COMPACTIFICATION OF STRONGLY COUNTABLE-DIMENSIONAL SPACES(1)

BY
ARLO W. SCHURLE

1. **Introduction.** In dimension theory it is well known [2, p. 65] that a finite-dimensional separable metrizable space may be topologically embedded in a compact metrizable space of the same dimension. A natural question is whether an analogous statement holds for the various types of infinite-dimensional spaces defined in [6, pp. 161–162]. Our main result is that strongly countable-dimensional absolute G_δ-spaces have strongly countable-dimensional compactifications. (A space is strongly countable-dimensional if it is a countable union of closed finite-dimensional subsets.) We restrict ourselves exclusively to separable metrizable spaces, assuming a fixed compatible metric to be given whenever necessary. Notation and definitions will be as in [3] and [6].

E. G. Sklyarenko has shown [7, p. 40] that weakly infinite-dimensional spaces have weakly infinite-dimensional compactifications. He also gives an example [7, p. 42] of a strongly countable-dimensional space which has neither a countable-dimensional nor a strongly countable-dimensional compactification. Hence, an added condition is necessary to insure that a countable-dimensional or strongly countable-dimensional space has the desired compactification. Lelek has shown [5] that if X is countable-dimensional and an absolute G_δ, then X has a countable-dimensional compactification. In the same paper he raises the question which Theorem 2 answers affirmatively, namely, whether his result is true with countable-dimensional replaced by strongly countable-dimensional (weakly countable-dimensional in [5]).

2. **Some preliminary propositions.** Throughout this paper letters such as i, j, k will denote positive integers. Also, unless stated otherwise, notation such as $\{A_i\}$ and $\bigcup A_i$ will always denote $\{A_i \mid i=1, 2, \ldots\}$ and $\bigcup_{i=1}^{\infty} A_i$. If there are double subscripts, then the variable one will be indicated. The following proposition may be proved easily by using the results and techniques of [2, pp. 53–55], and so the proof is omitted.

PROPOSITION 1. *Let $M \subset X$ with* dim $M \leqq n$. *Let $\{U_i\}$ be a collection of sets open in X and covering M. Then there is a collection $\{V_i\}$ of sets open in X and covering M*

Received by the editors October 17, 1967.

(1) This paper forms part of the author's doctoral dissertation written at the University of Kansas under the direction of Professor Charles J. Himmelberg while the author was supported by a NASA Pre-Doctoral Training Grant.

such that ord $\{V_i\} \leqq n+1$ *and such that* $V_{k(n+1)+j} \subset U_{k+1}$ *for* $k=0, 1, 2, \ldots$ *and* $j=1, 2, \ldots, n+1$.

Proposition 2. *Let G be an open subset of a totally bounded space Y and let $M_1, M_2, \ldots, M_r$ be relatively closed subsets of G with* $\dim M_i = m_i < \infty$ *for* $i=1, 2, \ldots, r$. *Let* $\varepsilon > 0$. *Then there is a collection* $\{G_i\}$ *such that*

(i) $G = \bigcup G_i$,
(ii) $\overline{G}_i \subset G$ *for* $i=1, 2, \ldots$,
(iii) G_i *is open in* Y *for* $i=1, 2, \ldots$,
(iv) diam $G_i < \varepsilon$ *for* $i=1, 2, \ldots$ *and* diam $G_i \to 0$ *as* $i \to \infty$,
(v) $\{G_i\}$ *is star-finite, and,*
(vi) ord $\{G_i \mid G_i$ *meets* $M_1 \cup M_2 \cup \cdots \cup M_k\} \leqq m_1+1+m_2+1+\cdots+m_k+1$ *for* $k=1, 2, \ldots, r$.

