A CHARACTERIZATION OF THE RIESZ SPACE OF MEASURABLE FUNCTIONS

BY

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1. **Introduction.** In [2], Kakutani has shown that any abstract (L)-space (Banach lattice such that $\|u+v\|=\|u\|+\|v\|$ for $u\geqq 0$ and $v\geqq 0$) can be represented as a space $L_1(X, \mathscr{S}, \mu)$ of equivalence classes of μ-summable functions, where μ is a completely additive measure on a σ-algebra of subsets of some set X. This result can also be viewed as a characterization of the class of spaces $L_1(X, \mathscr{S}, \mu)$ by properties of the norm and order.

The purpose of this note is to show that an analogous classification exists for the class of spaces $\mathscr{M}(X, \mathscr{S}, \mu)$ (the set of equivalence classes of almost-everywhere finite valued μ-measurable functions on the measure space $(X, \mathscr{S}, \mu)$) as long as μ is restricted to be a σ-finite measure. Since such spaces can not be normed, the characterization will involve only order properties. The above-mentioned classical theorem of Kakutani will be of significant value in obtaining our result.

The principal condition needed is a reflexivity property for Riesz spaces (vector lattices) which we will study in §2. Lastly, we examine the non σ-finite case.

2. **Riesz space preliminaries.** Let L be an Archimedean Riesz space throughout this paper. A linear functional φ defined on L is said to be a *normal integral* if $0\leqq u_\tau \downarrow 0$ in L implies that $\inf|\varphi(u_\tau)|=0$. The space $L_n^\sim$ of normal integrals is a closed ideal in the order-bound dual of L. (Further explanation of the definitions and notations used may be found in [3].)

In [3], the concept of the normal integral is generalized by considering functionals that are normal integrals for some dense ideal $D\subset L$. This gives rise to an extended dual space which we denote by $\Gamma(L)$. The space $\Gamma(L)$ is a Dedekind complete and universally complete Riesz space, containing $L_n^\sim$ as an ideal [3, Theorems 2.5 and 2.6].

If $\Gamma(L)$ is separating on L, then L is an order-dense Riesz subspace of $\Gamma^2(L)$. If $\Gamma^2(L)=L$, we say that L is *perfect in the extended sense*. This is the reflexivity condition referred to in the introduction. The following result [3, Theorem 2.7] is a characterization which will be used:

(A) *A Riesz space L, for which $\Gamma(L)$ is separating, is perfect in the extended sense if and only if* $0\leqq u_\tau \in L$, $u_\tau \uparrow$ *and*

$$\sup \varphi(u_\tau) < \infty$$

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for every $0 \leqq \varphi \in \Gamma(L)$ *which belongs to some order-dense ideal in* $\Gamma(L)$, *implies that* $\sup(u_\tau) = u$ *exists.*

Notice that the condition $\Gamma^2(L) = L$ implies that L is Dedekind complete but that in the general non σ-finite case, $\mathcal{M}(X, \mathcal{S}, \mu)$ need not be Dedekind complete. This gives rise to the restriction on the measure μ.

We recall some definitions and facts about order completions of a Riesz space. L is said to be *universally complete* if $0 \leqq u_\tau \in L$ and $\inf(u_{\tau_1}, u_{\tau_2}) = 0$ for $\tau_1 \neq \tau_2$ imply that $u = \sup(u_\tau)$ exists in L. Nakano has shown [5] that every Archimedean Riesz space can be embedded as a dense Riesz subspace, in a Dedekind complete and universally complete Riesz space L' in such a way that suprema taken in L are preserved when taken in L'. The latter space is unique (up to isomorphism) and is called the *universal completion* of L. Moreover, if L is Dedekind complete, then it is an ideal in L'. From the uniqueness it follows that if L is universally complete and Dedekind complete and D is a dense ideal in L, then L is the universal completion of D. Hence, if $\Gamma(L)$ is separating on L, implying L is dense in $\Gamma^2(L)$, then L is Dedekind complete and universally complete if and only if L is perfect in the extended sense.

3. **The characterization.** Given any element ψ in $\Gamma(L)$, there is a unique largest order-dense ideal to which $|\psi|$ can be extended finitely. We denote it by D_ψ. $\psi | D_\psi$ is a normal integral. Moreover, a positive element u in L is in D_ψ if and only if

$$\sup\{\psi(v) : 0 \leqq v \leqq u, v \in I_\psi\} < \infty,$$

where I_ψ is any dense ideal on which ψ is defined (that is, finitely defined) [3]. Moreover, Theorem 2.5 of the paper just cited says that if $\Gamma(L)$ is separating on L, there exists in $\Gamma(L)$ a strictly positive linear functional φ, that is, one such that $\varphi(u) > 0$ for all $u \in D_\varphi$, $u > 0$. We now state the central result.

