HAAR SERIES

BY

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1. **Introduction.** The orthonormal system introduced by A. Haar has been studied by P. L. Ul′janov [21]–[28] and numerous other authors. It has been our object to continue these investigations using a classical real variable approach. The main body of this paper deals with Haar series, although there are results on Walsh, trigonometric, and other orthonormal series.

Because of the large number of individual results we will not try to describe them all here but only state one theorem from each section. We will first give some definitions and basic properties of the Haar system.

DEFINITION 1.1. Haar's orthonormal system $\{\chi_m(t)\}$ is defined as follows on $[0, 1]$: $\chi_0(t) \equiv 1$ and for $m = 2^n + k$ with $0 \leqq k < 2^n$, $n = 0, 1, \ldots$

$$\begin{aligned} \chi_m(t) &= 2^{n/2}, & t &\in (k/2^n, (k+1/2)/2^n), \\ &= -2^{n/2}, & t &\in ((k+1/2)/2^n, (k+1)/2^n), \\ &= 0, & t &\in [0, 1]\backslash[k/2^n, (k+1)/2^n], \end{aligned}$$

and at the three remaining points we let $\chi_m(t)$ be equal to the average of the right and left hand limits.

DEFINITION 1.2. For $f(t) \in L(0, 1)$ we call $a_m(f) = \int_0^1 f(t)\chi_m(t)\,dt$, $m = 0, 1, \ldots$, the Haar-Fourier coefficients of f and $\sum_{m=0}^{\infty} a_m(f)\chi_m(t)$ the Haar-Fourier series of f.

It follows easily [1, p. 49] that for $S_m(t_0) = \sum_{i=0}^{m} a_i(f)\chi_i(t_0)$, $m = 2^n + k$, $0 \leqq k < 2^n$, $n = 0, 1, \ldots$, we have

(i) If t_0 is a dyadic irrational, then

$$S_m(t_0) = \frac{1}{|I_m|}\int_{I_m} f(t)\,dt \tag{1.1}$$

where $t_0 \in I_m$, which is an interval of length 2^{-n} or 2^{-n-1}.

(ii) If $t_0 = p/2^q$ is a dyadic rational, then for $m \geqq 2^q$

$$S_m(t_0) = \frac{1}{2|I_m|}\int_{I_m} f(t)\,dt + \frac{1}{2|J_m|}\int_{J_m} f(t)\,dt \tag{1.2}$$

where $I_m = (t_0 - \alpha, t_0)$, $J_m = (t_0, t_0 + \beta)$, and where $\alpha = \beta = 2^{-n-1}$ or $\alpha = 2^{-n-1}$ and $\beta = 2^{-n}$.

The above facts lead to the following results of A. Haar [1, p. 47]:

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THEOREM A. *If $f(t)$ is integrable on* (0, 1), *then the Haar-Fourier series of $f(t)$ converges to $f(t)$ a.e.*

THEOREM B. *If $f(t)$ is integrable on* (0, 1), *then the Haar-Fourier series of $f(t)$ converges to $f(t)$ at every point of continuity of $f(t)$, and converges uniformly in every interval in which $f(t)$ is uniformly continuous.*

In view of equations (1.1) and (1.2) and properties of approximately continuous functions [13, p. 292] we also have the following:

THEOREM 1.1. *If $f(t)$ is bounded and approximately continuous on* [0, 1], *then the Haar-Fourier series of $f(t)$ converges to $f(t)$ on* [0, 1].

THEOREM 1.2. *If t_0 is a dyadic irrational and $f(t)$ is integrable on* (0, 1) *and is bounded and has an approximate limit at t_0, then the Haar-Fourier series of $f(t)$ converges at t_0.*

THEOREM 1.3. *If t_0 is a dyadic rational and $f(t)$ is integrable on* (0, 1) *and is bounded and has an approximate right and left hand limit at t_0, then the Haar-Fourier series of $f(t)$ converges at t_0.*

In Theorem 1.1 the convergence is not uniform in general. In fact, since the Haar functions are continuous at dyadic irrationals and have right and left hand limits everywhere, we have

THEOREM 1.4. *If $\sum_{i=0}^{m} a_i\chi_i(t)$ converges to $f(t)$ uniformly on* [0, 1], *then $f(t)$ is continuous at dyadic irrationals and has right and left hand limits everywhere.*

REMARK 1.1. The condition of boundedness in Theorem 1.1 cannot be omitted since approximate continuity at t_0 does not imply that $S_m(t_0)$ converges to $f(t_0)$.

We now give a representative theorem from each of the sections which follow.

In §2 we shall study absolute convergence of Haar series and prove that the Haar-Fourier series of a function of bounded variation converges absolutely at all dyadic rationals and that for every dyadic irrational there exists an absolutely continuous function whose Haar-Fourier series diverges absolutely for that number.

In §3 we investigate relationships between the numbers a_m and the function $f(t)$ which represents the sum of the series $\sum a_m\chi_m(t)$ and prove that if $f(t)$ has the Darboux property and $a_m = o(m^{-3/2})$, then f is a constant.

In §4 we will give some applications of Haar series to general orthonormal series and show that if $\{\phi_m(t)\}$ is o.n. on $[a, b]$ and M_m denotes the supremum of the mth function, then

(i) $\int_a^b f(t)\phi_m(t)\,dt = o(M_m)$ for every integrable f.

(ii) For unbounded complete orthonormal systems this result may not in general be improved.

2. **Absolute convergence of Haar series.** We shall consider here some results concerning the convergence of the series

(2.1) $$\sum_{m=0}^{\infty} |a_m(f)\chi_m(t)|$$

and

(2.2) $$\sum_{m=0}^{\infty} |a_m(f)|.$$

Z. Ciesielski and J. Musielak proved [4, p. 62, Theorem 1]

THEOREM C. *If* $\sum_{n=1}^{\infty} \omega(f, 2^{-n}) < \infty$, *where* $\omega(f, \delta)$ *denotes the modulus of continuity of* f, *then* (2.1) *converges uniformly on* [0, 1].

If we assume that $f(t)$ is only continuous on [0, 1], then the conclusion in Theorem C is not true. In fact A. M. Olevskiĭ has shown [14, p. 1382] that for every complete orthonormal system there is a continuous function whose Fourier series is absolutely divergent a.e.

The following problem was posed by Ciesielski and Musielak [4, p. 65]: Does the convergence of (2.2) imply the uniform convergence of (2.1), $a_m(f)$ being the coefficients of a continuous function?

The answer to this question is negative (as Theorem 2.2 will show). However, in view of the fact that $|\chi_m(t)| \leqq m^{1/2}$, we have at once

THEOREM 2.1. *If* $\{a_m\}$ *is a sequence such that* $\sum |a_m| m^{1/2}$ *converges, then* $\sum |a_m \chi_m(t)|$ *converges uniformly on* [0, 1].

REMARK 2.1. Theorem 2.1 does not appear to be very sharp in view of the result proved by P. L. Ul'janov [21, p. 45]: If $\{a_m\}$ is a sequence such that $\sum |a_m| m^{-1/2}$ converges, then $\sum |a_m \chi_m(t)|$ converges a.e. on [0, 1]. However this condition is far from sufficient for the convergence of $\sum |a_m \chi_m(t)|$ everywhere on [0, 1]. In fact, Ciesielski and Musielak noted [4, p. 64]: "The uniform convergence of $\sum |a_m \chi_m(t)|$ does not imply the convergence of $\sum |a_m|$ and conversely."

In order to show that Theorem 2.1 is sharp and to answer the question of Ciesielski and Musielak we require

LEMMA 2.1. *If* $\{\varepsilon_m\}$ *is a sequence such that* $\lim \varepsilon_{2^m} = 0$, *then there exists a bounded function* f, *constant on each interval of a collection whose union is* [0, 1], *such that*

(2.3) $$\sum |a_m(f) m^{1/2} \varepsilon_m| < \infty$$

and

(2.4) $$\sum |a_m(f)\chi_m(0)| = \infty.$$

Proof. Choose a sequence $\{c_m\}$ such that $\sum c_m$ is conditionally convergent and $\sum |c_{m+1} \varepsilon_{2^m}| < \infty$. For example, if $|\varepsilon_{2^m}| < 2^{-n}$ for $m \geqq N_n$, we may let $c_{m+1} = (-1)^n/n$ if $m+1 = N_n$ and 0 elsewhere.

