CENTRALIZERS IN FREE ASSOCIATIVE ALGEBRAS(1)

BY
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Let R be the free associative algebra on some set of generators over a field k (equivalently: the tensor algebra on some k-vector-space). Though R is in general "very" noncommutative, it is easy to find pairs of commuting elements: If we take any $z \in R$, and polynomials P and Q in one indeterminate over k, then $P(z)$ and $Q(z)$ will commute. We shall here show that the centralizer, C, of any nonscalar element $u \in R$ (i.e. $C=\{x \in R \mid xu=ux\}$) is of the form $k[z]$ for some nonscalar $z \in R$. In particular, any pair of commuting elements of R can be written in the form $P(z)$, $Q(z)$. (This was conjectured by P. M. Cohn [8, p. 348].)

Let us first outline our proof, stating results as they apply to this problem, and not necessarily in the greatest generality in which they will be proved:

After setting up some general ring-theoretic tools in §1, we shall show in §2, by rather elementary arguments that our centralizer ring C is commutative, and in fact is a finite integral extension of $k[u]$. To complete our proof, it suffices to show that C is integrally closed, and is embeddable in a polynomial ring $k[x]$, because any subalgebra of a polynomial algebra $k[x]$ that is integrally closed (in its own field of fractions) and $\neq k$ is of the form $k[y]$ [9, Proposition 2.1]. (The proof uses Lüroth's theorem.)

In §3 we find that there exists a nonzero element $e \in R$, and a homomorphism f of the integral closure C' of C, into R, such that for all $x \in C$, $xe=ef(x)$. In other words, $f(C)$ is "conjugate" to C, and this conjugate, if not C itself, "can be integrally closed within R."

In §4, we show that $f(C')$ can be "pulled back across e," thus C itself has integral closure in R. This integral closure, being commutative and containing u, must coincide with our centralizer C. So C is integrally closed.

In §5, we show that any finitely generated nontrivial (i.e., $\neq k$) subalgebra of R can be mapped into a polynomial ring $k[x]$ so as to have nontrivial image. We apply this to C. Now the image will have transcendence degree 1 over k, and C has the same transcendence degree because it is a finite extension of $k[u]$. Hence the map will be 1-1 (see [16, Chapter II, §12, Theorem 29, p. 101]), giving the embedding required to complete our proof.

The remaining §§6–9, examine some related points—centralizers in some rings of differential operators (§9), generalizations of results of the preceding sections, and

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open questions and conjectures. At a few points in these last sections I call upon other results proved in my thesis, without giving the proofs, which are outside the scope of this paper but which will appear elsewhere.

1. **Basic tools.** In this section, matter in brackets constitutes background information or related concepts, not strictly necessary to the proof of our result.

A *ring* will here mean an associative ring with identity element. An *integral domain* will mean any ring without zero divisors. *Fields* (in contrast to *skew fields*) will be assumed commutative.

DEFINITION 1.1. *We shall say a ring R is a 2-fir if it is an integral domain, and if any two elements of R having a nonzero common right multiple generate together a principal right ideal.*

[REMARKS. P. M. Cohn introduced in [10] the concept of a free (right) ideal ring, or (right) *fir*: a ring in which all one-sided (respectively, right) ideals are free as modules, and all free modules have unique rank. For any cardinal n, one may similarly define a right *n-fir* as a ring in which all right ideals *generated by n or fewer elements* are free, and all free modules *of rank $\leqq n$* have unique rank. For n finite, this condition turns out to be left-right symmetric, and to have a number of interesting equivalent statements, among them: every right ideal generated by *$m \leqq n$ right linearly dependent* elements can be generated by $m-1$ elements. It is this condition that we have used above as a definition, for $n=2$. Cohn and the author will elsewhere (starting with [12]) study the general properties of n-firs.

A 1-fir is simply an integral domain. A 2-fir as defined above can be shown equivalent to what was previously [7, §5] called a *weak Bézout ring*.]

LEMMA 1.2 (BOWTELL). *In a 2-fir the intersection of two principal right ideals is again principal.*

Proof. Let aR and bR be right ideals with nonzero intersection. By the definition of a 2-fir, their sum will be principal, hence free; hence the obvious short exact sequence of right R-modules

$$0 \to aR \cap bR \to aR \oplus bR \to aR+bR \to 0$$

splits, and we can get a short exact sequence with arrows going the other way. This expresses $aR \cap bR$ as the image of $aR \oplus bR \cong R \oplus R$ under a homomorphism with nonzero kernel, in other words, as an ideal generated by two elements with nonzero common right multiple; hence it is principal. ▮

By a *Z-filtered ring*, we shall mean a ring R, expressed as the union of a chain of additive subgroups, $\cdots \subseteq R_{-1} \subseteq R_0 \subseteq R_1 \subseteq \cdots$, having intersection $\{0\}$, and satisfying $1 \in R_0$, and $R_iR_j \subseteq R_{i+j}$. Given $a \in R$, $v(a)$ will designate $\inf_{a \in R_i} i$, which is $-\infty$ if $a=0$, otherwise an integer, and will be called the *degree* of a. R will be called *positively filtered*, or simply, *filtered*, if $R_{-1}=\{0\}$; that is, if $v(a) \geqq 0$ for all $a \neq 0$.

When we speak of a filtered k-algebra R (k a field), it will be understood that all nonzero elements of k have degree 0 in R.

The reader is doubtless familiar with the Euclidean algorithm on filtered *commutative* rings such as the polynomial ring in one indeterminate over a field, which can be used to deduce the principal ideal domain property. A similar algorithm for not-necessarily-commutative rings implies the 2-fir condition:

DEFINITION 1.3. *A filtered ring will be said to satisfy* 2-term weak algorithm *if* (1) *the relation* $v(ab)=v(a)+v(b)$ *holds identically, and* (2) *given any two nonzero elements* a_1 *and* a_2, *with* $v(a_1)\leqq v(a_2)$ *such that there exist* b_1, b_2 *with* $v(a_1)+v(b_1)=v(a_2)+v(b_2) > v(a_1b_1+a_2b_2)$, *there will also exist* c_2 *such that* $v(a_2)+v(c_2)=v(a_1)>v(a_1-a_2c_2)$.

LEMMA 1.4. *A ring R satisfying* 2*-term weak algorithm with respect to a filtration is a* 2*-fir, and its subring* R_0 *is a skew field.*

Proof. That R is an integral domain follows from condition (1).

The argument to show that two elements with a common right multiple generate a principal ideal is exactly like the proof that a Euclidean domain is a principal ideal domain. Note that the existence of a common right multiple implies the hypothesis of (2) of the above definition.

