COMPLEMENTED B^*-ALGEBRAS

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1. **Introduction.** The theory of (left, right) complemented Banach algebras was developed in [6]. In the present paper we restrict our attention to complemented B^*-algebras. In §3 we show that a B^*-algebra is (left, right) complemented if and only if it is dual. Since a dual B^*-algebra is *-isomorphic to the $B^*(\infty)$-sum of its minimal closed two-sided ideals each of which is *-isomorphic to the algebra $LC(H)$ of all compact operators on a Hilbert space H, it is thus quite natural to focus our attention on the (left, right) complemented B^*-algebras of the form $LC(H)$. In fact, the greater part of this paper is devoted to the study of such algebras. In restricting our attention to $LC(H)$ we are also able to bring to the forefront the relationship that exists between right complementors on $LC(H)$ and the inner products in H which give rise to equivalent norms on H.

In §3 we also introduce the concept of a continuous right complementor $p: R \to R^p$ on a B^*-algebra A. We give the continuity of p in terms of the minimal idempotents in A. More precisely, let $\mathscr{E}$ be the set of all selfadjoint minimal idempotents e in A and $\mathscr{E}_p$ the set of all minimal idempotents f in A such that $(fA)^p = (1-f)A$. Let P be the mapping of $\mathscr{E}$ into $\mathscr{E}_p$ such that $P(e)A = eA$, and let $\mathscr{E}$ and $\mathscr{E}_p$ have the relative topologies induced by the norm on A. We say that p is continuous if the mapping P is continuous. It follows that if p is continuous then the induced complementor p_I on every closed two-sided ideal I of A is also continuous. In fact we show that p is continuous if and only if every p_{I_λ} is continuous for every minimal closed two-sided ideal I_λ of A, $\lambda \in \Lambda$.

In §4 we show that to every inner product $\langle \cdot \rangle$ in H giving rise to an equivalent norm on H there corresponds in a natural way a right complementor on $LC(H)$. In fact if $*_{\langle \cdot \rangle}$ denotes the involution on $LC(H)$ induced by $\langle \cdot \rangle$, then $R \to (R_l)^{*_{\langle \cdot \rangle}}$ is a right complementor on $LC(H)$, where R_l denotes the left annihilator of the closed right ideal R of $LC(H)$. It follows (§6) that if the dimension of H is at least three then all such complementors on $LC(H)$ are continuous, and in fact these are the only ones.

In §5 we define a mapping T of H onto itself corresponding to a given right complementor p on $LC(H)$. T is obtained by means of the mapping P and is such that, if we write $e \in \mathscr{E}$ in the form $e = x \otimes x$ with $x \in H$ and $\|x\| = 1$, then $P(e) = (x \otimes Tx)/(x, Tx)$. To obtain T we require that the dimension of H be at least three. In §6 we show that p is continuous if and only if T is linear. It turns out that if T is linear then it is a bounded positive operator with bounded inverse.

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We observe that every right complementor $p: R \to R^p$ on $LC(H)$ induces a complementor $S \to S'$ on the closed subspaces S of H in the sense of Theorem 1 in [3] (see §3). It is shown in [3] that if H is infinite dimensional then every complementor $S \to S'$ on the closed subspaces S of H defines an inner product in H which gives rise to an equivalent norm on H and with respect to which S and S' are orthogonal complements of each other. However, if H is finite dimensional this is not true in general, as is shown by examples in [3]. We show that if the dimension of H is at least three then, in order that $S \to S'$ define an inner product in H which gives rise to an equivalent norm on H and with respect to which S and S' are orthogonal complements of each other, it is necessary and sufficient that $S \to S'$ be induced by a continuous right complementor on $LC(H)$.

In §7 we return to the general complemented B^*-algebra A. We show that if A has no minimal left ideals of dimension less than three, then P is uniformly continuous if and only if there exists an involution $*'$ on A such that $R^p = (R_l)^{*'}$, for every closed right ideal R of A (Theorem 7.4).

We should like to thank Professor F. F. Bonsall for his general comments and advice. Some of the results appearing here come from the Ph.D. research of the first named author who has been working under his supervision. Theorem 3.6 was originally proved by one of the authors without the use of Theorem 3.5. We are grateful to Professor Bonsall for noticing that it is a simple consequence of this more general result. Corollaries 3.2–3.4, Theorems 3.5 and 5.2 as well as their proofs are due to him. We also wish to thank the referee for his many comments and for the suggestions which contributed to the shortening of the proofs of Theorem 5.3 and Lemma 6.2.

2. **Preliminaries.** Let A be a complex Banach algebra and let L_r be the set of all closed right ideals of A. Following [6] we shall say that A is a *right complemented Banach algebra* if there exists a mapping $p: R \to R^p$ of L_r into itself having the following properties:

(C_1) $R \cap R^p = (0)$, $(R \in L_r)$;

(C_2) $R + R^p = A$, $(R \in L_r)$;

(C_3) $(R^p)^p = R$, $(R \in L_r)$;

(C_4) if $R_1 \subset R_2$, then $R_2^p \subset R_1^p$, $(R_1, R_2 \in L_r)$.

The mapping p is called a *right complementor* on A. Analogously we define a *left complemented Banach algebra* and a *left complementor*. Thus a complex Banach algebra is left (right) complemented if and only if it has a left (right) complementor defined on it.

It is easy to see that property (C_3) may be replaced by one of the following:

(C_3') $(R^p)^p \supset R$, $(R \in L_r)$.

(C_3'') $(R^p)^p \subset R$, $(R \in L_r)$.

In fact, since $R \oplus R^p = A$ and $(R^p)^p \oplus R^p = A$, either $(R^p)^p \supset R$ or $(R^p)^p \subset R$ gives $(R^p)^p = R$.

A Banach algebra with an involution $x \to x^*$ such that $\|x^*x\| = \|x\|^2$ is called a B^*-algebra. If A is a B^*-algebra it is clear that there is a one-to-one correspondence between left complementors and right complementors on A. Thus it suffices in this case to limit our attention to right complemented B^*-algebras and, unless mentioned otherwise, a complementor on a B^*-algebra will always mean a right complementor.

An idempotent e in a Banach algebra A is said to be *minimal* if eAe is a division algebra. In case A is semisimple this is equivalent to saying that Ae (eA) is a minimal left (right) ideal of A. If S is any subset of a Banach algebra A, S_l (S_r) will denote the left (right) annihilator of S in A. A is dual if and only if for every closed right ideal R and every closed left ideal J of A we have $R_{lr} = R$ and $J_{rl} = J$.

Let $\{I_\lambda : \lambda \in \Lambda\}$ be a family of left (right) ideals I_λ in an algebra A. As usual $\sum_\lambda I_\lambda$ will denote the sum of the ideals I_λ, i.e., $\sum_\lambda I_\lambda$ is the set of all elements of A of the form $x_{\lambda_1} + x_{\lambda_2} + \cdots + x_{\lambda_n}$, with $x_{\lambda_i} \in I_{\lambda_i}$, $i = 1, 2, \ldots, n$. $\sum_\lambda I_\lambda$ is a left (right) ideal of A. If S is a subset of a topological space X, cl (S) will denote the closure of S in X.

We will write C for the field of complex numbers and $\alpha, \beta, \ldots$ for the elements of C.

Before continuing with other concepts and definitions we first state a lemma which will be useful to us in future.

LEMMA 2.1. *If A is a Banach algebra with a right complementor p and $\{R_\lambda : \lambda \in \Lambda\}$ is a family of closed right ideals of A, then*

$$\Big(\bigcap_\lambda R_\lambda^p\Big)^p = \mathrm{cl}\,\Big(\sum_\lambda R_\lambda\Big).$$

Proof. Since, for each $\lambda \in \Lambda$, $R_\lambda^p \supset \bigcap_\lambda R_\lambda^p$, we have $(\bigcap_\lambda R_\lambda^p)^p \supset R_\lambda$ and hence $(\bigcap_\lambda R_\lambda^p)^p \supset \sum_\lambda R_\lambda$. Thus $(\bigcap_\lambda R_\lambda^p)^p \supset \mathrm{cl}\,(\sum_\lambda R_\lambda)$. But $\mathrm{cl}\,(\sum_\lambda R_\lambda) \supset R_\lambda$ for each $\lambda \in \Lambda$. Hence $(\mathrm{cl}\,(\sum_\lambda R_\lambda))^p \subset \bigcap_\lambda R_\lambda^p$ and consequently $\mathrm{cl}\,(\sum_\lambda R_\lambda) \supset (\bigcap_\lambda R_\lambda^p)^p$. Therefore $(\bigcap_\lambda R_\lambda^p)^p = \mathrm{cl}\,(\sum_\lambda R_\lambda)$ and the proof is complete.

Let H be a Hilbert space with inner product $(\ \cdot\ )$. If x and y are elements of H, then $x \otimes y$ will denote the operator on H defined by the relation $(x \otimes y)(h) = (h, y)x$ for all h in H. For any $x \in H$, $[x]$ will stand for the linear subspace spanned by x. If T is a linear operator acting in H, $T(H)$ will denote the range of T, i.e., $T(H) = \{y \in H : y = Tx \text{ for some } x \in H\}$. If $\langle \cdot \rangle$ is an inner product in H which defines an equivalent norm on H, we shall say that $\langle \cdot \rangle$ is an *equivalent inner product* in H. It is well known that an inner product $\langle \cdot \rangle$ in a complex Hilbert space H defines an equivalent norm on H if and only if there exists a positive continuous linear operator T on H with continuous inverse such that $\langle x, y \rangle = (x, Ty)$ for all $x, y \in H$. All our Hilbert spaces are over the complex field C.

