

ON HANDLE DECOMPOSITIONS AND DIFFEOMORPHISMS

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In this paper we are concerned with two questions which arise naturally in the study of manifolds. One has to do with whether certain types of handle decompositions of a manifold are unique; and the other, under what circumstances they are preserved by diffeomorphisms. The corollaries to Theorem 1 and 2 are partial answers to these questions. Corollary 2 may prove useful for the general study of diffeomorphisms. We shall begin with some notation.

\mathbf{R}^k shall denote real k -dimensional vector space with unit disc D^k and unit sphere S^{k-1} . Let $I=[0, 1]$. All manifolds are assumed to be compact and C^∞ . Homology groups are taken with integer coefficients. \approx means either diffeomorphic or isotopic. By a *nice Morse function* on a manifold M^n (see [5, p. 44]) we shall mean a function $\eta: M^n \rightarrow \mathbf{R}$ which has only nondegenerate critical points, and if p is a critical point, then $\eta(p)=\text{index } p$. It is well known that in this case $M_k(\eta) \equiv \eta^{-1}[0, k+1/2]$ is a n -submanifold of M for $0 \leq k \leq n$, and

$$M_k(\eta) = M_{k-1}(\eta) \cup_{\phi_1} (D^k \times D^{n-k}) \cup \dots \cup_{\phi_p} (D^k \times D^{n-k}),$$

where $\phi_j: \partial D^k \times D^{n-k} \rightarrow \eta^{-1}[k-1/2]$ are imbeddings. Call η *minimal* if the number of critical points of index k is equal to $p(k)+q(k)+q(k-1)$, where $p(k)$ is the torsion-free rank of $H_k M$ and $q(k)$ is the rank of the torsion subgroup of $H_k M$ (see [6, §6]).

From now on M^n will be a closed simply-connected manifold with $n \geq 6$.

THEOREM 1. *Let η_j be minimal nice Morse functions on M , $j=1, 2$. Assume that $H_{i_s} M$ is torsion-free for $0=i_1 < i_2 < \dots < i_{t-1} < i_t=n$. Then there is a diffeomorphism $H: M \rightarrow M$ satisfying*

(i) $H \approx \text{identity}$,

(ii) $H(M_{i_s}(\eta_1)) = M_{i_s}(\eta_2)$, for $1 \leq s \leq t$.

THEOREM 2. *Let $F: M \rightarrow M$ be a diffeomorphism and η_1 any minimal nice Morse function on M . Assume that $F_*|_{\text{torsion}}(H_{i_s} M) = \text{identity}$ for $0=i_1 < i_2 < \dots < i_{t-1} < i_t=n$. Then F is isotopic to a diffeomorphism F' such that $F'(M_{i_s}(\eta_1)) = M_{i_s}(\eta_1)$, for $1 \leq s \leq t$.*

REMARK. The existence of minimal nice Morse functions on M is guaranteed by [6, §6].

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As an immediate consequence of Theorems 1 and 2 we get the following corollaries.

COROLLARY 1. *Assume that H_*M is torsion-free. Let η_j be minimal nice Morse functions on M , $j = 1, 2$. Then $M_i(\eta_1) \approx M_i(\eta_2)$, for $0 \leq i \leq n$ (i.e., minimal handle decompositions of M are unique up to diffeomorphisms).*

COROLLARY 2. *Let $F: M \rightarrow M$ be a diffeomorphism such that $F_*|_{\text{torsion } H_*M} = \text{identity}$. Let η be a minimal nice Morse function on M . Then $F \approx F'$, where $F'(M_i(\eta)) = M_i(\eta)$, for $0 \leq i \leq n$ (i.e., up to isotopy, most diffeomorphisms of M are "level-preserving").*

In order to prove Theorems 1 and 2 we begin with a lemma.

LEMMA. *Let η_1 and η_2 be minimal nice Morse functions on M . Define $M_l^i \equiv \eta_j^{-1}[0, i + 1/2]$ and suppose that $M_k^1 = M_k^2 = A$ for some fixed k , $0 \leq k < n$. Let $k < l < n - 2$.*

(a) *If $H_l M$ is torsion-free, then there is a diffeomorphism H of M such that*

- (i) $H \approx \text{identity}$,
- (ii) $H|_A = \text{identity}$,
- (iii) $H(M_l^1) = M_l^2$.

