisfies

$$\int_0^T q(s) ds < \eta_0 \quad \text{or} \quad q(t) < c\eta_0$$

or, more generally,

$$(1.5) \quad \int_0^t q(s)e^{-c(t-s)} ds, \quad \int_t^T q(s)e^{-c(s-t)} ds \leq \eta < \eta_0.$$

Then (1.1) is *disconjugate* on $0 \leq t < T$ ($\leq \infty$) [or on $0 \leq t \leq T$ ($< \infty$)].

Proof. Let $y = (u, u', \dots, u^{(n-1)})$ and write (1.1) as a linear first order system

$$(1.6) \quad y' = (A + Q(t))y,$$

where $A = (a_{jk})$ is the constant $n \times n$ matrix with the first $n-1$ rows given by $a_{jk} = 1$ or $a_{jk} = 0$ for $1 \leq j \leq n-1$, $k = j+1$ or $k \neq j+1$ and the last row is $(-a_0, \dots, -a_{n-1})$, and $Q(t)$ is the matrix with 0 in the first $n-1$ rows and the last row is $(q_0(t), \dots, q_{n-1}(t))$.

Let $\Lambda = \Lambda_n = (\lambda_{jk})$ be the constant $n \times n$ matrix with $\lambda_{jk} = \lambda_k^{j-1}$ for $j, k = 1, \dots, n$. Since the k th column $(1, \lambda_k, \dots, \lambda_k^{n-1})$ of Λ is an eigenvector of A belonging to the eigenvalue λ_k , we have

$$(1.7) \quad \Lambda^{-1}A\Lambda = J \equiv \text{diag}(\lambda_1, \dots, \lambda_n).$$

We make the change of variables

$$(1.8) \quad y = \Lambda z, \quad \text{i.e.,} \quad y_j = \sum_{m=1}^n \lambda_m^{j-1} z_m,$$

in (1.6) to obtain

$$(1.9) \quad z' = (J + G(t))z,$$

where

$$(1.10) \quad G(t) = (g_{jk}(t)) = \Lambda^{-1}Q(t)\Lambda.$$

It is clear that $g_{jk}(t)$ is a linear combination of $q_0(t), \dots, q_{n-1}(t)$ with coefficients depending on j, k and $\lambda_1, \dots, \lambda_n$. (The explicit formulas for $g_{jk}(t)$ will not be needed below, but a simple calculation shows that

$$(1.11) \quad g_{jk}(t) = \sum_{m=0}^{n-1} q_m(t) \lambda_k^m / \pi_j,$$

where π_j is defined by

$$\pi_j = (-1)^n \prod_{h>j} (\lambda_h - \lambda_j);$$

cf., e.g., [2, pp. 318–319].) Thus there exists a constant $M_0 = M_0(\lambda_1, \dots, \lambda_n) > 0$ with the property that

$$(1.12) \quad \left(\sum_{k=1}^n |g_{jk}(t)|^2 \right)^{1/2} \leq M_0 q(t) \quad \text{for } j = 1, \dots, n.$$

(In fact, by (1.11), M_0 can be chosen to be $\|\Lambda\|/\pi$, where $\|\Lambda\|$ is the bound of the matrix Λ as an operator from R^n to R^n and $\pi = \min(|\pi_1|, \dots, |\pi_n|)$.)

Define the functions

$$\sigma(t) = M_0 \int_0^t q(s) e^{-c(t-s)} ds, \quad \tau(t) = M_0 \int_t^T q(s) e^{-c(s-t)} ds$$

and assume that the bound η in (1.5) is so small that

$$(1.13) \quad 7M_0\eta < 1.$$

Then, for each k , [2, Lemma 4.2, p. 285] with $\varepsilon=0$ and $\psi(t)=M_0q(t)$ and its proof imply that (1.9) has a solution $z=(z_{k1}(t), \dots, z_{kn}(t)) \neq 0$ satisfying, for $0 \leq t < T$,

$$|z_{kj}(t)| \leq 7\sigma(t)z_{kk}(t) \quad \text{for } 1 \leq j < k,$$

$$|z_{kj}(t)| \leq 7\tau(t)z_{kk}(t) \quad \text{for } k < j \leq n;$$

cf. the Appendix below for a generalization. (Actually, the conditions of Lemma 4.2 in [2, p. 285] and its proof require that $\psi(t)=M_0q(t) \geq 0$ be positive, but it is clear that the validity of Lemma 4.2 with $\psi(t) > 0$ implies its validity for $\psi(t) \geq 0$.)

By (1.8), the corresponding solution $y = (y_{k1}(t), \dots, y_{kn}(t)) = \Lambda z$ of (1.6) satisfies

$$(1.14) \quad |y_{kj}(t) - z_{kk}(t)\lambda_k^{j-1}| \leq 7M_0\eta z_{kk}(t) \sum_{m \neq k} |\lambda_m|^{j-1}$$

for $0 \leq t < T$. The solution $y(t)$ of (1.6) corresponds to a solution $u = u_k(t)$ of (1.1) with the properties that if $e_{kj}(t)$ is defined by

$$e_{kj}(t)z_{kk}(t) = u_k^{(j-1)}(t) - \lambda_k^{j-1}z_{kk}(t),$$

then

$$|e_{kj}(t)| \leq C\eta, \quad \text{where } C = 7M_0 \max_j \sum_{m=1}^n |\lambda_m|^{j-1}.$$

Let $1 \leq m \leq n$ and let $W_m(t)$ be the Wronskian determinant of the m solutions $u_1(t), \dots, u_m(t)$ of (1.1). Then

$$W_m(t) = z_{11}(t) \cdots z_{mm}(t) \det(\lambda_k^{j-1} + e_{kj})$$

and, consequently, there exists a constant $K = K(\lambda_1, \dots, \lambda_n)$ with the property that

$$W_n(t) = z_{11}(t) \cdots z_{nn}(t)(\det \Lambda_m + \delta), \quad |\delta| \leq K\eta(1 + K\eta)^{n-1}$$

and $\det \Lambda_m = \prod_{1 \leq j < k \leq m} (\lambda_k - \lambda_j)$, $1 \leq j < k \leq m$. Hence, there exists an $\eta_0 > 0$ with the property that

$$W_m(t) \neq 0 \quad \text{for } 0 \leq t < T, m = 1, \dots, n \quad \text{if } 0 \leq \eta < \eta_0.$$

It follows from a theorem of Pólya [12, p. 317] that (1.1) is disconjugate on $0 < t < T$.

In order to see that (1.1) is disconjugate on $[0, T]$, that is, on every interval $[0, b]$, $0 < b < T$, it is only necessary to extend the definition of $q_k(t)$ to an interval $[-\epsilon, T]$, for a suitably small $\epsilon = \epsilon(b) > 0$, and apply the statement already proved with $(0, T)$ replaced by $(-\epsilon, b + \epsilon)$.

COROLLARY 1.1. *Let Assumption (A_1) hold. There exists a constant $\eta_0 = \eta_0(\lambda_1, \dots, \lambda_n) > 0$ with the property that if $q_0(t), \dots, q_{n-1}(t)$ are continuous for $-\infty < t < \infty$ and $q(t)$ in (1.4) satisfies $q(t) < c\eta_0$ for all t , then (1.1) is disconjugate on $-\infty < t < \infty$.*

This is a consequence of Theorem 1.1 which shows that the condition $q(t) < c\eta_0$ implies that (1.1) is disconjugate on every interval $[a, b]$, $-\infty < a < b < \infty$.