Define $f(t)$ on $[0, 1]$ as follows:

$$f(t) = c_0+c_1+\cdots+c_{n-1}-c_n \quad \text{on } (2^{-n}, 2^{-n+1}), \qquad n = 1, 2, \ldots, \tag{2.5}$$
$$= 0 \qquad \text{otherwise.}$$

Then

$$\int_0^1 f(t)\,dt = \lim_{n\to\infty} [(c_0-c_1)/2+(c_0+c_1-c_2)/4+\cdots+(c_0+\cdots+c_{n-1}-c_n)/2^n]$$
$$= \lim\,[(1-1/2^n)c_0-(c_1+c_2+\cdots+c_n)/2^n] = c_0,$$
$$\int_0^{1/2} f(t)\,dt-\int_{1/2}^1 f(t)\,dt = -(c_0-c_1)/2+(c_0+c_1-c_2)/4+\cdots = c_1,$$
$$\cdots$$
$$\int_0^{1/2^{n-1}} f(t)\,dt-\int_{1/2^{n-1}}^{1/2^n} f(t)\,dt$$
$$= -(c_0+\cdots+c_{n-1}-c_n)/2^{n-1}+\cdots = c_{n+1}/2^n, \qquad n = 1, 2, \ldots.$$

Consequently

$$a_0(f) = \int_0^1 f(t)\,dt = c_0,$$
$$a_1(f) = \int_0^{1/2} f(t)\,dt-\int_{1/2}^1 f(t)\,dt = c_1,$$
$$\cdots$$
$$a_{2^n}(f) = 2^{n/2}\left[\int_0^{1/2^{n+1}} f(t)\,dt-\int_{1/2^{n+1}}^{1/2^n} f(t)\,dt\right] = c_{n+1}/2^{n/2} \qquad (n = 0, 1, \ldots),$$

and $a_m(f)=0$ otherwise.

Therefore

$$\sum_{m=1}^{\infty} |a_m(f)m^{1/2}\varepsilon_m| = \sum_{m=0}^{\infty} |c_{m+1}\varepsilon_{2^m}| < \infty$$

and

$$\sum_{m=1}^{\infty} |a_m(f)\chi_m(0)| = \sum_{m=1}^{\infty} |c_m| = \infty$$

which establishes our lemma.

REMARK 2.2. A similar argument would show that Lemma 2.1 remains valid if we replace $t=0$ in condition (2.4) by any $t_0 \in [0, 1]$.

We shall now prove

THEOREM 2.2. *If $\varepsilon_m \downarrow 0$ and $\sum \varepsilon_m/m<\infty$, then there exists a continuous function g, linear on each interval of a collection whose union is* $[0, 1]$, *such that*

$$\sum |a_m(g)|m^{1/2}\varepsilon_m < \infty \tag{2.6}$$

and

$$\sum |a_m(g)\chi_m(0)| = \infty. \tag{2.7}$$

Proof. For clarity the proof will be divided into three parts. In part (i) we will show that the step function f defined in Lemma 2.1 can be "connected" at the jumps producing a continuous function h such that

$$2^{n/2}|a_{2^n}(f-h)| \leqq 2^n \int_0^{1/2^n} |f-h| \leqq K\cdot 2^{-n}, \qquad n = 1, 2, \ldots, \tag{2.8}$$

where K is an arbitrary positive constant.

In part (ii) we construct a continuous function g such that

$$f(t) \leqq g(t) \leqq h(t) \tag{2.9}$$

and

$$\sum |a_m(f-g)m^{1/2}\varepsilon_m| < \infty, \qquad m > 0, \quad m \neq 2^n, \quad n = 0, 1, \ldots. \tag{2.10}$$

In part (iii) we show how the above relations imply that g is our desired function.

Part (i). Let $\{c_m\}$ and f be as defined in Lemma 2.1 for the sequence $\{\varepsilon_m\}$ (actually $\sum c_m$ may be any conditionally convergent series). Assume (without loss of generality) that $\sum_{m=0}^{\infty} c_m = 0$.

Define $h(t)$ on $[2^{-n}, 2^{-n+1}]$ for $n = 1, 2, \ldots$ as follows:

$$\begin{aligned} h(t) &= 0, \qquad t = 2^n, 2^{-n+1}, \\ &= c_0 + \cdots + c_{n-1} - c_n, \qquad t \in I_j \equiv [2^{-n}+2^{-j_n}, 2^{-n+1}-2^{-j_n}] \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{where } j_n > n \geqq 1, \\ &= \text{linear otherwise.} \end{aligned} \tag{2.11}$$

Now defining $h(0) = 0$, we see that $h(t)$ is a continuous function.

We now choose the sequence $\{j_n\}$ so that (2.8) is satisfied.

Letting K be an arbitrary positive constant, we may choose $\{j_n^1\}_{n=1}^{\infty}$ such that $\int_0^1 |h_1 - f| \leqq K$, where h_1 is the function defined in (2.11) corresponding to $\{j_n^1\}$.

Choose $\{j_n^2\}_{n=2}^{\infty}$ such that $j_n^2 \geqq j_n^1$ for $n = 2, 3, \ldots$ and

$$2\int_0^{1/2} |h_2 - f| \leqq K/2,$$

where h_2 is the function defined in (2.11) corresponding to $\{j_n^2\}$.

Clearly, we may define for $m = 2, 3, 4, \ldots$ $\{j_n^m\}_{n=m}^{\infty}$ and the corresponding h_m such that $j_n^m \geqq j_n^{m-1}$ $(n = m, m+1, \ldots)$ and

$$2^{m-1}\int_0^{1/2^{m-1}} |h_m - f| \leqq K/2^{m-1}.$$

Setting now $\{j_n^n\} = \{j_n\}$, the corresponding h satisfies (2.8).

Part (ii). We now define g by the formula (2.11), replacing $\{j_n\}$ by a sequence of integers $\{k_n\}$ such that $k_n \geqq j_n$, which implies (2.9), the sequence to be chosen such that (2.10) is valid.

Setting now $E_n = \{m : \chi_m(t) = 0 \text{ if } t \notin [2^{-n}, 2^{-n+1}]\}$ for $n = 1, 2, \ldots,$ we shall show

$$\sum_{m\in E_n} |a_m(g)m^{1/2}\varepsilon_m| < K/2^n \tag{2.12}$$

for a suitable sequence $\{k_n\}$.

Since $2^n+1=\min\{m : m\in E_n\}$, it follows that

$$\begin{aligned} a_{2(2^n+1)}(g)/2^{(n+1)/2} &= a_{2^2(2^n+1)}(g)/2^{(n+2)/2} = \cdots \\ &= a_{2^{k_n-n-1}(2^n+1)}(g)/2^{(k_n-1)/2} \\ &= -[c_0+c_1\cdots+c_{n-1}-c_n]/2^{k_n+1}. \end{aligned}$$

Therefore, by symmetry considerations

$$\begin{aligned} &\sum_{m\in E_n, m<2^{k_n}} |a_m(g)m^{1/2}\varepsilon_m| \\ (2.13)\qquad &= \frac{2|c_0+\cdots+c_{n-1}-c_n|}{2^{k_n+1}}[2^{(n+1)/2}(2^{n+1}+2)^{1/2}\varepsilon_{2(2^n+1)}+\cdots] \\ &\leqq 2^{3/2}|c_0+\cdots|\cdot[(\varepsilon_{2^n+1})/2^{k_n-n}+(\varepsilon_{2^n+2})/2^{k_n-n-1}+\cdots+\varepsilon_{2^{k_n}-1}]. \end{aligned}$$

Letting now $m=2^{k_n-n}(2^n+1)$, we obtain

$$\begin{aligned} a_m(g) &= 2^{(k_n)/2}\left[\int_{2^{-n}}^{2^{-n}+2^{-k_n-1}} g(t)\,dt-\int_{2^{-n}+2^{-k_n-1}}^{2^{-n}+2^{-k_n}} g(t)\,dt\right] \\ &= -2^{(k_n)/2}[c_0+\cdots+c_{n-1}-c_n]/2^{k_n+2} \\ &= -[c_0+\cdots+c_{n-1}-c_n]/2^2\cdot 2^{(k_n)/2}, \\ a_{2m}(g) &= a_{2m+1}(g) = -[c_0+\cdots+c_{n-1}-c_n]/2^3\cdot 2^{(k_n+1)/2}, \\ a_{4m} = a_{4m+1} &= a_{4m+2} = a_{4m+3} = -[c_0+\cdots+c_{n-1}-c_n]/2^4\cdot 2^{(k_n+2)/2}, \\ &\cdots \end{aligned}$$

Hence by symmetry considerations,

$$(2.14)\qquad \sum_{m\in E_n, m\geqq 2^k} |a_m(g)m^{1/2}\varepsilon_m| \leqq 2^{-1/2}|c_0+\cdots+c_{n-1}-c_n|\cdot[\varepsilon_{2^{k_n}}+\varepsilon_{2^{k_n+1}}+\cdots].$$

Applying now the fact that $\sum \varepsilon_{2^m}$ converges (which is equivalent to $\sum \varepsilon_m/m$ converges) to (2.13) and (2.14), we may choose k_n sufficiently large and obtain (2.12).