Proof. By a very slight modification of [4, Proposition 5, p. 114] there exists a collection $\mathscr{G}_0 = \{G_{0,i}\}_i$ satisfying (i)–(v). Letting $M_0 = \varnothing$ and $m_0 = -1$, we see that $\mathscr{G}_0$ satisfies (vi) for $k=0$. We now inductively define collections satisfying (i)–(v) and (vi) for $k=1, 2, \ldots, r$ successively. Suppose $\mathscr{G}_n$ is a collection satisfying (i)–(v) and (vi) for $k=0, 1, \ldots, n$. Let $C = M_0 \cup M_1 \cup \cdots \cup M_n$. Then $\{G_{n,i} - C\}_i$ is an open cover of $G - C$, so by Proposition 1 there is an open collection $\{V_i\}$ covering $M_{n+1} - C$ such that ord $\{V_i\} \leqq m_{n+1}+1$ and such that

$$V_{t(m_{n+1}+1)+j} \subset G_{n,t+1} - C \tag{a}$$

for $t=0, 1, 2, \ldots$ and $j=1, 2, \ldots, m_{n+1}+1$.

Let

$$\begin{aligned}\mathscr{G}_{n+1} &= \{G_{n,i} - (C \cup M_{n+1})\}_i \cup \{V_i\} \cup \{G_{n,i} \mid G_{n,i} \text{ meets } C\} \\ &= \{G_{n+1,i}\}_i,\end{aligned}$$

where in the last sequence we rename the sets in the union. Using (a) we easily see that $\mathscr{G}_{n+1}$ satisfies (i)–(v), and it is clear that $\mathscr{G}_{n+1}$ satisfies (vi) for $k=0, 1, \ldots, n$. Let $\mathscr{H} = \mathscr{K} \cup \mathscr{L}$, where

$$\mathscr{K} = \{G_{n,i} \mid G_{n,i} \text{ meets } M_0 \cup M_1 \cup \cdots \cup M_n\},$$

and

$$\mathscr{L} = \{V_i \mid V_i \text{ meets } M_{n+1}\}.$$

Let $s = m_1+1+m_2+1+\cdots+m_{n+1}+1$, and let $\mathscr{P}$ be a set of $s+1$ distinct elements of $\mathscr{H}$. If $\mathscr{P} \cap \mathscr{K}$ contains $(m_0+1+m_1+1+\cdots+m_n+1)+1$ or more elements, then $\bigcap \mathscr{P} = \varnothing$, since ord $\mathscr{K} \leqq m_0+1+m_1+1+\cdots+m_n+1$. If this is not the case, then $P \cap \mathscr{L}$ must contain $(m_{n+1}+1)+1$ or more elements. Then again $\bigcap \mathscr{P} = \varnothing$ since ord $\mathscr{L} \leqq m_{n+1}+1$. Therefore ord $\mathscr{H} \leqq s$. The induction is complete, and the collection $\mathscr{G}_r$ satisfies the requirements of the proposition.

In our terminology the term "simplex" always denotes a closed simplex with the Euclidean topology. I^ω denotes the Hilbert cube, and if $x \in I^\omega$, then the first coordinate of x will be called the abscissa of x. The following proposition is essentially contained in [4, Proposition 2, p. 213] and is stated here for the sake of completeness.

PROPOSITION 3. *Let a sequence of points* $\{g_i\}$ *be given in* I^ω *such that the abscissa of each* g_i *is zero, and let a sequence of positive numbers* $\{\varepsilon_i\}$ *be given. Then there is a sequence of points* $\{p_i\}$ *in* I^ω *satisfying*

(i) $d(p_i, g_i) < \varepsilon_i$ *for* $i = 1, 2, \ldots$.

(ii) *The abscissa of* p_i *is positive for* $i = 1, 2, \ldots$.

(iii) *Each finite subset* $\{p_{i_0}, p_{i_1}, \ldots, p_{i_n}\}$ *of* $\{p_i\}$ *spans an n-simplex.*

(iv) *Each subspace of* I^ω *which is the union of a countable star-finite collection of simplexes spanned by finite subsets of* $\{p_i\}$ *is a polytope with the weak topology.*

3. **The main theorems.** The term "map" will be reserved for continuous functions. The first theorem is analogous to [4, Theorem 1, p. 215], but the conditions concerning dimension are new and will be crucial in the following theorem.

