

ON TOPOLOGICAL EQUIVALENCE OF \aleph_0 -DIMENSIONAL LINEAR SPACES⁽¹⁾

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1. Introduction. In this paper, l_2 will denote the separable Hilbert space of all square summable real sequences with the usual norm $\| \cdot \|$. s will denote the countable infinite product of real lines R . l_F and s_F will denote the subspaces of l_2 and s respectively consisting of all points $x = (x_1, x_2, \dots)$ such that $x_i = 0$ for all but finitely many i . In 1957 Long and Klee in [5] showed that all \aleph_0 -dimensional (algebraic) normed linear spaces are homeomorphic. Thus, in particular, they are all homeomorphic to l_F . Klee pointed out, however, that the assumption of normability cannot be completely abandoned, since there are \aleph_0 -dimensional locally convex topological linear spaces which are not metrizable. In 1963, C. Bessaga in [0] generalized Klee and Long's result (using their method) to some \aleph_0 -dimensional linear metric space which need not be normed. In particular, he shows that all \aleph_0 -dimensional locally convex metric linear spaces having a radially bounded neighborhood of zero (or equivalently, containing no subspace which is linearly homeomorphic with s_F) are homeomorphic [0, Proposition 3]. Therefore the ultimate question (classification problem) is whether all \aleph_0 -dimensional locally convex linear metric spaces are homeomorphic.

The purpose of this paper is to settle a special case of the above question raised by Fréchet [2, p. 83] in 1928, also by Klee, Bessaga [0, p. 163] and Pełczyński [6]. We shall prove

THEOREM I. l_F is homeomorphic with s_F .

COROLLARY I. $\bar{s}_p \sim \bar{s}_f \sim l_F$,

where " \sim " means homeomorphic to; see §2 for definitions of \bar{s}_p and \bar{s}_f .

(The proof in this paper is rather self-contained.)

2. Definition and notation. (All subsets inherit the subspace topology.)

$$(0) \quad s = \prod_{i=1}^{\infty} R_i \quad \text{where } R_i = R = \text{reals};$$

$$s' = \{x \in s : x_1 < 0\}.$$

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- (1) $l_+ = \{(x_1, x_2, \dots) \in l_F : x_i = 0 \text{ for all } i \text{ or } x_n > 0 \text{ when } n \text{ is the largest nonzero index of } x\};$
 $l_- = \{x \in l_F : -x \in l_+\}.$
- (2) $s_+ = \{(x_1, x_2, \dots) \in s_F : x_i = 0 \text{ for all } i \text{ or } x_n > 0 \text{ when } n \text{ is the largest nonzero index of } x\};$
 $s_- = \{x \in s_F : -x \in s_+\}.$
- (3) $S_F = \{x \in l_F : \|x\| = 1\};$
 $S_- = (S_F \cap l_-) \cap s'.$
- (4) $Q, Q' =$ Hilbert cubes represented by $\prod_{i=1}^{\infty} J_i$ and $\prod_{i=1}^{\infty} I_i$ respectively where each $J_i = [-1, 1] \subset R_i$ and each $I_i = [0, 1] \subset R_i.$
- (5) $s_f = \{x \in s_F : x_i \in (-1, 1) \text{ for all } i\};$
 $\bar{s}_f = \bigcup_{n=1}^{\infty} \left(\prod_{i=1}^n J_i \right).$
- (6) $s'_+ = s_+ \cap s_f.$
- (7) $s_p = \{0\} \cup \bigcup_{n=1}^{\infty} \left[\text{Int} \left(\prod_{i=1}^n I_i \right) \right];$
 $\bar{s}_p = \bigcup_{n=1}^{\infty} \left(\prod_{i=1}^n I_i \right).$

Let π_n, τ_n denote the projection functions defined by $\pi_n(x_1, x_2, \dots) = x_n$ and $\tau_n(x_1, x_2, \dots) = (x_1, \dots, x_n)$. Let $R^n = \prod_{i=1}^n R_i$ denote the Euclidean n -space with the usual norm. R^n is also considered as $R^n \times 0 \subset R^{n+1}$ and as $R^n \times 0 \times 0 \dots \subset s$. Let R_+^n and R_-^n denote respectively the subspaces $\{x \in R^n : x_n \geq 0\}$ and $\{x \in R^n : x_n \leq 0\}$. If f is an imbedding of R^n into R^n , then f' is said to be the imbedding of s into s induced by f if $f'(x) = (f(\tau_n(x)), x_{n+1}, x_{n+2}, \dots)$ for all $x \in s$.

3. Proof of Theorem I (§3–§6).

3.1 *A basic lemma.* Before introducing the lemma, we need the following definition.

DEFINITION. Let (X, d) be a metric space and $A \subset X$. Let f, g be imbeddings of A into X . For any $\varepsilon > 0$, let

$$\alpha(f, \varepsilon) = \inf \{d(f(x), f(y)) : x, y \in A \text{ and } d(x, y) > \varepsilon\},$$

$$\lambda(f, g) = \sup \{d(f(x), g(x)) : x \in A\},$$

where $\lambda(f, g)$ may be infinite. f is called an α -imbedding if $\alpha(f, \varepsilon) > 0$ for any $\varepsilon > 0$.

