CHEBYSHEV APPROXIMATION BY FAMILIES WITH THE BETWEENESS PROPERTY

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1. **Introduction.** In this note a theory of Chebyshev approximation is obtained for approximating families with a property which is a generalization of convexity, the betweeness property. This theory is of interest for several reasons. Most of the approximating families for which a tractable theory exists characterize best approximations by the extrema of their error curve. The betweeness property is the weakest easily verifiable condition giving such a characterization of best approximations. The development of the theory sheds considerable light on the well-known linear theory [2], [5] and rational theory [1], [2], [3]. A necessary and sufficient condition for the uniqueness of best approximations is obtained; it is the most general known necessary and sufficient condition for any theory.

Let X be a compact space and for a function g define $\|g\| = \sup\{|g(x)| : x \in X\}$.

Let $\mathcal{G}$ be a family of real continuous functions with elements $F, G, H, \ldots$. The Chebyshev problem is: given a continuous function f, to find an element G^* of $\mathcal{G}$ to minimize $e(G) = \|E(G, \cdot)\|$ where $E(G, x) = f(x) - G(x)$. Such an element G^* is called a best approximation in $\mathcal{G}$ to f on X. It will be assumed throughout the discussion that f is fixed, and mention of f is suppressed in the notation $e(G)$ and $E(G, \cdot)$.

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2. **The betweeness property.**

Definition. A family $\mathcal{G}$ of real continuous functions is said to have the *betweeness property* if for any two elements G_0 and G_1, there exists a *λ-set* $\{H_\lambda\}$ of elements of $\mathcal{G}$ such that $H_0 = G_0$, $H_1 = G_1$, and for all $x \in X$, $H_\lambda(x)$ is either a strictly monotonic continuous function of λ or a constant, $0 \leqq \lambda \leqq 1$. (It should be noted that $H_\lambda(x)$ can be monotone in different senses for different x.)

Lemma 1. *Let $\{G_k\}$ be a sequence of continuous functions on a compact space X such that $\{G_k\}$ converges pointwise to a continuous function G_0 and for any $x \in X$, $G_k(x)$ is a monotonic sequence, then $\{G_k\}$ converges uniformly to G_0.*

Proof. The sequence $|G_k(x) - G_0(x)|$ is a decreasing sequence of continuous functions, which converges to the continuous limit 0. By Dini's theorem, the convergence is uniform. From this lemma it can be seen that if $\{H_\lambda\}$ is a λ-set for G_0 and G_1 then the sequence $\{H_{1/k}\}$ converges uniformly to G_0.

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Any linear family $\mathscr{L}$ of continuous functions (and any convex subset of $\mathscr{L}$) has the betweeness property, for a λ-set is given by $H_\lambda = \lambda G_1 + (1-\lambda)G_0$.

More generally let $\mathscr{P}$ and $\mathscr{Q}$ be linear families and $\mathscr{F}$ a convex set of pairs (p, q), $p \in \mathscr{P}$, $q \in \mathscr{Q}$ such that $(p, q) \in \mathscr{F}$ implies $(\alpha p, \alpha q) \in \mathscr{F}$ for $\alpha > 0$. A function is an $\mathscr{F}$-admissible rational function if it is of the form p/q, $(p, q) \in \mathscr{F}$, $q > 0$. The set $\mathscr{R}(\mathscr{F})$ of $\mathscr{F}$-admissible rational functions has the betweeness property, for the λ-set corresponding to p_0/q_0 and p_1/q_1 is

$$H_\lambda = (\lambda p_1 + (1-\lambda)p_0)/(\lambda q_1 + (1-\lambda)q_0),$$

$$dH_\lambda/d\lambda = (p_1 q_0 - p_0 q_1)/(\lambda q_1 + (1-\lambda)q_0)^2$$

being of constant sign for a given point x and vanishing identically at any point x at which $p_0/q_0 - p_1/q_1$ vanishes. In the case where $\mathscr{F}$ consists of all pairs we obtain the family

$$\mathscr{R} = \{p/q : p \in \mathscr{P}, q \in \mathscr{Q}, q > 0\}$$

of admissible rational functions.

If ϕ is a continuous strictly monotonic function from the real line into the real line and $\mathscr{G}$ has the betweeness property, then the set of elements of the form $\phi(G)$, $G \in \mathscr{G}$ has the betweeness property, for if $\{H_\lambda\}$ is a λ-set for G_0 and G_1, $\{\phi(H_\lambda)\}$ is a λ-set for $\phi(G_0)$ and $\phi(G_1)$.