COROLLARY 1.2. *Let Assumption (A_1) hold. Let $q_0(t), \dots, q_{n-1}(t)$ be continuous for $0 \leq t < \infty$ and $q(t)$ be defined by (1.4). Let $u(t) \neq 0$ be a solution of (1.1) and $N = N(T)$ the number of zeros of $u(t)$, counting multiplicities, on $0 \leq t < T$. Then*

$$(1.15) \quad \eta_0 N(T) \leq (n-1) \int_0^T q(t) dt + \eta_0(n-1).$$

Proof. Let the nonnegative zeros (if any) of $u(t)$ be $0 \leq t_0 \leq t_1 \leq \dots$. Then

$$\int_s^t q(\rho) d\rho \geq \eta_0 \quad \text{if } s = t_j, t = t_{j+n-1}.$$

Thus if $t_{N-1} < T \leq t_N$,

$$\int_0^T q(\rho) d\rho \geq \eta_0[(N-1)/(n-1)] \geq \eta_0\{(N-1)/(n-1) - (n-2)/(n-1)\}.$$

This gives (1.15).

2. Coincident real roots. We now consider the case when Assumption (A₁) does not hold and, in fact, the extreme opposite case, $\lambda_1 = \dots = \lambda_n$, holds. We suppose that the value of $\lambda_1 = \dots = \lambda_n$ is 0; cf. the Remark at the end of this section.

THEOREM 2.1. *There exists a number $\eta_1 > 0$, independent of T , with the following property: In the differential equation*

$$(2.1) \quad u^{(n)} = q_{n-1}(t)u^{(n-1)} + \dots + q_0(t)u,$$

let $q_0(t), \dots, q_{n-1}(t)$ be continuous for $0 \leq t < T$ ($\leq \infty$) [or for $0 \leq t \leq T$ ($< \infty$)] and such that

$$(2.2) \quad q(t) = \left(\sum_{k=0}^{n-1} |q_k(t)t^{n-k-1}|^2 \right)^{1/2} \geq 0$$

satisfies

$$(2.3) \quad t^{-1} \int_0^t sq(s) ds, \quad t \int_t^T s^{-1}q(s) ds \leq \eta < \eta_1$$

(in particular, let

$$(2.4) \quad \int_0^T q(t) dt < \eta_1 \quad \text{or} \quad tq(t) < \eta_1$$

hold), then (2.1) is disconjugate on $0 \leq t < T$ ($\leq \infty$) [or on $0 \leq t \leq T$ ($< \infty$)].

If $n=2$ and $q_1(t) \equiv 0$, then the first criterion in (2.4) is known with $\eta_1 = 1$ (cf. [4, Theorem 5.1, p. 345] with $m(t) = t - a$) and the second is an analogue of A. Kneser's criterion $t^2q_0(t) \leq 1/4$. If $n \geq 2$, $q_1(t) \equiv \dots \equiv q_{n-1}(t) \equiv 0$ and $q_0(t)$ is of constant sign and monotone, then the first criterion in (2.4) has been proved by Nehari [12]. (For a related result of de la Vallée Poussin and generalizations, see Levin [9], [10], Nehari [11], and Hukuhara [7].) According to a theorem of Dunkel (cf. [2, Theorem 17.1, p. 315]), the convergence of the integral in (2.4) for $T = \infty$ implies asymptotic formulae for the solutions of (2.1) for large t . But these formulae do not imply a disconjugacy criterion for (2.1) on $t \geq 0$.

Proof. Under the change of independent variables

$$(2.5) \quad s = -\log t \quad \text{or} \quad t = e^{-s},$$

the interval $0 < t < T$ is changed to $-\log T < s < \infty$. Note that $u' = -e^s Du$, where $D = d/ds$. Thus

$$u^{(k)} = (-t)^{-k} D(D+1) \cdots (D+k-1)u.$$

Write this relation as

$$u^{(k)} = (-t)^{-k} \sum_{j=1}^k \beta_{kj} D^j, \quad \text{where } \beta_{kk} = 1,$$

so that

$$(2.6) \quad \sum_{j=1}^k \beta_{kj} \lambda^j = \lambda(\lambda+1) \cdots (\lambda+k-1).$$

Thus (2.1) becomes

$$(2.7) \quad D^n u + \sum_{j=1}^{n-1} \beta_{nj} D^j u = Q_{n-1}(s) D^{n-1} u + \cdots + Q_0(s) u,$$

where $t = e^{-s}$, $Q_0(s) = (-t)^n q_0(t)$, and

$$(2.8) \quad Q_j(s) = \sum_{k=j}^{n-1} (-t)^{n-k} q_k(t) \beta_{kj} \quad \text{for } j = 1, \dots, n-1.$$

Hence, there is a constant $C = C(n)$ such that

$$(2.9) \quad Q(s) = \left(\sum_{j=0}^{n-1} |Q_j(s)|^2 \right)^{1/2} \leq C q(t) t,$$

where $q(t)$ is defined by (2.2).

It is clear from (2.6) that the roots of the characteristic polynomial belonging to the left side of (2.7) are $0, -1, \dots, -(n-1)$. In view of (2.9), Theorem 2.1 follows from Theorem 1.1 after an obvious change of integration variables.

Theorem 2.1 will be combined with the method of [3] for the case $n=2$, $q_1(t) \equiv 0$ (cf. [2, pp. 346–347]) to obtain an analogue of Corollary 1.2.

COROLLARY 2.1. *There exist positive constants M_0, \dots, M_{n-1} with the following property: Let $q_0(t), \dots, q_{n-1}(t)$ be continuous for $t \geq 0$, $u(t) \not\equiv 0$ a solution of (2.1), and $N = N(T)$ the number of zeros of $u(t)$ on $0 \leq t < T$ ($< \infty$), counting multiplicities. Then $N \leq n-1$ or N satisfies the inequality*

$$(2.10) \quad \sum_{k=0}^{n-1} M_k \{N/(n-1) - 1\}^{k-n} \left\{ T^{n-1} \int_0^T |q_k(t)|^{n/(n-k)} dt \right\}^{1-k/n} > 1.$$

Since $k-n < 0$, (2.10) can be used to estimate $N = N(T)$ from above. An admissible choice for M_0, \dots, M_{n-1} is given by $M_k = 2^{k-n}$, $k=0, \dots, n-1$. This follows from the proof of Corollary 2.1 and a result of Nehari [11]. (Another choice of the constants may be given in the paper by Hukuhara [7] which is not available to me; the pertinent (last) result stated in the review in the Mathematical Reviews does

not seem correct.) Still another choice is, e.g., $M_0 = (n-2)^{n-2}/(n-1)^{n-1}(n-1)!$ and $M_k = 1/(n-k-1)!$ for $k=1, \dots, n-1$; cf. the argument in [2, Exercise 5.3, p. 570]. See also Levin [9], [10].