LEMMA (CONVERGENCE CRITERION). Let (X, d) be a metric space, $A \subset X$. If $\{f_i\}_{i \geq 1}$ is a sequence of imbedding of A into X satisfying (1) For each $x \in A$, $\lim_{n \geq 1} f_n(x)$ exists and belongs to X and (2) For all n , $\lambda(f_{n+1}, f_n)$

$\leq \min(1/3^{n+1}, 1/3^{n+1} \cdot \alpha(f_n, 1/3^n))$. Then the function f defined by $f(x) = \lim_{n \geq 1} f_n(x)$ is an imbedding of A into X .

Proof. If $x, y \in A$ such that $d(x, y) > 1/3^n$ for some n , it is easy to see from (2) that $d(f_m(x), f_m(y)) \geq 1/3 \cdot d(f_n(x), f_n(y))$ for all $m \geq n$. This shows f is 1-1. We claim this also shows f^{-1} is continuous for the following reason. If $\{x_i\}_{i \geq 1}$ is a sequence in X such that $d(x_i, x) > 1/3^n$ for some fixed n . Then $d(f_n(x_i), f_n(x)) \not\rightarrow 0$. Hence without loss of generality, we may assume that for all i , $d(f_n(x_i), f_n(x)) > \varepsilon$ for some $\varepsilon > 0$. But this implies for all i , $d(f(x_i), f(x)) \geq \varepsilon/3$. Thus $f(x_i) \not\rightarrow f(x)$. Finally f is continuous as a simple consequence of the fact $\lambda(f_{n+1}, f_n) \leq 1/3^{n+1}$ for all n .

3.2 Strategy for the proof of Theorem I. We shall prove Theorem I by constructing homeomorphisms h_1, \dots, h_6 , where

$$l_F \stackrel{h_1}{\sim} l_- \stackrel{h_2}{\sim} S_- \stackrel{h_3}{\sim} s_p \stackrel{h_4}{\sim} s'_+ \stackrel{h_5}{\sim} s_- \stackrel{h_6}{\sim} s_F.$$

We point out, first of all, that h_2 and h_5 are rather trivial; namely, define $h_2: l_- \rightarrow S_-$ by

$$h_2(x) = \left(- \left(1 - \left(\frac{\|x\|}{1 + \|x\|} \right)^2 \right)^{1/2}, \frac{x_1}{1 + \|x\|}, \frac{x_2}{1 + \|x\|}, \dots \right)$$

and define $h_5: s'_+ \rightarrow s_-$ by

$$h_5(x) = \left(\frac{-x_1}{1 - |x_1|}, \frac{-x_2}{1 - |x_2|}, \dots \right).$$

To see h_2 is a homeomorphism of l_- onto S_- , we decompose h_2 into $g_2 \cdot f_2$, where f_2 is the homeomorphism between l_- and the subspace $B = \{x \in l_- : \|x\| < 1\}$ defined by $f_2(x) = x/(1 + \|x\|)$, and g_2 is the homeomorphism of B onto S_- defined by $g_2(y) = (-(1 - \|y\|^2)^{1/2}, y_1, y_2, \dots)$. To see h_5 is a homeomorphism, note that

$$h'_5(y_1, y_2, \dots) = \frac{-y_1}{1 + |y_1|}, \frac{-y_2}{1 + |y_2|}, \dots$$

is the inverse of h_5 .

4. h_1 and h_6 .

4.1 LEMMA. If for some fixed $n \geq 1$, f_{n+1} is a mapping of R^n into R satisfying $f_{n+1}(b_j) \rightarrow 0$ whenever $\|b_j\| \rightarrow \infty$. Then the function F_{n+1} defined by

$$F_{n+1}(x_1, x_2, \dots) = (x_1, \dots, x_n, x_{n+1} + f_{n+1}(x_1, \dots, x_n), x_{n+2}, \dots)$$

induces an α -imbedding of l_F onto l_F .

Proof. F_{n+1} is clearly an imbedding of l_F onto l_F . To show F_{n+1} is an α -imbedding, let us assume the contrary. So for some $\varepsilon > 0$ and some sequence of pairs of points $\{(a_j, b_j)\}_{j \geq 1}$ we have $\|a_j - b_j\| > \varepsilon$ for all j but $\|F_{n+1}(a_j) - F_{n+1}(b_j)\| \rightarrow_j 0$. Thus for each $i \leq n$, $|\pi_i(a_j) - \pi_i(b_j)| \rightarrow_j 0$. We consider two cases. (1) $\{\|\tau_n(a_j)\|\}_{j \geq 1}$ is