To see that R_0 is a skew field, apply (2) to any pair $(a, 1)$ with $a \in R_0$. ∎

[The 2-term weak algorithm is equivalent to the statement that the *weak algorithm* defined by Cohn in [5] (there called the "generalized algorithm," but renamed in [8]) holds for relations of $\leqq 2$ terms. One similarly defines *n-term weak algorithm*, and can show that it implies the n-fir condition. (Cf. Cohn [11, Theorem 4.3.]. Rather than speaking of n-term weak algorithm, Cohn defines the "dependence number" of a filtered ring to be the least d for which what we call d-term weak algorithm *fails*.)]

By a *graded ring H* we shall mean a disjoint family of abelian groups $(H_i)_{i\in Z}$ with multiplication maps: $H_i \times H_j \to H_{i+j}$, satisfying the obvious associativity and distributivity laws, and such that H_0 contains an identity element 1. By $a \in H$ we shall mean $a \in \bigcup H_i$. Even the zero elements of the H_i are taken to be distinct; thus for every $a \in H$, there is a unique integer $d(a)$, which will be called the degree of a, such that $a \in H_{d(a)}$.

If R is a $\boldsymbol{Z}$-filtered ring, and for each i we define Gr R_i to be the abelian group R_i/R_{i-1}, this family has a natural structure of graded ring, Gr R. The quotient maps: $R_i \to \operatorname{Gr} R_i$ will be called gr_i; and for nonzero $a \in R$, $\mathrm{gr}_{v(a)}(a)$ will be called the *leading term* of a.

The definition of 2-term weak algorithm for a positively filtered ring R can now be stated more simply: (1) Gr R has no zero-divisors, and (2) if two elements x_1 and x_2 of Gr R have a nonzero common right multiple, and $d(x_2)\geqq d(x_1)$, then x_2 is a right multiple of x_1.

[For any n, the condition of n-term weak algorithm on R turns out to be equivalent to the following conditions on Gr R: (a) Gr R is a "graded n-fir," defined in the obvious way, and (b) if $n>1$, Gr R_0 is a skew field. It can be shown that (a) alone implies that R is an n-fir.]

The following result is a restricted case of a result of Cohn's ([5, Theorem 3.5; and cf. 3.4]).

LEMMA 1.5. *A free associative algebra R over a field k, on a set X of indeterminates, satisfies 2-term weak algorithm with respect to the natural filtration.*

Proof. We know that $\mathrm{Gr}\, R_i$ will have for a k-basis the monomials in X of degree i. Suppose that $x_1y_1=x_2y_2$ in $\mathrm{Gr}\, R$, with $x_i, y_i\neq 0$, and $d(x_2)\geqq d(x_1)$ (whence $d(y_2)\leqq d(y_1)$). Let u be any monomial occurring with nonzero coefficient in y_2. Adjusting the y's by scalars, we can assume this coefficient is 1.

Let ${}^{u)}$ designate the operation of taking the right cofactor of the monomial u in an expression—that is, the k-linear operator such that if v is a monomial of the form wu, then $v^{u)}=w$, while if v is any other monomial, $v^{u)}=0$. Then we have $(x_1y_1)^{u)}=(x_2y_2)^{u)}$, which yields $x_1(y_1{}^{u)})=x_2$. ∎

2. **A commutativity result.** We have seen that the condition of 2-term weak algorithm on a filtered ring R can be interpreted as saying that the nonzero elements of $\mathrm{Gr}\, R$ form a multiplicative semigroup with cancellation such that if x_1 and x_2 have a common right multiple and $d(x_2)\geqq d(x_1)$, then x_2 is a right multiple of x_1. Semigroups with degree function having this property are studied in [4]. Their structure is particularly simple when all the invertible elements lie in the center (corresponding to the case where R_0 is a field k, and R a k-algebra). The semigroup will then be the direct product of its group of invertible elements, and a free semigroup (with 1). (The generators of the free part, unique up to associates, are simply the irreducible elements. One uses the degree function to show that all elements factor into irreducibles.)

In a free semigroup with 1, let us associate to every element $\neq 1$ (thought of as a finite sequence of the generators) its "least repeating segment", that is, the minimal element of which it is a power. Clearly, two elements commute if and only if they have the same least repeating segment. Hence, commutability is an equivalence relation on elements $\neq 1$, and the equivalence classes, with 1 thrown in, are free cyclic semigroups.

Let R be a filtered ring with 2-term weak algorithm, such that $R_0=k$ lies in the center; and u a member of R of positive degree. The centralizer C of u in R inherits from R a structure of filtered k-algebra. We note that the leading term $\bar{u}$ of u in $\mathrm{Gr}\, R$ must commute with the leading terms of all members of C. But the centralizer of $\bar{u}$ in the semigroup of nonzero members of $\mathrm{Gr}\, R$ will be, modulo units, a free semigroup on one generator v. There is no more than one power of v in any degree. Hence for any n, $C_n/C_{n-1}=\mathrm{Gr}\, C_n$ is a k-vector space of dimension $\leqq 1$.

In this situation, the following result applies:

PROPOSITION 2.1. *Let C be a $\mathbf{Z}$-filtered algebra over a field k, in which the relation $v(ab)=v(a)+v(b)$ holds identically, and such that for all n, $\mathrm{Gr}\, C_n$ is 0 or 1-dimensional over k. Then C is commutative.*

Proof. Suppose the contrary. Let r be the nonnegative integer

$$\inf_{x,y\in C-\{0\}} (v(x)+v(y)-v(xy-yx)).$$

Then for each m, n, a well-defined k-bilinear map, $(\ ,\ )$: $\mathrm{Gr}\, C_m \times \mathrm{Gr}\, C_n \to \mathrm{Gr}\, C_{m+n-r}$ is given by $(\mathrm{gr}_m(x), \mathrm{gr}_n(y)) = \mathrm{gr}_{m+n-r}(xy-yx)$.

We verify from the corresponding result in C that for all $x, y, z \in \mathrm{Gr}\, C$,

(1) $(xy, z) = x(y, z)+(x, z)y$.

I.e., $(\cdot, z)$ is a derivation. In the same way we verify that for any $x \in \mathrm{Gr}\, C$ and nonnegative integer q, $(x^q, x)=0$. But if x is an arbitrary nonzero element of $\mathrm{Gr}\, C_n$, x^q will be nonzero element of $\mathrm{Gr}\, C_{nq}$, and since this is at most 1-dimensional over k, will span it. Hence:

(2) $(\ ,\ )$ is zero on $\mathrm{Gr}\, C_{nq} \times \mathrm{Gr}\, C_n$ (q nonnegative, n arbitrary).

We shall now prove that $(\ ,\ )$ is zero on all $\mathrm{Gr}\, C_m \times \mathrm{Gr}\, C_n$, which will contradict the definition of r, and hence our assumption that C is noncommutative.