THEOREM 1. *Let* C *be a closed subset of a compact space* Y *and let* $\varepsilon > 0$. *Let* $M_1, M_2, \ldots, M_r$ *be closed subsets of* Y *such that* $\dim M_i = m_i < \infty$ *for* $i = 1, 2, \ldots, r$. *Then there is an* ε*-map* f *from* Y *into* I^ω *such that*

(i) $f(C) \cap f(Y - C) = \varnothing$,

(ii) $f|C$ *is a homeomorphism,*

(iii) $f(Y - C)$ *is a countable polytope* P,

(iv) $P = \bigcup \Sigma_i$, *where* $\{\Sigma_i\}$ *is the collection of simplexes of* P *and* $\operatorname{diam} \Sigma_i \to 0$ *as* $i \to \infty$,

(v) $\dim f(M_i - C) \leqq m_1 + 1 + m_2 + 1 + \cdots + m_i$ *for* $i = 1, 2, \ldots, r$.

Proof. Without loss of generality we may assume that $Y \subset I^\omega$, and then that the abscissa of each point of Y is zero. If $C = Y$, then f may be taken to be the inclusion map of Y in I^ω, so we assume that $C \neq Y$. Let $G = Y - C$ and let $\mathscr{G} = \{G_i\}$ satisfy the conditions of Proposition 2 for G, Y, and $M_1 - C, M_2 - C, \ldots, M_r - C$, with $\operatorname{diam} G_i < \varepsilon/8$ for $i = 1, 2, \ldots$. In the remainder of the proof we consider only those integers i for which $G_i \neq \varnothing$, even though we retain the same notation and continue to write $i = 1, 2, \ldots$. For each $i = 1, 2, \ldots$ choose a $g_i \in G_i$. Now choose points $p_i \in I^\omega$ such that

$$d(p_i, g_i) < \min\{1/i, \varepsilon/8\}$$

and such that the sequence $\{p_i\}$ satisfies the conditions of Proposition 3.

Let N be the union of the collection of simplexes in I^ω defined as follows. Each simplex of the collection is spanned by a finite subset of $\{p_i\}$ and $\{p_{i_0}, p_{i_1}, \ldots, p_{i_n}\}$ spans a simplex in the collection if and only if $G_{i_0} \cap G_{i_1} \cap \cdots \cap G_{i_n} \neq \varnothing$. By Proposition 3, N is a polytope with the weak topology.

Define a function $f' \colon Y \to I^\omega$ by

$$\begin{aligned} f'(x) &= x && \text{if } x \in C, \\ &= \textstyle\sum_{i=1}^{\infty} d(x, Y - G_i) p_i \big/ \sum_{i=1}^{\infty} d(x, Y - G_i) && \text{if } x \in Y - C. \end{aligned}$$

A standard argument shows that f' is continuous, and that $d(y, f'(y)) < \varepsilon/4$ for each $y \in Y$. $f'|C$ is the identity map on C and $f'(Y-C) \subset N$. Each vertex of N, and hence each point of N, has abscissa greater than zero, so $f'(C) \cap f'(Y-C) = \varnothing$.

If N is finite, then by repeating the empty simplex we may write $N = \bigcup \Sigma_i$, where diam $\Sigma_i \to 0$ as $i \to \infty$. Suppose N is infinite, and let $\delta > 0$. Let $N = \bigcup \Sigma_i$. There is an integer j_0 such that $i \geqq j_0$ implies that diam $G_i < \delta/6$ and $1/i < \delta/6$. Since N is star-finite, there is an integer k_0 such that for $k > k_0$ none of the points $p_1, p_2, \ldots, p_{j_0}$ is a vertex of Σ_k. Suppose $k > k_0$ and let Σ_k be spanned by $p_{i_0}, p_{i_1}, \ldots, p_{i_t}$. Then for $s = 0, 1, \ldots, t$

$$d(p_{i_s}, g_{i_s}) < 1/i_s < \delta/6$$

and

$$d(g_{i_s}, g_{i_0}) \leqq \text{diam } G_{i_s} + \text{diam } G_{i_0} < \delta/3.$$

Hence $d(p_{i_s}, g_{i_0}) < \delta/2$, so that diam $\Sigma_k < \delta$. Therefore diam $\Sigma_j \to 0$ as $j \to \infty$.

As a subset of I^ω N is bounded, so by repeated barycentric subdivision N may be triangulated into simplexes of diameter less than $\varepsilon/4$. Since each "old" simplex of N yields only finitely many "new" ones, we may write $N = \bigcup \Sigma_i$ where diam $\Sigma_i \to 0$ as $i \to \infty$ and $\{\Sigma_i\}$ is the collection of all "new" simplexes of N. We use the "new" simplexes in the next construction.