unbounded. Without loss of generality, we assume $\|\tau_n(a_j)\| \rightarrow_j \infty$. This implies $\|\tau_n(b_j)\| \rightarrow_j \infty$. Then by the definition of f_{n+1} , $f_{n+1}(\tau_n(a_j)) \rightarrow_j 0$ and $f_{n+1}(\tau_n(b_j)) \rightarrow_j 0$. This means $\|F_{n+1}(a_j) - F_{n+1}(b_j)\| - \|a_j - b_j\| \rightarrow 0$ contradicting the assumption. (2) $\{\|\tau_n(a_j)\|\}_{j \geq 1}$ is bounded. This implies $\{\|\tau_n(b_j)\|\}_{j \geq 1}$ is also bounded. Without loss of generality we may assume $\tau_n(a_j) \rightarrow_j C$ and $\tau_n(b_j) \rightarrow_j C$ for some $C \in R^n$ since $|\pi_i(a_j) - \pi_i(b_j)| \rightarrow_j 0$ for all $i \leq n$. Hence $f_{n+1}(\tau_n(a_j)) - f_{n+1}(\tau_n(b_j)) \rightarrow_j 0$. This again implies $\|F_{n+1}(a_j) - F_{n+1}(b_j)\| - \|a_j - b_j\| \rightarrow 0$, a contradiction.

4.2 COROLLARY. *If for each $n \geq 1$, f_{n+1} is a mapping of R^n into R such that $f_{n+1}(a_i) \rightarrow 0$ whenever $\|a_i\| \rightarrow \infty$. Then for each $n \geq 1$, the function F_{n+1} defined by*

$$F_{n+1}(x) = (x_1, x_2 + f_2(x_1), \dots, x_{n+1} + f_{n+1}(x_1, \dots, x_n), x_{n+2}, \dots)$$

is an α -imbedding of l_F onto l_F .

Proof. It is clear that each F_{n+1} is a homeomorphism of l_F onto l_F . For $n=1$, F_2 is an α -imbedding according to §4.1. Suppose for $n \geq 1$, F_{n+1} is an α -imbedding. Clearly $F_{n+2}F_{n+1}^{-1}$ satisfies §4.1, hence is an α -imbedding. Thus

$$F_{n+2} = (F_{n+2}F_{n+1}^{-1})F_{n+1}$$

is an α -imbedding. This completes the induction.

4.3 Let G be a mapping of R into R defined by

$$G(x) = (|x| + x)/2(1 + x^2).$$

For each fixed $n > 1$, define a sequence of mappings $\{G_i^n\}_{1 \leq i \leq n}$ as follows: Each

$$G_1^n: R^{i-1} \rightarrow R,$$

$$G_2^n(x_1) = G(x_1)/n \text{ and inductively, if } 1 < k < n,$$

$$G_{k+1}^n((x_1, \dots, x_k)) = 1/2^{n-k} \cdot G(x_k + G_k^n((x_1, \dots, x_{k-1}))).$$

The following lemma is evident.

LEMMA. (1) $0 \leq G(x) \leq 1$ for all $x \in R$; $G(x) > 0$ if and only if $x > 0$; $G(x_i) \rightarrow 0$ whenever $|x_i| \rightarrow \infty$.

(2) For each $n > 1$ and each $1 < i \leq n$, $0 \leq G_i^n(x) \leq 1$ for all $x \in R^{i-1}$.

(3) If for some $1 < k < n$, $G_k^n(x) = 0$, then $G_n^n(x') = 0$ where $x' = (x, 0, \dots, 0) \in R^{n-1}$.

(4) $G_k^n(x) > 0$ if $\pi_{k-1}(x) > 0$.

4.4 For each $n > 1$, let $\{G_i^n\}_{1 \leq i \leq n}$ be defined as in §4.3. Define a sequence of mapping $\{Z_n\}_{n > 1}$ of R^{n-1} into R as follows:

$$Z_2(x_1) = G_2^2(x_1) \text{ and inductively for } k > 2,$$

$$(*) \quad Z_{k+1}((x_1, \dots, x_k)) = G_{k+1}^{k+1}((x_1, \dots, x_k))G(x_k - \frac{1}{2}Z_k((x_1, \dots, x_{k-1}))).$$

The following lemma is evident.

LEMMA. (1) $1 \geq Z_n(x) \geq 0$ for all $x \in R^{n-1}$,

(2) for all $n > 1$, $Z_n(x) = 0$ if $\pi_{n-1}(x) \leq 0$.

4.5 Homeomorphism h_6 . Let $\{Z_n\}_{n>1}$ be defined as above. Define $H: s \rightarrow s$ by

$$H(x) = (x_1, x_2 + Z_2(x_1), x_3 + Z_3(x_1, x_2), \dots).$$

THEOREM. $h_6 = H|_{s_-}$ is a homeomorphism of s_- onto s_F .