It will suffice to consider the case where m and n are of the same sign. For assuming this case, consider nonzero x and z in $\mathrm{Gr}\, C$ whose degrees are of different signs. Choose $q>0$ such that the degree of xz^q has the same sign as that of z. Then $0=(xz^q, z)=x(z^q, z)+(x, z)z^q=(x, z)z^q$, hence $0=(x, z)$.

For the argument when the signs are the same, we take $x \in \mathrm{Gr}\, C_m$, $y \in \mathrm{Gr}\, C_n$, and consider two possibilities:

Case a. $r=0$. In this case, $(x, y)=xy-yx$, so what we wish to prove is that $\mathrm{Gr}\, C$ is commutative.

Write m/n in lowest terms as p/q. Say $xy=\alpha yx$, where $\alpha \in k$. We find $x^p y=\alpha^p yx^p$. But $x^p \in \mathrm{Gr}\, C_{mp}=\mathrm{Gr}\, C_{nq}$, which by (2) commutes with $\mathrm{Gr}\, C_n \ni y$. So $\alpha^p=1$. Similarly, $\alpha^q=1$. As p and q are relatively prime, $\alpha=1$, so x and y commute.

Case b. $r>0$. Then by definition of r, $\mathrm{Gr}\, C$ will be commutative. Let p, q be as in Case a. Again, by (2), $0=(x^p, y)$. Applying the familiar calculation for a derivation in a commutative ring, this equals $px^{p-1}(x, y)$. Hence $p(x, y)=0$. Similarly, $q(x, y)=0$. Hence $(x, y)=0$. ∎

PROPOSITION 2.2. *Let R be a filtered ring with 2-term weak algorithm, such that R_0 is a field k in the center of R. Let $u \in R-k$, and let C be the centralizer of u in R. Then C will be commutative (hence, a maximal commutative subring of R), and will be a finite integral extension of $k[u]$, in which the valuation of $k[u]$ at ∞ is totally ramified.*

Proof. C is commutative by the preceding Proposition. To show it a finite integral extension of $k[u]$, let $t_1, \ldots, t_m \in C$ be chosen so that each congruence class *modulo* $v(u)$ that meets the set $v(C)$ contains exactly one $v(t_i)$, and this is equal to the minimal member of this class in $v(C)$. Then $t_1, \ldots, t_m$ is easily seen to be a module basis for C over $k[u]$.

The observation about the valuation at ∞ also follows from the fact that every C_n/C_{n-1} is of k-dimension $\leqq 1$. But we shall not need to use this fact. ∎

(In the case where R is a free associative algebra, the commutativity of C can also be shown by going to the completion of R and using results of Cohn's on centralizers in that ring—see §8.)

3. **Extension of module-structures, and conjugate embeddings.** Let C' be an integral domain, I a left ideal of C' which has nonzero intersection with every nonzero left ideal, and C a subring of C' containing I.

Note that given any nonzero $x \in C'$, Ix will be a nonzero left ideal of C', hence will meet I. In particular, Cx will meet C.

PROPOSITION 3.1. *Let $I \subseteq C \subseteq C'$ be as above, and M a torsion-free left C-module. Then the C-module structure on $IM \subseteq M$ extends uniquely to a C'-module structure; and this construction is functorial in M.*

Proof. We first note that if the structure of any torsion-free left C-module extends to a C'-module structure, the extension is unique. For given $a \in C'$, and x in the module, choose $b \in C$ such that $ba \in C$. Then the product, y, of a and x under our extended structure must satisfy $by = (ba)x$. If such a y exists, it is unique by torsion-freeness.

Now given $y = \sum a_i x_i \in IM$ ($a_i \in I$, $x_i \in M$) and $b \in C'$, define $b \cdot y = \sum (ba_i)x_i \in IM$. We claim this is independent of our choice of representation for y. For let $\sum a_i x_i = \sum a'_j x'_j$ be two such representations, and let us take nonzero $c \in C$ such that $cb \in C$. Then multiplying by cb, we get $\sum (cba_i)x_i = \sum (cba'_j)x'_j$. Now $(cba_i)x_i = (c(ba_i))x_i = c((ba_i)x_i)$. Rewriting these sums accordingly, and cancelling the c by torsion-freeness, we get $\sum (ba_i)x_i = \sum (ba'_j)x'_j$ as desired.

It is easy to see that this multiplication gives a module structure, and that the construction is functorial. ▮

If e is a left and right nonzero-divisor in a ring R, the set of pairs (x, y) of members of R such that $xe = ey$, forms a subring of $R \times R$, which is the graph of an isomorphism α_e between subrings of R, $R_e = \{x \mid xe \in eR\} \cong \{y \mid ey \in Re\} = {}_eR$. (Not to be confused with filtered-ring notation.)

We shall call a map f of a set into R right-conjugate to another map g, and g left-conjugate to f, if for some nonzero-divisor $e \in R$, $f = \alpha_e \circ g$; that is, if for all x in our set, $g(x)e = ef(x)$.

LEMMA 3.2. *Let $I \subseteq C \subseteq C'$ be as in Proposition 3.1. Assume C (but not C') embedded in a ring R, and that the right ideal IR of R is principal, and generated by a nonzero-divisor e.*

Then there exists a monomorphism $f: C' \to R$ such that $f \mid C$ is right-conjugate (by α_e) to the inclusion of C in R. (That is, $xe = ef(x)$ for all $x \in C$. This says that the inclusion of C in R extends to C' "modulo a conjugacy".)

Proof. If we consider R as a left C-module, then by 3.1 we get a structure of left C'-module on $IR = eR$ extending the C-module structure. Let us write this action $a \cdot x$ (for $a \in C'$, $x \in eR$). By functoriality, it will respect right multiplication by members of R.

Given $a \in C'$, we will have $a \cdot e \in eR$, say $a \cdot e = ef(a)$. The map f will be a homomorphism of C' into R. For, $ef(ab) = (ab) \cdot e = a \cdot (b \cdot e) = a \cdot (ef(b)) = (a \cdot e)f(b) = ef(a)f(b)$, whence f respects multiplication; while additivity is immediate. If a belongs to C, $ef(a) = a \cdot e$, so $f \mid C$ is right-conjugate to the inclusion of C in R. ∎

COROLLARY 3.3. *Let R be a 2-fir, C a commutative subring of R, and C' a finite integral extension of C in its field of fractions. Then there is an embedding f of C' in R such that $f \mid C$ is right conjugate to the inclusion of C.*

Proof. We can find a "common denominator" x for the elements of C' over C. Then $C'x$ will be a finitely generated ideal I of C, giving the situation of 3.1. Also, any two of our set of generators for I will have a nonzero common multiple in C, hence a common right multiple in R. Applying the 2-fir property repeatedly, we find that $(C'x)R$ will be principal, and 3.2 gives our desired result. ∎

In particular, this applies to the C of Proposition 2.2.

