TRIANGULATIONS OF THE 3-BALL WITH KNOTTED SPANNING 1-SIMPLEXES AND COLLAPSIBLE rTH DERIVED SUBDIVISIONS

BY
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It has been shown by R. H. Bing [1] that if K is a simplicial complex which triangulates a 3-ball, σ is a spanning 1-simplex of K (i.e. $\sigma \cap \partial K = \partial\sigma$) and $K^{(r)}$ simplicially collapses (where $K^{(r)}$ is an rth derived subdivision), then the bridge number of σ, br (σ), is less than or equal to 2^r+1. The proof is to be found in [3]. The following theorem shows that, in a sense, Bing's result is the best possible.

THEOREM. *Suppose that* κ *is a knot in* E^3 *and* br $(\kappa) \leq 2^r+1$. *Then there is a simplicial complex* K, *a triangulation* $\tau\colon |K| \to B^3$ *of a* 3-*ball, and a spanning* 1-*simplex* σ *of* K *such that*

(i) $K^{(r)}$ *simplicially collapses, and*

(ii) $\tau(\sigma)$ *has the same knot type as* κ(²).

REMARK. br (κ) is defined later. $\tau(\sigma)$ is said to have the same knot type as κ if, regarding B^3 as polyhedrally contained in E^3, and joining the end points of $\tau(\sigma)$ by an arc α in ∂B^3, $\tau(\sigma) \cup \alpha$ is a simple closed curve with the same knot type as κ. Then one can define bridge numbers of spanning arcs by br $(\sigma) =$ br $(\tau(\sigma)) =$ br (κ).

1. Introduction.

DEFINITIONS. Each polyhedral knot of S^1 in E^3 can, for some integer n, be represented as n straight linear arcs running from the top face of the unit 3-cube to its bottom face (and otherwise contained in the interior of the cube), together with n polyhedral arcs on the boundary of the cube. If κ is a knot of S^1 in E^3, then its *bridge number*, br (κ), is the smallest integer n for which such a representation is possible. If S^1 is a simple closed curve in a 3-ball B^3, S^1 is in *n-bridge position in* B^3 if there is a polyhedral homeomorphism of B^3 to the unit 3-cube sending S^1 to n straight spanning arcs of the cube and n arcs in its boundary, as described above. Schubert's paper [5] contains most of the fundamental work on bridge numbers. Note that a knot with bridge number one is always unknotted, but that there are many interesting knots with bridge number two (e.g. the trefoil and the four-knot).

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(²) This same result has been announced for $r=0$ by Hamstrom and Jerrard [4].

Throughout, the notation $X \searrow Y$ means that the polyhedron X polyhedrally collapses to a subpolyhedron Y. $K \overset{s}{\searrow} L$ means that the simplicial complex K simplicially collapses to a subcomplex L. Definitions of these concepts are to be found in [6].

The purpose of this paper is to prove the theorem stated above. The main idea of the proof is fairly simple although the details become a little involved. Figure 1 illustrates the principal idea when $r=0$. The theorem then says that given any knot with bridge number two, the 3-ball can be triangulated by a simplicially collapsible complex which contains a spanning 1-simplex knotted with the given knot type. Take the given knot S^1 in 2-bridge position in a cube C, with one of the arcs of $S^1 \cap \partial C$ in the top face of C, the other in the bottom face. Figure 1 shows the trefoil knot in this way. Remove from C the interior of a regular neighbourhood of the two spanning arcs, glue a second cube C' onto the bottom face of C (as shown) and remove from C' a neighbourhood of a standard (i.e. unknotted) U-shaped spanning arc of C', so that a knotted hole has now been bored out of $C \cup C'$. Insert a cylinder P to plug the hole, (see Figure 1), at the bottom of the U-shaped hole, to obtain a ball B. A straight arc σ in P from the left-hand face to the right-hand face is a spanning arc of B knotted in the required way. B collapses polyhedrally as follows. Collapse C' to its top face (less two discs), together with the boundary of the U-tube, P, and the disk D (see Figure 1 again). P can now be collapsed onto its two vertical disk faces, D_1 and D_2, say, plus $P \cap D$. D can now be removed. What remains is C, less two *standard* holes (as S^1 was in 2-bridge position in C), with a disk across an end of each hole (these disks are the boundaries of the "arms" of the U-tube together with D_1 and D_2) and this collapses. This polyhedral collapse can be triangulated, but the triangulation would (probably) not give a 1-simplex σ, as mentioned above, going straight across P. However, by a cone construction it is fairly simple to extend the triangulation of $\partial P - (\mathring{D}_1 \cup \mathring{D}_2)$ to a new triangulation of P which does have such a 1-simplex, so that the collapsing, in so far as it affects P, can still be performed in a simplicial way, and the remainder of the simplicial collapse is as before.

