THE C^*-ALGEBRA GENERATED BY AN ISOMETRY. II

BY

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I. **Introduction.** In [2] it was shown that the C^*-algebra generated by any nonunitary isometry is *-isomorphic to the C^*-algebra $\mathscr{A}(S)$ generated by the unilateral shift of multiplicity one, S. Further, the ideal theory of $\mathscr{A}(S)$ was determined. The main point was that $\mathscr{A}(S)$ contains the full algebra of compact operators $\mathscr{K}$ and $\mathscr{A}(S)/\mathscr{K}$ is *-isomorphic to $C(T)$, the algebra of all complex-valued continuous functions on the unit circle, T.

Here, I examine the structure of $\mathscr{A}(S)$ and certain related C^*-algebras in greater detail. In particular, the irreducible *-subalgebras of $\mathscr{A}(S)$ are characterized and reasonable necessary and sufficient conditions are given for an operator in $\mathscr{A}(S)$ to generate $\mathscr{A}(S)$. Finally, I provide an example of a C^*-algebra which is irreducible and has the same ideal structure as $\mathscr{A}(S)$, but which is not *-isomorphic to $\mathscr{A}(S)$.

II. **Preliminary results.** Henceforth all Hilbert spaces are over the complex numbers. If $\mathscr{H}$ is a Hilbert space and B is a bounded operator on $\mathscr{H}$ then the smallest C^*-algebra of operators containing B and 1 is denoted by $\mathscr{A}(B)$. The full algebra of bounded operators on $\mathscr{H}$ with the operator norm topology is called $\mathscr{B}(\mathscr{H})$ and the subalgebra of all compact operators on $\mathscr{H}$ is called $\mathscr{K}(\mathscr{H})$ (or just $\mathscr{K}$). We let π denote the usual quotient map from $\mathscr{B}(\mathscr{H})$ onto $\mathscr{B}(\mathscr{H})/\mathscr{K}$. In this paper, all ideals are closed and two-sided. We write $\sigma(B)$ for the spectrum of B in $\mathscr{B}(\mathscr{H})$.

In portions of this paper, we will be concerned with the algebra $C(X)$ of all complex-valued continuous functions on a compact Hausdorff space X with supremum norm. More particularly, let T be the unit circle and let A be the uniformly closed subalgebra of $C(T)$ consisting of those ϕ in $C(T)$ which are uniform limits of polynomials in the complex variable z. Further, let μ be normalized Haar measure on T and let L^2 denote the associated Hilbert space of square-integrable functions on T. As usual, we let H^2 denote the L^2-closure of A. In the following sections we will be interested in the C^*-algebras generated by Toeplitz operators [1] on H^2 associated with functions ϕ in $C(T)$ by the relation $T_\phi f = P(\phi f)$, where P is the orthogonal projection from L^2 onto H^2. We will also be concerned with the Laurent operators [1] on L^2 associated with ϕ in $C(T)$ by the relation $M_\phi f = \phi f$. It should be pointed out that in general [1], [3]

$$\|T_\phi\| = \|M_\phi\| = \|\pi(T_\phi)\| = \|\pi(M_\phi)\| = \|\phi\|.$$

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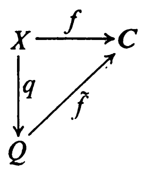
Further, the Toeplitz operator T_z is just the unilateral shift S on H^2 since $\{z^n \mid n=0, 1, 2, \ldots\}$ is a basis for H^2.

We will deal in the following sections with certain irreducible C^*-algebras and certain abelian C^*-algebras. The two propositions below will be among the necessary working tools.

PROPOSITION 1. *Let $\mathscr{A}$ be an irreducible C^*-algebra on $\mathscr{H}$ and suppose $\mathscr{A} \cap \mathscr{K} \neq (0)$. Then $\mathscr{K} \subset \mathscr{A}$ and $\mathscr{K} \subset \mathscr{I}$ for any nontrivial ideal $\mathscr{I}$ in $\mathscr{A}$.*