Proof. Let $N > n-1$ and $(0 \leq) t_0 \leq t_1 \leq \dots$ be the nonnegative zeros of $u(t)$; so that $t_{N-1} < T \leq t_N$. By Theorem 2.1, there exist positive constants M_0, \dots, M_{n-1} such that

$$\sum_{k=0}^{n-1} M_k (t-s)^{-k} \int_s^t |q_k| d\rho > (t-s)^{1-n} \quad \text{if } s = t_j, t = t_{j+n-1};$$

e.g., by (2.2) and (2.4), any set of constants $M_k > 1/\eta_1$ for $k=0, \dots, n-1$ is admissible. By [11], one can choose $M_k = 2^{-k}$ for $k=0, \dots, n-1$.

Letting $s = t_j$, $t = t_{j+n-1}$ and adding for $j=0, \dots, m$ gives

$$(2.11) \quad \sum_{k=0}^{n-1} M_k \sum_{j=0}^m (t-s)^{-k} \int_s^t |q_k| d\rho > \sum_{j=0}^m (t-s)^{1-n}.$$

Note that if $1 < \alpha < \infty$,

$$(t-s)^{-k} \int_s^t |q_k| d\rho \leq (t-s)^{1-k-1/\alpha} \left(\int_s^t |q_k|^\alpha d\rho \right)^{1/\alpha}.$$

Thus, if $1/\alpha + 1/\beta = 1$,

$$\sum_{j=0}^m (t-s)^{-k} \int_s^t |q_k| d\rho \leq \left(\sum_{j=0}^m (t-s)^{\beta(1-k-1/\alpha)} \right)^{1/\beta} \left(\int_{t_0}^{t_{m(n-1)}} |q_k|^\alpha d\rho \right)^{1/\alpha}.$$

For $1 \leq k \leq n-1$, choose $\alpha = n/(n-k)$ and $\beta = n/k$, so that $\beta(1-k-1/\alpha) = 1-n$ and the last product is

$$\left(\sum_{j=0}^m (t-s)^{1-n} \right)^{k/n} \left(\int_{t_0}^{t_{m(n-1)}} |q_k|^{n/(n-k)} d\rho \right)^{1-k/n}.$$

Thus, by (2.11),

$$\sum_{k=0}^{n-1} M_k \left(\sum_{j=0}^m (t-s)^{1-n} \right)^{-1+k/n} \left(\int_{t_0}^{t_{m(n-1)}} |q_k|^{n/(n-k)} d\rho \right)^{1-k/n} > 1.$$

From the inequality for the harmonic mean,

$$\left[\sum_{j=0}^m (t-s)^{-(n-1)} \right]^{-1/(n-1)} \leq (m+1)^{-n/(n-1)} \sum_{j=0}^m (t-s).$$

The last sum is $t_{m(n-1)} - t_0 < T$ if $t_{m(n-1)} \leq t_{N-1} < T$. Thus we obtain

$$\sum_{k=0}^{n-1} M_k \left[(m+1)^{-n} T^{n-1} \int_0^T |q_k|^{n/(n-k)} ds \right]^{1-k/n} > 1.$$

If we choose $m+1 = [(N-1)/(n-1)] \geq (N-1)/(n-1) - (n-2)/(n-1)$, then $m+1 \geq N/(n-1) - 1$. This, together with the last inequality, gives (2.10) and proves Corollary 2.1.

REMARK. The analogues of Theorem 2.1 and its corollary in which (2.1) is replaced by

$$(2.12) \quad (d/dt - \lambda)^n u = q_{n-1}(t)u^{(n-1)} + \cdots + q_0(t)u,$$

and (2.2) by

$$(2.13) \quad q(t) = \left(|q_0(t)t^{n-1}|^2 + \sum_{k=1}^{n-1} |q_k(t)t^{n-k-1}(1+|\lambda t|^k)|^2 \right)^{1/2},$$

are valid (and the corresponding number η_1 does not depend on λ).

In fact, the change of dependent variables

$$v = e^{-\lambda t}u$$

transforms (2.12) into an equation of the form

$$v^{(n)} = \rho_{n-1}(t)v^{(n-1)} + \cdots + \rho_0(t)v,$$

where

$$\rho_j(t) = \sum_{k=j}^{n-1} C_{kj} \lambda^{k-j} q_k(t), \quad \text{where } C_{kj} = k!/j!(k-j)!.$$

Hence, (2.13) satisfies

$$\sum_{k=0}^{n-1} |\rho_k(t)t^{n-k-1}|^2 \leq C^2 |q(t)|^2$$

for a suitable constant C , independent of λ .

3. Arbitrary real roots. We now consider the differential equation

$$(3.1) \quad u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_0u = q_{n-1}(t)u^{(n-1)} + \cdots + q_0(t)u,$$

when the following holds:

ASSUMPTION (A_2) . Let a_0, \dots, a_{n-1} be real numbers such that the polynomial

$$(3.2) \quad \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0$$

has only real roots, say, $\lambda(1), \dots, \lambda(g)$, with the respective multiplicities $h(1), \dots, h(g)$; so that $h(j) \geq 1$ and $h(1) + \cdots + h(g) = n$. Let

$$(3.3) \quad h_* = \max(h(1), \dots, h(g)).$$

THEOREM 3.1. *Let Assumption (A_2) hold. Then there exist numbers $C \geq 1$ and $\eta_2 > 0$, depending on a_0, \dots, a_{n-1} but independent of T , with the following property: If $q_0(t), \dots, q_{n-1}(t)$ are continuous on $0 \leq t < T$ ($\leq \infty$) [or on $0 \leq t \leq T$ ($< \infty$)] and*

$$(3.4) \quad q(t) = (1+t)^{h_*-1} \sum_{k=0}^{n-1} |q_k(t)|$$

satisfies

$$(3.5) \quad \int_0^t q(s)[(s+C)/(t+C)] ds, \int_t^T q(s)[(t+C)/(s+C)] ds \leq \eta < \eta_2$$

or, in particular,

$$(3.6) \quad \int_0^T q(t) dt < \eta_2 \quad \text{or} \quad (t+C)q(t) < \eta_2,$$

then (3.1) is *disconjugate* on $0 \leq t < T$ ($\leq \infty$) [or on $0 \leq t \leq T$ ($< \infty$)].

The proof depends on a method of Hartman and Wintner [5] for the asymptotic integration of linear systems with nearly constant coefficients. This method involves a change of variables which enables us to obtain rather precise information on the asymptotic behavior of each component of the solution vector.

Proof. We shall deal only with disconjugacy for $0 < t < T$. The passage to $0 \leq t < T$ can be made as in the proof of Theorem 1.1.

(a) Let $C = C(a_0, \dots, a_{n-1}) \geq 1$ be a number to be fixed below and introduce the abbreviation

$$(3.7) \quad s = t + C \geq 1 \quad \text{if} \quad 0 \leq t < T.$$

Assume, that for each j , there exist $h(j)$ solutions $u = u_{j1}(t), \dots, u_{jh(j)}(t)$ of (3.1) with the properties that $v_{j\kappa}(t) = u_{j\kappa}(t)e^{-\lambda(j)s}$ and its derivatives satisfy

$$(3.8) \quad \begin{aligned} |D^{i-1}v_{j\kappa}(t) - w_{\kappa}^j(t)s^{\kappa-i}/(\kappa-1)!| &\leq \gamma\eta|w_{\kappa}^j(t)|s^{\kappa-i} & \text{for } i = 1, \dots, \kappa, \\ |D^{i-1}v_{j\kappa}(t)| &\leq \gamma\eta|w_{\kappa}^j(t)|s^{\kappa-i} & \text{for } i = \kappa+1, \dots, h(j), \\ |D^{i-1}v_{j\kappa}(t)| &\leq \gamma\eta|w_{\kappa}^j(t)|s^{\kappa-h_*} & \text{for } h(j) < i \leq n, \end{aligned}$$

for $0 \leq t < T$, where $w_{\kappa}^j(t)$ is a continuous nonvanishing function; γ denotes a constant, not always the same, depending only on a_0, \dots, a_{n-1} ; and $D = d/dt$.