Let $L(H)$ be the algebra of all continuous linear operators on H into itself with the usual operator norm, and let $\langle \cdot \rangle$ be an equivalent inner product in H. Then every $T \in L(H)$ is continuous with respect to the topology given by the norm

$\|x\|_{\langle\cdot\rangle}=\langle x, x\rangle^{1/2}$ and the corresponding operator bound $\|T\|_{\langle\cdot\rangle}$ is equivalent to the given operator bound. Let $*_{\langle\cdot\rangle}$ denote the operation of taking the adjoint in $L(H)$ with respect to $\langle\cdot\rangle$. Then $*_{\langle\cdot\rangle}$ defines an involution in $L(H)$ and $L(H)$ is a B^*-algebra under the norm $\|T\|_{\langle\cdot\rangle}$ and the involution $T\to T^{*\langle\cdot\rangle}$. Let $LC(H)$ be the subalgebra of $L(H)$ consisting of all compact operators on H. Then $LC(H)$ is also a B^*-algebra under the norm $\|T\|_{\langle\cdot\rangle}$ and the involution $T\to T^{*\langle\cdot\rangle}$. We will denote orthogonality in H with respect to $\langle\cdot\rangle$ by $\perp_{\langle\cdot\rangle}$ and, for any subset S of H, $S^{\perp\langle\cdot\rangle}=\{h\in H : \langle h, h_0\rangle=0 \text{ for all } h_0\in S\}$. We shall often write $\perp$ for $\perp_{\langle\cdot\rangle}$ and $*$ for $*_{\langle\cdot\rangle}$. If S is any subset of $L(H)$, $S^{*\langle\cdot\rangle}=\{T^{*\langle\cdot\rangle} : T\in S\}$.

Let $\{A_\lambda : \lambda\in\Lambda\}$ be a family of Banach algebras A_λ, and let $(\sum A_\lambda)_0$ be the set of all functions f defined on Λ such that $f(\lambda)\in A_\lambda$ for each $\lambda\in\Lambda$ and such that, for arbitrary $\varepsilon>0$, the set $\{\lambda : \|f(\lambda)\|\geqq\varepsilon\}$ is finite (see [4, p. 106]). It is easy to see that $(\sum A_\lambda)_0$ is closed under the usual operations of addition, multiplication and multiplication by scalars for functions: $(f+g)(\lambda)=f(\lambda)+g(\lambda)$, $(fg)(\lambda)=f(\lambda)g(\lambda)$ and $(\alpha f)(\lambda)=\alpha f(\lambda)$ for all $f, g\in(\sum A_\lambda)_0$, $\lambda\in\Lambda$ and $\alpha\in C$. $(\sum A_\lambda)_0$ is a Banach algebra under the norm $\|f\|=\sup_\lambda \|f(\lambda)\|$. If each A_λ is a B^*-algebra, then $(\sum A_\lambda)_0$ is also a B^*-algebra under the above norm $\|f\|$ and the involution $f\to f^*$ given by $(f^*)(\lambda)=f(\lambda)^{*_\lambda}$, where $*_\lambda$ is the involution on A_λ; we call $(\sum A_\lambda)_0$ the $B^*(\infty)$-sum of A_λ.

3. **Complementors and duality in B^*-algebras.**

LEMMA 3.1. *Let A be a Banach algebra and R a modular closed right ideal of A. Suppose there exists a closed right ideal R' of A such that $R\cap R'=(0)$ and $R+R'=A$. Then there exists a unique idempotent j such that $jA=R'$ and $(1-j)A=R$.*

Proof. The existence of an idempotent j with the required properties is an immediate consequence of Lemma 2 in [6]. We show that j is unique. Suppose there is an idempotent k with $kA=R'$ and $(1-k)A=R$. Then $k\in R_l=Aj$ and so $k=kj$. But $(1-k)A=R$ and $k, j\in R'$. Hence $kj-j\in R\cap R'=(0)$ and so $kj=j$. Therefore $k=kj=j$.

COROLLARY 3.2. *If R is a modular closed right ideal in a Banach algebra A for which there exists a closed right ideal R' such that $R\cap R'=(0)$ and $R+R'=A$, then the left annihilator of R is nonzero.*

Proof. We have $R_l=Aj\neq(0)$ in the notation of Lemma 3.1.

COROLLARY 3.3. *If A is a right complemented Banach algebra with a complementor p, then for each modular closed right ideal M of A there exists a unique idempotent with the property that $M=(1-e)A$ and $M^p=eA$.*

COROLLARY 3.4. *If A is a B^*-algebra with a complementor p, then for every maximal closed right ideal M of A there exists a unique minimal idempotent e such that $M^p=eA$ and $M=(1-e)A$.*

Proof. By Theorem (2.9.5) (iii) in [2], M is a modular right ideal and so, by Lemma 3.1, there exists a unique idempotent e such that $M=(1-e)A$ and $M^p=eA$. Since M^p is a minimal right ideal, e is a minimal idempotent.

THEOREM 3.5. *Let A be a B^*-algebra with the property that for every maximal closed right ideal M there exists a closed right ideal M' such that $M \cap M'=(0)$ and $M+M'=A$. Then A is dual.*

Proof. Let R be a proper closed right ideal of A. Then, by Theorem (2.9.5) (iii) in [2], R is equal to the intersection of all the maximal modular right ideals which contain it. In particular, R is contained in a closed modular right ideal and so, by Corollary 3.2, $R_l \neq (0)$. Using the involution it is easily seen that every proper closed left ideal of A has a nonzero right annihilator. Thus A is an annihilator algebra and Corollary (4.10.26) in [4] completes the proof.

THEOREM 3.6. *A B^*-algebra A is* (*left*, *right*) *complemented if and only if it is dual.*

Proof. Suppose A is dual. Let R be any proper closed right ideal of A and let $R^p=(R_l)^*$. Then R^p is a closed right ideal of A. By Theorem 3 in [1] and the argument on p. 652 in [6], p is a complementor on A. The converse follows from Theorem 3.5.

DEFINITION 3.7. Let A be a B^*-algebra with a complementor p. Let $\mathscr{E}$ denote the set of all selfadjoint minimal idempotents e (i.e., $e^*=e$) and $\mathscr{E}_p$ the set of all minimal idempotents f in A such that $(fA)^p=(1-f)A$. The elements of $\mathscr{E}$ and $\mathscr{E}_p$ are called projections and p-projections respectively. For any $e \in \mathscr{E}$ let $P(e)$ be the unique element of $\mathscr{E}_p$ such that $P(e)A=eA$. We call P the p-derived mapping of $\mathscr{E}$ into $\mathscr{E}_p$. The complementor p is said to be continuous if P is continuous in the relative topologies of $\mathscr{E}$ and $\mathscr{E}_p$ induced by the given norm on A.

REMARK. Since, by Lemma (4.10.1) in [4], every minimal right ideal of A is of the form eA with unique $e \in \mathscr{E}$, it follows that P maps $\mathscr{E}$ onto $\mathscr{E}_p$. We shall show later on that if P is continuous, then it actually defines a homeomorphism between $\mathscr{E}$ and $\mathscr{E}_p$.

THEOREM 3.8. *Let A be a B^*-algebra with a complementor p, and let I be a closed two-sided ideal of A. Then p induces a complementor p_I on the B^*-algebra I. Moreover, if p is continuous then so is p_I.*

Proof. That I is a B^*-algebra follows from Theorem (4.9.2) in [4]. Since A is semisimple, $A_r=(0)$ and so, by Lemma 1 in [6], $I_l=I_r=I^p$. Let R be a closed right ideal of I. We have $RA \subset R(I+I_r)=RI \subset R$ which shows that R is also a closed right ideal of A. Hence for any closed right ideal R of I let $R^{p_I}=R^p \cap I$. Then R^{p_I} is a closed right ideal of I and it is straightforward to verify that the mapping $p_I\colon R \to R^{p_I}$ satisfies (C_1), (C_2), (C_3') and (C_4), and therefore is a complementor on I.

Let $\mathscr{E}^I$ and $\mathscr{E}_p^I$ be respectively the sets of projections and p_I-projections in I, and let P_I be the p_I-derived mapping of $\mathscr{E}^I$ onto $\mathscr{E}_p^I$. Since minimal right ideals of I are also minimal right ideals of A, it follows that an idempotent e in I is minimal in I if and only if it is minimal in A. It is now easy to verify that $\mathscr{E}^I = \mathscr{E} \cap I$, $\mathscr{E}_p^I = \mathscr{E}_p \cap I$ and $P_I = P \mid \mathscr{E}^I$, where $P \mid \mathscr{E}^I$ is the restriction of P to $\mathscr{E}^I$. Hence, if P is continuous, then so is P_I, and the proof is complete.

THEOREM 3.9. *Let A be a B^*-algebra with a complementor p, and let $\{I_\lambda : \lambda \in \Lambda\}$ be the family of all minimal closed two-sided ideals of A. Let p_λ be the complementor on I_λ induced by p. Then the following statements are true*:

(a) *A is isometrically *-isomorphic to $(\sum I_\lambda)_0$.*

(b) *Each I_λ is isometrically *-isomorphic to $LC(H_\lambda)$ for some Hilbert space H_λ.*

(c) *p is continuous if and only if each p_λ is continuous.*

Proof. (a) By Theorem 3.6, A is dual and hence, by Theorem (4.10.14) in [4], A is isometrically *-isomorphic to $(\sum I_\lambda)_0$.

(b) Since I_λ is complemented, it is dual and, since it is simple, by Corollary (4.10.20) in [4], it is isometrically *-isomorphic to $LC(H_\lambda)$ for some Hilbert space H_λ.

(c) Since A is isometrically *-isomorphic to $(\sum I_\lambda)_0$, it follows that if $e_1 \in \mathscr{E}^{I_\mu}$ and $e_2 \in \mathscr{E}^{I_\nu}$, $\mu \neq \nu$, then $\|e_1 - e_2\| \geqq 1$. Similarly if $f_1 \in \mathscr{E}_p^{I_\mu}$ and $f_2 \in \mathscr{E}_p^{I_\nu}$, $\mu \neq \nu$, then $\|f_1 - f_2\| \geqq 1$. Since every $e \in \mathscr{E}$ belongs to some I_μ, $\mathscr{E} = \bigcup_\lambda \mathscr{E}^{I_\lambda}$; similarly $\mathscr{E}_p = \bigcup_\lambda \mathscr{E}_p^{I_\lambda}$. It is straightforward now to verify that each $\mathscr{E}_p^{I_\lambda}$ is open and closed in $\mathscr{E}$; similarly each $\mathscr{E}_p^{I_\lambda}$ is open and closed in $\mathscr{E}_p$. Thus $\mathscr{E}$ and $\mathscr{E}_p$ are disjoint unions of open (and closed) sets $\mathscr{E}^{I_\lambda}$ and $\mathscr{E}_p^{I_\lambda}$ respectively. But, by the proof of Theorem 3.8, $P_{I_\lambda} = P \mid \mathscr{E}^{I_\lambda}$ with range in $\mathscr{E}_p^{I_\lambda}$, hence p is continuous if and only if p_λ is continuous. This completes the proof.