THEOREM. *Let L be a Riesz space which is Dedekind complete, universally complete and such that $\Gamma(L)$ is separating on L. Then L contains a dense ideal which is an abstract (L)-space.*

Proof. Let φ be a strictly positive element in $\Gamma(L)$. Let D be the (unique) largest order-dense ideal to which $|\varphi|$ can be extended finitely. For any u in L define $\|u\| = \varphi(|u|)$. Then $\|u\| \geqq 0$; equals zero if and only if $u = 0$ since φ is strictly positive. $\|au\| = |a| \, \|u\|$ for a scalar a and $\|u+v\| \leqq \|u\| + \|v\|$ follow from the linearity of φ. That $|u| \leqq |v|$ implies $\|u\| \leqq \|v\|$ results from the positivity of φ. So, with $\|\cdot\|$ as norm, D is a normed vector lattice. If $u \geqq 0$ and $v \geqq 0$, $\|u+v\| = \varphi(|u+v|) = \varphi(u+v)$ $= \varphi(u) + \varphi(v) = \varphi(|u|) + \varphi(|v|) = \|u\| + \|v\|$. To show that D is an abstract (L)-space then, it remains only to prove that D is a complete space under $\|\cdot\|$.

Let $\langle u_n \rangle$ be an absolutely convergent series in D. So, there is M such that $\sum_{n=1}^\infty \|u_n\| \leqq M < \infty$. But then $\|u_n^+\| \leqq \|u_n\|$ and $\|u_n^-\| \leqq \|u_n\|$ imply that $\langle u_n^+ \rangle$ and $\langle u_n^- \rangle$ are absolutely convergent series in D with the same absolute bound M.

Let $v_n = \sum_{k=1}^n u_k^+$. Then, $\langle v_n \rangle$ is an increasing sequence in D and $\|v_n\| \leqq M$, $n=1, 2, \ldots$. So, $\varphi(v_n) \leqq M$, $n=1, 2, \ldots$ and hence $\sup_n \varphi(v_n) < \infty$. For any ψ in I, then, where I is the principal ideal generated by φ, we have

$$\sup_n \psi(v_n) < \infty.$$

However, since φ is strictly positive, I is an order-dense ideal in $\Gamma(L)$. Hence, the conditions in the conclusion of (A) are satisfied for the sequence $\{v_n\}$. Since L is perfect in the extended sense (being Dedekind complete and universally complete) then, $v = \sup (v_n)$ must exist in L. Now, $\sup \varphi(v_n) \leqq M$ implies that

$$\sup \{\varphi(w) : 0 \leqq w \leqq v, w \in D\} < \infty.$$

(For if $0 \leqq w \leqq v$, $\sup (\inf (w, v_n)) = w$, hence $\varphi(w) \leqq M$.) By remarks made above, then, v is in D, and since φ is a normal integral on D, $\lim \varphi(v_n) = \varphi(v)$. So, $\lim \|v - v_n\| = 0$, hence $\lim \|v - \sum_{k=1}^n u_k^+\| = 0$. In the same way, we obtain v' in D such that $\lim \|v' - \sum_{k=1}^n u_k^-\| = 0$. But then, $\lim \|(v - v') - \sum_{k=1}^n u_k\| = 0$. We have shown that every absolutely summable series in D is summable. Hence, D is complete, finishing the proof.

Using Kakutani's representation theorem, the ideal D is isomorphic to a space $L_1 = L_1(X, \mathcal{S}, \mu)$ (μ may not be σ-finite). Let $\mathcal{M} = \mathcal{M}(X, \mathcal{S}, \mu)$ be the associated space of equivalence classes of measurable functions. Now D (or L_1) may be regarded as a dense ideal in both $\mathcal{M}$ and L. Hence, using [3, Theorems 2.6 and 2.3] (along with the hypotheses of the theorem) we have $\mathcal{M} \subset \Gamma^2(\mathcal{M}) = \Gamma^2(L) = L$. Moreover, L is the universal completion of $\mathcal{M}$. So,

COROLLARY 1. *Let L be a Riesz space which is Dedekind complete, universally complete and such that $\Gamma(L)$ is separating on L. Then, L contains a dense Riesz subspace which is isomorphic to $\mathcal{M}(X, \mathcal{S}, \mu)$.*

The Riesz space L is said to be *super Dedekind complete* if L is Dedekind complete and if any subset $A \subset L$ which is bounded above contains an, at most, countable subset having the same least upper bound as the whole set A.

This property was introduced by Luxemburg and Zaanen in [4] and studied in their later papers under the same title.