When $r>0$, the proof is similar, but one then has 2^r+1 tubes removed from C and 2^r tubes removed from C'. It is then expedient to have the "plug" P occupying most of the hole removed from $C \cup C'$. In the polyhedral collapsing, P is then collapsed onto 2^r+1 disks together with an arc.

The details of the proof will follow some preliminary lemmas.

2. **Preliminary results.**

LEMMA 1. *Let S^1 be a simple closed polyhedral curve in n-bridge position in a 3-ball B^3. Let $s_1, s_2, \ldots, s_n$ be the arcs of S^1 which span B^3 and for each $i=1, 2, \ldots, n$, let N_i be a regular neighbourhood of s_i in B^3 such that $N_i \cap N_j = \varnothing$ if $i \neq j$, and $N_i \cap \partial B^3$ is a pair of disks. Let D_i be one of these two disks. Then the closure of $(B^3 - \bigcup_{i=1}^n N_i) \cup \bigcup_{i=1}^n D_i$ collapses polyhedrally.*

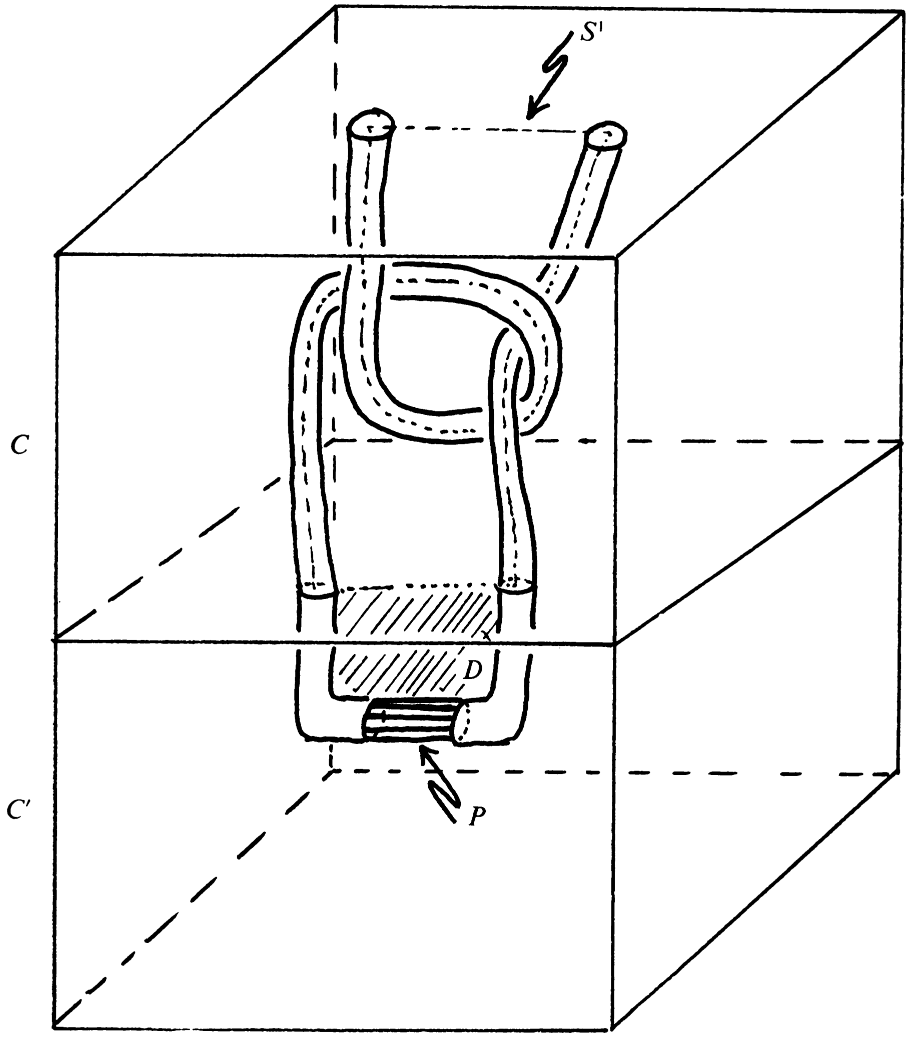

FIGURE 1

Proof. As S^1 is in n-bridge position it may be assumed that B^3 is the unit cube C, and that the s_i are straight spanning arcs of C. It may further be assumed, (after a homeomorphism of C), that the s_i are actually vertical, that the D_i lie in the bottom face, F^B, of the cube, and (by the uniqueness theorem for regular neighbourhoods) that $N_i = \pi^{-1} D_i$ where $\pi: C \to F^B$ is the vertical projection. Then clearly

$$\overline{\left(C - \bigcup_{i=1}^{n} N_i\right)} \cup \bigcup_{i=1}^{n} D_i \searrow F^B,$$

vertically, and $F^B \searrow 0$.

LEMMA 2. *Let D be a 2-simplex, r a nonnegative integer, and let $n = 2^r$. Suppose that $a \in \partial D$, $q \in D^\circ$ and that T is a simplicial subdivision of $\partial D \times [0, n]$ such that*

(i) *a subdivision of* $\{a\}\times[0, n]$ *is a subcomplex of* T,

(ii) *for each integer* i, $0\leqq i\leqq n$ *a subdivision of* $\partial D\times\{i\}$ *is a subcomplex of* T.

Then there is a simplicial subdivision L *of* $D\times[0, n]$ *extending* T, *and an* rth *derived subdivision* $L^{(r)}$ *of* L *such that*

(i) $q\times[0, n]$ *is a* 1-*simplex of* L,

(ii) *for each integer* i, $0\leqq i\leqq n$, *a subdivision of* $D\times\{i\}$ *is a subcomplex of* $L^{(r)}$,