Therefore

$$\begin{aligned} \sum_{m>0, m\neq 2^n, n=0,1,\ldots} |a_m(f-g)m^{1/2}\varepsilon_m| &= \sum_{m\in E_n, n=1,2,\ldots} |a_m(f-g)m^{1/2}\varepsilon_m| \\ &= \sum_{m\in E_n, n=1,2,\ldots} |a_m(g)m^{1/2}\varepsilon_m| < K \end{aligned}$$

which is exactly relation (2.10).

Part (iii). Relations (2.4), (2.8), and (2.9) imply

$$\sum |a_m(g)\chi_m(0)| = \sum |a_{2^n}(g)\cdot 2^{n/2}| = \infty.$$

Relations (2.3), (2.8), (2.9), and (2.10) imply

$$\sum |a_m(g)m^{1/2}\varepsilon_m| < \infty.$$

This completes the proof.

COROLLARY 2.1. *For every $\varepsilon > 0$, there exists a continuous function g, differentiable except on a denumerable set, such that*

$$\sum |a_m(g)|m^{1/2-\varepsilon} < \infty \quad \text{and} \quad \sum |a_m(g)\chi_m(0)| = \infty.$$

Proof. Let $\{\varepsilon_m\} = \{m^{-\varepsilon}\}$ in Theorem 2.2.

Also, since $|a_m(g)| \leqq 2^{1/2} \cdot M \cdot m^{-1/2}$ where $M = \max |g(t)|$, we obtain

COROLLARY 2.1′. *For every $\varepsilon > 0$, there exists a continuous function g, differentiable except on a denumerable set, such that*

$$\sum |a_m(g)|^{1+\varepsilon} m^{1/2} < \infty \quad \text{and} \quad \sum |a_m(g)\chi_m(0)| = \infty.$$

REMARK 2.3. A similar argument would show that Theorem 2.2 remains valid if we replace $t=0$ in condition (2.7) by any $t_0 \in [0, 1]$.

REMARK 2.4. It is known [5, p. 154] that if g is continuous on $I=[0, 1]$ and $|g'(t)| \leqq M$ on $I-D$ where D is a denumerable subset, then $g(t)$ satisfies a Lipschitz condition of order 1. Also it is easily seen from Theorem C [4, p. 62] that if g satisfies a Lipschitz condition of order $\alpha > 0$, then $\sum |a_m(g)\chi_m(t)|$ converges uniformly on [0, 1]. Thus for any function g satisfying Theorem 2.2 we must have $|g'(t)|$ unbounded on $I-D$.

REMARK 2.5. We do not know if the condition $\sum \varepsilon_m/m < \infty$ in Theorem 2.2 can be omitted.

REMARK 2.6. In Corollary 2.4 we shall prove that the function g in Theorem 2.2 cannot be differentiable at the origin, and in Theorem 2.11 we will show that the function g in Theorem 2.2 cannot be of bounded variation at the origin.

Letting V denote the class of functions of bounded variation on [0, 1], we now state the following theorem due to P. L. Ul′janov [23, p. 374].

THEOREM D. *If $f \in V$, then*

$$\sum |a_m(f)|m^\beta < \infty \quad \textit{for all } \beta < 1/2.$$

For $\beta = 1/2$ the theorem is false.

We note here that by modifying Ul′janov's proof slightly we obtain a stronger

THEOREM D′. (i) If $f \in V$, *then*

$$\sum |a_m(f)|m^{1/2}\varepsilon_m < \infty$$

for every sequence $\{\varepsilon_m\}$, $\varepsilon_m \downarrow 0$ such that $\sum \varepsilon_{2^n} < \infty$.

(ii) *If $\varepsilon_m \downarrow 0$ and $\sum \varepsilon_{2^n} = \infty$, then there exists $f \in V$ such that*

$$\sum |a_m(f)|m^{1/2}\varepsilon_m = \infty.$$

After we had solved the problem of Ciesielski and Musielak by Theorem 2.2, a paper appeared by Wang Si-Lei [18, p. 222] in which he proved the existence of an absolutely continuous function (and hence of bounded variation) whose Haar-Fourier series was not absolutely convergent at $t=2/3$. According to Theorem D with $\beta=0$, this produces a solution to the problem. We note here that in Theorem 2.2 we constructed a function of unbounded variation which was not absolutely convergent at $t=0$.

Wang's proof, although very interesting, relies upon a lemma of Bosanquet and Kestelman and is an existence proof only. Motivated by Wang's paper, we now apply the technique developed in the proof of Theorem 2.2 to actually exhibit such a function.

More precisely, we now prove (by construction)

THEOREM 2.3. *There exists a monotone and absolutely continuous function h, differentiable except at 2/3, such that*

$$\sum |a_m(h)| m^{1/2}\varepsilon_m < \infty \quad \textit{for every } \varepsilon_m \downarrow 0 \textit{ satisfying} \sum \varepsilon_{2^n} < \infty \tag{2.15}$$

and also

$$\sum |a_m(h)\chi_m(2/3)| = \infty. \tag{2.16}$$

Proof. According to Theorem D′, relation (2.15) is automatically satisfied for any absolutely continuous function. Hence we need only construct an absolutely continuous function of the given type which satisfies (2.16).

Now select a sequence $\{c_m\}$ such that $\sum c_m$ is conditionally convergent, and let $\{k_n\}$ be that sequence of integers satisfying

$$(k_n)/2^n \leqq 2/3 < (k_n+1)/2^n, \qquad n = 0, 1, \ldots.$$

We now define an infinite sequence of disjoint intervals as follows:

$$\begin{aligned} E_n &= \{t : (k_n)/2^n \leqq t < (k_n+1/2)/2^n\}, && n \text{ even}, \\ &= \{t : (k_n+1/2)/2^n < t \leqq (k_n+1)/2^n\}, && n \text{ odd}. \end{aligned}$$

Notice that for $n=0, 1, \ldots$ we have that E_{2n+2} is contiguous with and lies to the right of E_{2n}, and E_{2n+3} is contiguous with and lies to the left of E_{2n+1}.

Now define $f(t)$ on $[0, 1]$ as follows:

$$\begin{aligned} f(t) &= c_0+\cdots+c_{n-1}-c_n, \qquad t \in E_{n-1}, \quad n = 1, 2, \ldots, \\ &= \sum_{n=0}^{\infty} c_n, \qquad t = 2/3 = \sum_{0}^{\infty} 2^{-2n-1}. \end{aligned}$$

Then, as in the proof of Lemma 2.1, we obtain

$$\sum |a_m(f)\chi_m(2/3)| = \sum |c_n| = \infty. \tag{2.17}$$

If we further assume that $\sum c_n$ is an alternating series with $|c_n| \downarrow 0$, we obtain that $f(t)$ is actually monotone as can easily be seen by comparing values on adjacent intervals.

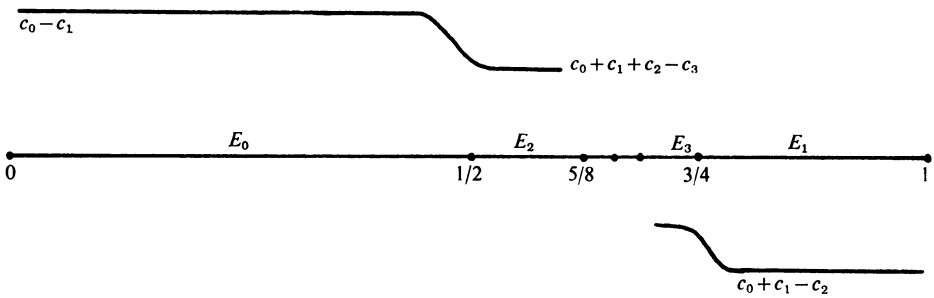

FIGURE 1

Analogous to part (i) of the proof of Theorem 2.2, we now "smooth" out our step function f at the jumps (see Figure 1) to give us a continuous function h, differentiable except at 2/3, satisfying (2.16), i.e. $\sum |a_m(h)\chi_m(2/3)| = \infty$.

Since $h(t)$ is monotone and absolutely continuous outside of every neighborhood of $t = 2/3$ our result follows.

We note here that the choice of 2/3 is not essential in Wang's argument and that his argument may be improved to include every dyadic irrational number. Thus we shall prove

THEOREM 2.4. *For every dyadic irrational number t_0 there exists an absolutely continuous function f such that* $\sum |a_m(f)\chi_m(t_0)| = \infty$.