Proof. It is clear that h_6 is 1-1, continuous and h_6^{-1} is continuous. To show $h_6(s_-) \supset s_F$, suppose $x \in s_F$. If $x=0$, it is trivial since $h_6(0)=0$ and $0 \in s_-$. If $x \neq 0$, let n be the largest nonzero index of x . Let

$$x' = (x_1, x_2 - Z_2(x_1), \dots, -Z_{n+1}(x_1, \dots, x_n), 0, 0, \dots).$$

By Lemma 4.4, $H(x') = x$. So h_6 is onto s_F if $x' \in s_-$. We consider four cases. (1) If $n=1$ and $x_1 > 0$, then $-Z_2(x_1) = -G_2^2(x_1) < 0$ by Lemma 4.3(4), hence $x' \in s_-$. (2) if $n=1$ and $x_1 < 0$, then by Lemma 4.4(2), $-Z_2(x_1) = 0$, hence $x' \in s_-$. (3) If $n > 1$ and $x_n < 0$, then $-Z_{n+1}(x_1, \dots, x_n) = 0$ by 4.4(2) and $x_n - Z_n(x_1, \dots, x_{n-1}) < 0$, hence $x' \in s_-$. (4) If $n > 1$ and $x_n > 0$, we have two cases. (1) If $-Z_{n+1}(x_1, \dots, x_n) < 0$, then we are finished since this implies $x' \in s_-$. If $-Z_{n+1}(x_1, \dots, x_n) = 0$, by (*) of §4.4, and by §4.3(4), $G(x_n - \frac{1}{2}Z_n((x_1, \dots, x_n))) = 0$. By §4.3(1), $x_n - \frac{1}{2}Z_n((x_1, \dots, x_n)) \leq 0$. Hence $x_n - Z_n((x_1, \dots, x_n)) < 0$. This implies $x' \in s_-$.

Finally we have to show $h_6(s_-) \subset s_F$. Suppose $x (\neq 0) \in s_-$, let x_n be the largest nonzero index of x . Hence $x_n < 0$. If $n=1$, then clearly $Z_{k+1}(x_1, \dots, x_k) = 0$ for all $k \geq 1$, thus $h_6(x) \in s_F$. If $n > 1$, for any $m > n$, consider

$$G_{n+1}^m(\tau_n(x)) = \frac{1}{2^{m-n}} G\left(x_n + \frac{1}{2^{m-n+1}} G(x_{n-1} + G_{n-1}^m(\tau_{n-1}(x)))\right).$$

Since $x_n < 0$ and

$$x_n + \frac{1}{2^{m-n+1}} G(x_{n-1} + G_{n-1}^m(\tau_{n-1}(x))) \leq x_n + \frac{1}{2^{m-n+1}}$$

by §4.3(1), there is a large enough m such that $G_{n+1}^{m'}(\tau_n(x)) = 0$ for all $m' \geq m$. By §4.3(3), $G_{m'}^{m'}(\tau_{m'-1}(x)) = 0$ for all $m' \geq m$. This implies $Z_{m'}(\tau_{m'-1}(x)) = 0$ for all $m' \geq m$. Hence $h_6(x) \in s_F$.

4.6 For each $n > 1$, let $\{G_i^n\}_{1 \leq i \leq n}$ and $\{Z_n\}_{n>1}$ be defined as in §4.3 and §4.4. Define a sequence of mapping $\{Z_n\}_{n>1}$ of R^{n-1} into R and a sequence of α -imbedding $\{F_n\}_{n>1}$ of l_F onto l_F as follows:

$$Z'_2(x_1) = Z_2(x_1); F_2(x) = (x_1, x_2 + Z'_2(x_1), x_3, \dots).$$

Inductively suppose for $k > 1$, $\{Z'_i\}_{1 \leq i \leq k}$ and $\{F_i\}_{1 \leq i \leq k}$ have been defined. Define

$$\begin{aligned} Z'_{k+1}(x_1, \dots, x_k) &= Z_{k+1}(x_1, \dots, x_k) \cdot G(1 + x_1^2 + \dots + x_k^2) \\ (**) \quad &\cdot \min\left(\frac{1}{3^{k+1}}, \frac{1}{3^{k+1}} \alpha\left(F_k, \frac{1}{3^k}\right)\right), \end{aligned}$$

and

$$F_{k+1}(x) = (x_1, x_2 + Z'_2(x_1), \dots, x_{k+1} + Z'_{k+1}(x_1, \dots, x_k), x_{k+2}, x_{k+3}, \dots).$$

LEMMA. (1) Each F_n is an α -imbedding.

(2) $\lambda(F_{n+1}, F_n) \leq \min(1/3^{k+1}, 1/3^{k+1} \cdot \alpha(F_k, 1/3^k))$.

(3) If $x \in l_2$, then $x' = (x_1, x_2 + Z'_2(x_1), x_3 + Z'_3(x_1, x_2), \dots) \in l_2$.