The following consequence is of independent interest:

PROPOSITION 3.4. *The center of a 2-fir is integrally closed (in its field of fractions).*

Proof. Apply the preceding Corollary to the center C of R and a finite integral extension thereof, C'. Any conjugacy map of R is the identity on the center C, hence the embedding f we get of C' in R will extend the inclusion of C itself; let us also identify C' with $f(C')$.

Any $x \in C'$ will satisfy an equation $ax = b$ $(a, b \in C)$, from which it can be deduced that x commutes with all members of R. So $C' = C$. ∎

4. **"Pulling back" the integral closure.** We shall prove a more general version of the following lemma elsewhere.

Let k be a field, $k[t]$ the polynomial ring in one indeterminate over k, and $k(t)$ the rational function field.

LEMMA 4.1 (LIFTING OF FACTORIZATIONS). *Let R be a k-algebra which is an integral domain, and remains one under tensoring with any algebraic extension of k. Then if $ab = c$ in the ring $R \otimes k(t)$, and c lies in the subring $R \otimes k[t]$, there exists nonzero $\lambda \in k(t)$ such that $\lambda^{-1}a$ and λb lie in $R \otimes k[t]$; and λ can be chosen as a function of a alone.*

Proof. Consider R as a k-vector-space. For every nonzero $a \in R \otimes k(t)$, there will exist a unique (up to units of k) $\lambda_a \in k(t)$ such that $\lambda_a^{-1}a$ belongs to $R \otimes k[t]$, and is not divisible by any nonunit of $k[t]$. Namely, we take for the denominator of λ_a the least common denominator of the coefficients in the expression for a in terms of any k-basis for R; and for its numerator, the greatest common divisor of their numerators. We can make λ_a unique by requiring numerator and denominator to be monic.

For a and b satisfying $\lambda_a = \lambda_b = 1$, we can see that $\lambda_{ab} \in k[t]$. But if λ_{ab} were divisible by any irreducible polynomial $P(t)$, the images of a and b in $R \otimes (k[t]/P(t))$ would have product zero, contradicting our hypothesis that such extensions are integral domains. So $\lambda_a = \lambda_b = 1$ implies $\lambda_{ab} = 1$, from which it follows that $\lambda_{ab} = \lambda_a \lambda_b$ identically. Hence if $ab = c$, and $c \in R \otimes k[t]$ (i.e., $\lambda_c \in k[t]$), then $\lambda_a^{-1}a$, and $\lambda_a b = \lambda_c(\lambda_b^{-1}b)$ both lie in $R \otimes k[t]$. ∎

Of the next two propositions, *either* will suffice for our application to centralizer rings. The argument for the second proposition is due to P. M. Cohn. It makes use of the right-left symmetry of the 2-fir condition, which we have not proved here, but we include it for interest. (The symmetry of the 2-fir condition is not difficult to prove: cf. [7, Proposition 5.1].)

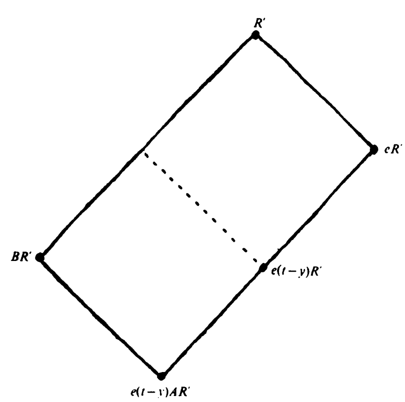
For any rings $Q \subseteq R$, we shall call an element $y \in R$ "right integral" over Q if the right Q-module generated in R by the powers of y is finitely generated; equivalently, if in the ring gotten by adjoining to R a commuting indeterminate t, there is a monic polynomial with coefficients in Q that is left divisible by $t - y$. We shall call Q right integrally closed in R if it contains all elements of R right integral over it.

PROPOSITION 4.2. *Let R be an algebra over a field k, such that $R \otimes k(t)$ is a 2-fir, and such that R remains an integral domain under tensoring with all finite algebraic extensions of k.*

Then for any nonzero $e \in R$, the ring ${}_eR = \{y \mid ey \in Re\}$ is right integrally closed in R.

Proof. Suppose $y \in R$ is right integral over ${}_eR$. Then we can find monic $A \in R \otimes k[t]$ such that $(t-y)A$ has coefficients in ${}_eR$. Then by the definition of ${}_eR$, we can write $e(t-y)A = Be$, where B lies in $R \otimes k[t]$ and is also monic.

Let $R' = R \otimes k(t)$. We claim that in this ring, e and B have no common left divisor, and $(t-y)A$ and e have no common right divisor. For by the preceding lemma (and its left-right dual) such a common right or left divisor could be taken to be a common divisor in $R \otimes k[t]$. Dividing e, it must lie in R, and dividing B (resp. $(t-y)A$) it must be monic. Hence it is 1.



Because R' is a 2-fir, and $Be=e(t-y)A$, the ideals $BR'+eR'$ and $BR' \cap eR'$ will both be principal. Since e and B have no common left divisor, the first ideal must equal R'. The second ideal contains $Be=e(t-y)A$; since e and $(t-y)A$ have no common right divisor, this must be its generator.

Our desired result $y \in {}_eR$ is trivial if $y \in k$. Assuming the contrary, we see by Lemma 4.1 that $t-y$ is not invertible in R' (cannot divide 1). Hence $e(t-y)R'$ is properly contained in eR'; hence from the lattice of ideals drawn above, we can see that $BR'+e(t-y)R'$ will be properly contained in R'.

This ideal will be principal by the existence of a common right multiple, its generator is thus a common left divisor of B and $e(t-y)$. Again by 4.1, this divisor can be taken to lie in $R \otimes k[t]$. Dividing B, it will be monic. Dividing $e(t-y)$, it will have degree 0 or 1. Being a nonunit, then, it must have degree 1. Let us write it $t-x$, and put $e(t-y)=(t-x)e'$. Clearly, $e'=e$, hence $ey=xe$, hence $y \in {}_eR$. ∎

Let us say a subring Q of an integral domain R is *left rationally* closed in R if it contains all elements y satisfying relations $ay=b$, where $a \neq 0$; $a, b \in Q$.

PROPOSITION 4.3. *Under the same hypotheses as Proposition* 4.2, ${}_eR$ *is left rationally closed in* R.

Proof. Say $y \in R$ with $ay=b$ as above. Left-multiplying by e, and recalling that $a, b \in {}_eR$, we get a relation $a'ey=b'e$. As ey and e have a common left multiple, $Rey \cap Re$ will be principal, say generated by $uey=ve$. Then u and v have no common left factor.