Proof. Following Wang we shall take advantage of the following result of L. S. Bosanquet and H. Kestelman [18, p. 222]:

LEMMA A. *Let* $\{\phi_n(t)\}$ $(n=1, 2, \ldots)$ *be a sequence of continuous functions on* $[0, 1]$. *If for each* $\psi \in L(0, 1)$

$$\sum_{n=1}^{\infty} \left| \int_0^1 \psi(t)\phi_n(t)\,dt \right| < \infty,$$

then $\sum_{n=1}^{\infty} |\phi_n(t)|$ *is bounded on* $[0, 1]$.

We now let $t_0 = \sum_{n=0}^{\infty} s_n 2^{-n}$ be a dyadic irrational (where $s_n = 0$ or 1),

$$\theta_m(t) = \int_0^t \chi_m(u)\,du, \qquad m = 0, 1, \ldots,$$

and $k_n = k_n(t_0)$ a positive integer satisfying $(k_n)/2^n \leqq t_0 < (k_n+1)/2^n$. Then for any absolutely continuous f,

$$a_m(f) = \int_0^1 f(t)\chi_m(t)\,dt = -\int_0^1 f'(t)\theta_m(t)\,dt.$$

If for every absolutely continuous f, we have

$$\sum |a_m(f)\chi_m(t_0)| = \sum \left| \int_0^1 f'(t)\theta_m(t)\,dt \right| \cdot |\chi_m(t_0)|$$
$$= \sum_n \left| \int_0^1 f'(t)\theta_{2^n+k_n}(t)\,dt \right| 2^{n/2} < \infty,$$

then by Lemma A the series

$$\sum |\theta_{2^n+k_n}(t)| \cdot 2^{n/2} = \sum \theta_{2^n+k_n}(t) \cdot 2^{n/2} \tag{2.18}$$

must be bounded on [0, 1].

Choose now a subsequence $\{s_{n_j}\}$ of the sequence $\{s_n\}$ such that $s_{n_j+1}=0$ and $s_{n_j+2}=1, j=1, 2, \ldots$. Then

$$0 < t_0 - \sum_{n=0}^{n_j} s_n 2^{-n} = \sum_{n_j+1}^{\infty} s_n 2^{-n} = \sum_{n_j+2}^{\infty} s_n 2^{-n} < 2^{-n_j-1}.$$

Hence,

$$\theta_{2^{n_j}+k_{n_j}}(t_0) = \left[t_0 - \sum_{n=0}^{n_j} s_n 2^{-n}\right] 2^{(n_j)/2} = 2^{(n_j)/2} \sum_{n_j+2}^{\infty} s_n 2^{-n}$$

and so

$$\sum_{j=1}^{\infty} 2^{(n_j)/2}\theta_{2^{n_j}+k_{n_j}}(t_0) = \sum_{j=1}^{\infty} 2^{n_j} \sum_{n=n_j+2}^{\infty} s_n 2^{-n} \geqq \sum_{j=1}^{\infty} 2^{n_j} \cdot 2^{-n_j-2} = \sum_{j=1}^{\infty} 1/4 = \infty.$$

Thus series (2.18) diverges at t_0.

REMARK 2.7. Theorem 2.4 is false for dyadic rationals. In Theorem 2.11 we will prove that the Haar-Fourier series of a function of bounded variation converges absolutely at all dyadic rationals.

It is easily shown [23, p. 360] that if $|f(t)| \leqq M$ for all $t \in [0, 1]$, then $|a_m(f)| \leqq 2^{1/2} M \cdot m^{-1/2}$. Although the converse of this result is not true, we do have

THEOREM 2.5. (i) *If* $|a_m| \leqq K\varepsilon_m m^{-1/2}$, *where* K *is a constant and* $\{\varepsilon_m\}$ *is a quasi-monotone sequence, i.e.* $(\varepsilon_m)/m^{\beta} \downarrow 0$ *for some* β; *such that* $\sum \varepsilon_{2^n} < \infty$, *then* $\sum a_m\chi_m(t)$ *is uniformly absolutely convergent and hence is the Haar-Fourier series of a bounded function.*

(ii) *If* $\{\varepsilon_m\}$ *is a quasi-monotone sequence such that* $\sum \varepsilon_{2^n} = \infty$, *then there exists a series* $\sum a_m X_m(t)$ *which is absolutely divergent everywhere and which is not the Fourier series of a bounded function although* $|a_m| \leqq K\varepsilon_m m^{-1/2}$.

Proof. (i) If $t_0 \in [0, 1]$, then from (2.20), which appears shortly, it follows that

$$\sum_{m=1}^{\infty} |a_m\chi_m(t_0)| \leqq \sum_{n=1}^{\infty} K\varepsilon_{2^n+k_n} \leqq K' \sum_{n=1}^{\infty} \varepsilon_{2^n} < \infty$$

where $\{k_n\}$ is some sequence such that $0 \leqq k_n < 2^n$.

(ii) Setting $a_m=(\varepsilon_{2^{n+1}})2^{-n/2}$, $2^n\leqq m<2^{n+1}$, for $n=0, 1, \ldots$, we obtain

$$\sum_{m=1}^{\infty} a_m\chi_m(t) = \sum_{n=1}^{\infty} \varepsilon_{2^{n+1}} r_n(t) \quad \text{on } (0, 1)$$

where $\{r_m(t)\}$ are the Rademacher functions [1, p. 51]. But then $\sum a_m\chi_m(t)$ is absolutely divergent everywhere and the Rademacher series above does not represent an essentially bounded function since $\sum \varepsilon_{2^n}$ diverges [9, p. 455]. But since the Haar-Fourier series of an integrable function f converges a.e. to f (Theorem A), we must have that the Haar-Fourier series of a bounded function is essentially bounded.

The relation on the numbers $|a_m|$ follows from (2.20).

REMARK 2.8. If $|a_m|\leqq K\varepsilon_m m^{-1/2}$ where $\{\varepsilon_m\}$ is quasi-monotone and $\sum (\varepsilon_{2^n})^2<\infty$, then $\sum a_m\chi_m(t)$ is a Haar-Fourier series since, by (2.21), we then have $\sum (a_m)^2<\infty$.

REMARK 2.9. The inequality $|a_m|\leqq K\varepsilon_m m^{-3/2}$ for $\varepsilon_m \downarrow 0$, $\sum \varepsilon_{2^n}=\infty$ does not imply that $\sum a_m\chi_m(t)$ is the Haar-Fourier series of a function of bounded variation. In fact, setting

$$a_m = (-1)^n\varepsilon_{2^n}2^{-3n/2} \quad \text{for } 2^n \leqq m < 2^{n+1}, \qquad n = 0, 1, \ldots,$$

we obtain that

$$\sum_{m=1}^{\infty} a_m\chi_m(t) = \sum_{n=1}^{\infty} (-1)^n\varepsilon_{2^n}2^{-n}r_n(t) \quad \text{on } (0, 1). \tag{2.19}$$

But, by a result of A. I. Rubinstein [17, p. 143], the function represented by series (2.19) is not of class Lip (1, 1), i.e., is not equivalent to a function of bounded variation.

P. L. Ul'janov has proved [23, p. 378], [21, p. 45]; see also [22, p. 439].

THEOREM E. *If* $a_m \downarrow 0$, *then for* $\sum |a_m\chi_m(t)|$ *to be uniformly convergent on* [0, 1] *it is both necessary and sufficient that one of the following conditions be satisfied*:

(a) $\sum a_m m^{-1/2}<\infty$,

(b) $\sum |a_m\chi_m(t)|$ *converges for at least one point* $t_0 \in [0, 1]$,

(c) $\sum a_m\chi_m(t)$ *is the Fourier series of a bounded function* f.

In order to generalize this result to quasi-monotone coefficients we shall prove the following (cf. [19, p. 203] where an equivalent definition of quasi-monotonic is used).

LEMMA 2.2. *If* $\{b_m\}$ *is quasi-monotonic, then*

$$b_{2^m+k_m} \leqq Kb_{2^m} \leqq K^2b_{2^{m-1}+k_{m-1}}, \tag{2.20}$$

$$\sum_{n=m+1}^{\infty} (b_{2^n+k_n})2^n \leqq K \sum_{n=2^m}^{\infty} b_n \leqq K^2 \sum_{n=m-1}^{\infty} (b_{2^n+k_n})2^n \tag{2.21}$$

for $m=1,2,\ldots$ *where* K *is a positive constant and* $\{k_n\}$ *is any sequence of integers such that* $0 \leqq k_n < 2^n$.

Proof.

$$\sum_{n=m+1}^{\infty} [(b_{2^n+k_n})/(2^n+k_n)^{\beta}]2^n(2^n+k_n)^{\beta}$$

$$\leqq \sum_{n=m+1}^{\infty}\left[\sum_{i=2^n-1}^{2^n-1}(b_i)/i^{\beta}\right]2(2^{n+1})^{\beta} \leqq 2^{2\beta+1}\sum_{i=2^m}^{\infty} b_i.$$

Relation (2.20) and the other half of inequality (2.21) are proved similarly.