(iii) $L^{(r)}\overset{s}{\searrow}(\{a\}\times[0, n])\cup(\bigcup_{i=0}^{n} D\times\{i\})$.

Proof. We subdivide $D\times[0, n]$ as follows: First we subdivide $D\times\{0\}$ by joining $q\times 0$ to the given subdivision of $\partial D\times\{0\}$. Then we subdivide $D\times[0, n]$ by joining the subdivision of $D\times\{0\}\cup\partial D\times[0, n]$ to the point $q\times n$. In this way we arrive at a subdivision L of $D\times[0, n]$. Notice that $\{q\}\times[0, n]$ is a 1-simplex of L. We now take a 1st derived subdivision of L, $L^{(1)}$, by starring each simplex at an interior point with the restriction that if $\gamma\in L$ and $\operatorname{int}\gamma\cap D\times\{n/2\}\neq\varnothing$, then we star γ from a point of $D\times\{n/2\}$. Hence in $L^{(1)}$, a subdivision of $D\times\{n/2\}$ is a subcomplex. Now it follows from a theorem of Chillingworth [2], applied to both $D\times[0, n/2]$ and $D\times[n/2, n]$, that $L^{(1)}$ collapses simplicially to $D\times\{0\}\cup D\times\{n/2\}\cup D\times\{1\}\cup\{a\}\times[0, n]$. Notice that if a simplex of $L^{(1)}$ intersects $D\times\{n/2\}$ in an interior point of the simplex, then that simplex lies in $D\times\{n/2\}$. We now take a 1st derived subdivision $L^{(2)}$ of $L^{(1)}$ by starring each simplex of $L^{(1)}$ at an interior point, with the restriction that if $\gamma\in L^{(1)}$ and $\operatorname{int}\gamma\cap D\times\{n/4\}\neq\varnothing$, then we star γ from a point of $D\times\{n/4\}$ and if $\operatorname{int}\gamma\cap D\times\{3n/4\}\neq\varnothing$, then we star γ from a point of $D\times\{3n/4\}$. Notice that no simplex of L^1 intersects both of $D\times\{n/4\}$ and $D\times\{3n/4\}$ in an interior point of the simplex. Now a subdivision of each of $D\times\{jn/4\}$, $0\leqq j\leqq 4$, is a subcomplex of L^2 and it follows from [2] that $L^{(2)}$ collapses simplicially to $\{a\}\times[0, n]\cup[\bigcup_{j=0}^{4} D\times\{jn/4\}]$. Continuing this process, we arrive at an rth derived $L^{(r)}$ of L which has a subdivision of each $D\times\{i\}$, $0\leqq i\leqq n$, as a subcomplex and which collapses simplicially to $\{a\}\times[0, n]\cup[\bigcup_{i=0}^{n} D\times\{i\}]$. This establishes the lemma.