Proof. Immediate from [4, 4.1.10; 2.11.3 (i); and 1.8.2.]. □

If $\mathscr{A}$ is a C^*-subalgebra of $C(X)$ which contains the constants then a natural equivalence relation is given on X by writing $x \sim y$ if and only if $f(x)=f(y)$ for all f in $\mathscr{A}$. Let Q be the quotient space $X/\sim$ and let q be the usual quotient map. To each f in $\mathscr{A}$ there corresponds a complex-valued function $\tilde{f}$ on Q by the commutative diagram



We will consider two topologies on Q. The first (Q, ω) is the weak topology induced by all the $\tilde{f}$ for f in $\mathscr{A}$. The second is the usual quotient topology, (Q, q). It is not hard to see that (Q, q) is stronger than (Q, ω). Whether (Q, q) and (Q, ω) are equivalent will be left open.

PROPOSITION 2. *The maximal ideal space of $\mathscr{A}$ is just (Q, ω) and $\mathscr{A}$ is $*$-isomorphic to $C(Q, \omega)$. A necessary and sufficient condition that (Q, q) and (Q, ω) are equivalent is that (Q, q) be Hausdorff.*

Proof. Since $\mathscr{A}$ is a closed symmetric subalgebra of $C(X)$, every irreducible representation of $\mathscr{A}$ extends to an irreducible representation of $C(X)$ [6, p. 305]. Since complex-homomorphisms of $\mathscr{A}$ are irreducible representations and the only irreducible representations of $C(X)$ are point-evaluations at points of X, we see that every complex-homomorphism of $\mathscr{A}$ is a point-evaluation at a point of X. Now since x and y in X induce the same homomorphism on $\mathscr{A}$ if and only if $x \sim y$, it is clear that Q can be identified with the maximal ideal space of $\mathscr{A}$. Under this identification, $\tilde{f}$ is the Gelfand transform of f in $\mathscr{A}$ and so (Q, ω) is the usual maximal ideal space with the weak topology. The mapping $f \to \tilde{f}$ is clearly a $*$-isomorphism from $\mathscr{A}$ into $C(Q, \omega)$. This mapping is also an isometry so the image $\tilde{\mathscr{A}}$ is a C^*-subalgebra of $C(Q, \omega)$. Since the $\tilde{f}$ separate points of Q, the Stone-Weierstrass theorem implies that $\tilde{\mathscr{A}} = C(Q, \omega)$.

If (Q, q) and (Q, ω) are to be equivalent then (Q, q) must be Hausdorff since (Q, ω) is Hausdorff. But the map $f \to \tilde{f}$ is an isometric $*$-isomorphism from $\mathscr{A}$ into

$C(Q,q)$ and if (Q,q) is Hausdorff the Stone-Weierstrass theorem again applies to show that $\tilde{\mathscr{A}}=C(Q,q)$. Hence $(Q,q)=(Q,\omega)$. □

III. **The algebra $\mathscr{A}(T_z)$.** We now focus on certain C^*-algebras on H^2. Since T_z is the unilateral shift on H^2, we will be interested in $\mathscr{A}(T_z)$ and its *-subalgebras. In this section we write $\mathscr{K}=\mathscr{K}(H^2)$ for the algebra of compact operators on H^2. We begin with a "functional" characterization of $\mathscr{A}(T_z)$. Recall (cf. [1]) that for Ψ in $C(T)$, $T_\Psi^*=T_{\bar{\Psi}}$. Also, if Ψ is in $C(T)$ and ϕ is in A, then $T_\Psi T_\phi=T_{\Psi\phi}$.

LEMMA 1. *If Ψ_1, Ψ_2 are in $C(T)$ then $T_{\Psi_1}T_{\Psi_2}=T_{\Psi_1\Psi_2}+K$ for some K in $\mathscr{K}$.*

Proof. It follows from standard results on polynomial approximation that there are sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ of polynomials in A such that

$$\alpha_n+\bar{\beta}_n\to\Psi_1,\qquad \gamma_n+\bar{\delta}_n\to\Psi_2.$$

It follows from the fact that $\|T_\phi\|=\|\phi\|$ that

$$T_{\alpha_n}+T_{\bar{\beta}_n}\to T_{\Psi_1}\quad\text{and}\quad T_{\gamma_n}+T_{\bar{\delta}_n}\to T_{\Psi_2}.$$