Theorem 3.9 thus enables us to restrict our attention to the complemented B^*-algebras of the form $LC(H)$. §§4–6 are devoted to the study of this class of B^*-algebras.

4. Complementors on $LC(H)$.

Throughout this section, H will denote a fixed complex Hilbert space with inner product $(\, , \,)$, and A the algebra $LC(H)$.

NOTATION. For every closed subspace S of H let $\mathscr{J}(S) = \{a \in A, a(H) \subset S\}$. For every closed right ideal R of A let $\mathscr{S}(R)$ be the smallest closed subspace of H that contains the range $a(H)$ of each operator a in R.

LEMMA 4.1. *For every closed right ideal R of A, $R = \mathscr{J}(\mathscr{S}(R))$; and for every closed subspace S of H, $\mathscr{J}(S)$ is a closed right ideal and $S = \mathscr{S}(\mathscr{J}(S))$.*

Proof. By the duality of A and Lemma (2.8.24) in [4], $R = \mathscr{J}(\mathscr{S}(R))$. Let S be a closed subspace of H. Then $\mathscr{J}(S)$ is a closed right ideal of A; in fact, let E be the orthogonal projection on S, $T \in \mathrm{cl}\,(\mathscr{J}(S))$ and $\{T_n\}$ a sequence in $\mathscr{J}(S)$ converging to T. Since $ET_n = T_n$ for all n, we obtain $T = ET$, which shows that $T(H) \subset S$ and hence $T \in \mathscr{J}(S)$. Thus $\mathscr{J}(S)$ is a closed right ideal of A and so $\mathscr{J}(S) = \mathscr{J}(\mathscr{S}(\mathscr{J}(S)))$.

It is clear that $\mathscr{S}(\mathscr{J}(S))\subset S$. Now if $m \in S$, then $m \otimes m \in \mathscr{J}(S)$ and therefore $m \in \mathscr{S}(\mathscr{J}(S))$. Hence $S\subset \mathscr{S}(\mathscr{J}(S))$ and so $\mathscr{S}(\mathscr{J}(S))=S$.

REMARK. Lemma 4.1 shows that $R \to \mathscr{S}(R)$ defines a one-to-one correspondence between the closed right ideals of $LC(H)$ and the closed subspaces of H. Moreover if p is a complementor on $LC(H)$, $S \to S' = \mathscr{S}((\mathscr{J}(S))^p)$ defines a complementor on the closed subspaces S of H in the sense of Theorem 1 in [3]. We shall say $S \to S'$ is the complementor on the closed subspaces of H induced by the complementor p on $LC(H)$. Conversely every complementor $S \to S'$ on the closed subspaces S of H induces a complementor p on $LC(H)$ given by the relation $R^p = \mathscr{J}(\mathscr{S}(R)')$ for every closed right ideal R of $LC(H)$.

We now show that the inner product in H defines a complementor on A in a natural way.

THEOREM 4.2. *Let R be a closed right ideal of A, $\mathscr{S}(R)^\perp$ the orthogonal complement of $\mathscr{S}(R)$ and $R^p = \mathscr{J}(\mathscr{S}(R)^\perp)$. Then the mapping $p\colon R \to R^p$ of L_r into itself is a complementor on A; $R^p = (R_l)^*$.*

Proof. If we show that $R^p = (R_l)^*$, then from the proof of Theorem 3.6 it will follow that p is a complementor on A. Now for every element $a \in A$ let $N(a)$ denote the null-space of a. We have $\mathrm{cl}\,(a(H)) = N(a^*)^\perp$. Since $R_l = \{a \in A : N(a) \supset \mathscr{S}(R)\}$,

$$(R_l)^* = \{a \in A : a(H) \subset S(R)^\perp\} = R^p,$$

which completes the proof.

Writing $\perp$ as $\perp_{\langle\cdot\rangle}$ and p as $p_{\langle\cdot\rangle}$, Theorem 4.2 can be rephrased for any equivalent inner product. This gives us the following corollary:

COROLLARY 4.3. *If $\langle\cdot\rangle$ is any equivalent inner product in H and $R^{p\langle\cdot\rangle}$ the closed right ideal $\mathscr{J}(\mathscr{S}(R)^{\perp\langle\cdot\rangle})$, then $p_{\langle\cdot\rangle}$ is a complementor on A and $R^{p\langle\cdot\rangle} = (R_l)^{*\langle\cdot\rangle}$.*

COROLLARY 4.4. *If e is a $p_{\langle\cdot\rangle}$-projection, then $e^{*\langle\cdot\rangle} = e$.*

Proof. Let $R = eA$. Then

$$R^{p\langle\cdot\rangle} = (1-e)A = ((eA)_l)^{*\langle\cdot\rangle} = (A(1-e))^{*\langle\cdot\rangle} = (1-e)^{*\langle\cdot\rangle}A$$

and consequently $e(1-e^{*\langle\cdot\rangle})A = (0)$. Since A is semisimple, $e(1-e^{*\langle\cdot\rangle}) = 0$ so that $e = ee^{*\langle\cdot\rangle}$ and, applying the involution $*\langle\cdot\rangle$, we see that $e^{*\langle\cdot\rangle} = ee^{*\langle\cdot\rangle}$. Hence $e^{*\langle\cdot\rangle} = e$ and the proof is complete.

We shall show in §6 that if the dimension of H is at least three then the complementor given in Corollary 4.3 is continuous, and that every continuous complementor on A is of this form.

In what follows we assume that there is given a complementor $p\colon R \to R^p$ on A.

LEMMA 4.5. *Let $e_1, e_2, \ldots, e_n$ be minimal idempotents in A. Then*

(i) $[e_i^*(H)] = (1-e_i)(H)$, $(i = 1, 2, \ldots, n)$.

(ii) $\mathscr{S}(\text{cl}\,(\sum_{i=1}^{n} e_i A)) = \sum_{i=1}^{n} e_i(H)$ and

$$\mathscr{I}\left(\sum_{i=1}^{n} e_i(H)\right) = \text{cl}\left(\sum_{i=1}^{n} e_i A\right).$$

(iii) $\mathscr{S}(\bigcap_{i=1}^{n} (1-e_i)A) = \bigcap_{i=1}^{n} (1-e_i)(H)$ and

$$\mathscr{I}\left(\bigcap_{i=1}^{n} (1-e_i)(H)\right) = \bigcap_{i=1}^{n} (1-e_i)A.$$

Proof. (i) Since $(e_i^* h_1, (e-1)h_2) = 0$ for all $h_1, h_2 \in H$, $[e_i^*(H)]^\perp \supset (1-e_i)(H)$. As e_i is one-dimensional, $(1-e_i)(H)$ is a maximal closed subspace of H, (i) follows.

(ii) Clearly the range of $\sum_{i=1}^{n} e_j A$ is contained in $\sum_{i=1}^{n} e_i(H)$ and so

$$\mathscr{S}\left(\text{cl}\left(\sum_{i=1}^{n} e_i A\right)\right) \subset \text{cl}\left(\sum_{i=1}^{n} e_i(H)\right) = \sum_{i=1}^{n} e_i(H).$$

But the irreducibility of A implies that

$$\mathscr{S}\left(\text{cl}\left(\sum_{i=1}^{n} e_i A\right)\right) \supset e_i(H) \qquad (i = 1, 2, \ldots, n).$$

Hence $\mathscr{S}(\text{cl}\,(\sum_{i=1}^{n} e_i A)) = \sum_{i=1}^{n} e_i(H)$. The second part of (ii) follows now from Lemma 4.1.

(iii) Again, by the irreducibility of A, $\mathscr{S}((1-e_i)A) = (1-e_i)(H)$. Hence

$$\mathscr{S}\left(\bigcap_{i=1}^{n} (1-e_i)A\right) = \bigcap_{i=1}^{n} \mathscr{S}((1-e_i)A) = \bigcap_{i=1}^{n} (1-e_i)(H).$$

The second part of (iii) is now a consequence of Lemma 4.1.

5. **The representing operator for a complementor on** $LC(H)$. Throughout this section we use the notation of §4. p is a given complementor on A. Furthermore we assume that the dimension of H is at least three.

LEMMA 5.1. *For any nonzero x in H there exist unique elements $y, z \in H$ such that $x \otimes y$, $z \otimes x$ are p-projections. If $x_1 \otimes y_1$, $x_2 \otimes y_2$ are p-projections, then $(x_1, y_2) = 0$ if and only if $(x_2, y_1) = 0$.*

Proof. $\mathscr{I}([x])$ and $(\mathscr{I}([x]^\perp))^p$ are minimal right ideals of A. By Corollary 3.4, there are unique minimal p-projections e_1, e_2 such that $e_1 \in \mathscr{I}([x])$ and $e_2 \in (\mathscr{I}([x]^\perp))^p$. Clearly $e_1 = x \otimes y$ for some $y \in H$. Also $(1-e_2)A = \mathscr{I}([x]^\perp)$ and hence, by Lemmas 4.1 and 4.5, $e_2^*(H) = [x]$; thus $e_2 = z \otimes x$ for some $z \in H$. Because of the uniqueness of e_1 and e_2, y and z are unique. Now write e_i for $x_i \otimes y_i$ and R_i for $e_i A$ $(i = 1, 2)$. Then

$$\begin{aligned}(x_i, y_j) = 0 &\Leftrightarrow (y_j, x_i)(h, x_j) = 0 \quad \text{for all } h \in H \Leftrightarrow e_i^* e_j^*(H) = 0 \\ &\Leftrightarrow e_i^* \in (e_j^* A)_l = A(1-e_j^*) \Leftrightarrow e_i \in (1-e_j)A \\ &\Leftrightarrow e_i A \subset (1-e_j)A \Leftrightarrow R_i \subset R_j^p.\end{aligned}$$

(C_4) now completes the proof.

NOTATION. In the rest of the paper, for any nonzero $x \in H$, e_x will denote the unique p-projection in the minimal right ideal $\mathscr{J}([x])$ and f_x the orthogonal projection $(x \otimes x)/(x, x)$; $e_x = x \otimes y$ for some nonzero $y \in H$.