(b) *Assume $\eta_2 = \eta_1 F^{-1}$, where F is a diffeomorphism of M with $F_*|_{\text{torsion } (H_l M)} = \text{identity}$. Then there is a diffeomorphism H of M satisfying (i)–(iii) in part (a).*

Proof. (a) The following easy general position argument shows that there is a diffeomorphism h_1 of M satisfying (i), (ii) and $h_1(M_l^2) \subset \text{int}(M_l^1) \equiv \text{interior of } M_l^1$: If $M_l^2 \subset \text{int } M_q^1$, for $q > l$, write $M_q^1 = M_{q-1}^1 \cup_{\psi_1} (D^q \times D^{n-q}) \cup \dots \cup_{\psi_p} (D^q \times D^{n-q})$, with respect to imbeddings $\psi_j: \partial D^q \times D^{n-q} \rightarrow \partial M_{q-1}^1$. Now as a cell complex M_l^2 has dimension at most l , so that $q > l$ implies that we may deform M_l^2 so that $M_l^2 \cap (0 \times D^{n-q})_j = \emptyset$; but once we have this, it is easy to push M_l^2 into $\text{int } M_{q-1}^1$. Repeating this process for each $q > l$ we get h_1 .

Let $B \equiv h_1(M_l^2)$ and let $f: B \rightarrow M$ be the inclusion map. Consider

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_{j+1}(M, B) & \xrightarrow{\partial_{j+1}} & H_j B & \longrightarrow & H_j M \longrightarrow H_j(M, B) \longrightarrow \cdots \\ & & \downarrow & & \downarrow f_* & & \downarrow \text{identity} \downarrow \\ \cdots & \longrightarrow & H_{j+1}(M, M_l^1) & \xrightarrow{\partial'_{j+1}} & H_j M_l^1 & \longrightarrow & H_j M \longrightarrow H_j(M, M_l^1) \longrightarrow \cdots \end{array}$$

Since $H_j(M, B) = 0 = H_j(M, M_l^1)$ for $j \leq l$, f_* is an isomorphism for $j < l$. But $\partial_{l+1} = 0 = \partial'_{l+1}$, because $H_l M$ is torsion-free. Thus f_* is an isomorphism for all j . By the Whitehead theorem (see [3, Theorem 10.1]), f is a homotopy equivalence.

CLAIM. $\pi_1 Y = \pi_1 \partial M_l^1 = \pi_1 \partial B = 0$, where $Y = \text{closure of } M_l^1 - B$.

Assuming this for the moment, we get by [5, Theorem 9.1] that $Y \approx \partial B \times I$. Thus, using the I -coordinate it should be clear how to define a diffeomorphism h_2 of M satisfying (i), (ii) and $h_2(B) = M_l^1$. Let $H = h_2 h_1$. Then H is the desired map.

To prove the claim, let $g: S^1 \rightarrow Y$. Since $\pi_1 M_l^1 = 0 = \pi_1 M$, there is a $\tilde{g}: D^2 \rightarrow M_l^1$ with $\tilde{g}|S^1 = g$. But $l+2 < n$ and so we can push \tilde{g} into Y by a general position argument similar to the one at the beginning of this proof. Therefore, $g \simeq 0$ in Y , i.e., $\pi_1 Y = 0$. To see that $\pi_1 \partial M_l^1 = 0$ we use induction ($\pi_1 \partial B = 0$ by an analogous argument). Clearly, $\pi_1 \partial M_0^1 = \pi_1 \partial M_1^1 = \pi_1 S^{n-1} = 0$, because $n > 2$ and $\pi_1 M = 0$. Now ∂M_j^1 is obtained from ∂M_{j-1}^1 by cutting out a certain number of disjoint $S^{j-1} \times D^{n-j}$ and adding an equal number of $D^j \times S^{n-j-1}$. If $\pi_1 \partial M_{j-1}^1 = 0$, for $1 < j < n$, then Van Kampen's theorem [2, Theorem 6.4.3] shows that $\pi_1 \partial M_j^1 = 0$. This proves that $\pi_1 \partial M_j^1 = 0$ for $0 \leq j \leq n$, and finishes the proof of part (a).