Hence, we have

$$T_{\alpha_n}T_{\gamma_n}+T_{\bar{\beta}_n}T_{\bar{\delta}_n}+T_{\alpha_n}T_{\bar{\delta}_n}+T_{\bar{\beta}_n}T_{\gamma_n}\to T_{\Psi_1}T_{\Psi_2}.$$

But $T_{\alpha_n}T_{\gamma_n}=T_{\alpha_n\gamma_n}$, $T_{\bar{\beta}_n}T_{\gamma_n}=T_{\bar{\beta}_n\gamma_n}$ and $T_{\bar{\beta}_n}T_{\bar{\delta}_n}=(T_{\delta_n}T_{\beta_n})^*=T_{\bar{\delta}_n\bar{\beta}_n}$. Further $T_{\bar{\delta}_n}T_{\alpha_n}=T_{\bar{\delta}_n\alpha_n}$ and

$$T_{\alpha_n}T_{\bar{\delta}_n}-T_{\bar{\delta}_n}T_{\alpha_n}$$

is in $\mathscr{K}$ since $\mathscr{A}(T_z)/\mathscr{K}$ is abelian. Putting the above remarks together, we see that there is a sequence of $\{K_n\}$ in $\mathscr{K}$ so that

$$T_{(\alpha_n+\bar{\beta}_n)(\gamma_n+\bar{\delta}_n)}+K_n\to T_{\Psi_1}T_{\Psi_2}.$$

But

$$T_{(\alpha_n+\bar{\beta}_n)(\gamma_n+\bar{\delta}_n)}\to T_{\Psi_1\Psi_2}$$

and it follows that there is a K in $\mathscr{K}$ with $K_n\to K$ such that

$$T_{\Psi_1\Psi_2}+K = T_{\Psi_1}T_{\Psi_2}.$$ □

THEOREM 1. *Let $\mathscr{B}=\{T_\Psi+K : \Psi\in C(T), K\in\mathscr{K}\}$. Then $\mathscr{A}(T_z)=\mathscr{B}$. Further, the representation as sums $T_\Psi+K$ is unique.*

Proof. Clearly, if Ψ is in $C(T)$ and $\varepsilon>0$ is arbitrary then there are polynomials α,β in A so that $\|\Psi-\alpha-\bar{\beta}\|<\varepsilon$. Since T_α and T_β^* are in $\mathscr{A}(T_z)$ it follows that T_Ψ is in $\mathscr{A}(T_z)$. Further, by Proposition 1 or directly as in [2] it is easily seen that $\mathscr{K}\subset\mathscr{A}(T_z)$. Thus, we have $\mathscr{B}\subset\mathscr{A}(T_z)$. To show the reverse inclusion, it suffices since T_z is in $\mathscr{B}$ to show that $\mathscr{B}$ is a closed *-subalgebra of bounded operators. Clearly, we have 1 in $\mathscr{B}$. Also, if B_1 and B_2 are in $\mathscr{B}$ then B_1^*, λB_1 and B_1+B_2 are in $\mathscr{B}$. It follows immediately from Lemma 1 that B_1B_2 is in $\mathscr{B}$. Thus, it remains to check that $\mathscr{B}$ is closed. To this end, suppose we have $T_{\Psi_n}+K_n \to B$. Then $\{\pi(T_{\Psi_n})\}$ is a Cauchy sequence. But $\|\Psi\|=\|T_\Psi\|=\|\pi(T_\Psi)\|$ so $\{\Psi_n\}$ is a Cauchy sequence in $C(T)$ and hence $\Psi_n \to \Psi$ for some Ψ in $C(T)$. It follows that $T_{\Psi_n} \to T_\Psi$ and hence $K_n \to K$ for some K in $\mathscr{K}$. Hence $B=T_\Psi+K$ and so $\mathscr{B}$ is closed. To check uniqueness, suppose $T_\Psi+K=T_{\Psi_1}+K_1$ then $\pi(T_{\Psi-\Psi_1})=0$ so $\Psi=\Psi_1$ and $K=K_1$. □

Corollary (1.1). *If B is in $\mathscr{A}(T_z)$ then either $\sigma(B)$ contains a nontrivial component or $\sigma(B)$ has at most one limit point.*