We now define a map F taking $f'(Y) \cap N$ onto a subpolytope P of N. Let $\mathscr{F}_0$ be the set of all simplexes Σ of N such that Σ is a face of no other simplex of N. Since N is star-finite, $N = \bigcup \mathscr{F}_0$. Let $\mathscr{F}_0 = \{\Sigma_i\}$. For each i define a map h_i from $f'(Y) \cap \Sigma_i$ into N as follows. If $\Sigma_i \subset f'(Y)$, then let h_i be the identity injection. If $\Sigma_i \not\subset f'(Y)$, then since $f'(Y)$ is compact, there is a point

$$q_i \in (\text{Int } \Sigma_i) - f'(Y);$$

in this case, let h_i project $f'(Y) \cap \Sigma_i$ into the boundary of Σ_i from q_i. Define a function f_0 from $f'(Y) \cap N$ into N by $f_0 = \bigcup h_i$. f_0 is clearly well defined, and since N has the weak topology, f_0 is a map.

Next, let $\mathscr{M}_1 = \{\Sigma \mid \Sigma \in \mathscr{F}_0$ and $\Sigma \subset f'(Y)$, or Σ is a face of a simplex Σ' in $\mathscr{F}_0$ such that $\Sigma' \not\subset f'(Y)\}$. Let $N_1 = \bigcup \mathscr{M}_1$ and let $\mathscr{F}_1$ be the set of all simplexes Σ of N_1 such that Σ is a face of no other simplex of N_1. We again have that $N_1 = \bigcup \mathscr{F}_1$. Clearly $f_0(f'(Y) \cap N) \subset N_1$. Define a map f_1 from $f_0(f'(Y) \cap N)$ into N_1 in the same way that f_0 was defined, replacing $f'(Y)$ by $f_0(f'(Y) \cap N)$.

Continue in this manner to obtain a decreasing sequence $N_0 = N, N_1, N_2, \ldots$ of subpolytopes of N and maps

$$\begin{aligned} f_0 &: f'(Y) \cap N \to N_1, \\ f_1 &: f_0(f'(Y) \cap N) \to N_2, \\ f_2 &: f_1(f_0(f'(Y) \cap N)) \to N_3, \\ &\cdots\cdots\cdots\cdots\cdots. \end{aligned}$$

These maps clearly have the property that for each $f'(y) \in N$ the sequence

$$\{f_n \circ f_{n-1} \circ \cdots \circ f_0 \circ f'(y) \mid n = 0, 1, 2, \ldots\}$$

is eventually constant. Define a function F from $f'(Y) \cap N$ into N by letting $F(f'(y))$ be the eventually constant value of the above sequence.

Let Σ be a simplex of N. It is clear that there is an integer n such that

$$f_m \circ f_{m-1} \circ \cdots \circ f_0 \circ f'(y) = f_n \circ f_{n-1} \circ \cdots \circ f_0 \circ f'(y)$$

for all $m \geqq n$ and all $f'(y) \in \Sigma$. Hence

$$F \mid (f'(Y) \cap \Sigma) = f_n \circ f_{n-1} \circ \cdots \circ f_0 \mid (f'(Y) \cap \Sigma).$$

Since N has the weak topology, F is a map. It is easily seen that $F(f'(y))$ is in the same simplexes as $f'(y)$ for each $f'(y) \in N$. Further, $F(f'(Y) \cap \Sigma)$ is the union of certain faces of Σ for each simplex Σ of N, so that $F(f'(Y) \cap N)$ is a subpolytope P of N.

Define a function f from Y into I^ω by

$$\begin{aligned} f(z) &= z & &\text{if } z \in C, \\ &= Ff'(z) & &\text{if } z \in Y-C. \end{aligned}$$

Since $f(z)$ and $f'(z)$ are in the same simplexes for $z \in Y-C$, a standard argument shows that f is continuous. Properties (i)–(iv) certainly hold for f. N was triangulated into simplexes of diameter less than $\varepsilon/4$, so that $d(f(y), f'(y)) < \varepsilon/4$ for each $y \in Y$. Therefore

$$\begin{aligned} d(f(y), y) &\leqq d(f(y), f'(y)) + d(f'(y), y) \\ &< \varepsilon/4 + \varepsilon/4 = \varepsilon/2. \end{aligned}$$

It is then clear that f is an ε-map.