Let D be a 2-simplex, a be a point of ∂D, and let D^+ denote D with a collar attached to ∂D; i.e. $D^+=D\cup(\partial D\times I)$. If $p\in\partial D$ we identify p with $(p, 0)\in\partial D\times I$. Let b denote the point $(a, 1)$ of $\partial D\times I$, and let α denote the arc $\{a\}\times I\subset\partial D\times I$. Throughout we let r be a nonnegative integer and let $n=2^r$. Now it is clear that $D\times[0, n]\searrow X$, where $X=(a\times[0, n])\cup[\bigcup_{i=0}^{n}(D\times i)]$. Now let $Y\subset D^+$ denote

$$(b\times[0, n])\cup\left[\bigcup_{i=0}^{n}(D^+\times i)\right]\cup\left[\bigcup_{i=0}^{n-1}(\partial D^+\times[i, i+1/2])\right].$$

See Figure 2.

LEMMA 3. $D^+\times[0, n]\searrow[(D\times[0, n])\cup(\alpha\times[0, n])\cup Y]\searrow[(\alpha\times[0, n])\cup Y]\searrow Y$.

Proof. For each integer i, $0\leqq i\leqq n-1$, we first collapse $D^+\times[i, i+1]$ onto $(D^+\times[i, i+1/2])\cup[(D\cup\alpha)\times[i+1/2, i+1]]\cup(D^+\times(i+1))$. Now collapsing in