Proof. (1) If $a_i \in R^{n-1}$ such that $\|a_i\| \rightarrow \infty$, then $G(1 + \|a_i\|^2) \rightarrow 0$ by §4.3(1). Hence $Z'_n(a_i) \rightarrow 0$. By Corollary 4.2, F_n is an α -imbedding. (2) Follows from §4.4(1) and §4.3(1). (3) Follows from (**). Specifically:

$$|Z'_{k+1}(x_1, \dots, x_k)| \leq 1/3^{k+1}.$$

4.7 Homeomorphism h_1 . Define $H': l_2 \rightarrow l_2$ by

$$H'(x) = (x_1, x_2 + Z'_2(x_1), x_3 + Z'_3(x_1, x_2), \dots).$$

THEOREM. $h_1 = H'|_{l_-}$ is a homeomorphism of l_- onto l_F .

Proof. It is clear that $h_1(x) = \lim_{n \rightarrow \infty} F_n(x)$, where $\{F_n\}$ as defined in §4.6. By Lemma 4.6(2) and Lemma 3.1, h_1 is an imbedding of l_- into l_2 . The proof that $h_1(l_-) = l_F$ is similar (the fact F_n is an α -imbedding is needed here) to the proof that $h_6(s_-) = s_F$ in §4.5.

5⁽³⁾. h_4 .

5.1 LEMMA. Let X be a compact Hausdorff space and $A \subset X$. If g is a mapping of X into X such that $f|_A$ is 1-1 and $f(X - A) \subset X - f(A)$, then $f|_A$ is an imbedding.

Proof. We have to show $f^{-1}: f(A) \rightarrow A$ is continuous. Suppose $x, \{x_\alpha\}_\alpha \subset A$ such that $f(x_\alpha) \rightarrow f(x)$. Since X is compact, we may assume $x_\alpha \rightarrow y \in X$. Hence $f(x_\alpha) \rightarrow f(y) (= f(x)) \in f(A)$. By hypothesis $y \in A$. Hence $y = x$ since f is 1-1 on A .

5.2 Let $D_n = J_n \times J_{n+1}$. Let $a = (1/n, 1)$, $b = (0, 1)$, $c = (-1, 0)$, $d = (0, -1)$ be points in D_n and let K = closure of the component of $\text{Bd}(D_n) - \{a, d\}$ that does not include b . For each $n \geq 1$, let f_n be a homeomorphism of D_n onto D_n such that $f_n|_K = \text{identity}$, $f_n(b) = c$ and $f_n(tx) = t(f_n(x))$ for each $0 \leq t \leq 1$ and each $x \in \text{Bd}(D_n)$.

LEMMA. If $x_{n+1} > 0$ and $\pi_n(f_n((x_n, x_{n+1}))) = 0$, then (1) $0 < x_n < 1/n$ (2) $x_{n+1} = \pi_{n+1}f_n((x_n, x_{n+1}))$ and (3) $x_{n+1} > x'_n$.

Proof. Clear.

5.3 Homeomorphism h_4 . Let $\{f_n\}_{n \geq 1}$ be defined as in §5.2. Define a sequence of homeomorphisms $\{F_n\}_{n \geq 1}$ of Q onto Q by $F_1(x) = (f_1(x_1, x_2), x_3, x_4, \dots)$ and for $n > 1$, $F_n(x) = (x_1, \dots, x_{n-1}, f_n(x_n, x_{n+1}), x_{n+2}, \dots)$. Let F be the mapping of Q into Q defined by $F(x) = \lim_{n \rightarrow \infty} F_n \cdots F_2 F_1(x)$.

THEOREM. $h_4 = F|_{s_p}$ is a homeomorphism of s_p onto s'_+ .

Proof. It is easy to check that $F(s_p) = s'_+$, $F(\bar{s}_p) = \bar{s}_f$, $s_p \subset \bar{s}_p$, $s'_+ \subset \bar{s}_f$ and F is a 1-1,

(3) The method employed in this section is due to Corson [1].

continuous mapping of \bar{s}_p onto \bar{s}_f . Therefore by means of §5.1, it is sufficient to show $F(Q - \bar{s}_p) \subset Q - \bar{s}_f$. Suppose $x \in Q$ and $x \notin \bar{s}_p$.

Case 1. For some i , $x_i < 0$. Then $\pi_j(F(x)) < 0$ for all $j > i$. Hence $F(x) \notin \bar{s}_f$.

Case 2. For all i , $x_i \geq 0$ and for some i , $x_i = 0$. Hence for infinite many i , $x_i > 0$. It is easy to check $\pi_j(F(x)) = -\pi_{j+1}(x)$ for all $j \geq i$. Hence $F(x) \notin \bar{s}_f$.

Case 3. For each i , $x_i > 0$. Let us assume $F(x) \in \bar{s}_f$. Hence there exists an $n > 1$ such that $\pi_m(F(x)) = 0$ for all $m \geq n$. It follows from Lemma 5.2(2) $x_{m+1} = \pi_{m+1}(F_m F_{m-1} \cdots F_1(x)) = x'_{m+1}$ for all $m \geq n$. By §5.2(1), $0 < x'_n = \pi_n(F_{n-1} \cdots F_1(x)) < 1/n$ and by repeating §5.2(3), $0 < x'_n < x_{n+1} = x'_{n+1} < x_{n+2} = x'_{n+2} < \cdots$. This is clearly impossible since $x'_{n+k} < 1/(n+k)$ for all k .