(b) The method of part (a) breaks down only at one point; namely, we do not know that $\partial_{l+1} = 0 = \partial'_{l+1}$. Hence we must give a special argument to make f into a homotopy equivalence. Specifically, assume that $F(M_l^1) \subset \text{int } M_l^1$ and let $f \equiv F| M_l^1 : M_l^1 \rightarrow M_l^1$. Again, f_* is an isomorphism for $j < l$. Consider

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_{l+1}(M, M_l^1) & \xrightarrow{\partial_{l+1}} & H_l M_l^1 & \xrightarrow{i_*} & H_l M \longrightarrow 0 \\ & & \downarrow F_* & & \downarrow f_* & & \downarrow F_* \\ \cdots & \longrightarrow & H_{l+1}(M, M_l^1) & \xrightarrow{\partial_{l+1}} & H_l M_l^1 & \xrightarrow{i_*} & H_l M \longrightarrow 0. \end{array}$$

Write

$$\begin{aligned} H_l M_l^1 &= \bigoplus \sum_{s=1}^{p(l)} \{x_s\} \oplus \sum_{s=1}^{q(l)} \{y_s\}, \\ H_l M &= \bigoplus \sum_{s=1}^{p(l)} \{x'_s\} \oplus \sum_{s=1}^{q(l)} \{y'_s\}, \quad k_s \equiv \text{order of } y'_s. \end{aligned}$$

Note that $\ker f_* \subset \ker i_* \subset \sum_{s=1}^{q(l)} \{y_s\}$ and $f_*(\sum_{s=1}^{q(l)} \{y_s\}) \subset \sum_{s=1}^{q(l)} \{y_s\}$. Let $f_* y_s = \sum_{j=1}^{q(l)} c_{sj} y'_j$, and set $C \equiv (c_{sj})$. We may assume without loss of generality that $i_* x_s = x'_s$ and $i_* y_s = y'_s$. Then our hypothesis shows that $y'_s = i_* y_s = F_* i_* y_s = i_* f_* y_s = i_* \sum_{j=1}^{q(l)} c_{sj} y'_j = \sum_{j=1}^{q(l)} c_{sj} y'_j$, i.e., $c_{sj} \equiv \delta_{sj}(k_j)$. Our task will be to modify F so that $c_{sj} = \delta_{sj}$.

Put $c_{sj} = \delta_{sj} + a_{sj} k_j$, and let

$$\begin{aligned} M_j^1 &= M_{j-1}^1 \cup_{\psi_{1,j}} (D^j \times D^{n-j}) \cup \cdots \cup_{\psi_{t_j,j}} (D^j \times D^{n-j}), \\ H_j(M_j^1, M_{j-1}^1) &= \bigoplus \sum_{s=1}^{t_j} \{z_s^j\}, \end{aligned}$$

where $t_j = p(j) + q(j) + q(j-1)$ and z_s^j corresponds to $(D^j \times 0)_{\psi_{s,j}}$. Consider

$$\begin{array}{ccc} H_{l+1}(M_{l+1}^1, M_l^1) & \longrightarrow & H_l(M_l^1, M_{l-1}^1) \\ & \searrow \partial' & \nearrow i'_* \\ & H_l M_l^1 & \end{array}$$

We may assume (see [6, §6]) that $i'_* x_s = z_s^l$, $i'_* y_s = z_{s+p(l)}^l$, and $\partial' z_{s+p(l)+1}^{l+1} + \cdots + z_{s+p(l)+q(l+1)}^{l+1} = k_s y_s$.

Let

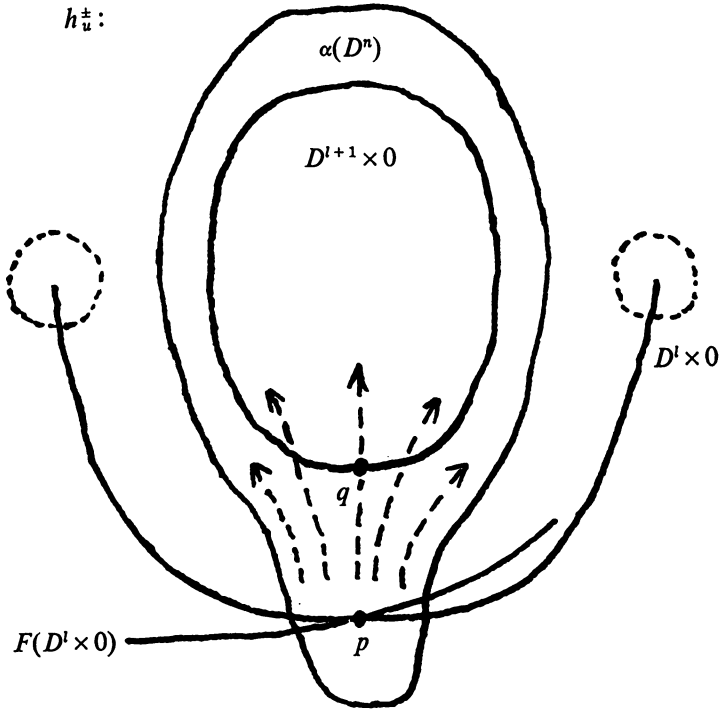
$$T^{s,t} \equiv F[(D^l \times 0)_{\psi_{s+p(l),l}}] \cap (0 \times D^{n-l})_{\psi_{t+p(l),l}},$$