Proof. By Theorem 1, B has the form $B=T_\Psi+K$. Now, it is known [8] that $\sigma(T_\Psi)$ is connected and [1] $\sigma(T_\Psi)$ consists of one point only if Ψ is constant. Further, $\sigma(T_\Psi)\subset\sigma(T_\Psi+K')$ for all compact K' [3]. Thus, if $\sigma(T_\Psi)$ is nontrivial then $\sigma(B)$ has a nontrivial component containing $\sigma(T_\Psi)$. If $\sigma(T_\Psi)$ is a one-point set, then $B=\lambda I+K$ so $\sigma(B)$ has at most one limit point. □

Corollary (1.2). *The mapping $\phi \to \pi(T_\phi)$ gives a *-isomorphism from $C(T)$ onto $\mathscr{A}(T_z)/\mathscr{K}$.*

Proof. Using Theorem 1 and the fact that $\|\phi\|=\|\pi(T_\phi)\|$, it is clear that the mapping $\phi \to \pi(T_\phi)$ is an additive *-isomorphism and isometry from $C(T)$ onto $\mathscr{A}(T_z)/\mathscr{K}$. To check multiplicativity, recall that by Lemma 1 $T_{\Psi_1}T_{\Psi_2}=T_{\Psi_1\Psi_2}+K$. Hence, we have

$$\pi(T_{\Psi_1})\pi(T_{\Psi_2}) = \pi(T_{\Psi_1}T_{\Psi_2}) = \pi(T_{\Psi_1\Psi_2}),$$

which is the desired result. □

Corollary (1.3). *We have $\sigma(\pi(T_\phi))=$* range *ϕ for ϕ in $C(T)$.*

Proof. In $C(T)$, $\sigma(\phi)=$ range ϕ. This fact and the isomorphism of Corollary (1.2) give the desired result. □

We are now in a position to classify the irreducible C^*-subalgebras of $\mathscr{A}(T_z)$. First note that if a closed *-subalgebra $\mathscr{D}$ is to be irreducible then $\mathscr{D}$ is clearly non-abelian so $\mathscr{D}$ contains a nontrivial commutator (i.e. element of the form $BD-DB$). But $\mathscr{A}(T_z)/\mathscr{K}$ is abelian so all commutators in $\mathscr{A}(T_z)$ are compact. Hence, $\mathscr{D}$ contains a nontrivial compact operator and it follows from Proposition 1 that $\mathscr{K}\subset\mathscr{D}$. For any irreducible C^*-subalgebra $\mathscr{D}$ of $\mathscr{A}(T_z)$, define $\mathscr{D}'$ by

$$\mathscr{D}' = \{\phi \text{ in } C(T) : T_\phi \text{ is in } \mathscr{D}\}.$$

Theorem 2. *$\mathscr{D}'$ is a closed *-subalgebra of $C(T)$.*

Proof. If Ψ_1, Ψ_2 are in $\mathscr{D}'$ then clearly $\lambda\Psi_1$, $\overline{\Psi}_1$, and $\Psi_1+\Psi_2$ are in $\mathscr{D}'$. Using Lemma 1 and the fact that $\mathscr{K}\subset\mathscr{D}$, we also see that $\Psi_1\Psi_2$ is in $\mathscr{D}'$. Finally, it follows from the fact that $\|T_\Psi\|=\|\Psi\|$ that $\mathscr{D}'$ is closed. □

In fact, the correspondence between irreducible C^*-subalgebras of $\mathscr{A}(T_z)$ and C^*-subalgebras of $C(T)$ induced by the map $\mathscr{D}\to\mathscr{D}'$ is 1-1 and onto.

THEOREM 3. *Let $\mathscr{A}$ be any C^*-subalgebra of $C(T)$. Then there is an irreducible C^*-subalgebra $\mathscr{D}$ of $\mathscr{A}(T_z)$ such that $\mathscr{D}'=\mathscr{A}$.*

Proof. Let $\mathscr{D}=\{T_\phi+K : \phi\in\mathscr{A}, K\in\mathscr{K}\}$. We need only check that $\mathscr{D}$ is closed under multiplication. This is a direct consequence of Lemma 1. □

The ideal theory of $\mathscr{D}$ can be determined by combining Proposition 1 and the following result.