It will now be shown that if $\eta > 0$ is sufficiently small and the n solutions $u_{11}, \dots, u_{1h(1)}, u_{21}, \dots, u_{gh(g)}$ are numbered as u_1, \dots, u_n , then the Wronskian determinant

$$W_m(t) = \det(D^{i-1}u_j), \quad \text{where } i, j = 1, \dots, m (\leq n),$$

does not vanish for $0 \leq t < T$.

Let $1 \leq m \leq n$; say, $m = h(1) + \dots + h(\mu-1) + \nu$, where $1 \leq \nu \leq h(\mu)$. Let $h'(j) = h(j)$ or ν according as $j \leq \mu-1$ or $j = \mu$. The entry $D^{i-1}u_{j\kappa}$ in $W_m(t)$ is

$$(3.9) \quad D^{i-1}u_{j\kappa} = D^{i-1}e^{\lambda(j)s}v_{j\kappa} = e^{\lambda(j)s} \sum_{r=1}^i C_{i-1, r-1} \lambda^{i-r}(j) D^{r-1}v_{j\kappa}.$$

If $i \leq h(j)$, let $e_{ih\kappa}(t) = 0$. If $i > h(j)$, let

$$(3.10) \quad e_{ij\kappa}(t) = \sum_{r=h(j)+1}^i C_{i-1, r-1} \lambda^{i-r}(j) D^{r-1}v_{j\kappa},$$

so that, by (3.8),

$$(3.11) \quad |e_{ij\kappa}(t)| \leq \gamma|w_{\kappa}^j(t)|\eta s^{\kappa-h_*} \quad \text{for } 0 \leq t < T; \kappa = 1, \dots, h(j); i = 1, \dots, n.$$

The relations (3.9), (3.10) give

$$(3.12) \quad D^{i-1}u_{j\kappa} = e^{\lambda(j)s} [f_{ij\kappa}(t) + e_{ij\kappa}(t)],$$

where

$$f_{ij\kappa} = \sum_{r=1}^{\min(i, h(j))} C_{i-1, r-1} \lambda^{i-r}(j) D^{r-1}v_{j\kappa}.$$

The $m \times m$ matrix $(f_{ij\kappa}(t))$, corresponding to $(D^{i-1}u_j)$, $i, j = 1, \dots, m$, can be written as the product $\Lambda_{(m)} W^m(t)$ of two $m \times m$ matrices. The matrix $\Lambda_{(m)}$ is a constant matrix which is the Wronskian determinant at $t=0$ of the first m functions of $\omega_{11}, \dots, \omega_{1h(1)}, \omega_{21}, \dots, \omega_{gh(g)}$, where $\omega_{jk} = e^{\lambda(j)t} t^{k-1} / (k-1)!$. By (3.9),

$$D^{i-1}\omega_{jr}(0) = \sum_k C_{i-1, k-1} \lambda^{i-k}(j) [t^{r-k} / (r-k)!]_{t=0},$$

and the sum is over the range $1 \leq k \leq \min(i, r)$; so that

$$D^{i-1}\omega_{jr}(0) = C_{i-1, r-1} \quad \text{or } 0, \text{ according as } r \leq i \text{ or } r > i.$$

Choose $W^m(t)$ to be the matrix

$$(3.14) \quad W^m(t) = \text{diag}(W_{(1)}(t), \dots, W_{(u)}(t)),$$

where $W_{(j)}(t)$ is an $h'(j) \times h'(j)$ matrix,

$$W_{(j)}(t) = (D^{i-i}v_{j\kappa}) \quad \text{for } i, \kappa = 1, \dots, h'(j).$$

Thus, we have $(f_{ij\kappa}(t)) = \Lambda_{(m)} W^m(t)$.

The matrix $\Lambda_{(m)}$ is nonsingular, since its determinant is the Wronskian determinant of m linearly independent solutions of an m th order linear equation $u^{(m)} + \dots = 0$ with constant coefficients. Thus, by (3.12), the matrix $(D^{i-1}u_j)$ can be written

$$(D^{i-1}u_{j\kappa}) = \Lambda_{(m)} [W^m(t) + \Lambda_{(m)}^{-1}(e_{ij\kappa})] D_0,$$

where $D_0 = \text{diag}(e^{\lambda(1)s}, \dots, e^{\lambda(u)s})$ in which $e^{\lambda(j)s}$ occurs $h'(j)$ times. Thus, the assertion $W_m(t) \neq 0$ is equivalent to

$$(3.15) \quad \det W^{(m)}(t) \neq 0, \quad \text{where } W^{(m)} = W^m(t) + \Lambda_{(m)}^{-1}(e_{ij\kappa}).$$

From (3.11), the absolute values of the elements in the columns of $\Lambda_{(m)}^{-1}(e_{ij\kappa})$ corresponding to $u_{j\kappa}$ are majorized by $\gamma |w_{\kappa}^j(t)| \eta s^{i-h_*}$. Divide the corresponding column of $W^{(m)}(t)$ by $s^{i-1} |w_{\kappa}^j(t)|$ and multiply the $[h(1) + \dots + h(j-1) + i]$ th row by s^{i-1} . In the resulting matrix, the elements which are not in the blocks corresponding to $W_{(1)}, \dots, W_{(u)}$ (cf. (3.14)) are majorized by $\gamma \eta s^{i-h_*}$, where $i - h_* \leq h(j) - h_* \leq 0$. Since $s \geq 1$ for $t \geq 0$, $\gamma \eta s^{i-h_*} \leq \gamma \eta$. By the same argument, the elements below the diagonals on the blocks $W_{(1)}, \dots, W_{(u)}$ are majorized by $\gamma \eta s^{i-\kappa} \leq \gamma \eta$ for $i \leq \kappa$. The elements on and above the diagonal become $1/(i-\kappa)! + \text{error}$, where $|\text{error}| \leq \gamma \eta$; in particular, the diagonal elements differ from 1 by at most $\gamma \eta$. Thus, it is clear that if η_2 is sufficiently small and $\eta < \eta_2$, then (3.15) holds.