It can now be shown that if this condition is added to the hypotheses of the theorem, the measure μ obtained in the corollary above must be σ-finite.

Let $\{X_\alpha\}$ be the class of all sets of finite μ-measure. We must have then, that $\chi_X = \sup (\chi_{X_\alpha})$ in L (χ_E being the class containing the characteristic function of E). There must then be an at most countable subset $\{\chi_{X_n}\}$ such that $\chi_X = \sup (\chi_{X_n})$. But then $X = \bigcup_{n=1}^\infty X_n$ (except possibly for a set of μ measure zero). Hence, μ is a σ-finite measure on S.

With μ σ-finite, we can show that $\mathcal{M} = L$. First of all, if μ is σ-finite, then $\mathcal{M}$ is Dedekind complete. [1, IV, p. 335.] The following two facts are also readily obtained

(a) $\mathcal{M}$ contains a complete element i.e., there is $0 \leqq e \in \mathcal{M}$ such that $\inf(e, x) = 0$ implies $x = 0$.

(b) $\mathcal{M}$ contains the supremum of any countable disjoint set of its elements.

Nakano has shown [5] that for a Dedekind complete Riesz space, (a) and (b) are necessary and sufficient for universal completeness. So, $\mathcal{M}$ is universally complete and hence, $\mathcal{M} = L$.

We have obtained one half of the characterization described in the introduction. The other half is a rather routine verification. We state the result:

COROLLARY 2. *Let L be a Riesz space. In order that there exist a completely additive σ-finite measure space $(X, \mathcal{S}, \mu)$ such that $L = \mathcal{M}(X, \mathcal{S}, \mu)$, the following conditions are necessary and sufficient:*

(1) *$\Gamma(L)$ is separating on L,*

(2) *L is universally complete,*

(3) *L is super Dedekind complete.*

4. **The non σ-finite case.** The last corollary cannot be modified to include any measure μ, since even the condition that $\Gamma(L)$ be separating restricts the measure μ to have the *finite subset property*, that is, any set of μ-positive measure has a subset of μ-finite measure (a fact we used in proving the last corollary).

As can be seen from the first two corollaries, a characterization in the non σ-finite case will obtain if a condition on μ can be found which guarantees that $\mathcal{M}(X, \mathcal{S}, \mu)$ is Dedekind complete. Such a condition appears in a paper of A. C. Zaanen [6] concerning an extension of the Radon-Nikodym Theorem. Namely, the measure μ is said to be *localizable* if the lattice of equivalence classes of μ-measurable sets is complete. (The term is originally due to I. E. Segal.) We discuss a characterization of this property.

Given a set E of finite μ-measure, let M_E^* be the collection of equivalence classes f^* of μ-measurable functions f vanishing almost everywhere off E. If for each such E we select f_E^* in M_E^* in such a way that for E and F of finite measure we have $(f_{E\cap F})^* = (f_E\chi_{E\cap F})^* = (f_F\chi_{E\cap F})^*$, then the collection $\{f_E^*\}$ is called a *cross-section* of X. Theorem 9.4 in [6] states that the measure μ is *localizable* if and only if for every cross-section $\{f_E^*\}$, there is a measurable f defined on all of X such that $(f\chi_E)^* = f_E^*$ for all E of finite measure.

The value of localizability is that it characterizes those measure spaces for which there is a Radon-Nikodym Theorem.

Assuming now that μ is localizable, let $\{f_\alpha^*\}$ be a collection of nonnegative elements in $\mathcal{M} = \mathcal{M}(X, \mathcal{S}, \mu)$, bounded above by g. Then, $(f_\alpha\chi_E)^*$ is in M_E^* for each set E of finite measure. But, M_E^* is Dedekind complete since the restriction of μ to E is a finite measure. Moreover, $(f_\alpha\chi_E)^* \leqq (g\chi_E)^*$. Hence, $\sup(f_\alpha\chi_E^*) = f_E^*$ (supremum taken in $\mathcal{M}$ since M_E^* is an ideal in $\mathcal{M}$) exists and is an element of M_E^*. Now it is easily verified that the collection $\{f_E^*\}$ is a cross-section, and hence there exists f^* in $\mathcal{M}$ such that $(f\chi_E)^* = f_E^*$ for each set E of finite measure. This last statement is true, however, only if $f^* = \sup(f_\alpha^*)$. So, $\mathscr{M}$ is complete and we have the following:

COROLLARY 3. *Let L be a Riesz space. The following conditions are necessary and sufficient that there exist a completely additive localizable measure μ such that $L = \mathscr{M}(X, \mathscr{S}, \mu)$:*

(1) *$\Gamma(L)$ is separating,*

(2) *L is Dedekind complete and universally complete.*

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