THEOREM 4. *The mapping $\phi\to\pi(T_\phi)$ gives a *-isomorphism from $\mathscr{D}'$ onto $\mathscr{D}/\mathscr{K}$.*

Proof. This follows from Corollary (1.2). □

From now on, we will assume that $\mathscr{D}$ is an irreducible C^*-subalgebra of $\mathscr{A}(T_z)$ and 1 is in $\mathscr{D}$.

THEOREM 5. *Let $\sim$ be the equivalence relation on T induced by $\mathscr{D}'$ and let (Q,ω) be the quotient space $T/\sim$ with the weak topology of* §II. *Then $\mathscr{D}/\mathscr{K}$ is *-isomorphic to $C(Q,\omega)$.*

Proof. This follows from Theorem 4 and Proposition 2. □

THEOREM 6. *If $\mathscr{D}=\mathscr{A}(T_\phi+K)$ for some fixed ϕ and K, then $\mathscr{D}'=\{\psi : \psi$ in the C^*-subalgebra of $C(T)$ generated by $\phi\}$.*

Proof. Since $\mathscr{D}$ is irreducible, $\mathscr{K}\subset\mathscr{D}$ and so T_ϕ is in $\mathscr{D}$. Hence, ϕ is in $\mathscr{D}'$ and so $\mathscr{D}'$ contains the C^*-subalgebra of $C(T)$ generated by ϕ because of Theorem 2. Conversely, for any "polynomial" $p(r,s)$ in two noncommuting indeterminates r, s, it follows from Lemma 1 that

$$p(T_\phi+K, T_\phi^*+K^*)-T_{p(\phi,\bar\phi)}$$

is in $\mathscr{K}$. Hence, for B in $\mathscr{A}(T_\phi+K)$ there is a sequence of elements $\{\psi_n\}$ in the C^*-algebra generated by ϕ and a sequence of compact operators K_n such that $T_{\psi_n}+K_n\to B$. The proof of Theorem 1 then shows that there is a ψ in the C^*-algebra generated by ϕ such that $B=T_\psi+K'$ for K' in $\mathscr{K}$. Since representation in the form $T_\psi+K'$ is unique for elements of $\mathscr{A}(T_z)$, we see that $\mathscr{D}'$ is contained in the C^*-algebra generated by ϕ. □

COROLLARY (6.1). *If $\mathscr{D}=\mathscr{A}(T_\phi+K)$ then $\mathscr{D}/\mathscr{K}$ is *-isomorphic to $C(\text{range }\phi)$.*

Proof. This follows from Corollary (1.3) since

$$\pi(T_\phi+K) = \pi(T_\phi)$$

and $\pi(T_\phi+K)$ generates $\mathscr{D}/\mathscr{K}$ as a C^*-algebra. □

COROLLARY (6.2). *$\mathscr{A}(T_\phi+K)=\mathscr{A}(T_z)$ if and only if $T_\phi+K$ is irreducible and ϕ is* 1-1 *on T.*

Proof. Clearly $\mathscr{A}(T_\phi+K)=\mathscr{D}=\mathscr{A}(T_z)$ if and only if $T_\phi+K$ is irreducible and $\mathscr{D}'=C(T)$. By Theorem 6, $\mathscr{D}'$ is the C^*-subalgebra of $C(T)$ generated by ϕ. Hence, by the Stone-Weierstrass theorem, $\mathscr{D}'=C(T)$ if and only if ϕ is 1-1. □

COROLLARY (6.3). *If ϕ is in A then $\mathscr{A}(T_\phi)=\mathscr{A}(T_z)$ if and only if ϕ is* 1-1 *on T.*

Proof. If ϕ is in A and ϕ is 1-1 on T then by a result in [7], T_ϕ is irreducible. Hence, by Corollary (6.2) $\mathscr{A}(T_\phi)=\mathscr{A}(T_z)$. Conversely, if $\mathscr{A}(T_\phi)=\mathscr{A}(T_z)$ then ϕ is 1-1 on T by Corollary (6.2). □

IV. **Extension of $\mathscr{K}$ by $C(T)$.** On the basis of the ideal theory developed in [2] and in the previous section, we can think of $\mathscr{A}(S)$ (or $\mathscr{A}(T_z)$) as an extension of $\mathscr{K}$ by $C(T)$. In this section, I construct a different extension of $\mathscr{K}$ by $C(T)$ and show that the new extension is not *-isomorphic to $\mathscr{A}(S)$. The construction is motivated by Theorem 1. Henceforth, we will deal with the Hilbert space L^2 described in §II and $\mathscr{K}=\mathscr{K}(L^2)$. Since H^2 and L^2 have the same dimension, we can think of $\mathscr{A}(S)$ as represented on L^2 by some spatial isomorphism.