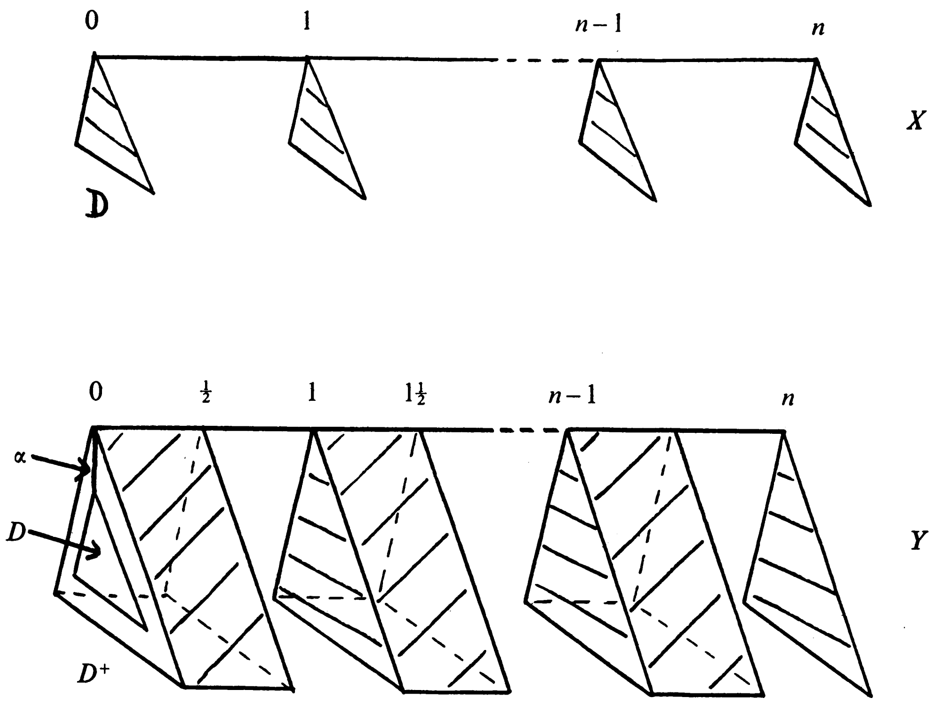

FIGURE 2

the cylinders, $D^+ \times [i, i+1/2]$ collapses to $(D^+ \times i) \cup [(D \cup \alpha \cup \partial D^+) \times [i, i+1/2]]$. Combining the above collapses we have

$$D^+ \times [0, n] \searrow (D \times [0, n]) \cup (\alpha \times [0, n]) \cup Y.$$

Now the second of the required collapses is induced by the collapse of $D \times [0, n]$ onto X, and the third follows because $(\alpha \times [0, n]) \searrow [\bigcup_{i=0}^{n} (\alpha \times i)] \cup (b \times [0, n])$. This establishes Lemma 3.

3. **Proof of the theorem.** Let κ be a knot of S^1 in E^3 with br $(\kappa) \leqq n+1$, where $n=2^r$. Let C be the unit 3-dimensional cube in E^3. Then κ may be regarded as being in $(n+1)$-bridge position in C, κ being the union of polyhedral spanning arcs $s_0, s_1, \ldots, s_n$ of C, together with polyhedral arcs $b_0, b_1, \ldots, b_n$ in ∂C. By a proper choice of notation we may assume that the arcs occur in the order $b_0, s_0, b_1, s_1, \ldots, b_n, s_n$ on κ. For each i, $0 \leqq i \leqq n$, let P_i be the point $b_i \cap s_i$, and let $P_{i+1/2}$ be the point $s_i \cap b_{i+1}$. An adjustment by a homeomorphism will insure that b_0 is in the top face of C, and that each of $b_1, b_2, \ldots, b_n$ is in the bottom face of C. Let C' be a second 3-dimensional cube, whose top face agrees with the bottom face of C.

Now for each i, $1 \leqq i \leqq n$, let E_i be a polyhedral disk in C' such that (1) $E_i \cap \partial C' = \partial E_i \cap \partial C' = b_i$, and (2) if $i \neq j$, $E_i \cap E_j = \varnothing$. Now for each i, $1 \leqq i \leqq n$, let c_i be the closure of $\partial E_i - b_i$. Figure 3 illustrates this notation.

