ORTHOGONAL REPRESENTATIONS OF ALGEBRAIC GROUPS

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Introduction. Let G_1 and G be connected semisimple algebraic groups defined over a field K of characteristic zero and assume that there is an isomorphism f of G_1 onto G which is defined over $\bar{K}$, the algebraic closure of K. If $\rho: G \to GL(V)$ is an absolutely irreducible (finite-dimensional) representation of G defined over K, then $\rho \circ f$ is an absolutely irreducible representation of G_1 defined over $\bar{K}$. Satake [7, p. 230] has shown that there is a field K_1 which is a finite extension of K, a (unique) central simple division algebra $K\#$ defined over K_1, a finite-dimensional right vector space V_1 over $K\#$, and a K_1-homomorphism $\rho_1: G_1 \to GL(V_1/K\#)$ (the group of all nonsingular $K\#$-linear endomorphisms of V_1) such that $(\rho \circ f)(g) = \theta_1(\rho_1(g))$ for all $g \in G_1$ where θ_1 is a unique absolutely irreducible representation of End $(V_1/K\#)$ (the algebra of all $K\#$-linear endomorphisms of V_1) onto End (V).

In this paper we are interested in the case where $K = K_1$ and where there are invariant forms on V and V_1. More precisely, we state the following two problems.

PROBLEM 1. Assume that $K\# = K$ and that there are invariant bilinear forms B on V and B_1 on V_1 which are defined over K. What is the relationship between these two forms over K? Of course, if B is alternating, so is B_1 and both are determined by dim V=dim V_1. Hence, we shall always take B and B_1 to be symmetric.

PROBLEM 2. Assume that $K\#$ is a nontrivial division algebra over K (i.e., $K\# \neq K$) and that there is an invariant bilinear form B on V and an invariant ε-hermitian form F ($\varepsilon = +1$ or -1) on V_1 both of which are defined over K. What is the relationship between these two forms over K?

We are especially interested in the case $K = Q_{\mathfrak{p}}$, a $\mathfrak{p}$-adic field. (In a future paper, we shall discuss the case $K = \boldsymbol{R}$.) Here, some simplifications are immediately available. In Problem 2, it can be shown [7, p. 232] that $K\#$ has an involution of the first kind; but over $Q_{\mathfrak{p}}$, it is known that the only such division algebra is the quaternion division algebra. Furthermore, it is known that a hermitian form on a finite-dimensional vector space over a quaternion division algebra defined over $Q_{\mathfrak{p}}$ is determined only by the dimension of the vector space. Therefore, in Problem 2 we shall always take F to be skew-hermitian; in the case where $K\#$ is a quaterion division algebra, this means that the form B is symmetric [7, p. 233].

If W is a vector space defined over K and if S is a symmetric form on W which is also defined over K, then three invariants can be associated with the pair (W, S),

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namely, (1) the dimension of W, dim W, (2) the discriminant of S, $\Delta(S)$, and (3) the Hasse invariant, $c(S)$. In answering Problem 1, we describe these three invariants of B_1 in terms of those of B. Over $Q_{\mathfrak{p}}$, these invariants completely describe a symmetric form.

Similarly, in Problem 2 we deal with two invariants of the space (V_1, F), namely, (1) the dimension of V_1 (over $K\#$), dim V_1, and (2) the discriminant of F, $\delta(F)$. We describe these invariants in terms of the invariants of B. Over $Q_{\mathfrak{p}}$, the two invariants above completely describe a skew-hermitian form.

The answers to the questions above fall into two main parts. In Part I, we assume that the isomorphism $f: G_1 \to G$ is of inner type, i.e., for each $\sigma \in \Gamma$ (the Galois group of $\bar{K}$ over K), $f^{-\sigma} \circ f = I_{g_\sigma}$ where $g_\sigma \in G_1$ and $I_{g_\sigma}(g) = g_\sigma g g_\sigma^{-1}$ for all $g \in G_1$. (By $f^{-\sigma}$, we shall always mean $(f^{-1})^\sigma$.)

For absolutely simple groups G_1, it is well known that there is a Chevalley group G defined over K and an isomorphism $f: G_1 \to G$ defined over $\bar{K}$ of inner type, except possibly when G_1 is of type A_n, D_n, or E_6. These last three cases are discussed in Part II.

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PART I

1.1. **The standard situation.** Throughout this part, we shall assume that f is of *inner type*, i.e. $f^{-\sigma} \circ f = I_{g_\sigma}$ for each $\sigma \in \Gamma$. The elements g_σ in G_1 are determined modulo the center of G_1, $Z(G_1)$, and so for $\sigma, \tau \in \Gamma$, the element $c_{\sigma,\tau} = g_\sigma^\tau g_\tau g_{\sigma\tau}^{-1}$ are in $Z(G_1)$. It follows that the cohomology class $(c_{\sigma,\tau})$ of the 2-cocycle $c_{\sigma,\tau}$ of Γ in $Z(G_1)$ is independent of the choice of elements g_σ. This 2-cocycle will play an important role in what follows.