6. h_3 .

6.1 LEMMA. Let X be a metric space, $A \subset X$ and f an imbedding of A into X . Then f is an α -imbedding if either (1) A is compact, or (2) $A = X$ and f is supported on a compact subset K of X ; i.e., $f|_{X-K} = \text{identity}$, or (3) $X = A = R^{n+m}$ and $f = f_n \times e$ where f_n is an α -imbedding of R^n into R^n and e is the identity function on $\prod_{i=n+1}^{n+m} R_i$.

Proof. Well known.

6.2 LEMMA. If for each n , $\{f_n\}_{n \geq 1}$ is an imbedding of R^n into R^n such that for $n > 1$ and any $x \in R^n$, $\pi_i(f_n(x)) = \pi_i(x)$ for all $i < n$. Then the function f defined by $f(x) = \lim_{n \geq 1} f'_1 \cdots f'_n(x)$ is an imbedding of s into s , where each f'_i is the imbedding of s into s induced by f_i .

Proof. It is evident that $f(x)$ exists for each $x \in s$ and is a 1-1, continuous mapping of s into s . To show f^{-1} is continuous, let us suppose $\{a_i\}_{i \geq 1}$ is a sequence in s such that $f(a_i) \rightarrow f(a) \in f(s)$. We want to show $\pi_j(a_i) \rightarrow \pi_j(a)$ for each j . For $j=1$, it is obvious since f_1 is an imbedding of R_1 into R_1 . Now suppose it is true for all $j \leq k$. From the definition of f_j , we observe that $\pi_j(f(x)) = \pi_j(f'_j(x))$ for any $x \in s$. Hence

$$\pi_{k+1}(f'_{k+1}(a_i)) = \pi_{k+1}(f(a_i)) \rightarrow \pi_{k+1}(f(a)).$$

On the other hand, since $\pi_j(a_i) \rightarrow \pi_j(a)$ for all $j \leq k$ by assumption, $\pi_j(f'_{k+1}(a_i)) = \pi_j(a_i) \rightarrow \pi_j(a)$ for all $j \leq k$. This shows that $f_{k+1}(\pi_{k+1}(a_i)) \rightarrow (\pi_1(a), \dots, \pi_k(a), \pi_{k+1}(f(a)))$. Therefore $\pi_{k+1}(a_i) \rightarrow \pi_{k+1}(a)$. This completes the induction.

6.3 Let B^n , S^{n-1} denote respectively the closed unit ball, unit sphere of R^n . Recall that R^n is also regarded as $R^n \times 0 \subset R^{n+1}$ and as $R^n \times 0 \times 0 \cdots \subset s$.

LEMMA. For $n > 1$ and any $\varepsilon > 0$, there is an ε -imbedding g_n of R^n onto R^n satisfying the following conditions:

- (1) $g_n|_{R^n} = \text{identity}$,
- (2) $\tau_{n-1}(g_n(B^n_+)) = B^{n-1}$,
- (3) $\pi_1(g_n(x)) = 0$ iff $\pi_1(x) = 0$,
- (4) each g_n is an α -imbedding of R^n onto R^n and

(5) for each $x \in g_n(B^{n-1}) = B^{n-1}$, $\{(x, t) : t \in R_n\} \cap g_n(B^n)$ is a nondegenerate closed line segment $[a, b]$ such that (1) if $x \in \text{Int}(B^{n-1})$, then $b \in g_n(S_+^{n-1})$, $a \in g_n(S_-^{n-1}) = S_-^{n-1}$ and $\text{Int}[a, b] \subset \text{Int } g_n(B^n)$, (2) if $x \in \text{Bd}(B^{n-1}) = S^{n-2}$, then $\|a - b\|$ is a constant r and $[a, b] \subset g_n(S_+^{n-1})$.

Proof. We shall prove this by actually constructing g_n . We may assume g_2 has been constructed for the given ε since g_2 obviously exists. To define g_n , write R^n as $R^{n-1} \times R_n$. For each $x \in S^{n-2}$, let $\mathcal{L}_x = \{tx : t \in R\}$ and $E_x = \mathcal{L}_x \times R_n$. We consider E_x as R^2 by identifying $(tx, r) \in \mathcal{L}_x \times R_n$ with $(t\|x\|, r) \in R^2$. Hence g_2 induces a homeomorphism g_x of E_x onto E_x . We assume g_2 is so chosen that the motion of g_2 is symmetric with respect to the R_2 -axis; that is, if $g_2(x_1, x_2) = (x'_1, x'_2)$, then $g_2(-x_1, x_2) = (-x'_1, x'_2)$. This implies $g_{(-x)} = g_x$. It is evident that defining g_x on each $\mathcal{L}_x \times R_n$ induces a homeomorphism g_n of R^n onto R^n and satisfies all the required conditions.