THEOREM 5.2. *Let $e_\alpha = x_\alpha \otimes y_\alpha$ be a set of p-projections of A. Then the set $\{x_\alpha\}$ is linearly independent if and only if the set $\{y_\alpha\}$ is linearly independent.*

Proof. Let $\{x_1, x_2, \ldots, x_{n-1}\}$ be any finite subset of $\{x_\alpha\}$ and $\{y_1, y_2, \ldots, y_{n-1}\}$ be the subset consisting of the corresponding elements of $\{y_\alpha\}$. Let X be the linear hull of $\{x_1, \ldots, x_{n-1}\}$ and Y the linear hull of $\{y_1, \ldots, y_{n-1}\}$. Let $x_n = x + u$, $y_n = y + v$, where $x \in X$, $y \in Y$, $u \perp X$ and $v \perp Y$. We show that $u = 0$ if and only if $v = 0$. Suppose $u \neq 0$. Then, by Lemma 5.1, there is a p-projection of the form $w \otimes u$. Then $(u, x_n) = (u, u) \neq 0$, and $(u, x_i) = 0$ for $i = 1, 2, \ldots, n-1$; hence, by Lemma 5.1, $(w, y_n) \neq 0$ and $(w, y_i) = 0$ for $i = 1, 2, \ldots, n-1$. Therefore $v \neq 0$. Similarly, if $v \neq 0$ there is a p-projection of the form $v \otimes z$ and using this we can show that $u \neq 0$.

THEOREM 5.3. *There exists a semilinear mapping T of H onto itself such that, for all nonzero x in H, $e_x = (x \otimes Tx)/(x, Tx)$.*

Proof. Let $[x]$ be any one-dimensional subspace of H and let $e_x = x \otimes y$ be the unique p-projection in $\mathscr{J}([x])$. Consider the mapping $A: [x] \to [y]$ of one-dimensional subspaces of H. Then A is a one-to-one mapping of the set of all one-dimensional subspaces of H into itself. Also A is onto; for, if y is any nonzero element of H, by Lemma 5.1, there is an $x \in H$ such that $A[x] = [y]$. Since, by Theorem 5.2, A preserves linear independence and the dimension of $H \geqq 3$, the Fundamental Theorem of Projective Geometry gives a transformation T of H onto itself which is additive, $[y] = [Tx]$ and $T(\alpha h) = \alpha' T(h)$ for all $\alpha \in C$, $h \in H$, where $\alpha \to \alpha'$ is an automorphism of C; that is, T is semilinear.

Now for any nonzero $x \in H$ we have $e_x = \lambda(x \otimes Tx)$ for some $\lambda \in C$. Since e_x is an idempotent, $\lambda = 1/(x, Tx)$ (and hence $(x, Tx) \neq 0$). Thus for all nonzero $x \in H$ we have $e_x = (x \otimes Tx)/(x, Tx)$, and the proof is complete.

Since a scalar multiple of T has all the properties of T, we can normalize T so that, for some fixed nonzero x_0 in H and $e_{x_0} = x_0 \otimes y_0$, we have $Tx_0 = y_0$, and hence that $(x_0, T_{x_0}) = 1$. With these observations in mind, we make the following definition:

DEFINITION 5.4. A semilinear mapping T of H onto itself is called a p-representing operator provided it has the following properties:

(i) $e_x = (x \otimes Tx)/(x, Tx)$ for all nonzero $x \in H$;

(ii) $(x_0, Tx_0) = 1$ for some nonzero $x_0 \in H$.

6. Continuous complementors on $LC(H)$.

The notation used in this section is the same as that of §5. As before we assume that the dimension of H is at least three. p is a complementor on A, T a p-representing operator on H and x_0 an element of H such that $(x_0, Tx_0) = 1$.

LEMMA 6.1. *If T is linear then it is a positive bounded hermitian operator with bounded inverse.*

Proof. Suppose T is linear. We break the proof up into several parts:

(i) $(x_1, Tx_2)=0$ *if and only if* $(x_2, Tx_1)=0$ $(x_1, x_2 \in H)$.

This is immediate if either x_1 or x_2 is zero, so assume that both x_1 and x_2 are nonzero. Let y_i be the element of H such that $e_{x_i}=x_i \otimes y_i$ $(i=1, 2)$. Since $e_{x_i} =\lambda_i(x_i \otimes Tx_i)$, $\lambda_i\neq 0$ *and* $y_i=\alpha_i Tx_i$ with the scalars $\alpha_i\neq 0$, we see that

$$(x_i, Tx_j) = 0 \Leftrightarrow (x_i, y_j) = 0 \qquad (i, j = 1, 2).$$

But, by Lemma 5.1, $(x_1, y_2)=0 \Leftrightarrow (x_2, y_1)=0$. Hence (i) is established.

(ii) $(x_1, Tx_2) = \overline{(x_2, Tx_1)}$ $(x_1, x_2 \in H)$.

This is clear if either $(x_1, Tx_2)=0$ or $(x_2, Tx_1)=0$ for then both terms are zero by (i). Hence assume that both terms are different from zero. (Notice that if $x\neq 0$ then $(x, Tx)\neq 0$ by the proof of Theorem 5.3.) Then there exist nonzero scalars α_1, α_2 such that

$$\alpha_1(x_1, Tx_1)+(x_1, Tx_2) = 0, \qquad \alpha_2(x_2, Tx_2)+(x_1, Tx_2) = 0. \tag{1}$$

Applying (i) to (1), using the linearity of T and taking complex conjugates we obtain

$$\alpha_1\overline{(x_1, Tx_1)}+\overline{(x_2, Tx_1)} = 0, \qquad \alpha_2\overline{(x_2, Tx_2)}+\overline{(x_2, Tx_1)} = 0. \tag{2}$$

From (1) we obtain

$$\alpha_1(x_1, Tx_1) = \alpha_2(x_2, Tx_2)$$

and from (2)

$$\alpha_1\overline{(x_1, Tx_1)} = \alpha_2\overline{(x_2, Tx_2)}.$$

As $(x_i, Tx_i)\neq 0$ for $i=1, 2$, it follows that

$$\frac{(x_1, Tx_1)}{\overline{(x_1, Tx_1)}} = \frac{(x_2, Tx_2)}{\overline{(x_2, Tx_2)}}.$$

Since $(x_0, Tx_0)=1$ (hence real), we see that (x, Tx) is real for all $x \in H$ for which $(x_0, Tx)\neq 0$. Now, if $(x_0, Tx)=0$, then $(x_0, T(x_0+x))\neq 0$ which implies that $(x_0+x, T(x_0+x))$ is real. Therefore $(x, Tx)=(x_0+x, T(x_0+x))-1$ is real for all $x \in H$, and hence from the first equations of (1) and (2) we obtain (ii). (Cf [3, p. 730].)

(iii) $(x, Tx)>0$ *for all nonzero* $x \in H$.

Suppose there is an $x \in H$ such that $(x, Tx)<0$, and consider the equation

$$0 = (\xi x+\eta x_0, T(\xi x+\eta x_0)) = \xi^2(x, Tx)+2\xi\eta \operatorname{Re}(x_0, Tx)+\eta^2(x_0, Tx_0)$$

in real variables ξ, η. It follows that this equation has real roots $\xi_0\neq 0$ and $\eta_0\neq 0$. Therefore $x=-\eta_0x_0/\xi_0$ and so $1=(x_0, Tx_0)=\eta_0^2(x, Tx)/\xi_0^2<0$, a contradiction.

To complete the proof we observe that, since T is linear, defined everywhere on H and, by (ii), is also hermitian, it is therefore bounded (by Théorème in [5, p. 293]).

Moreover, T is one-to-one and onto H. Hence T^{-1} exists and is bounded; T^{-1} is also hermitian.

LEMMA 6.2. *If p is continuous, then T is linear.*

Proof. Suppose p is continuous. We show first that $\alpha \to \alpha'$ is continuous. For nonzero elements $x, x' \in H$, we have

$$\|f_{x'} - f_x\| = \sup_{\|h\|=1} \left\| \frac{(x', h)x'}{(x', x')} - \frac{(x, h)x}{(x, x)} \right\|.$$

Since the mappings $x \to (x, h)$, for h fixed, and $x \to (x, x)$ of H into C are continuous, it follows that the mapping $x \to f_x$ of H into $\mathscr{E}$ is continuous with respect to the norm topology in H and the induced topology in $\mathscr{E}$. Also, $f_x \to e_x = P(f_x)$ is continuous since P is continuous. Hence the mapping $x \to e_x$ is continuous.

Now let x be a fixed nonzero element of H and let $y = Tx$. Let x' be any nonzero element of $[Ty]^\perp$ and let $y' = Tx'$. Since $(x', T(Tx)) = 0$, Lemma 5.1 implies that $(Tx, y') = 0$. Let $e_\alpha = e_{x+\alpha x'}$. Since $x \to e_x$ is continuous, it is easy to verify that $\alpha \to e_\alpha$ is continuous. By Lemma 4.5 (i) we have $(y + \alpha' y') \perp (1 - e_\alpha)(H)$ for all $\alpha \in C$ and so

$$(1) \qquad (y + \alpha' y', h - e_\alpha h) = 0 \qquad (h \in H, \alpha \in C).$$

We solve (1) for α' and obtain

$$\alpha' = \frac{(y, h - e_\alpha h)}{(y', h - e_\alpha h)}$$

whenever $(y', h - e_\alpha h) \neq 0$. Then, since $\alpha \to e_\alpha$ is continuous, it follows that $\alpha \to \alpha'$ is continuous at any point α_0 where $(y', h - e_{\alpha_0} h) \neq 0$. Since it is sufficient to prove continuity at $\alpha_0 = 0$, take $h = y'$ and obtain

$$(y', y' - e_0 y') = (y', y') - (y', (y', Tx)x) = (y', y') \neq 0.$$

Hence $\alpha \to \alpha'$ is continuous. Since the only continuous automorphisms of the complex field C are the identity and the conjugacy we need only show that $\alpha' = \bar{\alpha}$ is impossible. So suppose $\alpha' = \bar{\alpha}$ and let x' be any nonzero element of $X = [x_0]^\perp$. Then clearly the equation

$$(x_0 + \alpha x', y_0 + \bar{\alpha} T x') = 0$$

has a solution $\alpha_0 \neq 0$. But this is a contradiction since

$$(x_0 + \alpha_0 x', y_0 + \bar{\alpha}_0 T x') = (x_0 + \alpha_0 x', T(x_0 + \alpha_0 x')) \neq 0$$

for all $x' \in X$ and all $\alpha \in C$. (See proof of Theorem 5.3.)