In $R'=R \otimes k(t)$, we have $ue(t-y)=(ut-v)e$. Hence $ueR'+(ut-v)R'$ will be principal. Its generator can, by Lemma 4.1, be taken to be a common left divisor of ue and $ut-v$ in $R \otimes k[t]$. Dividing the former, it must lie in R; dividing $ut-v$, it must thus divide u and v, hence it is a unit. So $ueR'+(ut-v)R'=R'$. Let us express 1 as a right linear combination of ue and $(ut-v)$ in R', and then clear denominators in $k(t)$: we get a relation of the form

$$ueP+(ut-v)Q = \phi,$$

where $P, Q \in R \otimes k[t]$, and ϕ is a nonzero element of $k[t]$, which we may assume monic, of degree n.

Now suppose P were of degree $m>n$, having leading term pt^m. Then the leading term of Q will be $-ept^{m-1}$. The product of $ut-v$ with this term will be left divisible by ue, since $ve=uey$; hence we can "transfer" this term to the expression ueP, with the effect of lowering the degrees of P and Q. (Explicitly, we rewrite our equation as $ue(P-pt^m-ypt^{m-1})+(ut-v)(Q-ept^{m-1})=\phi$.) Hence, inductively, we can reduce to the case where P has degree $\leqq n$, and Q degree $\leqq n-1$. Then, looking at coefficients of t^n, we get an equation $uep+uq=1$. Hence u is a unit, hence from the equation $uey=ve$, we get $ey=(u^{-1}v)e \in Re$, so $y \in {}_eR$. ∎

Either Proposition 4.2 or Proposition 4.3 yields:

COROLLARY 4.4. *Let R be as in* 4.2, *C a commutative subring of R, and $\bar{C}$ the integral closure of C in its field of fractions. Then the inclusion of C in R extends uniquely to an embedding of $\bar{C}$ in R.*

Proof. Let C' be any finitely generated subextension of C in $\bar{C}$. Then (by Corollary 3.3) we can find a nonzero $e \in R$ and an embedding $f: C' \to R$, such that f agrees with α_e on C.

Hence $f(C) \subseteq_e R$, and $f(C')$ is integral (and also rational) over $f(C)$. So by Proposition 4.2 (or 4.3), $f(C') \subseteq_e R$. But then $\alpha_e^{-1} \circ f$ will be an embedding of C' in R that extends the inclusion of C. The uniqueness of such an extension is clear because C' is rational over C.

An embedding of $\bar{C}$ can now be obtained as the direct limit of the embeddings of finitely generated subextensions, since these, by uniqueness, agree on their intersections. ▮

PROPOSITION 4.5. *Let R be a filtered ring with 2-term weak algorithm, such that R_0 is a field k in the center of R. Suppose R remains a 2-fir under tensoring with $k(t)$, and remains an integral domain under tensoring with all algebraic extensions of k. Then the centralizer subring of any nonunit u (already known to be a commutative finite integral extension of $k[u]$) will be integrally closed in its field of fractions.*

Proof. If C is the centralizer and $\bar{C}$ its integral closure, we can embed $\bar{C}$ in R by the preceding result. Being a commutative ring containing u, it will be contained in the centralizer of u, so $C = \bar{C}$. ▮

Further observations: if two maximal commutative subrings (equivalently: centralizer rings of nonunits), C and D, of an R as in 4.4 contain conjugate elements: $ue = ev$ ($u \in C, v \in D, e \in R - \{0\}$), then they must themselves be conjugate: $D = \alpha_e(C)$. For we note that C and D are integral extensions of $k[u]$ and $k[v]$, and then apply 4.2 as we did above. (More precisely, 4.2 and its left-right dual, for which we need the left-right symmetry of the 2-fir condition.)

A particularly interesting example may possibly arise as follows: If the centralizer C of a nonunit u of R as above is *not* of the form $k[z]$, as is presumably possible in some R's, one can show that it will be a Dedekind domain but not a principal ideal domain. (One here needs the "total ramification" observation of Proposition 2.2.) If I is a nonprincipal ideal of C, and we write $IR = eR$, we must have $e \notin C$. We find $C \subseteq R_e$, but α_e is not the identity on C, as that would imply that e commutes with u. Letting I run over representatives of the ideal class group of C, we get a *family* of conjugates of C, and an action of the ideal class group on this family!

5. **Mapping into polynomial rings—a lexicographic fantasy.** Let A be a totally ordered set, W the free semigroup, with identity 1, on the set A, represented as the set of all "words" in A, and let $\bar{W}$ be the set of all *right infinite* words in A (infinite sequences of elements of A). Given $u \in W - \{1\}$, let $u^\infty \in \bar{W}$ designate the word obtained by repeating u indefinitely: $uuu\ldots$. Let $\bar{W}$ be ordered lexicographically.

LEMMA 5.1. *Let $u, v \in W-\{1\}$. If $u^\infty > w^\infty$, then $u^\infty > (uv)^\infty > (vu)^\infty > v^\infty$, and similarly with "$>$" replaced by "$=$" or "$<$" throughout.*

Proof. It will suffice to show that the triple inequality shown is implied by $(uv)^\infty > (vu)^\infty$, and similarly for the corresponding equality and reversed inequality.

Suppose $(uv)^\infty > (vu)^\infty$. Then we note

$$(vu)^\infty = v(uv)^\infty > v(vu)^\infty = v^2(uv)^\infty > v^2(vu)^\infty = \cdots \to v^\infty,$$

giving us one of our desired inequalities, since lexicographic order is "continuous". We similarly find $(uv) < u^\infty$, QED. The cases of "$=$" and "$<$" are proved in exactly the same way. ▮

(**Another method of proof.** Reduce to the case where the ordered set A is finite, and identify it with one of the form $\{0, \dots, n-2\}$. Associate to every $u = a_1 a_2 \dots \in \overline{W} \cup W$ the "decimal fraction" $\boldsymbol{u} = \sum a_i n^{-i}$, and get a formula for $\boldsymbol{u}^\infty$ in terms of $\boldsymbol{u}$ and the length of u. We then find that we can express $(\boldsymbol{uv})^\infty$ as a convex linear combination of $\boldsymbol{u}^\infty$ and $\boldsymbol{v}^\infty$, with the former given higher weight.)

Let R be the semigroup algebra on W, over a field k. This is the free associative k-algebra on A. Given any *periodic* word $z \in \overline{W}$, let us define $R_{(z)}$ to be the k-subspace of R spanned by words u satisfying $u = 1$ or $u^\infty \leqq z$, and $I_{(z)}$ as the k-subspace spanned by words such that $u \neq 1$ and $u^\infty < z$. Using the above lemma, we can see that $R_{(z)}$ will be a subring of R, in which $I_{(z)}$ is a two-sided ideal.