$$T_0^{s,t} \equiv (\partial D^{l+1} \times 0)_{\psi_{s+p(l+1)+q(l+1),l+1}} \cap (0 \times \partial D^{n-l})_{\psi_{t+p(l),l}}.$$

By a general position argument we can make $T^{s,t}$ and $T_0^{s,t}$ consist of only a finite number of points. In fact, using methods of Whitney (e.g., see [1, §7]), we may suppose that $|T^{s,t}| = |c_{st}|$, $|T_0^{s,s}| = k_s$, and $T_0^{s,t} = \emptyset$, for $s \neq t$ ($|X|$ denotes the cardinality of X).

We shall now describe isotopies h_u^\pm , with $h_0^\pm = \text{identity} = h_u^\pm|_A$, so that if we let $F' = h_1^\pm F$ and define $C' = (c'_{st})$ similarly to C , then $c'_{s_1 t_1} = c_{s_1 t_1}$, for $s_1 \neq s$, $t_1 \neq t$, and $c'_{st} = c_{st} \pm k_t$. Using such isotopies it should be clear how to deform F so that $c_{st} = \delta_{st}$.

THE CONSTRUCTION OF h_u^\pm . Let $p \in T^{s,t}$ and $q \in T_0^{t,t}$. Let $\alpha: D^n \rightarrow M^n$ be an



imbedding with the following properties:

- (1) $(D^{l+1} \times 0)_{\psi_{t+p(l+1)+q(l+1),l+1}} \subset \text{int } \alpha(D_0^{l+1})$, where

$$D_0^{l+1} \equiv \{(x_1, \dots, x_{l+1}) \in D^{l+1} \mid x_{l+1} \geq 0\},$$

- (2) $\alpha(D^{l+1}) \cap F[(D^l \times 0)_{\psi_{s+p(l),l}}] = \alpha(D^l)$,

- (3) If $U \equiv F(M_1^l) \cap \alpha(D^n)$, then U is a ball neighborhood of p with $U \subset (D^l \times D^{n-l})_{\psi_{t+p(l),l}}$ and $U \cap T^{s,t} = \{p\}$,

(4) $\alpha(D^n) \cap (D^l \times D^{n-l})_{\psi_{f+p(l),l}} = \emptyset$, for $j \neq l$,

(5) $\alpha(\partial D^{l+1}) \subset \text{int } M_l^1$.

h_u^+ is then the isotopy which pushes $\alpha(D^l)$ to $\alpha(D_+^l)$, where $D_+^l \equiv D_0^{l+1} \cap S^l$. h_u^- is defined in essentially the same manner, except that one must give $\alpha(D^l)$ first a small twist in order to change the orientation.

Thus, assume now that $c_{st} = \delta_{st}$. By previous observations we get that f_* is a monomorphism. A little diagram chasing shows that it is also onto, since it is already onto $\sum_{s=1}^{q(l)} \{y_s\}$. Therefore f_* is an isomorphism and f is a homotopy equivalence. The rest of the proof goes as in part (a). This finishes the lemma. \square

We are now ready to prove our theorems, keeping the notation of the lemma.

Proof of Theorem 1. By [4] we can find a diffeomorphism H_{i_1} of M isotopic to the identity and $H_{i_1}(M_0^1) = M_0^2$. Suppose inductively that we have defined a sequence of diffeomorphisms, $\{H_{i_s} \mid s = 1, 2, \dots, k\}$, of M satisfying

(1) $H_{i_s} \approx H_{i_{s-1}}$,

(2) $H_{i_s} \mid M_{i_{s-1}}^1 = H_{i_{s-1}} \mid M_{i_{s-1}}^1$,

(3) $H_{i_s}(M_{i_s}^1) = M_{i_s}^2$.