We need now only verify (v). Let $y \in M_i - C$, and let $n_i = m_1 + 1 + m_2 + 1 + \cdots + m_i$. By the properties of the open cover $\mathscr{G}$, y is in at most $n_i + 1$ elements of $\mathscr{G}$. Hence $f'(y)$ is in a simplex of dimension not greater than n_i. Since $f'(y)$ and $f(y)$ are in the same simplexes and since subdividing a simplex does not raise its dimension, $f(y)$ is in a simplex of dimension not greater than n_i. Thus $f(M_i - C) \subset \bigcup \{\Sigma_j \mid \Sigma_j \text{ is a simplex of } P \text{ and } \dim \Sigma_j \leqq n_i\}$. By the countable sum theorem [2, p. 30] and the subspace theorem [2, p. 26]

$$\dim f(M_i - C) \leqq n_i$$

for $i = 1, 2, \ldots, r$.

The goal of this paper is reached by the following theorem. In fact, the theorem yields the desired compactification in such a way that the remainder is a pseudopolytope. (A metric space is called a pseudopolytope if it is the union of a sequence $\{\Sigma_i\}$ of simplexes such that for any positive integers i and j $\Sigma_i \cap \Sigma_j$ is either empty or a face of both Σ_i and Σ_j and such that diam $\Sigma_i \to 0$ as $i \to \infty$.)

THEOREM 2. *Let X be strongly countable-dimensional and an absolute G_δ. Then there is a strongly countable-dimensional compactification dX of X such that $dX - X$ is a pseudopolytope.*

Proof. Let $X=\bigcup F_i$, where F_i is closed and $\dim F_i=m_i<\infty$ for $i=1, 2, \ldots$. By a result of Hurewicz [1, p. 207] there is a compactification cX of X such that $\dim \bar{F}_i^{cX}=m_i$ for $i=1, 2, \ldots$. Since X is an absolute G_δ, there are compact subsets $Y_1, Y_2, \ldots$ of cX such that $Y_i\subset Y_{i+1}$ and $cX-X=\bigcup Y_i$. Let $Y_0=\varnothing$, and let

$$n_k = m_1+1+m_2+1+\cdots+m_k$$

for $k=1, 2, \ldots$. Let i be any positive integer. By Theorem 1 there is a $1/i$-map f_i from Y_i into I^ω such that

(i) $f_i(Y_{i-1})\cap f_i(Y_i-Y_{i-1})=\varnothing$,

(ii) $f_i| Y_{i-1}$ is a homeomorphism,

(iii) $f_i(Y_i-Y_{i-1})$ is a countable polytope N_i,

(iv) $N_i=\bigcup_j \Sigma_{ij}$ where $\operatorname{diam}\Sigma_{ij}\to 0$ as $j\to\infty$,

(v) $\dim f_i(\bar{F}_i^{cX}\cap(Y_i-Y_{i-1}))\leqq n_k$ for $k=1, 2, \ldots, i$.

Decompose cX into sets $f_i^{-1}(z)$ for $z\in f_i(Y_i-Y_{i-1})$ and into individual points $x\in X$. Let the quotient space be dX and the quotient map be f.

We show that the decomposition of cX is upper semicontinuous. Let U be an open set in cX containing a point x of X. There is a positive ε such that $x\in S_\varepsilon(x)\subset U$ and there is a positive integer i_0 such that $1/i_0<\varepsilon/2$. Then the set

$$\bigcup\{f^{-1}(z)\mid f^{-1}(z)\subset cX-Y_{i_0} \text{ and } f^{-1}(z) \text{ meets } S_{\varepsilon/2}(x)\}$$

may easily be shown to be an inverse set which is a neighborhood of x contained in U.