(b) In order to complete the proof of Theorem 3.1, it is necessary to prove the existence of the solutions $u = u_{jk}(t)$ assumed in part (a). Let j be fixed, $1 \leq j \leq g$. For the sake of definiteness let $j = 1$, $\lambda = \lambda(1)$, $h = h(1)$ and put $\mu(j) = \lambda(j) - \lambda$ for $j = 1, \dots, g$. In particular $\mu(1) = 0$. In (3.1), introduce the new dependent variable

$$(3.16) \quad v = e^{-\lambda t} u.$$

Then (3.1) takes the form

$$(3.17) \quad v^{(n)} + b_{n-1}v^{(n-1)} + \dots + b_h v^{(h)} = \sum_{i=0}^{n-1} \rho_i(t) v^{(i)}.$$

Here b_h, \dots, b_{n-1} are constants such that the roots of

$$\tau^n + b_{n-1}\tau^{n-1} + \dots + b_h\tau^h = 0$$

are $0 = \mu(1), \mu(2), \dots, \mu(g)$ with the respective multiplicities $h(1), \dots, h(g)$. Also

$$(3.18) \quad \rho_i(t) = \sum_{k=i}^{n-1} C_{k,i} \lambda^{k-i} q_k(t), \quad \text{where } C_{k,i} = k! / i!(k-i)!.$$

Write (3.17) as a first order system

$$(3.19) \quad y' = By + P(t)y$$

for the vector $y = (v, v', \dots, v^{(n-1)})$; cf. (1.1) and (1.6).

For a given j , let J_j denote the $h(j) \times h(j)$ Jordan matrix with the diagonal elements $\mu(j)$ and, if $h(j) > 1$, superdiagonal elements 1. The first h columns of B constitute an $h \times n$ matrix with J_1 in the upper portion and the zero matrix $0_{h \times (n-h)}$ below. Thus, there exists a nonsingular constant matrix of the form

$$(3.20) \quad \Lambda = \begin{pmatrix} I_h & \cdots \\ 0 & \cdots \end{pmatrix}, \quad \Lambda^{-1} = \begin{pmatrix} I_h & \cdots \\ 0 & \cdots \end{pmatrix},$$

such that I_h is the unit $h \times h$ matrix and

$$(3.21) \quad \Lambda^{-1}B\Lambda = J \equiv \text{diag}(J_1, \dots, J_g).$$

The linear change of variables

$$(3.22) \quad y = \Lambda z$$

sends (3.19) into

$$(3.23) \quad z' = Jz + \Lambda^{-1}P(t)\Lambda z.$$

Write $z = (z^1, \dots, z^g)$, where $z^j = (z_1^j, \dots, z_{h(j)}^j)$ is a vector of dimension $h(j)$, and (3.23) as

$$(3.24) \quad z^{j'} = J_j z^j + \sum_{k=1}^g P^{jk}(t) z^k \quad \text{for } j = 1, \dots, g,$$

where $P^{jk}(t)$ is an $h(j) \times h(k)$ matrix.

Since $P(t)$ is an $n \times n$ matrix with 0 entries except for the last row $(-\rho_0(t), \dots, -\rho_{n-1}(t))$, it follows from (3.20), (2.23) that the i th component of $P^{j1}(t)z^1$ is of the form

$$(3.25) \quad (P^{j1}(t)z^1)_i = \alpha_i^1 \sum_{m=0}^{h-1} \rho_m(t) z_{m+1}^1,$$

where $(-\alpha_1^1, \dots, -\alpha_{h(g)}^1)$ is the last column of Λ^{-1} . Furthermore the elements of $P^{jk}(t)$ are linear combinations of $\rho_0(t), \dots, \rho_{n-1}(t)$ with constant coefficients.

Let $C \geq 1$, $\varepsilon > 0$ be constants to be specified below and introduce the abbreviation $s = t + C$, as in (a) above. (Note that t is still the independent variable.) Make the linear change of variables $z \rightarrow w$:

$$(3.26) \quad z^1 = (\exp J_1 s) w^1 \quad \text{and} \quad z_i^j = \varepsilon^{h(j)-i} w_i^j$$

for $j=2, \dots, g$, to obtain

$$(3.27) \quad w^{1'} = \sum_{k=1}^g Q^{1k}(t) w^k \quad \text{and} \quad w^{j'} = J_{j\varepsilon} w^j + \sum_{k=1}^g Q^{jk}(t) w^k,$$

where $J_{j\varepsilon}$ is obtained by replacing the superdiagonal elements 1 in J_j by ε . If (3.26) is written as

$$z = Q_0(t)w, \quad Q_0(t) = \text{diag}(\exp J_1 s, D_1),$$

where $D_1 = D_1(\varepsilon)$ is a diagonal matrix, then (3.27) is

$$w' = \text{diag}(0, J_{2\varepsilon}, \dots, J_{g\varepsilon})w + Q_0^{-1} \Lambda^{-1} P \Lambda Q_0 w.$$

From this, we can write (3.27) in detail as

$$(3.28) \quad w_i^{1'} = \left(\sum_{r=i}^h c_{ir} s^{r-i} \right) \sum_{k=1}^h \sum_{m=0}^{k-1} d_{km} \rho_m(t) s^{k-1-m} w_k^1 + \sum_{i=2}^g \sum_{m=1}^{h(1)} \sum_{r=i}^h L_{1i,lmr}(t) s^{r-i} w_m^l,$$

$i=1, \dots, h$, and

$$(3.29) \quad w_i^{j'} = \mu_j w_i^j + \varepsilon_{ji} w_{i+1}^j + \sum_{k=1}^h \sum_{m=0}^{k-1} d_{ji,km} \rho_m(t) s^{k-1-m} w_k^1 + \sum_{i=2}^g \sum_{m=1}^{h(1)} L_{ji,lm}(t) w_m^l,$$

for $j=2, \dots, g$ and $i=1, \dots, h(j)$, where c_{ir} , d_{km} , $d_{ji,km}$ are constants; $L_{1i,lmr}(t)$ and $L_{ji,lm}(t)$ are linear functions of $\rho_0(t), \dots, \rho_{n-1}(t)$ with constant coefficients; and $\varepsilon_{ji} = \varepsilon$ or 0 according as $i < h(j)$ or $i = h(j)$.

Let $1 \leq \kappa \leq h$. Make the last change of variables $w \rightarrow x$:

$$(3.30) \quad x_i^1 = s^{i-\kappa} w_i^1 \quad \text{and} \quad x_i^j = s^{h_*-\kappa} w_i^j, \quad j > 1.$$

Then (3.28)–(3.29) becomes

$$(3.31) \quad x_i^{1'} = (i-\kappa) s^{-1} x_i^1 + \sum \sum \sum c_{ir} d_{km} \rho_m(t) s^{r-m-1} x_k^1 + \sum \sum \sum L_{1i,lmr}(t) s^{r-h_*} x_m^l,$$

$$(3.32) \quad x_i^{j'} = [\mu(j) + (h_* - \kappa) s^{-1}] x_i^j + \varepsilon_{ji} x_{i+1}^j + \sum \sum d_{ji,km} \rho_m(t) s^{h_*-m-1} + \sum \sum L_{ji,lm}(t) x_m^l,$$

where the ranges for the sums are the same as in (3.28), (3.29), respectively.