The set of words in W with $u^\infty = z$, together with 1, will form the set of non-negative powers of an element v, the "least repeating segment" of z. It follows that $R_{(z)}/I_{(z)}$ will be isomorphic to a polynomial ring $k[v]$.

PROPOSITION 5.2. *Let C be a finitely generated subalgebra of a free associative algebra R over a field k. Then if $C \neq k$, there is a homomorphism f of C into the polynomial algebra over k in one indeterminate, such that $f(C) \neq k$.*

Proof. Let us totally order a set of free generators A of the algebra R, as above. Let G be a finite set of generators for C, and let z be the maximum over all monomials $u \neq 1$ occurring with nonzero coefficient in members of G, of u^∞. Then $G \subseteq R_{(z)}$, hence $C \subseteq R_{(z)}$, and the quotient map $f\colon R_{(z)} \to R_{(z)}/I_{(z)} \cong k[v]$ is nontrivial on C. ▮

This result is false if we do not assume the subalgebra C finitely generated. For instance, it can be shown that in any ring R, if I is the commutator ideal $R[R, R]R$, then $I^4 \subseteq I[I, I]$. Hence any *commutative* homomorphic image of the subalgebra $k+I$ of R is generated over k by elements whose fourth powers are zero, and hence cannot be a nontrivial subalgebra of $k[v]$.

(Here I^4 means $IIII$, that is, the abelian group generated by all products $ABCD$ ($A, B, C, D \in I$). The inclusion $I^4 \subseteq I[I, I]$ follows from $I^3 \subseteq R[I, I]$, which is proved as follows: we note that any element of I^3 is a sum of elements $A[b, c]D$ ($A, D \in I$; $b, c \in R$). Now modulo $R[I, I]$, we have $AbcD \equiv cDAb \equiv cAbD \equiv cbDA \equiv AcbD$.

For, each of these congruences is of the form $xYZ = xZY$ ($x \in R$, $Y, Z \in I$)—e.g., in the first step, $x=1$, $Y=Ab$, $Z=cD$. Hence $A(bc-cb)D \equiv 0$.)

THEOREM 5.3. *Let R be a free associative algebra over a field k. Then the centralizer ring of any nonscalar element of R is isomorphic to a polynomial algebra over k in one indeterminate.*

Proof. The centralizer C will be a finitely generated k-algebra by 1.5 and 2.2, hence by the above it may be mapped nontrivially into a polynomial algebra. The results of 2.2 and 4.5 (C a finite extension of $k[u]$ and integrally closed) together with the result of Cohn's quoted in our original sketch of the proof, now give the stated result. ▮

6. **Extensions of 5.2 and 5.3.** The method of the preceding section can be used to get results analogous to Proposition 5.2 (nontrivial map into $k[v]$) for a great many other rings than free algebras. The most general result is not easy to formulate; let us look at some examples:

Suppose that instead of taking a free algebra over a field k, we let k be an arbitrary ring, and R the ring generated over k by a set A of elements with relations $\alpha a = a\alpha^{\bar{a}}$ ($a \in A$, $\alpha \in k$), where for each $a \in A$, $\bar{a}$ is an endomorphism of k. Then we can deduce that any finitely generated subring of R can be mapped nontrivially into a ring generated over k by a single element v, with relations $\alpha v = v\alpha^r$, where r is a member of the semigroup of endomorphisms of k generated by the $\bar{a}$.

Let k be a field, ∂ a derivation on k, and R the ring having for right k-basis all monomials in x and y, multiplication being determined by equations $\alpha x = x\alpha + y\partial(\alpha)$, $\alpha y = y\alpha$. If we order $\{x, y\} = A$ by making $x > y$, we find that a product $(u\alpha)(v\beta)$ of monomial terms equals $uv\alpha\beta$ plus a sum of terms involving monomials w of the same length as uv, but lexicographically inferior, hence satisfying $w^\infty < (uv)^\infty$. We can here obtain the conclusion of 5.2 unchanged. We could also have set $\alpha x = x\alpha + y^2\partial(\alpha)$, for the operation of replacing x at some point in a word u by y^2 still gives a word w with $w^\infty < u^\infty$.

Suppose we set $\alpha x = x\alpha + \partial(\alpha)$ and again let $y < x$ commute with k. We note that u^∞ is decreased by dropping an x from u *unless* u is a power of x. We can deduce that every finitely generated subring of this R is either mappable into $k[v]$ or *contained* in the subring generated over k by x, which has a different structure. On the other hand, we cannot handle the ring generated by x and y with relations $\alpha x = x\alpha + \partial(\alpha)$, $\alpha y = y\alpha + \partial'(\alpha)$, for we cannot order $\overline{W}$ so that the operations of dropping an x from a word u and dropping a y from u both decrease u^∞.

However, because the rings constructed by the above methods do not have k in their centers, the methods of the preceding sections (especially 2 and 4) are not applicable to them, and it is not even too easy to conjecture what the centralizers of elements of such rings should be like (but cf. §9).

However, by a somewhat different route we can get a generalization of Proposition 5.2 which *can* be used to extend our main Theorem to some other rings!

Let B (for "Birkhoff-Witt") be a totally ordered set, and let us define an *ascending monomial* in B to be a formal product $x_1 \cdots x_n$ of elements of B, with $n>0$ and $x_1 \leqq \cdots \leqq x_n$. Given an arbitrary monomial u in B other than the empty product 1, let u^* designate the ascending monomial obtained from u by reordering the terms.

If $u=x_1 \cdots x_n$ is an ascending monomial, let ind u, "the index of u", designate the nondecreasing step function from the interval $[0, 1) \subseteq \boldsymbol{R}$ to B, sending $[0, 1/n)$ to $x_1, \ldots, [(n-1)/n, 1)$ to x_n. Let us order such functions "lexicographically"; i.e., ind $u <$ ind v if, for the least $r \in [0, 1)$ at which u and v differ, $(\text{ind } u)(r) < (\text{ind } v)(r)$.

Note that if u and v are ascending monomials, with ind $u >$ ind v, we will have ind $u >$ ind $(uv)^* =$ ind $(vu)^* >$ ind v, and similarly with "$>$" replaced by "$=$" and "$<$". We can also see that the set of ascending monomials u such that ind u is equal to a given ascending step function will (if nonempty) be of the form $\{(v^n)^*\}_{n=1,\ldots}$ for some v.

The peculiar way in which ind u was defined; in particular, the way the length was involved, makes it rather complicated to predict, given u, v and w, how ind $(wu)^*$ will compare with ind $(wv)^*$. Let us say that an ascending monomial u is "absolutely greater than" an ascending monomial v if, for any monomial w (including 1), ind $(wu)^* >$ ind $(wv)^*$. The reader can work out the explicit criterion for when this holds; it divides into three cases, according to the relative lengths of u and v. Our application (Corollary 6.2) will involve only one case.