Let $\eta_3 = \eta_1 H_{i_k}^{-1}$ and $M_i^3 \equiv \eta_3^{-1}[0, i+1/2]$. Then $M_{i_k}^3 = M_{i_k}^2$. Assume that $1 \leq i_k < i_{k+1} < n-2$. Applying part (a) of the lemma to η_3 and η_2 we obtain a diffeomorphism h of M with $h \approx \text{identity}$, $h \mid M_{i_k}^2 = \text{identity}$, and $h(M_{i_{k+1}}^3) = M_{i_{k+1}}^2$. Define $H_{i_{k+1}} \equiv h H_{i_k}$. It is easily checked that $H_{i_{k+1}}$ satisfies (1)–(3).

If $n-2 \leq i_{k+1} \leq n$, let $\eta_j^* \equiv n - \eta_j$ and $M_i^{j*} \equiv \eta_j^{*-1}[0, i+1/2]$. (Observe that η_j^* is also a minimal nice Morse function.) Using [4] we can find a diffeomorphism h_1 of M such that $h_1 \approx \text{identity}$, $h_1 \mid M_{i_k}^2 = \text{identity}$, and $h_1(M_0^{3*}) = M_0^{2*}$. If $i_{k+1} = n-1$, define $H_{i_{k+1}} = h_1 H_{i_k}$. If $i_{k+1} = n-2$, apply part (a) of the lemma to $\eta_3^* h_1^{-1}$ and η_2^* with $l=1$ to get a diffeomorphism h_2 of M with the property that $h_2 \approx \text{identity}$, $h_2 \mid M_0^{2*} = \text{identity}$, and $h_2(h_1(M_1^{3*})) = M_1^{2*}$. Furthermore, the proof of the lemma shows that we may suppose that $h_2 \mid M_{i_k}^2 = \text{identity}$. Define $H_{i_{k+1}} = h_2 h_1 H_{i_k}$. Finally, if $i_{k+1} = n$, define $H_{i_{k+1}} = H_{i_k}$. In all cases, $H_{i_{k+1}}$ satisfies (1)–(3). This finishes our inductive definition of the H_{i_s} and $H \equiv H_n$ satisfies the conclusion of the theorem. \square

Proof of Theorem 2. Let $\eta_2 \equiv \eta_1 F^{-1}$ and apply part (b) of the lemma in the proof of Theorem 1 instead of part (a) to obtain a diffeomorphism $H: M \rightarrow M$ such that $H \approx \text{identity}$ and $H(M_{i_s}^1) = M_{i_s}^2$. Let $F' = H^{-1}F$. \square

At first glance, one might be led to conjecture that the condition, $F_* \mid \text{torsion } H_1 M = \text{identity}$, in the lemma is unnecessary; however, the following example shows that it is not:

Let $M^8 \equiv \text{double of } (S^2 \times D^6 \cup_\alpha (D^3 \times D^5))$, where $\alpha: S^2 \times D^5 \rightarrow S^2 \times S^5$ is an imbedding with $[\alpha(S^2 \times 0)] = 5[S^2 \times 0] \in H_2(S^2 \times D^6)$. Let $\eta: M \rightarrow \mathbb{R}$ be a minimal nice Morse function such that $M_2 = S^2 \times D^6$ and $M_3 = S^2 \times D^6 \cup_\alpha (D^3 \times D^5)$, where $M_i \equiv M_i(\eta)$. Construct an imbedding $h: M_2 \rightarrow M_2$ such that $[h_2(S^2 \times 0)] = 3[S^2 \times 0] \in H_2 M_2$. One can extend h to an imbedding $M_2 \cup_\alpha (D^3 \times 0) \rightarrow M_3$. But $\pi_2 SO_5 = 0$, and so we can thicken the handle to get an imbedding $h: M_3 \rightarrow M_3$.

Now $H_2M_3 \approx H_2M \approx (a)$, where a has order 5, and $h_*(a) = 3a$ by construction. Therefore h is a homotopy equivalence and we may assume h is a diffeomorphism of M_3 . Let $F: M \rightarrow M$ be the double of h . Then $F_* | H_2M \neq \text{identity}$. However, if $F \approx F'$ and $F'(M_2) = M_2$, then we must have $F_* | H_2M = F'_* | H_2M = \pm \text{identity}$ because $H_2M_2 \approx \mathbb{Z}$. Thus (M, F) is the example that we wanted. In fact, this also shows that Theorem 1 is the best possible (set $\eta_1 = \eta$ and $\eta_2 = \eta F$).

Finally, although we just saw that Theorem 1 is the best possible, this still leaves open the interesting question whether, despite this, the manifolds $M_i(\eta_1)$ and $M_i(\eta_2)$ are diffeomorphic anyway.

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