Choose $C \geq 1$ so large and $\varepsilon > 0$ so small that

$$(3.33) \quad [\operatorname{sgn} \mu(j)] J_{j\varepsilon} x^j \cdot x^j - (h_* - \kappa) s^{-1} |x^j|^2 \geq s^{-1} |x^j|^2 \quad \text{for } j > 1.$$

Since $s = t + C \geq 1$ when $t \geq 0$, it is clear that there exists a constant

$$C_1 = C_1(a_0, \dots, a_{n-1})$$

such that if

$$(3.34) \quad \psi(t) = C_1 \sum_{m=0}^{n-1} |\rho_m(t)| s^{h_*-1},$$

then

$$|\{\dots\}| \leq \psi(t)|x| \quad \text{if } |x|^2 = \sum_{j=1}^g |x^j|^2 = \sum_{j=1}^g \sum_{k=1}^{h(j)} |x_k^j|^2$$

and $\{\dots\}$ denotes either the sum of the two triple sums in (3.31) or the sum of the two double sums in (3.32). Thus, (3.31)–(3.32) imply that

$$\begin{aligned} \sum_{i=1}^{\kappa-1} x_i^1 x_i^{1'} &\leq -s^{-1} \sum_{i=1}^{\kappa-1} |x_i^1|^2 + \psi(t)|x| \left(\sum_{i=1}^{\kappa-1} |x_i^1|^2 \right)^{1/2}, \\ x^j \cdot x^{j'} &\leq -s^{-1} |x^j|^2 + \psi(t)|x| \cdot |x^j| \quad \text{if } \mu(j) < 0, \\ |x_k^1 x_k^{1'}| &\leq \psi(t)|x| \cdot |x_k^1|, \\ \sum_{i=\kappa+1}^h x_i^1 x_i^{1'} &\geq s^{-1} \sum_{i=\kappa+1}^h |x_i^1|^2 - \psi(t)|x| \left(\sum_{i=\kappa+1}^h |x_i^1|^2 \right)^{1/2}, \\ x^j \cdot x^{j'} &\geq s^{-1} |x^j|^2 - \psi(t)|x| \cdot |x^j| \quad \text{if } \mu(j) > 0. \end{aligned}$$

We can now apply Theorem (*) of the Appendix below (with y^κ corresponding to the 1-dimensional x_κ^1 with $\alpha^\kappa = \alpha_\kappa = 0$; the vectors $y^1, \dots, y^{\kappa-1}$ correspond to $y^1 = (x_1^1, \dots, x_{\kappa-1}^1)$ and the x^j for which $\mu(j) < 0$ with the respective $\alpha_1 = \alpha_2 = \dots = \alpha_{\kappa-1} = -s^{-1}$; the vectors $y^{\kappa+1}, \dots, y^n$ correspond to $y^{\kappa+1} = (x_{\kappa+1}^1, \dots, x_h^1)$ and the x^j for which $\mu(j) > 0$ with the respective $\alpha_{\kappa+1} = \alpha_{\kappa+2} = \dots = s^{-1}$).

Thus, we have to examine the functions

$$\sigma_1 = \int_0^t \psi(\rho)(\rho + C)/(t + C) d\rho, \quad \sigma_{\kappa+1} = \int_t^T \psi(\rho)(t + C)/(\rho + C) d\rho.$$

According to Theorem (*), if σ_1 and $\sigma_{\kappa+1}$ exist and have a sufficiently small bound η , then (3.31)–(3.32) has a solution $x(t) \neq 0$ such that

$$|x_i^1| \leq \gamma\eta |x_\kappa^1| \quad \text{for } i \neq \kappa, \quad |x^j| \leq \gamma\eta |x_\kappa^1| \quad \text{for } j \neq 1,$$

$0 \leq t < T$, where γ denotes a constant independent of η , but not always the same. In view of (3.30), this means that (3.28)–(3.29) has a solution $w(t) \neq 0$ such that

$$|w_i^1(t)| \leq \gamma\eta s^{\kappa-i} |w_\kappa^1(t)| \quad \text{and} \quad |w^j(t)| \leq \gamma\eta s^{\kappa-h_*} |w_\kappa^1(t)|$$

for $i \neq \kappa, j \neq 1$, respectively. By (3.26), (3.24) has a solution $z(t) \neq 0$ satisfying

$$\begin{aligned} |z_i^1(t) - w_\kappa^1(t)s^{\kappa-i}/(\kappa-i)!| &\leq \gamma\eta|w_\kappa^1(t)|s^{\kappa-i} & \text{for } i = 1, \dots, \kappa, \\ |z_i^1(t)| &\leq \gamma\eta|w_\kappa^1(t)|s^{\kappa-i} & \text{for } i = \kappa+1, \dots, h, \\ |z^j(t)| &\leq \gamma\eta|w_\kappa^1(t)|s^{\kappa-h} & \text{for } j = 2, \dots, g. \end{aligned}$$

Finally, by (3.20) and (3.22), (3.19) has a solution $y=(v, v', \dots, v^{(n-1)})$ satisfying (3.8) if $j=1$ and $v=v_{j\kappa}$. This completes the proof of Theorem 3.1.

4. Nonlinear interpolation problems. The results and methods of Lasota and Opial [8], combined with those above, make it possible to treat nonlinear boundary value or interpolation problems associated with certain nonlinear differential equations

$$(4.1) \quad u^{(n)} + a_{n-1}u^{(n-1)} + \dots + a_0u = f(t, u, u', \dots, u^{(n-1)}),$$

$$(4.2) \quad u^{i-1}(t_k) = c_{ik} \quad \text{for } i = 1, \dots, m(k) \text{ and } k = 1, \dots, \nu,$$

$$(4.3) \quad (0=t_1 < \dots < t_\nu (=T) \quad \text{and} \quad m(1) + \dots + m(\nu) = n.$$

ASSUMPTION (B₁). Let a_0, \dots, a_{n-1} be real numbers satisfying Assumption (A₁) of Theorem 1.1, or the assumption $a_0 = \dots = a_{n-1} = 0$ of Theorem 2.1, or Assumption (A₂) of Theorem 3.1. Correspondingly, let $0 < T < \infty$ and let $q_0(t), \dots, q_{n-1}(t)$ be nonnegative, continuous functions on $0 \leq t \leq T$ such that $q(t)$ defined in (1.4) satisfies (1.5), or $q(t)$ defined in (2.2) satisfies (2.3), or $q(t)$ defined by (3.4) satisfies (3.5).

ASSUMPTION (B₂). Let $f(t, y_0, \dots, y_{n-1})$ be a continuous function for $0 \leq t \leq T$ and arbitrary (y_0, \dots, y_{n-1}) satisfying

$$(4.4) \quad |f(t, y_0, \dots, y_{n-1})| \leq \sum_{k=0}^{n-1} q_k(t)|y_k|.$$

THEOREM 4.1. Let a_0, \dots, a_{n-1} and $q_0(t), \dots, q_{n-1}(t)$ satisfy Assumption (B₁) and $f(t, y_0, \dots, y_{n-1})$ satisfy Assumption (B₂). Then the interpolation problem (4.1)–(4.3), in which $c_{11}, \dots, c_{m(\nu)\nu}$ are arbitrary constants, has at least one solution. If, in addition, $f(t, y_0, \dots, y_{n-1})$ satisfies

$$(4.5) \quad |f(t, y_0, \dots, y_{n-1}) - f(t, z_0, \dots, z_{n-1})| \leq \sum_{k=0}^{n-1} q_k(t)|y_k - z_k|,$$

then the solution of (4.1)–(4.3) is unique.