PROPOSITION 6.1. *Let R be an algebra over a field k having a totally ordered set of generators B, such that* 1 *and the ascending monomials in B form a basis for R as a k-vector-space. Suppose that for every $y > x$ in B, yx, written in terms of this basis, has the form $\alpha_{yx}xy$ ($\alpha_{xy} \in k-\{0\}$) plus a k-linear combination of ascending monomials absolutely less than xy.*

Then any finitely generated subalgebra $C \neq k$ of R can be mapped into a polynomial algebra $k[v]$, so that $f(C) \neq k$.

Proof. It is easily seen by induction that for every monomial u in B, the expression for u in terms of the basis of ascending monomials will be $\alpha_u u^* +$ linear combination of monomials absolutely less than u.

Given a step function s with its steps at rational points, let $R_{(s)}$ designate the k-subspace of R spanned by 1 and the ascending monomials u with ind $u \leqq s$, and $I_{(s)}$ the subspace spanned by ascending monomials u with ind $u < s$. Again, $I_{(s)}$ is a 2-sided ideal of $R_{(s)}$.

Let v be the ascending monomial such that $\{v^{n*}\}_{n=1,\ldots} = \{u \mid \text{ind } u = s\}$. Then we see that the elements v^n ($n=0, 1, \ldots$. These are not ascending monomials!) will span $R_{(s)}$ over $I_{(s)}$ and be linearly independent modulo $I_{(s)}$. Hence the map of $k[v]$ into $R_{(s)}/I_{(s)}$ sending v to its image in this quotient is an isomorphism; so $R_{(s)}/I_{(s)}$ is a polynomial ring.

The proof is now completed exactly like that of 5.2. ∎

Now let us note that if we delete from an ascending monomial u some letter *not* equal to the first (minimal) letter, ind u is decreased (because the subinterval of $[0, 1)$ on which ind u has its minimal value is lengthened). From this it can be deduced that deleting a nonminimal letter from a monomial *absolutely* decreases its value. It is also clear that of two ascending monomials of the same length, the lexicographically smaller one is absolutely smaller. From this it follows that given $z \leqq x < y$ in B, z is absolutely less than xy. We can now prove:

COROLLARY 6.2. *Let L be a Lie algebra over a field k, having a k-basis B which can be partially ordered so that for all $x, y \in B$, the element $[x, y]$ is a k-linear combination of elements of B less than or equal to both x and y.*

Let R be the universal enveloping algebra of L. Then any finitely generated subalgebra $C \neq k$ of R can be mapped nontrivially into a polynomial algebra $k[v]$.

Proof. Let us strengthen the given partial ordering on B to a total ordering, in an arbitrary manner. The stated property is preserved.

By the Birkhoff-Witt theorem [3, §2, No. 7, Corollary 3, p. 33], the ascending monomials and 1 form a basis for R over k. Given $y > x$ in B, the expression for yx in terms of this basis is $xy - [x, y]$, and the terms of $[x, y]$, by our hypothesis and the above observations, will be absolutely less than xy. Hence Proposition 6.1 can be applied. ∎

In particular, if L is a Lie algebra graded by the positive integers, this grading, with sign changed, will give us an appropriate partial ordering on any basis of L consisting of homogeneous elements so the above corollary applies.

Now we recall that the free k-algebra R on a set A is the universal enveloping algebra of the free Lie algebra on A, which has a natural graded structure. Applying the above result to this Lie algebra, we obtain again Proposition 5.2.

Let k be a field and consider the k-algebra R with generators $x_1, \ldots, x_n, y_1, \ldots, y_n$ $(n \geqq 1)$ and a single relation, $\sum^n x_i y_i - y_i x_i = 0$. This will be the universal enveloping algebra of the graded Lie algebra with the relation $\sum [x_i, y_i] = 0$. One can show (although the proof is quite long, and we shall not give it here) that this ring satisfies $2n-2$-term weak algorithm. Since the above conditions put no restriction on k, this property will be retained under all base field extensions, and for $n \geqq 2$ we can prove as for free algebras that the centralizer of any nonscalar element of R will be a polynomial ring in one indeterminate over k.

The maps of a subring C of R into a polynomial ring constructed in 5.2 and 6.1 "only scratch the surface" of C, in a rather literal sense! One feels, in a free algebra at least, that if the map we get into $k[v]$ is not 1-1, one should be able to go "deeper", and get *something like* the following: Let us call a subring S of a free associative algebra R on n generators "full" if the image of S in R-made-abelian, i.e., $k[v_1, \ldots, v_n]$, has transcendence degree n over k.

CONJECTURE *Any finitely generated subalgebra of a free associative algebra is isomorphic to a full subalgebra of some free associative algebra.*

7. **The question for other rings—some examples to investigate.** It would be of interest to examine centralizers C in rings that satisfy only the conditions of Proposition 4.4, or the weaker conditions of 2.2, for examples which are not isomorphic to polynomial rings. These Propositions still put strong restrictions on C.

A number of constructions for rings with n-term weak algorithm (where we may take $n \geqq 2$) are given by Cohn in [11, see Theorem 5.2] and [13, last section]. The rings $U_{m,n}$ and $V_{m,n}$ of the first paper are notable for the fact that they cannot be mapped by *any* homomorphism into a "nonpathological" ring (e.g., any commutative ring—see Proposition 2.4 and Theorem 2.6 of that paper), so that they cannot possibly satisfy a result like our Proposition 5.2. These ring constructions each use a fixed presentation over an arbitrary base field; hence they will retain their properties under tensoring with any extension of k; hence they satisfy the hypotheses of 4.4.

We can find algebras which change their properties under base extension with the help of a result which I shall not prove here:

Let A and B be vector spaces over a field k. The "rank" of a member of $A \otimes B$ will mean the minimum number of summands needed to express it in the form $\sum a_i \otimes b_i$. If M is a subspace of $A \otimes B$, $r(M)$ will be defined as the minimum of the ranks of nonzero members of M.

Let U be the tensor algebra on $A \oplus B$, M a subspace of $A \otimes B$, and R the quotient of the ring U by the two-sided ideal generated by M (considered as contained in the degree-two part of U). *Then U satisfies $r(M)$-1-term, but not $r(M)$-term weak algorithm, and in fact is not an $r(M)$-fir.*

Now the function $r(M)$ is sensitive to base-field extension. That is, if we tensor with an extension field K/k, the K-subspace $M_K \subseteq A_K \otimes_K B_K$ may have $r_K(M_K) < r(M)$. The behavior of the function $r(M)$ is studied in [17]; in particular, it is seen that it may drop from $\geqq 3$ to 1 on tensoring with an appropriate extension. (Take $3 \leqq [K:k] < \infty$; for A and B use copies of the dual K^* of the k-space K, and for $M \subseteq A \otimes B$, the image of the comultiplication map: $K^* \to K^* \otimes K^*$.) The algebra R constructed as above from such M will satisfy 2-term weak algorithm, but $R \otimes K$ will fail to be a 1-fir, i.e., an integral domain, giving the desired example.