Suppose that $f^{-1}(z_0)\subset Y_{i_0}-Y_{i_0-1}$ and that $f^{-1}(z_0)\subset U$, where U is open in cX. Since $f^{-1}(z_0)$ is compact, there is a positive ε such that $S_\varepsilon(f^{-1}(z_0))\subset U$, and then there is an integer $j_0\geqq i_0$ such that $1/j_0<\varepsilon/2$. Let

$$\begin{aligned} A &= \bigcup\{f^{-1}(z)\mid f^{-1}(z)\subset cX-Y_{j_0} \text{ and } f^{-1}(z) \text{ meets } S_{\varepsilon/2}(f^{-1}(z_0))\},\\ B &= \bigcup\{f^{-1}(z)\mid f^{-1}(z)\subset Y_{j_0}\cap S_\varepsilon(f^{-1}(z_0))\},\\ C &= A\cup B.\end{aligned}$$

Then C is clearly an inverse set containing $f^{-1}(z_0)$ and contained in U. If the decomposition of Y_k is upper semicontinuous for each k, then $B=V\cap Y_{j_0}$, where V is open in cX. Then $V\cap S_{\varepsilon/2}(f^{-1}(z_0))\subset C$, so that C is a neighborhood of $f^{-1}(z_0)$.

We show by induction that the decomposition of Y_k is upper semicontinuous for each k. f_i is a closed map on Y_i for $i=1, 2, \ldots$. Since the decompositions of Y_1 induced by f and f_1 are the same, the decomposition of Y_1 is upper semicontinuous. Suppose this is true of Y_{k-1}, and let U be open in Y_k with $f^{-1}(z_0)\subset U$. If $f^{-1}(z_0)\subset Y_k-Y_{k-1}$, then since f and f_k induce the same decompositions of Y_k-Y_{k-1} and $f_k|(Y_k-Y_{k-1})$ is clearly a closed map,

$$\bigcup\{f^{-1}(z)\mid f^{-1}(z)\subset U\cap(Y_k-Y_{k-1})\}$$

is a neighborhood of $f^{-1}(z_0)$. Suppose $f^{-1}(z_0)\subset Y_{k-1}$. Since the decomposition of Y_{k-1} is upper semicontinuous,

$$\bigcup\{f^{-1}(z)\mid f^{-1}(z)\subset U\cap Y_{k-1}\} = W\cap Y_{k-1},$$

where $W\subset U$ and W is open in Y_k. Since f_k is closed on Y_k,

$$A = \bigcup\{f_k^{-1}(z)\mid f_k^{-1}(z)\subset W\}$$

is open in Y_k. But $A\cap Y_{k-1}=W\cap Y_{k-1}$, since $f_k|Y_{k-1}$ is a homeomorphism. Therefore A is an inverse set with respect to f, and clearly

$$f^{-1}(z_0)\subset A\subset W\subset U.$$

Hence the decomposition of Y_k is upper semicontinuous.

Since the decomposition of cX is upper semicontinuous, $f\colon cX\to dX$ is a closed map, and dX is a compact metrizable space. X is an inverse set for f and $f|X$ is one-one, so that $f|X$ is a homeomorphism. Hence dX is a compactification of X. For each i define a function $g_i : f_i(Y_i)\to f(Y_i)$ by $g_i(f_i(y))=f(y)$. It is easy to show that $g_i|f_i(Y_i-Y_{i-1})$ is a uniformly continuous homeomorphism.

$f_i(Y_i-Y_{i-1})$ is a polytope $N_i\subset I^\omega$, where $N_i=\bigcup_j\Sigma_{ij}$, with diam $\Sigma_{ij}\to 0$ as $j\to\infty$. Since g_i is uniformly continuous and N_i is bounded, we may triangulate N_i into simplexes which we also call Σ_{ij} in such a way that diam $g_i(\Sigma_{ij})<1/i$ for all j and diam $g_i(\Sigma_{ij})\to 0$ as $j\to\infty$. Now

$$dX-X = \bigcup_i f(Y_i-Y_{i-1}) = \bigcup_i\Big[\bigcup_j g_i(\Sigma_{ij})\Big].$$

If $i\neq k$, then $g_i(\Sigma_{ij})$ and $g_k(\Sigma_{kl})$ are clearly disjoint. Since $g_i|f_i(Y_i-Y_{i-1})$ is a homeomorphism, $f(Y_i-Y_{i-1})=g_i(f_i(Y_i-Y_{i-1}))$ is a polytope. Hence by the property of the diameters of the $g_i(\Sigma_{ij})$ it is clear that $dX-X$ is a pseudopolytope.