After the theory of this paper had been obtained it was noticed that Meinardus and Schwedt had used a condition [4, p. 304] quite close to the betweeness property, but developed a different type of theory.

3. Characterization of best approximation.

The points at which $E(G, \cdot)$ attains its norm $e(G)$ will be denoted by $M(G)$. By compactness of X and continuity of $E(G, \cdot)$, $M(G)$ is nonempty and closed.

THEOREM 1. *Let $\mathscr{G}$ have the betweeness property. An element G_0 of $\mathscr{G}$ is a best approximation if and only if there exists no element $G_1 \in \mathscr{G}$ such that $|E(G_1, x)| < e(G_0)$ for all $x \in M(G_0)$.*

Proof. The condition is obviously sufficient for G_0 to be a best approximation (we do not need the betweeness property). We now prove necessity. Let us suppose that $|E(G_1, x)| < e(G_0)$ for all $x \in M(G_0)$ then by continuity of $E(G_1, \cdot)$ there exists an open cover U of $M(G_0)$ on which this inequality holds. Let $V = X \sim U$, then if V is empty it is immediate that G_0 is not best. We therefore suppose that V is nonempty. Let H_λ be a λ-set corresponding to G_0 and G_1, $H_0 = G_0$, $H_1 = G_1$. On the set U we have $E(H_\lambda, x)$ on the open interval between $E(G_0, x)$ and $E(G_1, x)$ for $0 < \lambda < 1$, hence

$$E(H_\lambda, x) < e(G_0), \qquad 0 < \lambda < 1, x \in U.$$

Let $\eta = e(G_0) - \sup\{|E(G_0, x)| : x \in V\}$. As V is compact and $E(G_0, \cdot)$ is continuous, $E(G_0, \cdot)$ attains its supremum on V and this supremum cannot be $e(G_0)$, as

$M(G) \cap V$ is empty, hence $\eta > 0$. The sequence $\{H_{1/k}\}$ converges uniformly to G_0. Choose $\delta > 0$ such that $\|G_0 - H_\delta\| < \eta$. It follows that for $x \in V$,

$$|E(H_\delta, x)| = |f(x) - H_\delta(x)| \leqq |f(x) - G_0(x)| + |G_0(x) - H_\delta(x)| < e(G_0) - \eta + \eta = e(G_0).$$

Combining this inequality and the previous one for $x \in U$, we have

$$|E(H_\delta, x)| < e(G_0), \qquad x \in X = U \cup V.$$

and G_0 is not best, proving necessity. The theorem is proven.

Let us suppose that $E(G_0, x) \cdot (G_1(x) - G_0(x)) > 0$ for all $x \in M(G_0)$ and $\{H_\lambda\}$ is a λ-set for G_0 and G_1. For λ sufficiently small, $|E(H_\lambda, x)| < e(G_0)$ for all $x \in M(G_0)$. We then apply Theorem 1 to get

COROLLARY. *Let $\mathcal{G}$ have the betweeness property. An element G_0 of $\mathcal{G}$ is a best approximation if and only if there exists no element $G_1 \in \mathcal{G}$ such that $E(G_0, x) \cdot (G_1(x) - G_0(x)) > 0$ for all $x \in M(G_0)$.*

4. **An error determining set on which best approximations agree.** Let $\mathcal{G}^*$ be the set of best approximations to f and $N = \bigcap M(G)$, $G \in \mathcal{G}^*$. We will show in this section that if $\mathcal{G}^*$ is nonempty then N is nonempty, best approximations must agree on N and that N is an error determining set, that is, there exists no approximant F such that $|E(F, x)| < \inf\{e(G) : G \in \mathcal{G}\}$ for $x \in N$. In the cases of approximation by linear or rational families of finite dimension, it can easily be shown that if $\mathcal{G}^*$ is nonempty, there exists an element $F \in \mathcal{G}$ such that $M(F) = N$; in the linear case any element of the convex interior of $\mathcal{G}^*$ is such an F. This is not true in general, for let $X = [0, 1]$ and $\mathcal{G}$ be the set of monotonic continuous functions G with G zero in a neighborhood of the point zero. In the approximation of $f = 1$ any element G of $\mathcal{G}$ such that $\|1 - G\| = 1$ is a best approximation and $N = \{0\}$, but there is no element G such that $M(G) = N$.