6.4 Let $\{g_n\}_{n>1}$ be inductively defined as in §6.3 subject to the following ε -condition.

For $k \geq 1$, let g_n^k denote the homeomorphism of R^{n+k} onto R^{n+k} induced by g_n ; i.e., $g_n^k(x_1, \dots, x_{n+k}) = (g_n(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+k})$. First we define g_2 as in §6.3 for $\varepsilon = \frac{1}{2}$. Suppose $\{g_i\}_{i=2}^n$ has been defined, let $\tilde{g}_n = g_2^{n-2} \cdots g_{n-1}^1 g_n$ and let \tilde{g}_n^1 be defined as g_n^1 . We then choose ε to be so small for g_{n+1} so that

$$(\#) \quad \lambda(\tilde{g}_n^1 \cdot g_{n+1}, \tilde{g}_n^1) \leq \min \left(\frac{1}{3^{n+1}}, \frac{1}{3^{n+1}} \alpha \left(\tilde{g}_n^1, \frac{1}{3^n} \right) \right).$$

Let G_n denote the homeomorphism of s onto s induced by g_n and let $G'_n = G_2, \dots, G_n$.

LEMMA. (1) $G(x) = \lim_{n \geq 2} G'_n(x)$ is an imbedding of s into s , (2) for $n > 1$, $G(B^n) = G'_n(B^n)$, (3) $G|_{B^1} = \text{identity}$.

Proof. (1) $G(x)$ exists since for each i , $\{\pi_i(G'_n(x))\}_{n \geq 2}$ is a Cauchy sequence in R_i . The fact that G is an imbedding is a rather straightforward consequence of (#). (The proof can be reconstructed similar to that in §3.1.)

(2) If $x \in B^n$ then $x \in R^{n+k}$ for all $k \geq 1$, hence by §6.3(1), $g_{n+k}(x) = x$. This implies $G'_{n+k}(x) = G'_n(x)$ for all $k \geq 1$. Thus $G(x) = G'_n(x)$.

(3) By (2), $G|_{B^1} = G'_2|_{B^1} = G_2|_{B^1} = g_2|_{B^1} = \text{identity}$.

6.5 Let $G, \{G'_n\}_{n \geq 2}$ be defined as in §6.4. For each $A \subset s$, we shall denote $G(A)$ by \tilde{A} . Hence $\tilde{B}^n = G(B^n) = G'_n(B^n) = \tilde{g}_n(B^n) = g_2^{n-2} \cdots g_{n-1}^1(g_n(B^n))$ by §6.4. Now by applying Lemma 6.3 and the known properties of R^n , we have

LEMMA. There is a homeomorphism ϕ_n of R^n onto R^n , $n > 1$, such that

- (1) $\phi_n(\tilde{B}^n) = \tilde{B}^{n-1} \times I_n$, where $I_n = [0, 1] \subset R_n$,
- (2) $\phi_n(\tilde{S}^{n-1}) = \tilde{B}^{n-1}$,
- (3) $\phi_n|_{\tilde{S}^{n-2}} = \text{identity}$ and
- (4) for any $x \in R^n$, $\pi_i(\phi_n(x)) = \pi_i(x)$ for all $i < n$.

6.6 Let $\{\phi_n\}_{n>1}$ be defined as above. Let ϕ'_n be the homeomorphism of s onto s induced by ϕ_n . Define $\phi: s \rightarrow s$ by $\phi(x) = \lim_{n>1} \tilde{\phi}_n(x)$ where $\tilde{\phi}_n = \phi'_1 \phi'_2 \cdots \phi'_n$. Then according to §6.2, ϕ is an imbedding of s into s . Furthermore, for $n > 1$, we have the following Lemma:

LEMMA. (1) $\tilde{\phi}_n(\tilde{B}^n) = B^1 \times \prod_{j=2}^n I_j$, where $I_j = [0, 1] \subset R_j$,

(2) $\phi(\tilde{S}^{n-1}) = \text{Bd}(B^1 \times \prod_{j=2}^n I_j)$,

(3) $\phi(\tilde{S}^1) = B^1$ and $\phi(\tilde{S}^{n-1}) = B^1 \times \prod_{j=2}^{n-1} I_j$ if $n > 2$,

and

(4) $\phi(\text{Int } \tilde{S}^1) = \text{Int } B^1$ and $\phi(\text{Int } (\tilde{S}^{n-1})) = \text{Int}(B^1 \times \prod_{j=2}^{n-1} I_j)$ if $n > 2$.

Proof. (1) For $n=2$, this follows from Lemma 6.5(1) and Lemma 6.4(3). Suppose it is true for all $i \leq k$. $\tilde{\phi}_{n+1}(\tilde{B}^{n+1}) = \tilde{\phi}_n(\phi'_{n+1}(\tilde{B}^{n+1}))$. By Lemma 6.5(1), $\phi'_{n+1}(\tilde{B}^{n+1}) = \tilde{B}^n \times I_{n+1}$. Hence

$$\tilde{\phi}_{n+1}(\tilde{B}^{n+1}) = \tilde{\phi}_n(\tilde{B}^n \times I_{n+1}) = \tilde{\phi}_n(\tilde{B}^n) \times I_{n+1} = B^1 \times \prod_{j=2}^{n+1} I_j.$$

This completes the induction.