LEMMA 6.3. *If T is linear, then p is continuous.*

Proof. Suppose T is linear. Then, by Lemma 6.1, T is a positive bounded hermitian operator with bounded inverse T^{-1}. Let

$$K_1 = \inf\{(x, Tx) : \|x\| = 1\} \quad \text{and} \quad K_2 = \sup\{(x, Tx) : \|x\| = 1\}.$$

We have $0 < K_1 \leqq K_2$, $\|T\| = K_2$ and $\|T^{-1}\| = K_1^{-1}$.

We recall that every element of $\mathscr{E}$ is of the form $x \otimes x$ with $\|x\|=1$ and every element of $\mathscr{E}_p$ is of the form $(x \otimes Tx)/(x, Tx)$ with $x \neq 0$ (in fact we may assume $\|x\|=1$). Let $x, x' \in H$ with $\|x\|=\|x'\|=1$. Then

$$(1)\quad \begin{aligned} \left\| \frac{x \otimes Tx}{(x, Tx)} - \frac{x' \otimes Tx'}{(x', Tx')} \right\| &= \sup\left\{ \left\| \frac{x \otimes x}{(x, Tx)} - \frac{x' \otimes x'}{(x', Tx')}(Th) \right\| : h \in H \text{ and } \|h\| = 1 \right\} \\ &\leqq \left\| \frac{x \otimes x}{(x, Tx)} - \frac{x' \otimes x'}{(x', Tx')} \right\| \|T\|. \end{aligned}$$

Let $\langle x, y\rangle = (x, Ty)$ for all $x, y \in H$. Then clearly $\langle\cdot\rangle$ is an inner product in H giving rise to an equivalent norm on H. Hence if $\| \quad \|_{\langle\cdot\rangle}$ denotes the operator bound with respect to $\langle\cdot\rangle$, then $\| \quad \|_{\langle\cdot\rangle}$ is equivalent to the given operator bound $\| \quad \|$. This means that there exist nonzero positive constants K_1', K_2' such that

$$K_1'\|a\| \leqq \|a\|_{\langle\cdot\rangle} \leqq K_2'\|a\| \qquad (a \in L(H)).$$

Consider now the operators $x \otimes x$ and $x' \otimes x'$. Then, writing $h \otimes' k$ for the operator $h \otimes' k(h') = \langle h', k\rangle h$, $h' \in H$, we have

$$\|x \otimes' x\|_{\langle\cdot\rangle} = \langle x, x\rangle = (x, Tx), \qquad \|x \otimes' x\| = \langle x', x'\rangle = (x', Tx')$$

and so

$$\begin{aligned} |(x, Tx)-(x', Tx')| \leqq \|x \otimes' x - x' \otimes' x'\| &\leqq K_2'\|x \otimes Tx - x' \otimes Tx'\| \\ &\leqq K_2'\|T\| \, \|x \otimes x - x' \otimes x'\|. \end{aligned}$$

Therefore (x, Tx) is a continuous function of $x \otimes x$. Since $0 < K_1\|x\| \leqq (x,Tx) \leqq K_2\|x\|$ for all $x \in H$, $[(x, Tx)]^{-1}$ is a continuous function of $x \otimes x$ for all nonzero $x \otimes x$; in particular $[(x, Tx)]^{-1}$ is continuous on the set of $x \otimes x$ with $\|x \otimes x\|=1$. From inequalities (1) it follows now that the mapping $x \otimes x \to (x \otimes Tx)/(x, Tx)$ of $\mathscr{E}$ into $\mathscr{E}_p$ is continuous. But $P(x \otimes x) = (x \otimes Tx)/(x, Tx)$. Hence P is continuous and therefore p is continuous.

We summarize the above results for p and T in the following theorem:

THEOREM 6.4. *The following statements are equivalent*:

(i) *p is continuous.*

(ii) *The automorphism $\alpha \to \alpha'$ is continuous and $\alpha = \alpha'$ for all $\alpha \in C$.*

(iii) *T is a continuous positive linear operator with continuous inverse.*

Proof. (i) ⇒ (ii). This is immediate from the proof of Lemma 6.2.

(ii) ⇒ (iii). This is Lemma 6.1.

(iii) ⇒ (i). This is Lemma 6.3.

COROLLARY 6.5. *If p is continuous, then the bilinear hermitian form $\langle x, y\rangle = (x, Ty)$ defines an equivalent inner product in H with respect to which every p-projection e_x is selfadjoint.*

Proof. Suppose p is continuous. Then, by Theorem 6.4, $\langle x, y\rangle=(x, Ty)$ defines an equivalent inner product on H. It is straightforward to verify that

$$\langle (x \otimes Tx)(h_1), h_2\rangle = \langle h_1, (x \otimes Tx)(h_2)\rangle$$

for all $h_1, h_2 \in H$, i.e., $x \otimes Tx$ is self-adjoint with respect to $\langle\cdot\rangle$. Since (x, Tx) is real, it follows that e_x is also selfadjoint with respect to $\langle\cdot\rangle$.

Corollary 6.6. *If the p-derived mapping P is continuous then it is uniformly continuous.*

Proof. Suppose P is continuous. Then p is continuous and so, by Theorem 6.4, for every $x, x' \in H$, we have

$$\|x \otimes Tx - x' \otimes Tx'\| \leqq \|x \otimes x - x' \otimes x'\| \, \|T\|,$$

which shows that $x \otimes x \to x \otimes Tx$ is uniformly continuous. Moreover, since

$$|(x, Tx) - (x', Tx')| \leqq K_2'\|T\| \, \|x \otimes x - x' \otimes x'\|,$$

$x \otimes x \to (x, Tx)$ is also uniformly continuous (see proof of Lemma 6.3). But

$$0 < \|T^{-1}\|^{-1} \leqq (x, Tx) \leqq \|T\|$$

for all $x \in H$ with $\|x\|=1$. Hence $x \otimes x \to 1/(x, Tx)$ is uniformly continuous for all x with $\|x\|=1$. Since $P(x \otimes x)=(x \otimes Tx)/(x, Tx)$ for all x with $\|x\|=1$, it follows that P is uniformly continuous.

Corollary 6.7. *If the p-derived mapping P is continuous, then it defines a homeomorphism between $\mathscr{E}$ and $\mathscr{E}_p$.*

Proof. Suppose P is continuous. It is clear that P is one-to-one and onto (see Remark following Lemma 4.1). Let P^{-1} be the inverse mapping of P. Since p is continuous, by Theorem 6.4, $\langle x, y\rangle=(x, Ty)$ is an equivalent inner product in H. Then, with respect to $\langle\cdot\rangle$, every $f \in \mathscr{E}_p$ is of the form $f=x \otimes' x/\langle x, x\rangle$ and every $e \in \mathscr{E}$ is of the form $e=(x \otimes' T^{-1}x)/\langle x, T^{-1}x\rangle$, where $h \otimes' k$ is the operator on H given by $(h \otimes' k)(h')=\langle h', k\rangle h$ for all $h' \in H$. In particular

$$P^{-1}(f) = (x \otimes' T^{-1}x)/\langle x, T^{-1}x\rangle.$$

Hence, by the proof of Lemma 6.3, P^{-1} is continuous. From the proof of Corollary 6.6 we see that P^{-1} is also uniformly continuous.

Theorem 6.8. *If H is infinite dimensional, then every complementor p on A $(=LC(H))$ is continuous.*

Proof. Suppose H is infinite dimensional and let T be a p-representing operator on H. We show that T maps closed maximal subspaces onto closed maximal subspaces of H. In fact, let X' be a maximal closed subspace of H and Y' its image under T. As T is a semilinear operator Y' is a linear subspace of H. If Y' is not of codimension one, then there are linearly independent elements y_1, y_2, both orthogonal to Y'. Let $x_1=T^{-1}y_1$, $x_2=T^{-1}y_2$. Then x_1, x_2 are linearly independent. (In fact $T(\alpha_1x_1+\alpha_2x_2)=\alpha_1'Tx_1+\alpha_2'Tx_2=\alpha_1'y_1+\alpha_2'y_2$, hence $\alpha_1x_1+\alpha_2x_2=0$ if and only if $\alpha_1'y_1+\alpha_2'y_2=0$ if and only if $\alpha_1'=\alpha_2'=0$.) Moreover $x_1, x_2 \notin X'$. Hence X' is not a maximal closed subspace of H, a contradiction. Thus Y' is of codimension one.

Suppose $y\in \mathrm{cl}\,(Y')$, but $y\notin Y'$. Then $T^{-1}y=x\notin X'$. Thus x and X' span H and so Tx and $T(X')$ span H since T maps H onto H. This shows that $\mathrm{cl}\,(Y')=H$, which is a contradiction since the orthogonal complement of Y' is one-dimensional. Thus $\mathrm{cl}\,(Y')=Y'$. But, by Lemma 2 in [3], $\alpha\to\alpha'$ is continuous and, by the proof of Lemma 6.2, $\alpha'=\alpha$. Hence p is continuous, and the proof is complete.