LEMMA 3. *Let $\mathcal{G}$ have the betweeness property and $\mathcal{G}^*$ be nonempty. Given a finite number $G_1, \ldots, G_n$ of elements of $\mathcal{G}^*$ there exists an element G_0 of $\mathcal{G}^*$ such that $\bigcap_{k=1}^{n} M(G_k) \supset M(G_0)$.*

Proof. Let G_1 and G_2 be any two best approximations and $\bar{G}_1$ be any element of the λ-set corresponding to G_1 and G_2, $0 < \lambda < 1$, then for all $x \in X$, $\bar{G}_1(x)$ lies between $G_1(x)$ and $G_2(x)$,

$$|E(\bar{G}_1, x)| \leqq \sup\{|E(G_1, x)|, |E(G_2, x)|\}$$

with equality only if $G_1(x) = G_2(x)$. It follows that $\bar{G}_1$ is a best approximation and $M(\bar{G}_1) \subset M(G_1) \cap M(G_2)$. Similarly, there exists $\bar{G}_k \in \mathcal{G}^*$ such that $\bar{G}_k$ is in the λ-set corresponding to $\bar{G}_{k-1}$ and G_{k+1}, $0 < \lambda < 1$, and $M(\bar{G}_k) \subset \bigcap_{j=1}^{k+1} M(G_j)$, $k = 2, \ldots, n-1$. The required approximant in $\mathcal{G}^*$ is $\bar{G}_{n-1}$ and the lemma is proven.

COROLLARY. *Let $G_0, G_1 \in \mathcal{G}^*$, then the λ-set $\{H_\lambda\}$ for G_0 and G_1 is contained in $\mathcal{G}^*$.*

LEMMA 4. *Let $\mathscr{G}$ have the betweeness property. If $\mathscr{G}^*$ is nonempty, N is nonempty.*

Proof. If N, an intersection of a nonempty family of closed sets, were empty, it could be expressed as a finite intersection of these sets.

$$N = \bigcap_{k=1}^{n} M(G_k), \qquad G_k \in \mathscr{G}^*.$$

By the previous lemma there exists $G_0 \in \mathscr{G}^*$ such that $\bigcap_{k=1}^{n} M(G_k) \supset M(G_0)$. It follows from the definition of N that $N = M(G_0)$. But $M(G_0)$ is nonempty and so we have a contradiction proving the lemma.

LEMMA 5. *Let the family $\mathscr{G}$ of real continuous functions have the betweeness property. Let $G_0, G_1 \in \mathscr{G}^*$, then $G_0(x) = G_1(x)$ for all $x \in N$.*

Proof. Let $G_0, G_1 \in \mathscr{G}^*$ be given and select a λ-set $\{H_\lambda\}$ corresponding to G_0 and G_1, $0 < \lambda < 1$. If $G_0(x) \neq G_1(x)$ for some x, then

$$|E(H_\lambda, x)| < \max\{|E(G_0, x)|, |E(G_1, x)|\}$$

for $0 < \lambda < 1$ and since $\{H_\lambda\} \in \mathscr{G}^*$, $x \notin N$.

LEMMA 6. *Let $\mathscr{G}$ have the betweeness property. If $\mathscr{G}^*$ is nonempty there exists no approximant G such that $|E(G, x)| < \inf\{e(G) : G \in \mathscr{G}\}$ for all $x \in N$.*

Proof. Suppose such a G exists, then by continuity of $E(G, \cdot)$, the inequality

$$|E(G, x)| < \inf\{e(G) : G \in \mathscr{G}\} = \Delta(f, \mathscr{G})$$

holds on an open cover U of N. Let $V = X \sim U$, then V is nonempty (for otherwise $e(G) < \Delta(f, \mathscr{G})$, which is impossible).

Since

$$\{\bigcap (V \cap M(F)) : F \in \mathscr{G}^*\} = V \cap N = \varnothing$$

is an intersection of closed sets in a compact space, there exists a finite set $G_1, \ldots, G_n$ of elements of $\mathscr{G}^*$ such that $\bigcap_{k=1}^{n} (V \cap M(G_k)) = \varnothing$. Applying Lemma 3, there exists $G_0 \in \mathscr{G}^*$ such that $M(G_0) \subset \bigcap_{k=1}^{n} M(G_k) \subset U$. Now let $\{H_\lambda\}$ be a λ-set corresponding to G_0 and G, $H_0 = G_0$, $H_1 = G$. Since $E(H_\lambda, x)$ is between $E(G_0, x)$ and $E(G, x)$ for $0 < \lambda < 1$ and $x \in U$,