(2) By (1), $\tilde{\phi}_n(\tilde{S}^{n-1}) = \text{Bd}(B^1 \times \prod_{j=2}^n I_j)$. Hence we only have to show that for each $x \in \tilde{S}^{n-1}$, $\phi(x) = \tilde{\phi}_n(x)$. $x \in \tilde{S}^{n-1}$ implies $x \in \tilde{S}^{k-2}$ for all $k-2 \geq n-1$. Hence by Lemma 6.5(3), $\phi'_k(x) = x$ for all $k \geq n+1$. Thus $\tilde{\phi}_n(x) = \tilde{\phi}_n \phi'_{n+1} \cdots \phi'_{n+j}(x) = \tilde{\phi}_{n+j}(x)$ for all $j \geq 1$. This shows $\tilde{\phi}_n(x) = \phi(x)$.

(3) Since $\tilde{S}^{n-1} \subset \tilde{S}^{n-1}$, by the proof of (2) above $\phi(\tilde{S}^{n-1}) = \tilde{\phi}_n(\tilde{S}^{n-1})$. Now for $n=2$, $\phi_2(\tilde{S}^1) = \tilde{B}^1 = B^1$ by Lemma 6.5(2) and Lemma 6.4(3). For $n > 2$,

$$\phi(\tilde{S}^{n-1}) = \tilde{\phi}_n(\tilde{S}^{n-1}) = \tilde{\phi}_{n-1}(\phi'_n(\tilde{S}^{n-1})) = \tilde{\phi}_{n-1}(\tilde{B}^{n-1})$$

by Lemma 6.5(2). By (1), $\phi_{n-1}(\tilde{B}^{n-1}) = B^1 \times \prod_{j=2}^{n-1} I_j$. Thus $\phi(\tilde{S}^{n-1}) = B^1 \times \prod_{j=2}^{n-1} I_j$. This completes the proof.

(4) This is the consequence of (3).

6.7 *Homeomorphism h_3 .* Let G be defined as in §6.4 and ϕ as in §6.6. It is easy to see that by means of §6.3(3) and §6.5(4), we have $G(s') \subset s'$ and $\phi(s') \subset s'$ (see §2(0) for definition of s'). Furthermore they also imply $\pi_1(x) = 0$ iff $\pi_1(G(x)) = 0$ and $\pi_1(x) = 0$ iff $\pi_1(\phi(x)) = 0$. Let $a = (-1, 0, 0, \dots) \in s$. By §6.4(3), $G(a) = a$. By the proof of §6.6(2), $\phi(a) = \phi_2(a) = \phi'_2(a) = \phi_2(a)$. By §6.5(3) $\phi_2(a) = a$. Hence $\phi(a) = a$. As a consequence of §6.6(3) and (4),

$$\begin{aligned} \phi\left(\bigcup_{n>1} \tilde{S}^{n-1}\right) &= \bigcup_{n>1} \left(B^1 \times \prod_{j=2}^n I_j\right); \\ \phi\left(\bigcup_{n>1} \text{Int } (\tilde{S}^{n-1})\right) &= \text{Int } (B^1) \cup \left(\bigcup_{n>2} \text{Int} \left(B^1 \times \prod_{j=2}^{n-1} I_j\right)\right). \end{aligned}$$

Evidently,

$$S_- = (S_F \cap I_-) \cap s' = \{a\} \cup \left(\left(\bigcup_{n>1} \text{Int } (\tilde{S}^{n-1})\right) \cap s'\right).$$

Hence

$$\phi G(S_-) = \{a\} \cup \text{Int}(I'_1) \cup \left(\bigcup_{n \geq 2} \text{Int} \left(I'_1 \times \prod_{j=2}^{n-1} I_j \right) \right)$$

where $I'_1 = [-1, 0]$. Let T be the homeomorphism of s onto s defined by $T(x) = (x_1 + 1, x_2, x_3, \dots)$. Obviously

$$T\phi G(S_-) = \{0\} \cup \bigcup_{n \geq 1} \text{Int} \left(\prod_{j=1}^n I_j \right) = s_p.$$

Thus we have

THEOREM. $h_3 = T\phi G|_{S_-}$ is a homeomorphism of S_- onto s_p .

7. Proof of Corollary I. It is well known that I_F is homeomorphic to its unit sphere S_F [3], [4]. According to §6.7,

$$\phi G(S_F) = \phi \left(\bigcup_{n \geq 1} \tilde{S}^{n-1} \right) = \bigcup_{n \geq 2} \text{Bd} \left(B^1 \times \prod_{j=2}^n I_j \right) = \bigcup_{n \geq 2} \left(B^1 \times \prod_{j=2}^n I_j \right) \sim \bar{s}_p.$$

By the proof of §5.3, $\bar{s}_p \sim \bar{s}_f$. This completes the proof.

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