LEMMA 6.9. *Let $\langle\cdot\rangle$ be an equivalent inner product on H and let Q be the positive selfadjoint operator on H such that $\langle x, y\rangle=(x, Qy)$ for all $x, y\in H$. Let T be a $p_{\langle\cdot\rangle}$-representing operator on H. Then $T=\kappa Q$, for some positive real number κ.*

Proof. By Corollary 4.3, $R^{p_{\langle\cdot\rangle}}=\mathscr{J}(\mathscr{S}(R)^{\perp_{\langle\cdot\rangle}})$. Let x be any nonzero element of H, and let $R=\mathscr{J}([x])$. Then $R^{p_{\langle\cdot\rangle}}=\mathscr{J}(\mathscr{S}(R)^{\perp_{\langle\cdot\rangle}})$ is a maximal closed right ideal, and we have $e_xA=R$ and $(1-e_x)A=R^p$. By Corollary 4.4, $e_x^{*_{\langle\cdot\rangle}}=e_x$ and so $e_x(h)=\langle h, x\rangle x/\langle x, x\rangle$ for every $h\in H$. But with respect to the given inner product $(\ \cdot\ )$ in H, $e_x=\lambda x\otimes Tx$. Hence

$$e_x(h) = \frac{\langle h, x\rangle x}{\langle x, x\rangle} = \frac{(h, Qx)x}{\langle x, x\rangle} = \lambda(h, Tx)x,$$

for all $h\in H$. This means that $(h, Tx-\rho Qx)=0$ for all $h\in H$, or $Tx=\rho Qx$, for some scalar ρ. Therefore $Tx=\rho(x)Qx$ for all nonzero $x\in H$, where $\rho(x)$ is a complex-valued function of x. We show now that $\rho(x)$ is actually a constant. In fact for any nonzero x_1, x_2 in H we have

$$T(x_1+x_2) = \rho(x_1+x_2)Q(x_1+x_2) = \rho(x_1+x_2)Qx_1+\rho(x_1+x_2)Qx_2,$$
$$Tx_1+Tx_2 = \rho(x_1)Qx_1+\rho(x_2)Qx_2,$$

and

$$T(x_1+x_2) = Tx_1+Tx_2.$$

Therefore if x_1, x_2 are linearly independent, Qx_1, Qx_2 are also so, and hence $\rho(x_1)=\rho(x_2)=\rho(x_1+x_2)$. In particular, $\rho(\alpha x_0)=\rho(x')$ for all nonzero scalars $\alpha\in C$ and all nonzero $x'\in X=[x_0]^{\perp}$. Let $\rho(0)=\rho(x_0)$. Since every $x\in H$ can be written in the form $x=\alpha x_0+x'$ with $x'\in X$ and $\alpha\in C$, we have

$$\rho(x)Q(x) = Tx = T(\alpha x_0)+Tx' = \rho(\alpha x_0)Q(\alpha x_0)+\rho(x')Q(x') = \rho(x_0)Q(x).$$

Therefore $Tx=\rho(x_0)Qx$ for all $x\in H$, i.e., $T=\kappa Q$, where $\kappa=\rho(x_0)$. Since T and Q are positive operators, κ is a positive real number.

COROLLARY 6.10. *If p is continuous then a p-representing operator is unique up to a positive real multiplicative constant.*

THEOREM 6.11. *The following statements are equivalent:*

(i) *p is continuous.*

(ii) *There is an equivalent inner product $\langle\cdot\rangle$ in H such that $p=p_{\langle\cdot\rangle}$.*

(iii) *R^p can be expressed as $(R_l)^{*'}$ for some involution $*'$ in A (and hence there is an equivalent norm on A which satisfies the $B*$-condition for $*'$).*

Proof. (i) ⇒ (ii). Suppose p is continuous. Then, by Theorem 6.4, $\langle x, y\rangle =(x, Ty)$ defines an equivalent inner product in H and so, by Corollary 4.3, $p_{\langle\cdot\rangle}$ is a complementor on A. We shall prove that $R^{p_{\langle\cdot\rangle}}=R^p$. So let M be a maximal closed right ideal of A and let e be the p-projection contained in M^p. By Corollary 6.5, e is selfadjoint with respect to $\langle\cdot\rangle$. But if f is the $p_{\langle\cdot\rangle}$-projection in M^p, then, by Corollary 4.4, f is also selfadjoint with respect to $\langle\cdot\rangle$. Hence, since $\mathscr{S}(M^p)$ is one-dimensional, we have $e=f$. Thus for maximal closed right ideals M, $M^p=M^{p_{\langle\cdot\rangle}}$. Now let R be any closed right ideal of A. Then R is equal to the intersection of all maximal closed right ideals M of A containing R. Let $\mathscr{M}$ be the set of M such that $R\subset M$. Then, by (C_4) and Lemma 2.1, we obtain

$$R^p = \mathrm{cl}\left(\sum_{M\in\mathscr{M}} M^p\right) = \mathrm{cl}\left(\sum_{M\in\mathscr{M}} M^{p_{\langle\cdot\rangle}}\right) = R^{p_{\langle\cdot\rangle}}.$$

(ii) ⇒ (i). This is immediate from Lemma 6.9 and Theorem 6.4.

(ii) ⇒ (iii). This follows at once by letting $*'=*_{\langle\cdot\rangle}$.

(iii) ⇒ (ii). We show first that $a^{*'}a=0$ implies $a=0$. Let R_a be the closed right ideal of A generated by a. If $a^{*'}a=0$, then $a^{*'}\in(R_a)_l$ and hence $a\in(R_a)^p\cap R_a=(0)$. Now if e is any p-projection then, by the proof of Corollary 4.4, $e^{*'}=e$ and therefore $(Ae)^{*'}=eA$. Since $A=LC(H)$, e is of the form $e=u\otimes v$, $u, v\in H$, and so $Ae=\{h\otimes v : h\in H\}$ and $eA=\{u\otimes h : h\in H\}$. Hence $(h\otimes v)^{*'}=u\otimes h'$ for some $h'\in H$. Let Q be the mapping on H such that $Qh=h'$. It is easy to verify that Q is linear one-to-one and onto; also

$$\begin{aligned}(x, Qy)e &= (u\otimes Qy)(x\otimes v) = (y\otimes v)^{*'}(x\otimes v) = [(x\otimes v)^{*'}(y\otimes v)]^{*'}\\ &= [(u\otimes Qx)(y\otimes v)]^{*'} = \overline{(y, Qx)}e = (Qx, y)e,\end{aligned}$$

so that Q is hermitian. It follows now that Q as well as its inverse Q^{-1} are bounded. Moreover, if $(Qx, x)=0$ then $(x\otimes v)^{*'}(x\otimes v)=0$ and hence $x\otimes v=0$ by the first part. Thus $(Qx, x)=0$ implies $x=0$. Since $Qu=v$ and $(u, v)e=e^2=e$ we see that $(Qu, u)>0$ and it is now straightforward to verify that Q is positive definite (cf. part (iii) of the proof of Lemma 6.1). Thus $\langle x, y\rangle=(x, Qy)$ is an equivalent inner product in H.

For every $a\in A$, we have

$$\begin{aligned}\langle ax, y\rangle e &= (u\otimes Qy)(ax\otimes v) = (u\otimes Qy)a(x\otimes v) = (y\otimes v)^{*'}a(x\otimes v)\\ &= a^{*'}(y\otimes v)^{*'}(x\otimes v) = (a^{*'}y\otimes v)^{*'}(x\otimes v) = \langle x, a^{*'}y\rangle e\end{aligned}$$

so that $a\to a^{*'}$ is the involution given by $\langle\cdot\rangle$. Hence if $\|\ \|'$ denotes the operator bound on A with respect to the norm on H defined by $\langle\ \rangle$, then $\|\ \|'$, which is equivalent to the given norm on A, satisfies the B^*-condition for $*'$. Also $R^p = (R_l)^{*'} = R^p$.

COROLLARY 6.12. *Let p be continuous and let T be any p-representing operator on H. Then the hermitian bilinear form $\langle x, y\rangle = (x, Ty)$, $x, y \in H$, defines an equivalent inner product in H and $p = p_{\langle\cdot\rangle}$. In particular $\mathscr{S}(R^p) = \mathscr{S}(R)^{\perp\langle\cdot\rangle}$ for every closed right ideal R of A and the induced complementor $S \to S' = \mathscr{S}(\mathscr{J}(S)^p)$ on the closed subspaces S of H satisfies $S' = S^{\perp\langle\cdot\rangle}$.*

Proof. By the proof of (i) $\Rightarrow$ (ii) of Theorem 6.11, $p = p_{\langle\cdot\rangle}$ and hence, by Corollary 4.3, $R^p = (R_l)^{*\langle\cdot\rangle}$. Therefore $\mathscr{S}(R)^{\perp\langle\cdot\rangle} = \mathscr{S}(R^p)$, and the Remark following Lemma 4.1 completes the proof.

COROLLARY 6.13. *If H is infinite dimensional then every complementor p on $LC(H)$ defines an equivalent inner product $\langle\cdot\rangle$ in H such that $p = p_{\langle\cdot\rangle}$ and $\mathscr{S}(R)^{\perp\langle\cdot\rangle} = \mathscr{S}(R^p)$.*

Proof. This follows from the continuity of p when H is infinite dimensional.

COROLLARY 6.14. *If $*'$ is an involution on A for which $R \to (R_l)^{*'}$ is a complementor on A, then there exists a bicontinuous positive linear operator Q mapping H onto itself such that with respect to the inner product $\langle x, y\rangle = (x, Qy)$ in H we have $\langle ax, y\rangle = \langle x, a^{*'}y\rangle$ for all $x, y \in H$; in particular $a^{*'} = Q^{-1}a^*Q$.*

Proof. The proof of (iii) $\Rightarrow$ (ii) of Theorem 6.11 gives the existence of Q and establishes the relation: $\langle ax, y\rangle = \langle x, a^{*'}y\rangle$ for all $x, y \in H$. We have

$$\langle x, a^{*'}y\rangle = (ax, Qy) = (x, a^*Qy) = (Qx, Q^{-1}a^*Qy) = \langle x, Q^{-1}a^*Qy\rangle.$$

From Corollaries 6.12 and 6.14 we see that if p is continuous and T is any p-representing operator on H, then $a^{*\langle\cdot\rangle} = T^{-1}a^*T$, for every $a \in LC(H)$, where $\langle\cdot\rangle$ is the inner product in H given by $\langle x, y\rangle = (x, Ty)$.

THEOREM 6.15. *Let H be a Hilbert space of dimension at least three and let $S \to S'$ be a complementor on the closed subspaces S of H. Then, in order that there exist an equivalent inner product $\langle\cdot\rangle$ in H with respect to which S and S' are orthogonal complements of each other, it is necessary and sufficient that $S \to S'$ be induced by a continuous complementor p on $LC(H)$.*

Proof. The sufficiency of the condition follows at once from Corollary 6.12. To prove the necessity of the condition, suppose that $\langle\cdot\rangle$ is an equivalent inner product in H with respect to which $S' = S^{\perp\langle\cdot\rangle}$ for every closed subspace S of H. For every closed right ideal R of A, let $R^p = \mathscr{J}(\mathscr{S}(R)^{\perp\langle\cdot\rangle})$. Then, by Corollary 4.3, $R \to R^p$ is a right complementor on A such that $R^p = (R_l)^{*\langle\cdot\rangle}$. Hence, by Theorem 6.11, p is continuous. Moreover, if S is a closed subspace of H, then $S = \mathscr{S}(R)$ and

$S'=S^{\perp\langle\cdot\rangle}=\mathscr{S}((R_l)^{*\langle\cdot\rangle})$ for some closed right ideal R of A. This completes the proof.