$$E(H_\lambda, x) < e(G_0), \qquad 0 < \lambda < 1, x \in U.$$

Now let $\eta = e(G_0) - \sup\{|E(G_0, x)| : x \in V\}$. As the sequence $\{H_{1/k}\}$ converges uniformly to G_0, there exists $\delta > 0$ such that $\|G_0 - H_\delta\| < \eta$. For $x \in V$ we have

$$\begin{aligned}|E(H_\delta, x)| &= |f(x) - H_\delta(x)| \\ &\leqq |f(x) - G_0(x)| + |G_0(x) - H_\delta(x)| < e(G_0) - \eta + \eta = e(G_0).\end{aligned}$$

Combining this inequality for $x \in V$ with the earlier one for $x \in U$, we have $E(H_\delta, x) < e(G_0)$, $x \in X$, and so

$$e(H_\delta) < \inf\{e(G) : G \in \mathscr{G}\}.$$

This is a contradiction and the lemma is proven.

5. **Uniqueness results.** Lemmas 5 and 6 are very powerful results. Using them we can obtain many uniqueness results. We give below the most general uniqueness result, a generalization of Haar's classical result concerning necessary and sufficient conditions for best linear approximations to be unique. After this result was obtained it was noted that it includes a uniqueness result of Singer [6] for approximation by arbitrary linear subspaces of $C(X)$.

DEFINITION. A family $\mathcal{G}$ of real continuous functions is said to have *zero-sign compatibility* if for any two distinct elements G and H, any closed subset Z of the zeros of $G-H$, and any continuous function s which takes the values $+1$ or -1 on Z, there exists $F \in \mathcal{G}$ such that

$$(*) \qquad \operatorname{sgn}(F(x)-G(x)) = s(x), \qquad x \in Z.$$

Without loss of generality we can assume $\|s\|=1$.

THEOREM 2. *Let $\mathcal{G}$ have the betweeness property. A necessary and sufficient condition that for every continuous function a best approximation is unique is that $\mathcal{G}$ have zero-sign compatibility.*

Proof. Suppose that for two distinct elements G and H, a closed subset Z of the zeros of $G-H$, and a continuous function s, $\|s\|=1$, which takes the values $+1$ or -1 on Z, there exists no element F for which $(*)$ holds.

Define:

$$f(x) = G(x)+s(x)[\|G-H\|-|G(x)-H(x)|],$$

then

$$E(G,\cdot) = s(x)[\|G-H\|-|G(x)-H(x)|].$$

For $x \in Z$ we have $E(G,x)=s(x)\|G-H\|$, hence $Z \subset M(G)$. If a better approximant F existed it would satisfy

$$\operatorname{sgn}(F(x)-G(x)) = s(x), \qquad x \in Z,$$

which is impossible by hypothesis. Hence G is a best approximation to f and since

$$\begin{aligned} |f(x)-H(x)| &\leqq |f(x)-G(x)|+|G(x)-H(x)| \\ &\leqq \|G-H\|-|G(x)-H(x)|+|G(x)-H(x)| = \|G-H\|, \end{aligned}$$

H is also a best approximation to f, proving necessity.

REMARK. The proof of necessity assumes nothing about $\mathcal{G}$ and therefore shows that zero-sign compatibility is necessary for uniqueness, $\mathcal{G}$ any approximating family.

Suppose now that $\mathcal{G}$ has zero-sign compatibility and G, G_1 are distinct best approximations. Therefore $G(x)=G_1(x)$ for $x \in N$ by Lemma 5. Let the function s be $E(G,\cdot)/\|E(G,\cdot)\|$ then by zero-sign compatibility there exists an element F such that $\operatorname{sgn}(F(x)-G(x))=\operatorname{sgn}(E(G,x))$ for $x \in N$. Let $\{H_\lambda\}$ be a λ-set for G and F, $H_0=G$, $H_1=F$. The sequence $\{H_{1/k}\}$ converges uniformly to G so for some $\delta>0$, $E(H_\delta,x)$ will be between $E(G,x)$ and $-E(G,x)$ for all $x \in N$, hence $|E(H_\delta,x)| < |E(G,x)|=e(G)$ for all $x \in N$. This contradicts Lemma 6 so sufficiency is proven.