We shall now apply (2.21) with $b_m = a_m m^{-1/2}$ and prove

THEOREM 2.6. *In Theorem* E *we may replace the condition* $\{a_m\}$ *is monotonic by the condition* $\{a_m\}$ *is quasi-monotonic.*

Proof. Assuming $t_0 \in [0, 1]$ and $\sum a_m m^{-1/2} < \infty$ we obtain

$$\sum_{m=2^n}^{\infty} |a_m\chi_m(t_0)| \leqq \sum_{m=n}^{\infty}(a_{2^m+k_m})2^{m/2} \leqq K\sum_{m=2^n-1}^{\infty} a_m m^{-1/2} \to 0$$

as $n \to \infty$, where $0 \leqq k_n = k_n(t_0) < 2^n$ and K is a positive constant.

We now show that condition (b) implies condition (a). Assuming

$$\sum_{m=1}^{\infty}|a_m\chi_m(t_0)| = D < \infty,$$

we obtain

$$D = \sum_{m=1}^{\infty}|a_m\chi_m(t_0)| \geqq \sum_{n=N}^{\infty}(a_{2^n+k_n})2^{n/2} \geqq K\sum_{n=2^N+1}^{\infty} a_n n^{-1/2}$$

where $0 \leqq k_n < 2^n$, $N = N(t_0)$, and K is a positive constant.

Assuming now condition (c) we have that

$$\sup_n\left|\sum_0^n a_m(f)\chi_m(0)\right| = \sup_n\sum_0^n|a_m(f)\chi_m(0)| < \infty$$

and hence condition (b) is satisfied. This completes our proof.

In Theorem 2.2 we showed that $\sum |a_m(f)| < \infty$ does not imply $\sum |a_m(f)\chi_m(t)| < \infty$ for every $t \in [0, 1]$, for a continuous f. Hence the question naturally arises as to whether $\sum |a_m(f)\chi_m(t)| < \infty$ on $[0, 1]$ implies that $\sum |a_m(f)| < \infty$ for continuous or differentiable f? The answer is no and we now prove

THEOREM 2.7. *There exists a differentiable function* $f(t)$ *on* $[0, 1]$ *such that* $\sum |a_m(f)\chi_m(t)| < \infty$ *on* $[0, 1]$ *and* $\sum |a_m(f)|^{2-\varepsilon} = \infty$ *for every* $\varepsilon > 0$.

Proof. A. M. Olevskiĭ has shown [24, p. 4] that for every complete orthonormal system there exists a differentiable function f such that $\sum |a_m(f)|^{2-\varepsilon} = \infty$ for every

$\varepsilon>0$. The fact that the Haar-Fourier series of f converges absolutely on [0, 1] follows immediately from Corollary 2.4, which follows presently.

Applying now the fact that $|a_m(f)| \leqq 2^{1/2}Mm^{-1/2}$, where $M = \max|f(t)|$, we obtain

COROLLARY 2.2. *There exists a differentiable function f on* [0, 1] *such that* $\sum |a_m(f)\chi_m(t)| < \infty$ *on* [0, 1] *and for every* $\varepsilon>0$

$$\sum |a_m|^{1-\varepsilon}m^{-1/2} = \sum |a_m|m^{-1/2}m^{\varepsilon} = \infty.$$

REMARK 2.10. Olevskiĭ has shown [21, p. 44] that if $\sum |a_m|^{2-\varepsilon} < \infty$ for some ε satisfying $0<\varepsilon<2$, then $\sum |a_m\chi_m(t)|$ converges a.e. on [0, 1].

REMARK 2.11. By utilizing the fact [7, p. 620; 8, p. 1281] that there exists a function $f \in \mathrm{Lip}\,(1/2)$ such that $\sum |a_m(f)|$ diverges, it follows immediately from Theorem C that there exists a continuous function f such that $\sum |a_m(f)\chi_m(t)|$ converges uniformly on [0, 1] and yet $\sum |a_m(f)| = \infty$.

It may be of interest to note that we have a theorem analogous to Theorem 2.2 for the trigonometric system. Thus we shall now prove

THEOREM 2.8. *For every null sequence $\{\varepsilon_k\}$ there exists a continuous function f such that*

(a) $\sum [a_k(f)\cos kt + b_k(f)\sin kt]$ *converges to $f(t)$ uniformly,*

(b) $\sum [a_k(f)\cos kt + b_k(f)\sin kt]$ *is nowhere absolutely convergent,*

(c) $\sum [|a_k(f)| + |b_k(f)|]\varepsilon_k < \infty$,

where $a_k(f)$, $b_n(f)$ denote the trigonometric Fourier coefficients of f.

Proof. Let $\{N_m\}$ be an increasing sequence of integers such that $|\varepsilon_k| < 2^{-m}$ if $N_m \leqq k < N_{m+1}$. Then setting

$$a_k = (N_{m+1} - N_m)^{-1} \quad \text{if } N_m \leqq k < N_{m+1}$$

we obtain for a fixed $\varepsilon>0$

$$\begin{aligned}\sum_{k=N_1}^{\infty} a_k^2(\log k)^{1+\varepsilon} &= \sum_{N_1}^{N_2-1} + \sum_{N_2}^{N_3-1} + \cdots \\ &\leqq \frac{(\log N_2)^{1+\varepsilon}}{N_2-N_1} + \frac{(\log N_3)^{1+\varepsilon}}{N_3-N_2} + \cdots \\ &\leqq \frac{2(\log N_2)^{1+\varepsilon}}{N_2} + \frac{2(\log N_3)^{1+\varepsilon}}{N_3} + \cdots < \infty\end{aligned} \tag{2.22}$$

for a suitable sequence $\{N_m\}$, chosen such that $|a_k| \downarrow 0$.

But Payley and Zygmund have shown [29, p. 219] that since series (2.22) is convergent, we then have $\sum \pm a_k \cos kt$ uniformly convergent for some alternation of signs. Also, since $|a_k| \downarrow 0$ and $\sum |a_k| = \infty$, it follows [29, p. 232] that $\sum |a_k \cos kt|$ diverges everywhere on $[0, 2\pi]$.

Condition (c) is satisfied by construction.

For the Haar functions the principle of localization holds, i.e. if two functions coincide in some interval, then their Fourier series converge or diverge simultaneously at any interior point. With respect to absolute convergence the principle remains true for the Haar functions although it fails for the trigonometric system [3, II, p. 187].

In view of this it seems appropriate to consider further absolute convergence at specific points. We shall need the following definitions in which $f(t)=0$ if $t \notin [0,1]$:

$$\omega(f, t_0, h) = \sup_{|t-t_0|\leqq h} |f(t)-f(t_0)|;$$

$$\omega_1(f, t_0, h) = \int_{t_0-h}^{t_0+h} |f(t)-f(t_0)|\,dt;$$

$$f \in \operatorname{Lip}(\alpha, t_0) \quad \text{if } |f(t)-f(t_0)| \leqq k|t-t_0|^{\alpha},$$

where $0<\alpha\leqq 1$, in a neighborhood of t_0 (K is a positive constant).

We now prove (cf. Theorem C)

THEOREM 2.9. *If* $\sum_{n=1}^{\infty} 2^n\omega_1(f, t_0, 2^{-n})<\infty$ *for some* $t_0 \in [0, 1]$, *then*

$$\sum |a_m(f)\chi_m(t_0)| < \infty.$$

Proof. Choose integers $k_n(t_0)$ and $k_n'(t_0)$ such that $(k_n)/2^n<t_0\leqq(k_n+1)/2^n$ and $(k_n')/2^n\leqq t_0<(k_n'+1)/2^n$. Then

$$\begin{aligned}\sum |a_m(f)\chi_m(t_0)| &\leqq \sum [|a_{2^n+k_n}(f)|+|a_{2^n+k_n'}(f)|]2^{n/2}\\ &= \sum_{n=1}^{\infty} 2^n\left\{\left|\int_{(k_n)/2^n}^{(k_n+1/2)/2^n} [f(t)-f(t+2^{-n-1})]\,dt\right|\right.\\ &\qquad\qquad \left.+\left|\int_{(k_n')/2}^{(k_n'+1/2)/2^n} [f(t)-f(t+2^{-n-1})]\,dt\right|\right\}\\ &\leqq 4\sum_{n=1}^{\infty} 2^n\omega_1(f, t_0, 2^{-n}).\end{aligned}$$

We have immediately the following

COROLLARY 2.3. *If* $\sum \omega(f, t_0, 2^{-n})<\infty$ *for some* $t_0 \in [0, 1]$, *then*

$$\sum |a_m(f)\chi_m(t_0)| < \infty.$$

By a slight modification of the proof of Theorem 2.9 we also obtain

THEOREM 2.9′. *If* $f \in \operatorname{Lip}(\alpha, t_0)$ *for some* $\alpha>0$ *and* $t_0 \in [0, 1]$, *then*

$$\sum |m^{\alpha-\varepsilon}a_m(f)\chi_m(t_0)|$$

converges for every $\varepsilon>0$.