Let H be finite dimensional and of dimension at least three. Then Theorem 6.15 enables us to construct a complementor on A which is not continuous. In fact, the example in [3, p. 732] shows that there exists a complementor $S\to S'$ on the closed subspaces S of H for which there does not exist an inner product in H with respect to which S and S' are orthogonal complements of each other. Let $p: R\to R^p=\mathscr{J}(\mathscr{S}(R)')$ be the complementor on A induced by $S\to S'$. Since $S\to S'$ is induced by p, it follows from Theorem 6.15 that p is not continuous and therefore is not of the form $p_{\langle\cdot\rangle}$. This evidently cannot happen if H is infinite dimensional by Theorem 6.8.

7. **Complementors on a general B^*-algebra.** Throughout this section A is a general B^*-algebra with a complementor p. By Theorem 3.6, A is dual and therefore is isometrically *-isomorphic to $(\sum I_\lambda)_0$, where $\{I_\lambda : \lambda\in\Lambda\}$ is the family of all minimal closed two-sided ideals of A. Thus every element x of A may be regarded to be of the form $x=(x_\lambda)$ with $x_\lambda\in I_\lambda$ for each $\lambda\in\Lambda$. As before we denote by p_λ the complementor on I_λ induced by p; $R_\lambda^{p_\lambda}=R_\lambda^p\cap I_\lambda$ for every closed right ideal R_λ of I_λ, which is also a closed right ideal of A. Let $LC(H_\lambda)$ be the algebra of compact operators on the Hilbert space H_λ which is isometrically *-isomorphic to I_λ for each $\lambda\in\Lambda$. For any subset S of A we will denote by $S_{l_\lambda}(S_{r_\lambda})$ the left (right) annihilator of S in I_λ; $S_{l_\lambda}=\{x\in I_\lambda : xy=0 \text{ for every } y\in S\}$.

LEMMA 7.1. *If we identify A with $(\sum I_\lambda)_0$, then for every closed right ideal R of A the following statements are true*:

(i) $R=(\sum R\cap I_\lambda)_0$ *and* $R^p=(\sum [(R\cap I_\lambda)^p\cap I_\lambda])_0$.

(ii) $(R_l)^*\cap I_\lambda=((R\cap I_\lambda)_{l_\lambda})^*$.

(iii) $(R_l)^*=(\sum ((R\cap I_\lambda)_{l_\lambda})^*)_0$.

Proof. All three statements of the lemma are obviously true for $R=(0)$. Suppose that $R\neq(0)$.

(i) Since A is dual, R contains a minimal idempotent and hence $R\cap I_\lambda\neq(0)$ for at least one $\lambda\in\Lambda$. In fact it is easy to see that $R'=\sum_\lambda R\cap I_\lambda$ is dense in R. Now identifying A with $(\sum I_\lambda)_0$, R' can be identified with a dense subset of $(\sum R\cap I_\lambda)_0$ and, since $(\sum R\cap I_\lambda)_0$ is closed, we obtain $(\sum R\cap I_\lambda)_0=\text{cl}\,(R')=R$.

Since $R\cap I_\lambda\subset R$, by property (C_4), $(R\cap I_\lambda)^p\supset R^p$ and therefore

$$(1)\quad \text{cl}\left(\sum_\lambda [(R\cap I_\lambda)^p\cap I_\lambda]\right) = \left(\sum [(R\cap I_\lambda)^p\cap I_\lambda]\right)_0 \supset \left(\sum R^p\cap I_\lambda\right)_0 = R^p.$$

Now, by Lemma 2.1,

$$(R\cap I_\lambda)^p\cap I_\lambda = [\text{cl}\,((R\cap I_\lambda)+I_\lambda^p)]^p$$

and, by Lemma 1 in [6], $I_\lambda^p=(I_\lambda)_l\supset I_\mu$ for $\mu\neq\lambda$. Hence, identifying A with $(\sum I_\lambda)_0$,

R with $(\sum R\cap I_\lambda)_0$ and I_λ^p with $\{a=(a_\mu) : a_\mu=0$ for $\mu=\lambda\}$, we obtain $R \subset \mathrm{cl}\,((R\cap I_\lambda)+I_\lambda^p)$, or equivalently, $R^p \supset (R\cap I_\lambda)^p \cap I_\lambda$ for every $\lambda\in\Lambda$. Thus

$$R^p \supset \mathrm{cl}\left(\sum_\lambda (R\cap I_\lambda)^p \cap I_\lambda\right)$$

and so from (1) we obtain $R^p=(\sum (R\cap I_\lambda)^p\cap I_\lambda)_0$.

(ii) We have $R_l\cap I_\lambda \subset (R\cap I_\lambda)_{l_\lambda}$ for each $\lambda\in\Lambda$. Now suppose $a\in(R\cap I_\lambda)_{l_\lambda}$. Since $I_\lambda I_\mu=(0)$, for $\lambda\neq\mu$, and $R=(\sum R\cap I_\lambda)_0$, we see that $a\in R_l$ and hence $a\in R_l\cap I_\lambda$. Thus $(R\cap I_\lambda)_{l_\lambda}\subset R_l\cap I_\lambda$ and therefore $(R\cap I_\lambda)_{l_\lambda}=R_l\cap I_\lambda$. But for every left (right) ideal I of A we have $I^*\cap I_\lambda=(I\cap I_\lambda)^*$. Hence

$$(R_l)^*\cap I_\lambda = (R_l\cap I_\lambda)^* = ((R\cap I_\lambda)_{l_\lambda})^* \quad \text{for each } \lambda\in\Lambda.$$

(iii) This follows immediately from (i) and (ii).

Corollary 7.2. *Let $*'$ be any involution on A such that $a^{*'}a=0$ implies $a=0$. Then for every closed right ideal R of A we have*:

(ii)′ $(R_l)^{*'}\cap I_\lambda=((R\cap I_\lambda)_{l_\lambda})^{*'}$;

(iii)′ $(R_l)^{*'}=(\sum((R\cap I_\lambda)_{l_\lambda})^{*'})_0$.

Proof. Let $\mathfrak{S}$ be the socle of A. Since A is dual, $a\mathfrak{S}=0$ implies $a=0$ and so, by Corollary (4.10.4) in [4], A is an A^*-algebra with auxiliary norm $\|\cdot\|'$ which satisfies the B^*-condition for $*'$. Therefore, by Theorem (4.1.15) in [4], $*'$ is continuous with respect to both norms $\|\cdot\|'$ and $\|\cdot\|$ and consequently $*'$ takes closed sets onto closed sets (sets closed with respect to the topology of the given norm). Moreover, by Lemma (4.10.13) in [4], $I_\lambda^{*'}=I_\lambda$ for every $\lambda\in\Lambda$. We can now apply the proofs of (ii) and (iii) of Lemma 7.1 to complete the proof.

Lemma 7.3. *If there exists an involution $*'$ on A for which $R^p=(R_l)^{*'}$ for every closed right ideal R of A, then there exists an equivalent norm $\|\cdot\|'$ on A which satisfies the B^*-condition for $*'$.*

Proof. Since $R^p=(R_l)^{*'}$, we have $a^{*'}a=0$ implies $a=0$ (see proof of (iii) $\Rightarrow$ (ii) of Theorem 6.11). Since $a\mathfrak{S}=0$ implies $a=0$, where $\mathfrak{S}$ is the socle of A, by the proof of Corollary 7.2, A is an A^*-algebra under the involution $*'$. Let $\|\cdot\|'$ be the auxiliary norm on A. By Corollary (4.1.16) in [4] there exists a constant β such that $\|a\|'\leqq\beta\|a\|$ for all $a\in A$. But, by Corollary (4.8.4) in [4],

$$\|a\|^2 \leqq \|a^*\|'\|a\|'$$

for all $a\in A$, hence

$$(1/\beta)\|a\| \leqq \|a\|' \leqq \beta\|a\| \qquad (a\in A).$$

Thus the two norms are equivalent and so A is a B^*-algebra under the involution $*'$ and the norm $\|\cdot\|'$.

THEOREM 7.4. *If A has no minimal left ideals of dimension less than three, then the following statements are equivalent*:

(i) *The p-derived mapping P is uniformly continuous.*

(ii) *There exists an involution $*'$ on A for which $R^p = (R_\cdot)^{*'}$ for every closed right ideal R of A (and hence there exists an equivalent norm $\|\cdot\|'$ on A which satisfies the B^*-condition for $*'$).*

REMARK. The assumption in the statement of Theorem 7.4 that A does not contain any minimal left ideals of dimension less than three is obviously equivalent to the assumption that each H_λ is of dimension at least three. To prove the equivalence of (i) and (ii) of Theorem 7.4 we will have to identify each I_λ with $LC(H_\lambda)$ and then apply Theorem 6.11 (or Corollary 6.12); and in order to do so we must make this dimensionality condition on the minimal left ideals of A.