We now consider approximation on a compact subset Y of X. If the betweeness property holds on X, it holds on Y.

LEMMA 7. *Let X be a compact normal space and Y a compact subset of X. If $\mathcal{G}$ has zero-sign compatibility on X, $\mathcal{G}$ has zero-sign compatibility on Y.*

Proof. Let (G, H) be a pair of distinct elements of $\mathcal{G}$. Let Z be a closed subset of $Y \cap \{x : G(x)-H(x)=0\}$ then Z is a closed subset in X of the zeros of $G-H$. Let s' be a continuous mapping of Y into $[-1, 1]$ taking values $+1$ or -1 on Z. Since X is a normal space, there exists by the Tietze extension theorem $s \in C(X)$, $\|s\|=1$, $s(x)=s'(x)$ for $x \in Y$. Let $\mathcal{G}$ have zero-sign compatibility on X; then there exists $F \in \mathcal{G}$ such that

$$\operatorname{sgn}(F(x)-G(x)) = s(x) = s'(x), \qquad x \in Z.$$

From the lemma and Theorem 2 we obtain

COROLLARY. *Let X be a compact normal space. Let $\mathcal{G}$ have the betweeness property and best approximations on X to any continuous function be unique, then best approximations on any compact subset of X are unique to any continuous function.*

We now consider approximation by an open subset $\mathcal{G}'$ of $\mathcal{G}$. If $\mathcal{G}$ has the betweeness property, it follows that the function F in the definition of zero-sign compatibility can be chosen arbitrarily close to the function G of that definition. If $G \in \mathcal{G}'$ it follows that F can be chosen such that $F \in \mathcal{G}'$. It follows that if $\mathcal{G}$ has zero-sign compatibility, so does $\mathcal{G}'$. We obtain:

COROLLARY. *Let both $\mathcal{G}$ and $\mathcal{G}'$, an open subset of $\mathcal{G}$, have the betweeness property. If every continuous function has at most one best approximation from $\mathcal{G}$, every continuous function has at most one best approximation from $\mathcal{G}'$.*

Less general but simpler uniqueness results can be developed in terms of the sign changing property and property Z.

DEFINITION. $\mathcal{G}$ has the sign changing property of degree n at G if for any n distinct points $\{x_1, \ldots, x_n\}$ and n real numbers $w_1, \ldots, w_n$ which are either $+1$ or -1, there exists an approximant F such that

$$\operatorname{sgn}(F(x_k)-G(x_k)) = w_k, \qquad k = 1, \ldots, n.$$

We need not specify the closeness of F to G in the above definition since if such an F exists, there exists with the betweeness property such an F arbitrarily close to G.

DEFINITION. $\mathcal{G}$ has property Z of degree n at G if $G-F$ having n zeros implies $F=G$.

Let $\mathcal{G}$ have the betweeness property. The F in the definition of the sign changing property can be chosen such that for given $\varepsilon>0$, $\|F-G\|<\varepsilon$. Let $G \in \mathcal{G}^*$. If $\mathcal{G}$ has the sign changing property of degree n at G then G either coincides with the function f being approximated or N has at least $n+1$ points, for if it had less we could find F

such that $|E(F, x)| < e(G)$ for $x \in N(X)$, which contradicts Lemma 6. If $\mathcal{G}$ has property Z of degree n at G then by Lemma 5 best approximations must be identical if N has n or more points. We therefore have:

THEOREM 3. *Let $\mathcal{G}$ have the betweeness property and $G \in \mathcal{G}^*$. If $\mathcal{G}$ has property Z of degree $n+1$ at G and the sign changing property of degree n at G then G is a unique best approximation.*

BIBLIOGRAPHY

1. B. Brosowski, *Über die Eindeutigkeit der rationalen Tschebyscheff Approximationen*, Numer. Math. **7** (1965), 176–186.

2. E. W. Cheney, *Introduction to approximation theory*, McGraw-Hill, New York, 1966.

3. E. W. Cheney and H. L. Loeb, *Generalized rational approximation*, SIAM J. Numer. Anal. **1** (1964), 11–25.

4. G. Meinardus and D. Schwedt, *Nicht-lineare Approximation*, Arch. Rational Mech. Anal. **9** (1964), 329–351.

5. John Rice, *Tchebycheff approximation in several variables*, Trans. Amer. Math. Soc. **109** (1963), 444–466.

6. Ivan Singer, *On best approximation of continuous functions*, Math. Ann. **140** (1960), 165–168.

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