Proof. If $\nu = n$ and $m(1) = \dots = m(n) = 1$, then this theorem follows from [8]. In fact, since T is finite and the main conditions in (1.5), (2.3), (3.5) involve a strict inequality, it is clear that these conditions hold whenever $q_k(t)$ is replaced by a continuous function $Q_k(t)$ satisfying $|Q_k(t)| < q_k(t) + (\eta_* - \eta)/nT$, where $\eta_* = \eta_0, \eta_1, \eta_2$, respectively. The proof for the general case (4.1)–(4.3) is similar.

APPENDIX

Let $y^k = (y_1^k, \dots, y_{m(k)}^k)$ be a real Euclidean vector of dimension $m(k)$ and $y = (y^1, \dots, y^N) = (y_1^1, \dots, y_{m(N)}^N)$ the corresponding Euclidean vector of dimension $M = m(1) + \dots + m(N)$. Consider a linear differential equation

$$(1) \quad y' = A(t)y,$$

where $A(t)$ is an $M \times M$ matrix with real continuous entries for $0 \leq t < T$ ($\leq \infty$). Let κ be fixed, $1 \leq \kappa \leq N$. Assume that there exists continuous functions $\psi_1(t) \geq 0, \dots, \psi_\kappa(t) \geq 0$ and $\alpha_1(t), \dots, \alpha_\kappa(t)$ and $\alpha^\kappa(t), \dots, \alpha^N(t)$ such that (1) implies

$$(2) \quad y^k \cdot y^{k'} \leq \alpha_k(t)|y^k|^2 + \psi_k(t)|y^k| \cdot |y| \quad \text{for } k = 1, \dots, \kappa,$$

$$(3) \quad y^k \cdot y^{k'} \geq \alpha^k(t)|y^k|^2 - \psi_k(t)|y^k| \cdot |y| \quad \text{for } k = \kappa, \dots, N.$$

Assume that the integrals

$$(4) \quad \sigma_k(t) = \int_0^t [\psi_k(s) + \psi_\kappa(s)] \left\{ \exp \int_s^t [\alpha_k(\rho) - \alpha^\kappa(\rho)] d\rho \right\} ds \quad \text{for } k = 1, \dots, \kappa - 1,$$

$$(5) \quad \sigma_k(t) = \int_t^T [\psi_k(s) + \psi_\kappa(s)] \left\{ \exp - \int_t^s (\alpha^k(\rho) - \alpha_\kappa(\rho)) d\rho \right\} ds$$

for $k = \kappa + 1, \dots, N$,

exist and are bounded for $0 \leq t < T$. Suppose finally that there exist positive constants γ, η with the properties that

$$(6) \quad y \sigma_k(t) \leq \eta \quad \text{for } 0 \leq t < T, k \neq \kappa,$$

$$(7) \quad [1 + (N-1)\eta^2]^{1/2} \max(1, \eta) < \gamma.$$

THEOREM (*). *Under the conditions enumerated above, there exists an $m(1) + \dots + m(\kappa)$ parameter family of solutions of (1) such that, for $0 \leq t < T$, we have $y^\kappa(t) \neq 0$ and*

$$(8) \quad |y^k(t)| \leq \gamma \sigma_k(t) |y^\kappa(t)| \quad \text{for } k \neq \kappa.$$

REMARK 1. The proof will show that one obtains a solution of (1) satisfying (8) for which one can assign an arbitrary partial set of initial conditions $y^k(0)$, $k = 1, \dots, \kappa$ such that $y^\kappa(0) \neq 0$ and $|y^k(0)|/|y^\kappa(0)|$ is small for $k = 1, \dots, \kappa - 1$.

REMARK 2. Theorem (*) has an analogue if the components of y and the entries of $A(t)$ are complex-valued. It is only necessary to replace $y^k \cdot y^{k'}$ by $\operatorname{Re} \bar{y}^k \cdot y^{k'}$ in (2), (3).

REMARK 3. Note that there is no assumption about the signs of the functions $\alpha_k, \alpha^\kappa, \alpha^k - \alpha_\kappa, \alpha^k - \alpha_\kappa$.

For particular cases of this theorem, see [2, pp. 284–290; especially, Lemmas 4.1, 4.2 and Exercise 4.3].

Proof. Theorem (*) will be deduced from a result of Ważewski [15]. We shall use the formulation and notation of [2, Theorem 3.1, p. 282]. In order to be able to use

this theorem, we shall suppose that $\psi_k(t) > 0$ for $0 < t < T$ and prove the existence of solutions satisfying $y^*(t) \neq 0$ and (8) for $t_0 \leq t < T$, where t_0 is an arbitrary point of $(0, T)$. The passage to the case $\psi(t) \geq 0$ and $t_0 = 0$ will be clear.

Let Ω be the open (t, y) -set $\Omega = \{(t, y) : 0 < t < T, y \neq 0 \text{ arbitrary}\}$. Define an open subset by

$$(9) \quad \Omega^0 = \{(t, y) \in \Omega : u_j(t, y) < 0, v_k(t, y) < 0\}$$

for $1 \leq k < \kappa < j \leq N$, where

$$\begin{aligned} u_b(t, y) &= |y^b|^2 - \gamma^2 \sigma_b^2(t) |y^*|^2 \quad \text{for } b = \kappa + 1, \dots, N, \\ v_a(t, y) &= |y^a|^2 - \gamma^2 \sigma_a^2(t) |y^*|^2 \quad \text{for } a = 1, \dots, \kappa - 1. \end{aligned}$$

Correspondingly, define the subsets of Ω

$$\begin{aligned} U_b &= \{u_b(t, y) = 0, u_j(t, y) \leq 0, v_k(t, y) \leq 0 \quad \text{for } 1 \leq k < \kappa < j \leq N\}, \\ V_a &= \{v_a(t, y) = 0, u_j(t, y) \leq 0, v_k(t, y) \leq 0 \quad \text{for } 1 \leq k < \kappa < j \leq N\}, \end{aligned}$$

where $a = 1, \dots, \kappa - 1$ and $b = \kappa + 1, \dots, N$.

If $f(t, y)$ is a function of class C^1 , let $\dot{f}(t, y)$ be the trajectory derivative of f relative to (1); i.e.,

$$\dot{f}(t, y) = \partial f / \partial t + (\text{grad}_y f) \cdot A(t)y.$$

Then we can verify

$$(10) \quad \dot{v}_a(t, y) > 0 \quad \text{for } (t, y) \in V_a, a = 1, \dots, \kappa - 1,$$

$$(11) \quad \dot{u}_b(t, y) < 0 \quad \text{for } (t, y) \in U_b, b = \kappa + 1, \dots, N.$$

For example, in order to obtain (11), note that

$$\dot{u}_b = 2(y^b \cdot y^{b'} - \gamma^2 \sigma_b^2 y^* \cdot y^{*'} - \gamma^2 \sigma_b \sigma_b' |y^*|^2).$$

Using (2) and (3), it is seen that

$$\frac{1}{2} \dot{u}_b \geq \alpha^b |y^b|^2 - \psi_b(t) |y^b| \cdot |y| - \gamma^2 \sigma_b^2 (\alpha_\kappa |y^*|^2 + \psi_\kappa(t) |y^*| \cdot |y|) - \gamma^2 \sigma_b \sigma_b' |y^*|^2.$$