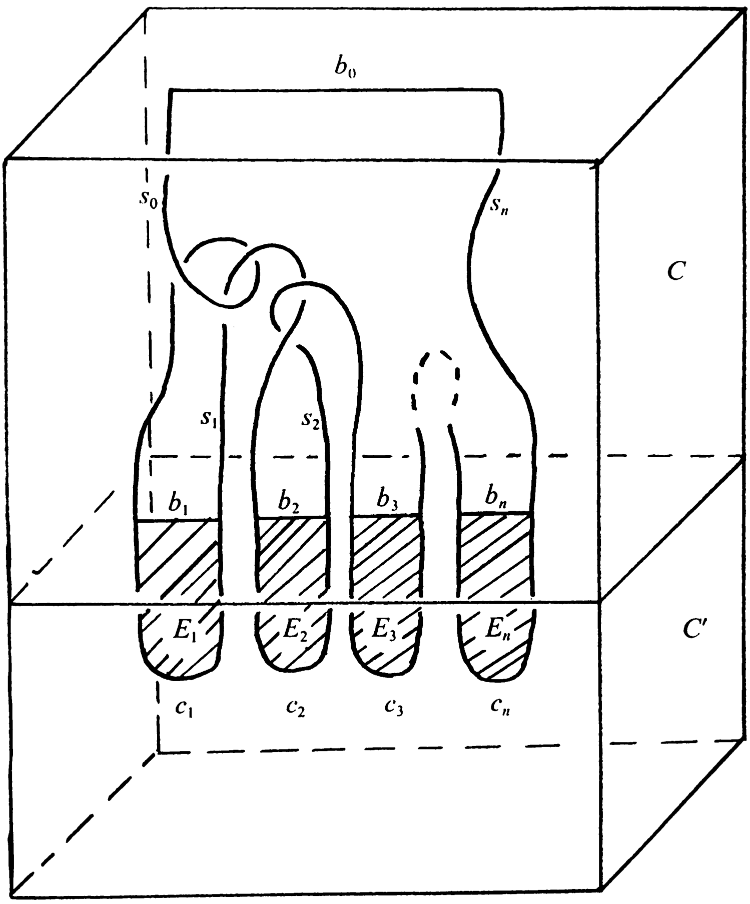

FIGURE 3

We now choose a triangulation of $C \cup C'$ with respect to which all of the polyhedra which have previously been defined become subcomplexes. If Z_1 is such a subcomplex, and Z_2 is a subcomplex of the second derived subdivision of this triangulation, with $|Z_1| \subset |Z_2|$, let $N(Z_1, Z_2)$ denote the polyhedron underlying the simplicial neighbourhood of $|Z_1|$ in Z_2. This particular triangulation will be used only to define various regular neighbourhoods.

Let M be the closure in $C \cup C'$ of $(C \cup C') - N(s_n, C)$. Then M is a polyhedral 3-ball. Let T be $N(s_0 \cup c_1 \cup s_1 \cup c_2 \cup \cdots \cup c_n, M)$, and for each i, $0 \leqq i \leqq n$, let D_i be $N(P_i, \partial C)$, and let $D_{i+1/2}$ be $N(P_{i+1/2}, \partial C)$. Recall the disk D^+ of Lemma 3, with $D \subset D^+$. Now there is a polyhedral homeomorphism $H: D^+ \times [0, n] \rightarrow T$ such that (1) if $0 \leqq i \leqq n$, $H(D^+ \times i) = D_i$, (2) if $0 \leqq i \leqq n-1$, $H(D^+ \times \{i+1/2\}) = D_{i+1/2}$, and (3) if $1 \leqq i \leqq n$, $H(b \times [i-1/2, i]) = T \cap E_i$. Now for each i, $1 \leqq i \leqq n$, let c_i' denote $T \cap E_i$, and let F_i be the closure of $E_i - T$. Notice that F_i is a disk whose boundary is $c_i' \cup b_i'$ where b_i' is a subarc of b_i.

We now describe a polyhedral collapse of M. Let C^- denote the closure in C of $C-\bigcup_{i=0}^n N(s_i, C)$.

First we notice that we have the collapse $M \searrow C^- \cup T \cup [\bigcup_{i=1}^n F_i]$. Now notice that T intersects $C^- \cup [\bigcup_{i=1}^n F_i] \cup [\bigcup_{i=1}^n D_i]$ in $H(Y)$. (Recall the set Y from Lemma 3.) Now the polyhedral collapse $D^+ \times [0, n] \searrow Y$, given by Lemma 3 can be transferred under the polyhedral homeomorphism H so as to obtain the collapse $C^- \cup T \cup [\bigcup_{i=1}^n F_i] \searrow C^- \cup [\bigcup_{i=0}^n D_i] \cup [\bigcup_{i=1}^n F_i]$. Each F_i now has the free edge c_i', and so F_i can be collapsed onto b_i'. Combining these collapses we have $M \searrow C^- \cup [\bigcup_{i=0}^n D_i]$, which collapses by Lemma 1.