Now $dX-X$ is a pseudopolytope and $X\subset\bigcup_i F_i\subset\bigcup_i \bar F_i^{dX}$. Hence we can finish the proof by showing that $\bar F_k^{dX}$ is strongly countable-dimensional for all k. Let k be fixed.

Since f is a closed map, $\bar F_k^{dX}=f(\bar F_k^{cX})$. Since $f(Y_i-Y_{i-1})$ is homeomorphic to $f_i(Y_i-Y_{i-1})$ for all i,

$$\dim f(\bar F_k^{cX}\cap(Y_i-Y_{i-1}))\leqq n_k$$

for all $i\geqq k$.

Let

$$E_i = \bigcup_{j=k}^{i} f(\bar F_k^{cX}\cap(Y_j-Y_{j-1}))$$

for $i\geqq k$, and let $D_k=\bigcup_{i=k}^\infty E_i$. It is easily shown that E_i is closed in D_k for all $i\geqq k$. Now dim $E_k\leqq n_k$, so suppose dim $E_i\leqq n_k$. Since

$$E_{i+1} = E_i\cup f(\bar F_k^{cX}\cap(Y_{i+1}-Y_i))$$

and

$$\dim f(\bar F_k^{cX}\cap(Y_{i+1}-Y_i))\leqq n_k,$$

dim $E_{i+1} \leqq n_k$ by [2, Corollary 1, p. 32]. By induction dim $E_i \leqq n_k$ for all $i \geqq k$, so that dim $D_k \leqq n_k$ by the countable sum theorem [2, p. 30]. By [2, Proposition B, p. 28]

$$\dim D_k \cup F_k \leqq n_k + m_k + 1.$$

Now $f(\bar{F}_k^{cX} \cap Y_{k-1})$ is closed in $dX - X$, so that $f(\bar{F}_k^{cX} \cap Y_{k-1})$ is a countable union of compact finite-dimensional sets. Since

$$D_k \cup F_k = \bar{F}_k^{dX} - f(\bar{F}_k^{cX} \cap Y_{k-1}),$$

$D_k \cup F_k$ is an open subset of the compact space $\bar{F}_k^{dX}$. By Proposition 2, $D_k \cup F_k = \bigcup_i G_{ki}$, where the closure H_{ki} of G_{ki} in $\bar{F}_k^{dX}$ is contained in $D_k \cup F_k$ for all i. But

$$H_{ki} = \bar{G}_{ki}^{dX} \cap \bar{F}_k^{dX} = \bar{G}_{ki}^{dX}.$$

Therefore dim $\bar{G}_{ki}^{dX} \leqq n_k + m_k + 1$, so that $D_k \cup F_k$, and hence $\bar{F}_k^{dX}$, is a countable union of compact finite-dimensional sets.

Recall that a completion of a space X is an absolute G_δ containing X as a dense subspace. Theorem 2 then immediately yields the following.

COROLLARY 1. *A space X has a strongly countable-dimensional compactification if and only if it has a strongly countable-dimensional completion.*

BIBLIOGRAPHY

1. W. Hurewicz, *Über Einbettung separabler Räume in gleichdimensionale kompakten Räume*, Monatsh. Math. Phys. **37** (1930), 199–208.

2. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Univ. Press, Princeton, N. J., 1941.

3. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955.

4. Casimir Kuratowski, *Topologie*. I, 4th ed., Monogr. Mat., Warsaw, 1958.

5. A. Lelek, *On the dimension of remainders in compact extensions*, Dokl. Akad. Nauk SSSR **160** (1965), 534–537 = Soviet Math. Dokl. **6** (1965), 136–140.

6. J. Nagata, *Modern dimension theory*, Bibliotheca Mathematica **6**, Wiley, New York, 1965.

7. E. G. Sklyarenko, *On dimensional properties of infinite dimensional spaces*, Amer. Math. Soc. Transl. (2) **21** (1962), 35–50.

INDIANA UNIVERSITY,
BLOOMINGTON, INDIANA