THEOREM 7. *Let $\mathscr{C}=\{M_\phi+K : \phi\in C(T), K\in\mathscr{K}\}$. Then $\mathscr{C}$ is a C^*-subalgebra of $\mathscr{B}(L^2)$ and $\mathscr{C}/\mathscr{K}$ is *-isomorphic to $C(T)$.*

Proof. Immediate from [4, 1.8.4]. □

Theorem 7 combined with Proposition 1 shows that $\mathscr{C}$ is an irreducible C^*-algebra with the same ideal theory as $\mathscr{A}(S)$. To show that $\mathscr{A}(S)$ and $\mathscr{C}$ are not *-isomorphic we need some machinery from the theory of Fredholm operators [5]. Recall that an operator B is semi-Fredholm if B has closed range $R(B)$ and either the null space $n(B)$ or $R(B)^\perp$ is of finite dimension. If B is semi-Fredholm, the index $\kappa(B)$ is defined by

$$\kappa(B) = \dim R(B)^\perp - \dim n(B).$$

It is well known that if B is semi-Fredholm and K is compact then $K+B$ is semi-Fredholm and $\kappa(B)=\kappa(B+K)$.

THEOREM 8. *There is no *-isomorphism from $\mathscr{A}(S)$ onto $\mathscr{C}$.*

Proof. Suppose Φ is a *-isomorphism from $\mathscr{A}(S)$ onto $\mathscr{C}$. Then $\Phi(S)^*\Phi(S)=1$ so $\Phi(S)$ must be a noninvertible isometry. Thus $\Phi(S)$ must be a semi-Fredholm operator with index other than zero. But M_ϕ is a normal operator for all ϕ in $C(T)$ so

$$\dim n(M_\phi) = \dim R(M_\phi)^\perp.$$

It follows that if $\Phi(S)=M_\phi+K$ then M_ϕ must be semi-Fredholm and

$$\kappa(\Phi(S)) = \kappa(M_\phi+K)=\kappa(M_\phi) = 0.$$

This contradiction completes the proof. □

V. **Some remarks.** It is clear from the ideal theory of $\mathscr{A}(S)$ that $\mathscr{A}(S)$ is postliminaire [4, 4.3.1] with a composition series of length 2. It is not hard to describe the usual spaces of primitive ideals, irreducible representations and pure states of $\mathscr{A}(S)$. Denoting these spaces as usual [4, §3] by Prim ($\mathscr{A}(S)$), $\mathscr{A}(S)^\wedge$, and $P(\mathscr{A}(S))$, we note that since $\mathscr{A}(S)$ is postliminaire, $\mathscr{A}(S)^\wedge = \text{Prim}\ (\mathscr{A}(S))$ [4, 4.4.1]. Further, Prim $(\mathscr{A}(S))$ is the union of a circle and a point, the point being dense in the Jacobson topology. This is because $\{0\}$ is primitive and by Proposition 1, any other primitive ideal contains $\mathscr{K}$ and so corresponds to a maximal ideal in $C(T)$. The relative topology on the circle part of Prim $(\mathscr{A}(S))$ is the usual one for T. Note also that for $\mathscr{C}$ as in §IV, $\hat{\mathscr{C}} = \text{Prim}\ (\mathscr{C}) = \text{Prim}\ (\mathscr{A}(S))$ since $\mathscr{C}$ has the same ideal structure as $\mathscr{A}(S)$. Similarly, it is not hard to see that $P(\mathscr{A}(S))$ consists of vector states of $\mathscr{K}$ extended to $\mathscr{A}(S)$ [4, 2.11.8] and multiplicative states corresponding to the points of T. Again $P(\mathscr{C}) = P(\mathscr{A}(S))$.

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