For $(t, y) \in U_b$, we have $|y^b| = \gamma \sigma_b |y^*|$ and

$$|y| \leq |y^*| \left(1 + \sum_{j \neq \kappa} \gamma^2 \sigma_j^2 \right)^{1/2} \leq c |y^*|,$$

where, by (6), $c = [1 + (N-1)\eta^2]^{1/2} > 0$. Thus we have

$$\frac{1}{2} \dot{u}_b \geq \gamma^2 \sigma_b |y^*|^2 \{ (\alpha^b - \alpha_\kappa) \sigma_b - \sigma_b' - (c/\gamma) [\psi_b + \psi_\kappa \gamma \sigma_b] \}.$$

Note that, by (5),

$$\sigma_b' = -(\psi_b + \psi_\kappa) + (\alpha^b - \alpha_\kappa) \sigma_b;$$

hence, for $(t, y) \in U_b$,

$$\frac{1}{2} \dot{u}_b \geq \gamma \sigma_b |y^*|^2 \{ (\gamma - c) \psi_b + \psi_\kappa (\gamma - c \gamma \sigma_b) \}.$$

Thus $\frac{1}{2} \dot{u}_b > 0$ if $\gamma > c$ and $\gamma > c \gamma \sigma_b$, while $\gamma > c \gamma \sigma_b$ if $\gamma > c \eta$. (Note that $y^* \neq 0$, for

otherwise $(t, y) \in U_b \subset \Omega$ implies that $y=0$, which cannot hold for $(t, y) \in \Omega$. Similarly (10) is verified.

Thus, in the terminology of [2, pp. 281–282], Ω^0 is a (u, v) -subset of Ω and, by a theorem of Ważewski [13] (cf. [2, Lemma 3.1, p. 281]), the set of egress points Ω_e^0 of Ω^0 consists only of strict egress points and

$$(12) \quad \Omega_e^0 = \bigcup_{b=\kappa+1}^N U_b - \bigcup_{a=1}^{\kappa-1} V_a.$$

If $0 < t_0 < T$, let $y^1(t_0), \dots, y^\kappa(t_0)$ be arbitrary vectors satisfying $y^\kappa(t_0) \neq 0$, $|y^a(t_0)| < \gamma \sigma_a(t_0) |y^\kappa(t_0)|$ for $a=1, \dots, \kappa-1$. Let S be the set of points $(t, y) = (t_0, y^1(t_0), \dots, y^\kappa(t_0), y^{\kappa+1}, \dots, y^N) \in \Omega$ satisfying $|y^b| \leq \gamma \sigma_b(t_0) |y^\kappa(t_0)|$ for $b=\kappa+1, \dots, N$. Topologically, S is a ball of dimension $m(\kappa+1) + \dots + m(N)$ and $S \cap Q_e^0$ is its boundary. Thus $S \cap \Omega_e^0$ is not a retract of S . But $S \cap \Omega_e^0$ is a retract of Ω_e^0 . In fact, a retraction is given by the map $\pi: \Omega_e^0 \rightarrow S \cap \Omega_e^0$ defined by $\pi(t, y) = (t_0, z)$, where $z^a = y^a(t_0)$ for $a=1, \dots, \kappa$ and $z^b = y^b \sigma_b(t_0) |y^\kappa(t_0)| / \sigma_a(t) |y^\kappa|$ for $b=\kappa+1, \dots, N$. The map π is continuous (since $y \neq 0$, hence $y^\kappa \neq 0$, on Ω_e^0 and $\sigma_b(t) > 0$), $\pi(\Omega_e^0) \subset S \cap \Omega_e^0$ and $\pi|_{S \cap \Omega_e^0}$ is the identity.

By a theorem of Ważewski [13] (cf. [2, Theorem 3.1, p. 282]), it follows that there exists a point $(t_0, y(t_0)) \in S$ such that the solution of (1) determined by this initial condition is in Ω^0 for $t_0 \leq t < T$, i.e., satisfies (8). This proves Theorem (*).

REMARK. It is clear from the proof of this theorem that the conclusion is valid if the functions (4)–(5) and conditions (6)–(7) are replaced by a set of functions $\sigma_a(t)$ and $\sigma_b(t)$, $1 \leq a < \kappa < b \leq N$, which are nonnegative, continuously differentiable for $0 \leq t < T$, and satisfy the following system of differential inequalities

$$\sigma'_b = [\alpha^b(t) - \alpha_\kappa(t)]\sigma_b - \chi_b(t), \quad \sigma'_a = [\alpha_a(t) - \alpha^\kappa(t)]\sigma_a + \chi_a(t),$$

where χ_a, χ_b are continuous functions satisfying

$$\gamma \chi_k > \left(1 + \gamma^2 \sum_{j \neq \kappa} \sigma_j^2\right)^{1/2} [\psi_k + \psi_\kappa \gamma \sigma_\kappa], \quad k \neq \kappa.$$

REFERENCES

1. P. Hartman, *Unrestricted n-parameter families*, Rend. Circ. Mat. Palermo (2) 7 (1958), 123–142.
2. ———, *Ordinary differential equations*, Wiley, New York, 1964.
3. P. Hartman and A. Wintner, *A criterion for the non-degeneracy of the wave equation*, Amer. J. Math. 71 (1949), 206–213.
4. ———, *On an oscillation criterion of Liapounoff*, Amer. J. Math. 73 (1951), 885–890.
5. ———, *Asymptotic integrations of linear differential equations*, Amer. J. Math. 77 (1955), 45–87.
6. D. B. Hinton, *Disconjugate properties of a system of differential equations*, J. Differential Equations 2 (1966), 420–437.
7. M. Hukuhara, *On the zeros of solutions of linear ordinary differential equations*, Sûgaku 15 (1963), 108–109 (Japanese). Cf. Math. Reviews 29 (1965), 709, #3704, and reference there to M. Nagumo, T. Sato and M. Tumura.

8. A. Lasota and Z. Opial, *L'existence et l'unicité des solutions du problèmes d'interpolation pour l'équation différentielle ordinaire d'ordre n* , Ann. Polon. Math. **15** (1964), 253–271.
9. A. Ju. Levin, *On some boundary value problems*, Naučn. Dokl. Vysš. Školy Fiz.-Mat. Nauki **1958**, No. 5, 34–37. (Russian)
10. ———, *Some estimates of a differentiable function*, Dokl. Akad. Nauk SSSR **138** (1961), 37–38 = Soviet Math. Dokl. **2** (1961), 523–524.
11. Z. Nehari, *On an inequality of Lyapunov*, pp. 256–261, Studies in Mathematical Analysis and Related Topics, Stanford Univ. Press, Stanford, Calif., 1962.
12. ———, *Disconjugacy criteria for linear differential equations*, Carnegie Inst. Tech. Rep., Pittsburgh, Pa., 1967.
13. Z. Opial, *On a theorem of O. Aramă*, J. Differential Equations **3** (1967), 88–91.
14. G. Pólya, *On the mean value theorem corresponding to a given linear homogeneous differential equation*, Trans. Amer. Math. Soc. **24** (1922), 312–324.
15. T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Polon. Math. **20** (1947), 279–313.
16. A. Wintner, *On the non-existence of conjugate points*, Amer. J. Math. **73** (1951), 368–380.

THE JOHNS HOPKINS UNIVERSITY,
BALTIMORE, MARYLAND