8. **The free algebra and its completion.** To an element of a free associative algebra R, we can associate not only its degree—the maximum of the degrees of its nonzero homogeneous components—but also its "order", the minimum of the degrees of these components. The order function will be a valuation on R—equivalently, its negative will be a ($\leqq 0$-valued) filtration. We can complete R with respect to this function, getting the ring $\hat{R}$ of "noncommuting formal power series".

The centralizer of an element of this ring can be studied by much more straightforward methods than in the noncompleted case. Cohn proves in [6] that the centralizer of any nonscalar element is a formal power series ring in one indeterminate.

(His proof works for any complete k-algebra R with valuation satisfying *inverse* 2-term weak algorithm (see [12], or cf. [6]), and such that k maps *onto* the residue ring: $k \xrightarrow{\sim} \mathrm{Gr}\, R_0$.

In the absence of the latter condition, one can still conclude that such a centralizer will be a complete not-necessarily-commutative discrete valuation ring—the last condition meaning that each element right and left divides all elements of the same or higher order.)

A question we have not been able to answer is:

Will the centralizer in $\hat{R}$ of an element $u \in R$ be the closure in $\hat{R}$ of its centralizer in R?

An equivalent question: Suppose $v \in \hat{R}$ is of positive order, and some nontrivial formal power series in v (nonscalar member of $k[[v]] \subseteq \hat{R}$) lies in R. Must there then exist a member of $k[[v]] \cap R$ whose order in v is 1, i.e., which has the same order in R as does v?

We have proved a lemma which might be useful in approaching this question: Given any equation $ab=c$, with $a, b \in \hat{R}-\{0\}$, $c \in R$, there exists an invertible (i.e., order-zero) $\alpha \in \hat{R}$ such that $a\alpha^{-1}$ and $\alpha b \in R$. In a stronger form: If we define a Galois connection [2, Chapter 4, §6] on $\hat{R} \times \hat{R}$ by the relation of "having product in R", then all pairs of corresponding *closed sets*, other than 0, $\hat{R}$ and $\hat{R}$, 0, are of the form $\hat{R}\alpha$, $\alpha^{-1}\hat{R}$, for invertible $\alpha \in \hat{R}$. Tarasov obtains some similar results in [15].

9. Centralizers in rings of differential operators.

Let K be a field of characteristic zero, ∂ a nonzero derivation on K, and k the subfield of constants; that is, the kernel of ∂. Let R be the ring gotten by adjoining to K an indeterminate y with the property $yc = cy + \partial(c)$ for all c in K. R can be considered the ring of formal differential operators generated by ∂ (represented by y) and the elements of K (representing multiplication operators). Such a ring satisfies the full weak algorithm with respect to the filtration by degree in y (cf. [5, Theorem 2.1]).

Proposition 2.2 cannot be applied to this ring because $R_0 = K$ does not lie in the center. However, some similar arguments are applicable:

The structure of R is such that $v(pq-qp) < v(p)+v(q)$. So if elements p and q respectively have leading terms by^m and cy^n, the term of degree $m+n-1$ in $pq-qp$ will equal the corresponding term of $(by^m)(cy^n)-(cy^n)(by^m)$, which comes to $(mb\partial(c)-nc\partial(b))y^{m+n-1} = \partial(c^m/b^n)b^{1+m}c^{1-n}y^{m+n-1}$. So if p and q commute, this must be zero, i.e., c^m/b^n must lie in k.

Let us fix p, and consider the consequences of this relation for its centralizer C.

If $p \in R-K$, so that $m>0$, then given n, we see that c is determined to within a factor in k. In other words, the k-dimension of C_n/C_{n-1} is $\leqq 1$. This allows us to apply Proposition 2.1 with k as the base-field, and conclude as in 2.2 that C is commutative, and in fact a finite algebraic extension of $k[p]$—a result first proved (in this generality) by Amitsur [1].

On the other hand, if $p \in K-k$, we see that $c^m/b^n=1/p^n \in k$, which entails $n=0$, whence we can deduce that $C=K$. Finally, if $q \in k$, C is of course R.

Now suppose K contains an element x such that $\partial(x)=1$. For instance, K might be the field $k(x)$, or the field of meromorphic functions in a neighborhood of zero in the complex plane, and ∂ ordinary differentiation. Let $p=x^2y-x$. Since $v(p)=1$, it is clear that the centralizer of p is precisely $k[p]$.

One calculates that $p^2=x^4y^2$. Hence for all $n>1$, $p^n \in Ry \subseteq {}_yR$. But p itself does not lie in ${}_yR$, for $yp=x^2y^2+xy+1 \notin Ry$. So ${}_yR$, containing p^2 and p^3 but not p, is not right integrally (nor left rationally) closed in R. Now, applying α_y^{-1} to $k[p] \cap {}_yR=k[p^2,p^3]$, we get what is easily seen to be the centralizer of $\alpha_y^{-1}(p^2)$, which is thus nonintegrally closed.

Let us, for definiteness, take $K=k(x)$. Which hypothesis of Proposition 4.2 (integral closure of ${}_eR$) fails? There can be no difficulty when we tensor with an algebraic extension of k: $K \otimes k'$ will simply be $k'(x)$, whence $R \otimes k'$ is a ring of the same sort as R, and so still an integral domain. So R must fail to remain a 2-fir on tensoring with $k(t)$. (And indeed, $k(x) \otimes k(t)$ is not a field, so we no longer have a ring of the "same sort".)

Let R' be the k-subalgebra of the above R generated by x and p. It inherits from R the property that the centralizer of any $u \notin K$ is a finite integral extension of $k[u]$. In fact, because $R' \cap K=k[x]$, one can show that this holds for all $u \notin k$. Now the defining relation between x and p in R' is $px=xp+x^2$. We can apply Proposition 6.1 to this ring and deduce that all centralizers of nonscalar elements are embeddable in polynomial rings in one indeterminate. But again these centralizers are not all integrally closed: one computes $(p-x)(p+x)=p^2$, whence $k[p^2,p^3] \subseteq {}_{p+x}R$, but $p \notin {}_{p+x}R$, and the preceding argument repeats itself.

For further discussion of centralizers in rings of differential operators $R=K(y;\partial)$, the reader should see Amitsur [1] and Flanders [14]. §7 of the latter contains an interesting example in which K is a field of Weierstrass $\wp$-functions, and a centralizer ring C has field of fractions isomorphic to K, hence is not embeddable in a polynomial ring.

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