COROLLARY 2.4. *If f has a right and left hand derivative at* t_0, *then* $\sum |a_m(f)\chi_m(t_0)|$ *converges.*

REMARK 2.12. Theorem 2.9′ is false for $\alpha=1$, $\varepsilon=0$ since if $f(t)=(1-2t)$ we obtain $a_m(f)=2^{-1-3n/2}$ if $2^n \leqq m < 2^{n+1}$ and consequently $\sum |ma_m(f)\chi_m(t)|$ diverges on [0, 1].

Corollary 2.3 is best possible in the sense of

THEOREM 2.10. *For every positive null sequence* $\{\varepsilon_n\}$ *there exists a continuous function* f *such that*

$$\sum \omega(f, 0, 2^{-n})\cdot \varepsilon_n < \infty \tag{2.23}$$

and

$$\sum |a_m(f)\chi_m(0)| = \infty. \tag{2.24}$$

Proof. Choose a sequence $\{c_m\}$ satisfying:

(i) $\sum c_n$ is an alternating conditionally convergent series,

(ii) $|c_n| \leqq |c_{n-1}|$,

(iii) $\sum |c_n \varepsilon_n| < \infty$.

For example, we may let $c_m=(-1)^m/n(N_n-N_{n-1})$ for $N_{n-1} \leqq m < N_n$, where $\{N_n\}$ is chosen such that $|c_n| \downarrow 0$ and $|\varepsilon_m| < 2^{-n}$ for $m \geqq N_{n-1}$.

Define $f(t)$ on $[2^{-n}, 2^{-n+1}]$ for $n=1, 2, \ldots$ as follows:

$$\begin{aligned} f(t) &= 0, \qquad t = 2^{-n}, 3\cdot 2^{-n-1}, 2^{-n+1}, \\ &= c_n, \qquad t = 7\cdot 2^{-n-2} \\ &= \text{linear and continuous on the contiguous intervals.} \end{aligned}$$

Defining now $f(0)=0$, we obtain from (ii) that $\omega(f, 0, 2^{-n}) \leqq |c_n|$ if $n \geqq 1$ which together with (iii) implies (2.23). Also

$$|a_{2^n}(f)| = 2^{n/2}|c_{n+1}2^{-n-3}-(c_{n+2}2^{-n-4}+c_{n+3}2^{-n-5}+\cdots)|,$$

and hence

$$2^{n/2}|a_{2^n}(f)| = 2^{-3}|c_{n+1}-(c_{n+2}2^{-1}+c_{n+3}2^{-2}+\cdots)| \geqq 2^{-3}|c_{n+1}|$$

for $n=1, 2, \ldots$ by (i) and (ii) which implies (2.24).

REMARK 2.13. A similar proof would suffice for any dyadic rational.

Since a function f of bounded variation has right and left hand limits everywhere, it follows from Theorem 1.3 that the Haar-Fourier series of f converges at all dyadic rationals (cf. [23, p. 368, Theorem 3]). We will strengthen this result and prove

THEOREM 2.11. *If* t_0 *is a dyadic rational and a point of bounded variation for* $f(t)$, *then* $\sum |a_m(f)\chi_m(t_0)| < \infty$.

Proof. We shall prove the theorem for $t_0=0$. A similar argument will suffice for any dyadic rational.

Assume that

$$\sum |a_m(f)\chi_m(0)| = \sum 2^n \left| \int_0^{1/2^{n+1}} f(t)\,dt - \int_{1/2^{n+1}}^{1/2^n} f(t)\,dt \right| \equiv \sum |c_n| = \infty.$$

Then

$$\begin{aligned} 2c_{n-1} - c_n &= 2^n \left[\int_0^{1/2^n} - \int_{1/2^n}^{1/2^{n-1}} \right] - 2^n \left[\int_0^{1/2^{n+1}} - \int_{1/2^{n+1}}^{1/2^n} \right] \\ &= 2^{n+1} \int_{1/2^{n+1}}^{1/2^n} f(t)\,dt - 2^n \int_{1/2^n}^{1/2^{n-1}} f(t)\,dt \end{aligned}$$

and thus

$$|2c_{n-1} - c_n| \leqq V(f; [2^{-n-1}, 2^{-n+1}]) \equiv V(2^{-n-1}, 2^{-n+1}),$$

the variation of f on $[2^{-n-1}, 2^{-n+1}]$.

Further

$$\begin{aligned} 2V(0, 1) &= V(1/4, 1) + [V(1/4, 1) + V(1/8, 1/4)] + [V(1/8, 1/4) + V(1/16, 1/8)] + \cdots \\ &\geqq V(1/4, 1) + V(1/8, 1/2) + V(1/16, 1/4) + \cdots \\ &\geqq \sum_{n=1}^{\infty} |2c_{n-1} - c_n| \geqq \sum_{n=1}^{\infty} (|2c_{n-1}| - |c_n|) = \sum |c_n| = \infty. \end{aligned}$$

But this is a contradiction if f is of bounded variation on $[0, 1]$. By an obvious adoption of this proof we see that we need only require that f is of bounded variation in a neighborhood of 0.

Remark 2.14. In the proof of Theorem 2.11 we did not actually need the hypothesis that 0 was a point of bounded variation for $f(t)$; we only used the fact that if

$$F(t) = 2^n \int_{1/2^n}^{1/2^{n-1}} f(x)\,dx \quad \text{for } t \in (2^{-n}, 2^{-n+1}), \qquad n = 1, 2, \ldots,$$

then 0 was a point of bounded variation for $F(t)$.

Remark 2.15. $\sum |a_m(f)\chi_m(0)| < \infty$ does not imply that $f(t)$ is of bounded variation at the origin.

Remark 2.16. It is known (see [1, p. 256]; [23, p. 368]) that if $f(t)$, a function of bounded variation, has a discontinuity at a dyadic irrational, then its Haar-Fourier series diverges there.

Remark 2.17. Since a function of bounded variation is differentiable a.e., we have by Corollary 2.4 and Theorem 2.11 that $\sum |a_m(f)\chi_m(t)|$ converges except possibly on a set of measure zero of dyadic irrationals.

We now show that the converse of Corollary 2.3 is false. More precisely, we now prove

Theorem 2.12. *If f is continuous on* $[0, 1]$, *then* $\sum |a_m(f)\chi_m(t)| < \infty$ *on* $[0, 1]$ *does not imply* $\sum \omega(f, 0, 2^{-n}) < \infty$.

Proof. Let $\omega(2^{-n}) \downarrow 0$ arbitrarily slowly as $n \to \infty$. Set

$$\begin{aligned} f(t) &= \omega(2^{-n}), \qquad t = 2^{-n} \quad \text{for } n = 0, 1, \ldots, \\ &= 0, \qquad\qquad\quad t = 0, \\ &= \text{linear and continuous on } [2^{-n-1}, 2^{-n}]. \end{aligned}$$

Then $f(t)$ is monotone and is differentiable at all dyadic irrationals. Hence by Theorem 2.11 and Corollary 2.4 we obtain that $\sum |a_m(f)\chi_m(t)|$ converges for every $t \in [0, 1]$.

Since $\omega(f, 0, 2^{-n}) = \omega(2^{-n})$, this completes our proof.

We now investigate the sharpness of Theorem C and prove

THEOREM 2.13. *If $\omega(\delta) \downarrow 0$, $\omega(\delta)/\delta \uparrow$ as $\delta \downarrow 0$ and $\sum \omega(n^{-1})n^{-1} = \infty$, then there exists a continuous function f such that*

$$\text{(a) } \omega(\delta, f) = O[\omega(\delta)]; \qquad \text{(b) } \sum |a_m(f)\chi_m(0)| = \infty.$$

Proof. Setting $c_n = (-1)^m \omega(2^{-n})$ for $n \geqq 1$ we obtain

(i) $\sum c_n$ is an alternating conditionally convergent series,

(ii) $|c_n| \leqq |c_{n-1}|$,

(iii) $2|c_n| \geqq |c_{n-1}|$.