Proof. (i) $\Rightarrow$ (ii). Suppose that P is uniformly continuous. Then in particular p is continuous and therefore, by Theorem 3.9, each p_λ is continuous. Hence, by Theorem 6.4, every p_λ-representing operator is positive linear with continuous inverse. Let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of p_λ-representing operators such that $\|T_\lambda^{-1}\| = 1$ for all $\lambda \in \Lambda$. (Such a choice of T_λ is always possible since a real positive scalar multiple of a p_λ-representing operator is also a p_λ-representing operator.) We observe that $\|T_\lambda^{-1/2}\| = 1$ and $\|T_\lambda\| \geqq 1$ for all $\lambda \in \Lambda$. We claim that the norms $\|T_\lambda\|$ are bounded above. For suppose that this is not so. Then we can choose a sequence $\{T_{\lambda_n}\}$ of the T_λ such that $\|T_{\lambda_n}^{1/2}\| > n$. For convenience write $T_n = T_{\lambda_n}$, $H_n = H_{\lambda_n}$ $(n = 1, 2, \ldots)$. Then, since $\|T_n^{-1}\| = 1$, there exists $x_n \in H_n$, $\|x_n\| = 1$, such that $\|T_n x_n\| \leqq 2$ (and hence $\|T_n^{1/2} x_n\| \leqq 2$). Also, since $\|T_n^{1/2}\| > n$, there exists $z_n \in H_n$ orthogonal to x_n such that $\|z_n\| = 1$ and $\|T_n^{1/2} z_n\| = \kappa_n > n$. Let $y_n = \kappa_n^{-1} z_n + x_n$ and e_{y_n} the corresponding p_{λ_n}-projection in $LC(H_n)$.

$$|(x_n, T_n z_n)| = |(T_n x_n, z_n)| \leqq 2, \tag{1}$$

$$\|y_n\| = (1 + 1/\kappa_n^2)^{1/2} \geqq 1, \tag{2}$$

and

$$|(z_n, T_n y_n)| \geqq \kappa_n^{-1}(z_n, T_n z_n) - |(z_n, T_n x_n)| \geqq n - 2. \tag{3}$$

From (1) we get

$$(y_n, T_n y_n) \leqq 1 + 4/\kappa_n + 4 \leqq 9. \tag{4}$$

From (2), (3) and (4) it follows that

$$\|e_{y_n}\| \geqq \|e_{y_n} z_n\| = \frac{|(z_n, T_n y_n)|\ \|y_n\|}{(y_n, T_n y_n)} \geqq \frac{n-2}{9}. \tag{5}$$

Since $\|T_n^{-1/2}\| = 1$ and $(x_n, T_n x_n) \geqq 1$,

$$\|e_{x_n}\| = \|x_n \otimes T_n x_n\| / (x_n, T_n x_n) \leqq \|T_n x_n\|\ \|x_n\| \leqq 2.$$

Thus

$$\|e_{y_n} - e_{x_n}\| \geqq \|e_{y_n}\| - 2. \tag{6}$$

Now, for every $h \in H$, we have

$$f_{y_n}(h)-f_{x_n}(h) = \kappa_n^{-1}\frac{(h, y_n)}{(y_n, y_n)} z_n + \left(h, \frac{y_n}{(y_n, y_n)} - x_n\right)x_n$$

and therefore, by (2) and Schwarz's inequality, we obtain

$$\|f_{y_n}-f_{x_n}\| \leqq \frac{\|z_n\|}{\kappa_n} + \left\|\frac{y_n}{1+1/\kappa_n^2} - x_n\right\| < \frac{3}{\kappa_n}.$$

Hence $\|f_{y_n}-f_{x_n}\| \to 0$ as $n \to \infty$. But $e_{y_n}=P(f_{y_n})$ and $e_{x_n}=P(f_{x_n})$ and, from (6), $\|P(f_{y_n})-P(f_{x_n})\| \to \infty$ as $n \to \infty$, contradicting the uniform continuity of P. Therefore there must exist a constant K such that $\|T_\lambda\| \leqq K$ for all $\lambda \in \Lambda$.

For each $\lambda \in \Lambda$, let $\langle\cdot\rangle_\lambda$ be the inner product in H_λ given by $\langle x, y\rangle_\lambda = (x, T_\lambda y)$, $x, y \in H_\lambda$. Let $*'_\lambda$ be the adjoint operation in I_λ with respect to $\langle\cdot\rangle_\lambda$. Then, by Corollary 6.12, $p_\lambda = p_{\langle\cdot\rangle_\lambda}$ and

$$R_\lambda^{p_\lambda} = ((R_\lambda)_{l_\lambda})^{*'_\lambda} \qquad (\lambda \in \Lambda). \tag{7}$$

Identifying A with $(\sum I_\lambda)_0$ and using the fact that $\|T_\lambda\| \leqq K$ for all $\lambda \in \Lambda$, we see that $a \to a^{*'} = (a_\lambda^{*'_\lambda})$ is an involution on A and that $a^{*'}a=0$ implies $a=0$. Hence, by Corollary 7.2,

$$(R_l)^{*'} = \left(\sum ((R \cap I_\lambda)_{l_\lambda})^{*'}\right)_0 \tag{8}$$

for every closed right ideal R of A. But, by Lemma 7.1 (i),

$$R^p = \left(\sum [(R \cap I_\lambda)^p \cap I_\lambda]\right)_0$$

and so, if we let $R_\lambda = R \cap I_\lambda$ then, from (7) and (8) and the fact that for any subset S_λ of I_λ, $S_\lambda^{*'} = S_\lambda^{*'_\lambda}$, we obtain

$$R^p = \left(\sum R_\lambda^{p_\lambda}\right)_0 = \left(\sum ((R_\lambda)_{l_\lambda})^{*'}\right)_0 = \left(\sum ((R \cap I_\lambda)_{l_\lambda})^{*'}\right)_0 = (R_l)^{*'}.$$

The existence of the norm $\|\cdot\|'$ is given by Lemma 7.3; in fact $\|a\|' = \sup_\lambda \|a_\lambda\|_\lambda$, where $\|\cdot\|_\lambda$ denotes the operator bound in I_λ with respect to $\langle\cdot\rangle_\lambda$.

(ii) $\Rightarrow$ (i). Suppose that (ii) holds. Then, by Lemma 7.3, there exists an equivalent norm $\|\cdot\|'$ on A which satisfies the B^*-condition for $*'$. Hence, by Lemma 7.1 (or Corollary 7.2), we have

$$R_\lambda^{p_\lambda} = ((R_\lambda)_l)^{*'} \cap I_\lambda = ((R \cap I_\lambda)_{l_\lambda})^{*'} = ((R_\lambda)_{l_\lambda})^{*'}$$

for every closed right ideal R_λ of I_λ. Thus, by Theorem 6.11, p_λ is continuous and therefore, by Theorem 3.9, p is continuous which means that P is continuous. It remains to show that P is uniformly continuous.

Now $*'$ is an involution on I_λ for which $R_\lambda \to ((R_\lambda)_{l_\lambda})^{*'}$ is a complementor on I_λ. Therefore, by Corollary 6.14, there exists a positive hermitian linear operator Q_λ of H_λ onto itself such that $*'$ is the adjoint operation with respect to the equivalent inner product $\langle\cdot\rangle_\lambda$ in H_λ given by $\langle x, y\rangle_\lambda=(x, Q_\lambda y)$, for all $x, y \in H_\lambda$. Also, since $R_\lambda^{p_\lambda}=((R_\lambda)_{l_\lambda})^{*'}$, by Corollary 4.3, we have that $p_\lambda=p_{\langle\cdot\rangle_\lambda}$ and hence, by Lemma 6.9, that Q_λ is a p_λ-representing operator. We have

$$a_\lambda^{*'} = a_\lambda^{*\langle\cdot\rangle_\lambda}$$

and

$$a_\lambda^{*'} = Q_\lambda^{-1}a_\lambda^* Q_\lambda \qquad (a_\lambda \in I_\lambda, \lambda \in \Lambda). \tag{9}$$

(See remark following Corollary 6.14.)

Let $\|\cdot\|_\lambda$ be the operator bound on I_λ with respect to $\langle\cdot\rangle_\lambda$. Since $\|\cdot\|'$ and $\|\cdot\|_\lambda$ satisfy the B^*-condition for $*'$ in I_λ and since both are equivalent to the given norm, we have that $\|\cdot\|'=\|\cdot\|_\lambda$ on each I_λ. Therefore, if we identify A with $(\sum I_\lambda)_0$, we obtain

$$a^{*'} = (a_\lambda^{*\langle\cdot\rangle_\lambda}) \quad \text{for every } a \in A.$$

Since for every positive scalar α, αQ_λ is a p_λ-representing operator and since the operator bound $\|\cdot\|_\lambda$ remains the same with respect to the inner product $(x, \alpha Q_\lambda y)$, $x, y \in H_\lambda$, it follows that we may assume that $\|Q_\lambda\|=1$ for all $\lambda \in \Lambda$. We claim that $\|Q_\lambda^{-1}\|$ are also bounded above. For suppose on the contrary that $\|Q_\lambda^{-1}\|$ are not bounded above. Then we can find a sequence $\{Q_{\lambda_n}\}$ of the Q_λ such that $\|Q_{\lambda_n}^{-1}\| \to \infty$ as $n \to \infty$. Write $Q_n=Q_{\lambda_n}$ and $H_n=H_{\lambda_n}$ $(n=1, 2, \ldots)$. Then, for each n, there is an x_n in H_n, $\|x_n\|=1$, such that

$$(Q_n x_n, x_n) < 1/n \qquad (n = 1, 2, \ldots).$$

Since $\|Q_n\|=1$, there is a $y_n \in H_n$, $\|y_n\|=1$, such that

$$\|Q_n y_n\| > 1-1/n \qquad (n = 1, 2, \ldots).$$

Let a_n be the element of A defined by $a_n=x_n \otimes Q_n y_n$ $(n=1, 2, \ldots)$. Then, by (9), $a_n^{*'}=y_n \otimes Q_n x_n$ $(n=1, 2, \ldots)$. Hence $\|a_n^{*'}\| \to 0$ as $n \to \infty$ and therefore $\|a_n^{*'}\|' \to 0$ as $n \to \infty$. But $\|a_n\|=\|a_n^*\| \to 1$ as $n \to \infty$, contradicting the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|'$. Thus there exists a constant K such that $\|Q_\lambda^{-1}\| \leqq K$ for all $\lambda \in \Lambda$. Let $H=\bigoplus_\lambda H_\lambda$, the direct sum of H_λ, and let $Q=(Q_\lambda)$ be the operator on H defined by $Qx=(Q_\lambda x_\lambda)$ for every $x=(x_\lambda) \in H$. Then Q is a continuous positive linear operator mapping H onto H with continuous inverse.

Now, since every $e \in \mathscr{E}$ is of the form $x_\lambda \otimes x_\lambda$ for $x_\lambda \in H_\lambda$ with $\|x_\lambda\|=1$, we have

$$P(x_\lambda \otimes x_\lambda) = \frac{x_\lambda \otimes Q_\lambda x_\lambda}{(x_\lambda, Q_\lambda x_\lambda)}.$$

Identifying H_λ as a subspace of H, we may write $P(x_\lambda \otimes x_\lambda)=(x_\lambda \otimes Qx_\lambda)/(x_\lambda, Qx_\lambda)$ and therefore, by the proof of Corollary 6.6, we may conclude that P is uniformly continuous.

COROLLARY 7.5. *If the p-derived mapping P is uniformly continuous, then it defines a homeomorphism between* $\mathscr{E}$ *and* $\mathscr{E}_p$.

Proof. This follows immediately from the proofs of Corollary 6.7 and Theorem 7.4.

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