Our task now is to triangulate M so that the collapse described above can be carried out simplicially in an rth derived of the triangulation, and so that the triangulation has a spanning 1-simplex which is of the same knot type as κ. To this end, let K be a simplicial complex and $\tau\colon |K| \to M$ be a triangulation such that each subpolyhedron of M already mentioned in the proof, and $H(D \times [0, n])$ is the image under τ of a subcomplex of K. After a subdivision of K we may assume that $H^{-1}\tau$ maps the subcomplex $\tau^{-1}H(D \times [0, n])$ isomorphically onto some subdivision of the convex linear cell structure on $D \times [0, n]$. We may also assume, by standard results of [6] (after further subdivision), that τ triangulates the polyhedral collapsing process described above; i.e. if $X_i \overset{e}{\searrow} X_{i+1}$ is an elementary collapse in the polyhedral collapsing sequence, then there are subcomplexes K_i and K_{i+1} of K such that $\tau(K_i) = X_i$, $\tau(K_{i+1}) = X_{i+1}$, and $K_i \overset{s}{\searrow} K_{i+1}$.

Now let L be a simplicial complex subdividing $D \times [0, n]$ which extends the subdivision of $\partial D \times [0, n]$ that is isomorphic under $\tau^{-1}H$ to a subcomplex of K, and which has the properties in the conclusion of Lemma 2. A new triangulation of M is now given by $\tau \mid |K_1| \to M$ where K_1 is the subcomplex of K such that $|K_1|$ is the inverse image under τ of the closure of $M - H(D \times [0, n])$, together with $H\colon |L| \to H(D \times [0, n])$. Notice that this triangulation contains the 1-simplex $H(q \times [0, n])$ and that this 1-simplex has the same knot type as κ.

The final step in the argument is to show that the rth derived of this triangulation collapses simplicially. Now, since K collapses, $K^{(r)}$, an rth derived of K, collapses, [2] or [6], and hence $\tau|K^r| \to M$ triangulates the previously described polyhedral collapse of M. Hence the polyhedral collapse can be followed simplicially in $K_1^{(r)}$ until we reach simplexes in $H(D \times [0, n])$. However, at this stage, the polyhedral collapse collapses $H(D \times [0, n])$ onto $H(X)$. But the complex L has been chosen so that $L^{(r)}$ has a subcomplex L_1 such that $|L_1| = X$ and $L^{(r)} \overset{s}{\searrow} L_1$. (Note: The sub division $K_1^{(r)}$ of K_1 can be chosen to be compatible with $L^{(r)}$.) Hence in our triangulation of M we may follow the polyhedral collapse as far as $M \searrow C^- \cup [\bigcup_{i=0}^n D_i]$. C^- is triangulated by a subcomplex K_2 of $K_1^{(r)}$ and the disks D_i by subcomplexes of $L^{(r)}$. We may now collapse K_2 simplicially to a subcomplex K_3 which is 2-dimensional and such that $\tau(K_3) \cup [\bigcup_{i=0}^n D_i]$ is polyhedrally collapsible. But any polyhedrally collapsible 2-complex is simplicially collapsible and so the triangulation on the $\bigcup_{i=0}^n D_i$ is irrelevant. Hence an rth derived of the triangulation we have described is simplicially collapsible. This establishes the theorem.

References

1. R. H. Bing, *Some aspects of the topology of 3-manifolds related to the Poincaré conjecture*, Vol. 2, Lectures on modern mathematics, Wiley, New York, 1964.

2. D. R. J. Chillingworth, *Collapsing three-dimensional convex polyhedra*, Proc. Cambridge Philos. Soc. **63** (1967), 353–357.

3. R. Goodrick, *Non-simplicially collapsible triangulations of I^n*, Proc. Cambridge Philos. Soc. **64** (1968), 31–36.

4. M. E. Hamstrom and R. P. Jerrard, *Collapsing a triangulation of a "knotted" cell*, Notices Amer. Math. Soc. **15** (1968), 368.

5. H. Schubert, *Uber eine numerische Knoteninvariante*, Math. Z. **61** (1954), 245–288.

6. E. C. Zeeman, Seminar on combinatorial topology, Mimeographed notes, Inst. Hautes Études Sci., Paris, 1963–1966.

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