Now defining $f(t)$ as in Theorem 2.10, we obtain

$$\begin{aligned} \omega(f, 2^{-n}) &\leqq 2 \max (|c_{n-2}|, |c_{n-3}|2^{-1}, \ldots, |c_1|2^{-n+3}) \\ &\leqq 2|c_{n-2}| \leqq 8|c_n| = 8\omega(2^{-n}) \end{aligned}$$

if $n \geqq 3$ by (ii) and (iii), and

$$2^{n/2}|a_{2^n}(f)| \geqq |c_{n+1}|2^{-3} \quad \text{by (i) and (ii).}$$

Consequently

$$\sum |a_m(f)\chi_m(0)| = \sum |a_{2^n}(f)|2^{n/2} \geqq 2^{-3} \sum |c_n| = \infty.$$

REMARK 2.18. Ul'janov has independently proved Theorem 2.13 [26, p. 982]; [28, p. 200].

REMARK 2.19. Theorem 2.13 is similar to a result of B. I. Golubov for the Haar-Fourier coefficients of a continuous function [8, p. 1277].

Letting

$$E_n(f) = \inf_{\{a_m\}} \sup_t \left| f(t) - \sum_{m=1}^{n} a_m \chi_m(t) \right|,$$

where the infimum is extended over all real sequences $\{a_m\}$, we shall prove the following:

COROLLARY 2.5. *If f is continuous, then $\sum E_n(f)n^{-1} < \infty$ implies the uniform convergence of $\sum |a_m(f)\chi_m(t)|$ on $[0, 1]$.*

COROLLARY 2.6. *If* $\{E_n\}$ *is a sequence such that* $E_n \downarrow 0$, $nE_n \uparrow$, *and* $\sum E_n n^{-1} = \infty$, *then there exists a continuous f such that*

$$\text{(a) } E_n(f) = O(E_n), \qquad \text{(b) } \sum |a_m(f)\chi_m(0)| = \infty.$$

Proof of Corollary 2.5. By a lemma of Golubov [8, p. 1274] we have that if $f(t)$ is continuous on [0, 1], then

$$E_n(f) \leqq \omega(n^{-1}, f) \leqq 6E_n(f), \qquad n \geqq 2. \tag{2.25}$$

Hence our desired result follows immediately from Theorem C.

Proof of Corollary 2.6. Let $\omega(n^{-1}) = E_n$ in Theorem 2.13. Then (b) follows from Theorem 2.13 and (a) follows from inequality (2.25) and Theorem 2.13.

REMARK 2.20. Corollaries 2.5 and 2.6 are similar to theorems of S. N. Bernstein for the trigonometric system [3, II, pp. 154, 165] and to theorems of B. I. Golubov for convergence of the Haar-Fourier coefficients [7, p. 621]; [8, p. 1281].

3. **Functions represented by Haar and Walsh series.** The relationships between the coefficients of Haar (and Walsh) series and the functions which the series represent have been much investigated (see [8], [10], [11], [18] and [23]). This section is concerned with these relationships.

L. A. Balasov has shown [2, p. 631] that a necessary and sufficient condition that $f(t) = \sum a_m r_m(t)$ have a derivative at at least one point is that the limit of $a_m 2^m$ exist. We shall prove

THEOREM 3.1. *If* $\lim a_m m^{3/2} = 0$, *then* $f(t) = \sum a_m \chi_m(t)$ *has a derivative on a dense set.*

Proof. Let $t_0 = .1010\ldots$, where the expansion is given in binary form, and let $\{t_k\}$ be a sequence of numbers such that $t_k \neq t_0$, $\lim t_k = t_0$. Assuming that the first p places of t_k are the same as t_0 and the $(p+1)$th place differs, we have that $p \to \infty$ as $k \to \infty$. Then (cf. [2, p. 632]) $|t_k - t_0| \geqq 2^{-p-3}$ and

$$\lim_{t_k \to t_0} \left| \frac{f(t_k) - f(t_0)}{t_k - t_0} \right| = \lim_{t_k \to t_0} \left| \frac{\sum_{m=2^{p-2}}^{\infty} a_m[\chi_m(t_k) - \chi_m(t_0)]}{t_k - t_0} \right|$$

$$\leqq \lim_{p \to \infty} \left| \frac{\varepsilon_p \sum_{n=p-2}^{\infty} 2^{-3n/2} \cdot 2^{n/2}}{2^{-p-3}} \right|$$

where $\varepsilon_p \to 0$ as $p \to \infty$, and this last expression equals

$$\lim_{p \to \infty} |16 \cdot \varepsilon_p| = 0.$$

Hence $f'(t)$ exists at $t_0 = .1010\ldots$.

In order to complete the proof we need only notice that this argument is actually valid for t_0 plus any dyadic rational number.

By similar reasoning applied to $\{w_m(t)\}$, the Walsh orthonormal system [1, p. 59], we obtain the following

THEOREM 3.2. *If* $\lim a_m m^2 = 0$, *then* $f(t) = \sum a_m w_m(t)$ *has a derivative on a dense set.*

In order to obtain our next results we need the following lemma which follows from Minkowski's inequality:

LEMMA 3.1. *If* $V_p(f_m)$ *denotes the pth variation of* $f_m(t)$, *then*

(i) *if* $0 < p \leqq 1$,

$$V_p^p\left(\sum_{n=1}^{\infty} f_m\right) \leqq \sum_{m=1}^{\infty} V_p^p(f_m);$$

(ii) *if* $p \geqq 1$,

$$V_p\left(\sum_{m=1}^{\infty} f_m\right) \leqq \sum_{m=1}^{\infty} V_p(f_m).$$

We now state two theorems which are immediate consequences of Lemma 3.1.

THEOREM 3.3. (i) *If* $0 < p \leqq 1$, *then* $\sum |a_m|^p m < \infty$ *implies* $f(t) = \sum a_m w_m(t)$ *is of bounded pth variation.*

(ii) *If* $p \geqq 1$, *then* $\sum |a_m| m^{1/p} < \infty$ *implies* $f(t) = \sum a_m w_m(t)$ *is of bounded pth variation.*

THEOREM 3.4. (i) *If* $0 < p \leqq 1$, *then* $\sum |a_m|^p m^{p/2} < \infty$ *implies* $f(t) = \sum a_m \chi_m(t)$ *is of bounded pth variation.*

(ii) *If* $p \geqq 1$, *then* $\sum |a_m| m^{1/2} < \infty$ *implies* $f(t) = \sum a_m \chi_m(t)$ *is of bounded pth variation.*

REMARK 3.1. By applying Theorem (4.1) in [12] we obtain:

(i) if $\lim \varepsilon_m = 0$, then $\sum |a_m m \varepsilon_m| < \infty$ does not imply $f(t) = \sum a_m w_m(t)$ is differentiable a.e.

(ii) if $\lim \varepsilon_m = 0$, then $\sum |a_m m^{1/2} \varepsilon_m| < \infty$ does not imply $f(t) = \sum a_m \chi_m(t)$ is differentiable a.e.

REMARK 3.2. Part (i) of Theorems 3.3 and 3.4 are best possible by part (iii) of Theorem (3.1) in [12].

REMARK 3.3. Part (ii) of Theorem 3.4 is best possible in the sense that if $\lim \varepsilon_m = 0$, then $\sum |a_m| m^{1/2} \varepsilon_m < \infty$ does not imply $f(t) = \sum a_m \chi_m(t)$ is bounded. This is easily seen by setting $a_{2^n} = 2^{-n/2}$ at selected n's and $a_m = 0$ otherwise.

REMARK 3.4. V. A. Matveev has proved [11, p. 1405] that if $f \in L$ and $\sum |a_m(f)| m^{1/2} < \infty$, then $f(t)$ is equivalent to a function of bounded variation. However, since $f(t) = \sum a_m(f) \chi_m(t)$ a.e. whenever $f \in L$, this result follows immediately from Theorem 3.4 for $p = 1$.

B. I. Golubov has shown ([8, p. 1296]; see also Wang [18, p. 224]) that if f is continuous and $a_m(f) = o(m^{-3/2})$ with respect to the Haar system, then f is a constant. We shall prove, using a simpler method, THEOREM 3.5. *If $f(t)=\sum a_m\chi_m(t)$ has the Darboux property and $a_m=o(m^{-3/2})$, then f is a constant.*

Proof. Assume $|a_m|\leqq(m^{-3/2})\varepsilon/4$ if $m\geqq 2^N$. Then $x,y\in((i-1)/2^N, i/2^N)$, $i=1, 2,\ldots, 2^N$ implies

$$|f(x)-f(y)| \leqq (\varepsilon/2)\sum_{m=0}^{\infty} 2^{-N-m} = \varepsilon/2^N. \tag{3.1}$$

Hence, letting $x\to(i-1)/2^N$, $y\to i/2^N$ in inequality (4.1) we obtain by the Darboux property

$$|f(x)-f(y)| \leqq \varepsilon/2^N \quad \text{if } x,y\in[(i-1)/2^N, i/2^N].$$

But this implies that

$$|f(x)-f(y)| \leqq \varepsilon \quad \text{if } x,y\in[0,1],$$

which completes the demonstration.

By utilizing Theorem 1.1 and the fact that an approximately continuous function has the Darboux property (or just by noticing that the proof of Theorem 3.5 goes through easily for approximately continuous functions) we obtain

COROLLARY 3.1. *If f is bounded and approximately continuous and $a_m(f)=o(m^{-3/2})$, then f is a constant.*

Golubov has further proved ([8, p. 1295]; see also Wang [18, p. 226] and Matveev [10, p. 108]) that if f is continuous on $[0, 1]$, $0<\alpha\leqq 1$, and $a_m(f)=O(m^{-1/2-\alpha})$ with respect to the Haar system, then $f\in\operatorname{Lip}\alpha$. We will prove a somewhat more general result by a simpler method.

THEOREM 3.6. *If $f(t)=\sum a_m\chi_m(t)$ has the Darboux property and $a_m=O(m^{-1/2-\alpha})$, then $f\in\operatorname{Lip}\alpha$.*

Proof. Analogous to the proof of Theorem 3.5 we obtain

$$|f(x)-f(y)| \leqq K'(2^{-n\alpha}+2^{-(n+1)\alpha}+\cdots) = K/2^{n\alpha}$$

whenever $x,y\in[i/2^n, (i+1)/2^n]$.

Consequently, if $2^{-n-1}<|x-y|\leqq 2^{-n}$, then

$$|f(x)-f(y)| \leqq 2K\cdot 2^{-n\alpha} < 4K|x-y|^{\alpha},$$

which produces our desired result.

Applying Theorem 1.1 we immediately obtain

COROLLARY 3.2. *If f is bounded and approximately continuous and $a_m(f)=O(m^{-1/2-\alpha})$, then $f\in\operatorname{Lip}\alpha$.*

4. **Applications of Haar series to general orthonormal series.** The system of Haar functions is an easily defined example of a complete system of bounded orthonormal functions which is not uniformly bounded. Hence, when examining

any conjecture about general orthonormal series, it is often advantageous to consider the special case of the Haar system. In many cases such consideration will also show that certain results are best possible for unbounded complete orthonormal systems. For example, if $\{\phi_m\}$ is a sequence of functions with variations $\{V_m\}$, then $\sum |a_m| V_m < \infty$ implies that $f(t) = \sum a_m \phi_m(t)$ is a function of bounded variation; and as Remark 3.2 shows, the theorem is best possible for the Haar system. We now proceed to some other results.

For $\{\phi_m\}$ orthonormal on $[a, b]$, we have that if $\sum |a_m|$ converges, then $\sum |a_n \phi_m(t)|$ converges a.e. there [1, p. 63]. We now state

THEOREM 4.1. (i) *If* $\sup_{a \leq t \leq b} |\phi_m(t)| = M_m$, *then* $\sum |a_m| M_m < \infty$ *implies that* $\sum |a_m \phi_m(t)| < \infty$ *on* $[a, b]$.

(ii) *For unbounded complete orthonormal systems the condition in* (i) *may not be relaxed even though the series may be a uniformly convergent Fourier series.*

Proof. Part (i) is obvious.

Part (ii) follows by combining the fact that $M_m = O(m^{1/2})$ for the Haar system with Lemma 2.1.

For uniformly bounded orthonormal systems we have that the Fourier coefficients of any integrable function tend to zero [3, I, p. 66]. We now prove

THEOREM 4.2. (i) *If* $\{\phi_m\}$ *is orthonormal on* $[a, b]$ *and* $\sup_{a \leq t \leq b} |\phi_m(t)| = M_m$, *then*

$$\int_a^b f(t)\phi_m(t)\, dt = o(M_m) \quad \text{for every } f \in L. \tag{4.1}$$

(ii) *For unbounded complete orthonormal systems* (4.1) *may not in general be improved.*

Proof. (i) Let $M = M_m$ in the proof given in Bary's book [3, I, p. 66].

(ii) Combine the fact that $M_m = O(m^{1/2})$ for the Haar system with the result of Ul'janov [23, p. 363] that $a_m(f) = o(m^{1/2})$ cannot be improved.

For a uniformly bounded orthonormal system $\{\phi_m\}$ on $[a, b]$, we have that the relation $a_m \to 0$ is a necessary condition for the convergence a.e. of the orthogonal series $\sum a_m \phi_m(t)$ [1, p. 7]. We shall prove

THEOREM 4.3. (i) *If* $\sup |\phi_m(t)| = M_m$ *and* $\sum a_m \phi_m(t)$ *is uniformly convergent, then* $a_m \cdot M_m = o(1)$.

(ii) *For unbounded complete orthonormal systems* (i) *may not in general be improved even though the series may be a uniformly convergent Fourier series of a continuous function.*

Proof. Part (i) is immediate.

(ii) Use the result of Ul'janov [23, p. 365] that for continuous f, the relation $a_m(f) \cdot m^{1/2} = o(1)$ cannot be improved for the Haar system.

REMARK 4.1. The uniformity condition in part (i) may not be omitted [1, p. 8].

We now prove

THEOREM 4.4. (i) *If* $\{\phi_m(t)\}$ *is a normal system* (*i.e.* $\int_a^b \phi_m^2(t)\,dt=1, m=0, 1, \ldots$) *and* $\sup_{a\leqq t\leqq b}|\phi_m(t)|=M_m$, *then* $\sum |a_m|/M_m$ *converges whenever* $\sum a_m\phi_m(t)$ *is uniformly absolutely convergent.*

(ii) *For unbounded complete orthonormal systems the conclusion in* (i) *may not in general be strengthened.*

Proof.

(i)
$$\sum_m^n |a_i|/M_i = \sum_m^n (|a_i|/M_i)\int_a^b \phi_i^2(t)\,dt$$
$$\leqq \sum_m^n |a_i|\int_a^b |\phi_i(t)|\,dt = \int_a^b \left(\sum_m^n |a_i\phi_i(t)|\right)dt.$$

(ii) If $\delta_n \uparrow \infty$ arbitrarily slowly, choose a nonnegative sequence $\{\varepsilon_m\}$ such that

$$\sum \varepsilon_n < \infty \quad \text{and} \quad \sum \varepsilon_n\delta_{2^n} = \infty.$$

For example, if $\{n_k\}$ is chosen such that $\delta_{2^{n_k}}\geqq 2^k$, then let $\varepsilon_{n_k}=2^{-k}$, $k=1, 2, \ldots$, and $\varepsilon_m=0$ otherwise.

Setting now $a_m=2^{-n/2}\varepsilon_n$ for $2^n\leqq m<2^{n+1}$, we obtain that $\sum |a_m\chi_m(t)|\leqq\sum\varepsilon_m$ and

$$\sum (|a_m|/m^{1/2})\delta_m \geqq 2^{-1/2}\sum \varepsilon_n\delta_{2^n} = \infty.$$

REMARK 4.2. In part (i) the uniformity condition may not be omitted for the Haar system because by choosing $\{a_m\}$ such that $a_m=0$ if $m\neq 2^n+1$ $(n=0, 1, \ldots)$ we obtain that $\sum a_m\chi_m(t)$ is absolutely convergent on [0, 1].

REMARK 4.3. In part (i) the condition of normality may not be omitted. To see this we consider the orthogonal system $\{\phi_n(t)\}$ where $\phi_n(t)=n^{1/2}$ on E_n and 0 elsewhere, where $\{E_n\}$ is a mutually disjoint infinite sequence of subsets of [a, b]. Then $\sum |a_m\phi_m(t)|$ is uniformly convergent whenever $a_m m^{1/2}\to 0$. However, this condition does not imply that $\sum a_m m^{-1/2}$ converges.

Of course, if $M_m\to 0$, then the fact that $\sum a_m\phi_m(t)$ is uniformly absolutely convergent does not imply that $\sum |a_m|/M_m$ converges.

W. Orlicz has proved [16, p. 232]

THEOREM G. *Let* $d_n\geqq 0$ $(n=1, 2, \ldots)$ *and* $\sum d_n^2=\infty$. *If* $\{\phi_n(t)\}$ *is a uniformly bounded orthonormal system on* [0, 1], *then there exists a continuous function f for which the Fourier coefficients* a_n *with respect to the system* $\{\phi_m(t)\}$ *satisfy the condition*

$$\sum d_n|a_n| = \infty.$$

A. M. Olevskiĭ has noted [15, p. 654] that the conclusion of Orlicz's theorem is not valid for the Haar system. We notice, however, that the conclusion is still valid for the Haar system if $\{d_n\}$ is quasi-monotonic. This follows easily since by Lemma

2.2 we then have $\sum d_m^2 \chi_m^2(t) = \infty$ a.e. and our desired conclusion is obtained by a remark of Orlicz [16, p. 233].

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