Let $\rho: G \to SO(V, B)$ be an absolutely irreducible orthogonal representation defined over K and assume that B is also defined over K. In general, such a representation will be denoted by the triple (V, ρ, B) and will be called an *orthogonal representation of G defined over K*. Then $\rho \circ f$ is an orthogonal representation of G_1 defined over $\bar{K}$ and, setting $A_\sigma = (\rho \circ f)(g_\sigma^{-1})$, we have that for each $\sigma \in \Gamma$

$$(\rho \circ f)^\sigma(g) = A_\sigma(\rho \circ f)(g)A_\sigma^{-1} \tag{1}$$

for all $g \in G_1$. Also, by definition of A_σ and (1), it follows that

$$A_\sigma^\tau A_\tau = (\rho \circ f)(c_{\sigma,\tau}^{-1})A_{\sigma\tau} \tag{2}$$

for all $\sigma, \tau \in \Gamma$. The continuous 2-cocycle $(\rho \circ f)(c_{\sigma,\tau})$ defines $K\#$ as a normal division algebra if we require that $c(K\#) \sim ((\rho \circ f)(c_{\sigma,\tau}))$ [7, p. 227].

1.2. **Problem 1.** Our concern in this section is the case where $((\rho \circ f)(c_{\sigma,\tau})) \sim 1$. As we shall see, this is the case of Problem 1. However, before proving the theorem describing completely this situation, we need two lemmas.

LEMMA I.1. *Assume that* $((\rho \circ f)(c_{\sigma,\tau})) \sim 1$. *Then there exist elements* h_σ *in* G_1 *such that* $h_\sigma \equiv g_\sigma \bmod Z(G_1)$ *and* $(\rho \circ f)(h_{\sigma,\tau}^{-1} h_\sigma^\tau h_\tau) = 1$ *for all* $\sigma, \tau \in \Gamma$.

Proof. We set $d_{\sigma,\tau} = (\rho \circ f)(c_{\sigma,\tau})$ for all $\sigma, \tau \in \Gamma$. Then, as is well known, since $d_{\sigma,\tau}$ is a 2-cocycle of Γ in $\{+1, -1\}$ which is equivalent to 1, there exist elements a_σ in $\{+1, -1\}$ for each $\sigma \in \Gamma$ such that $d_{\sigma,\tau} = a_\sigma^\tau a_\tau a_{\sigma,\tau}^{-1}$.

If dim $V \equiv 1$ (2), it is immediate that the elements $d_{\sigma,\tau}$ are always 1 as can be seen by taking determinants of both sides of (2). The case where dim $V \equiv 0$ (2) is harder; however, if $d_{\sigma,\tau}$ is always 1 then there is nothing to prove. Therefore, we may assume that there is an element $z \in Z(G_1)$ such that $(\rho \circ f)(z) = -1$. In particular, for each $\sigma \in \Gamma$, there is an element $z_\sigma \in Z(G_1)$ such that $(\rho \circ f)(z_\sigma) = a_\sigma$. Using these z_σ, we define h_σ to be $g_\sigma z_\sigma$. It is easy to see that these h_σ satisfy the conditions above and so this lemma is proved.

From now on, we shall assume that the g_σ are chosen so that $(\rho \circ f)(c_{\sigma,\tau}) = 1$ for all $\sigma, \tau \in \Gamma$. Actually, in practice this choice is frequently trivial, for in many cases $(\rho \circ f)(Z(G_1)) = \{1\}$. Also, we shall assume that G_1 is simply connected. This assumption will be removed following the proof of Theorem I.1.

Denote the "spin group" of B by Spin (B) and let π be the canonical mapping from Spin (B) onto $SO(V, B)$. It is known that π is defined over K and that its kernel is $\{+1, -1\}$. Since G_1 is simply connected, there is a (polynomial) map $\rho_s: G_1 \to \text{Spin}(B)$ such that $\pi \circ \rho_s = \rho \circ f$. We define elements $\bar{A}_\sigma \in \text{Spin}(B)$ by $\bar{A}_\sigma = \rho_s(g_\sigma^{-1})$. Then $\pi(\bar{A}_\sigma) = A_\sigma$ and the system $\{\bar{A}_\sigma\}$ satisfies the relation $\bar{A}_\sigma^\tau \bar{A}_\tau = e_{\sigma,\tau} \bar{A}_{\sigma\tau}$ where each $e_{\sigma,\tau}$ is $+1$ or -1.

LEMMA I.2. *Let* $\rho_s: G_1 \to \text{Spin}(B)$ *be such that* $\pi \circ \rho_s = \rho \circ f$ *and assume that each* $(\rho \circ f)(c_{\sigma,\tau}) = 1$. *Then the* $e_{\sigma,\tau}$ *above are given as follows*: $e_{\sigma,\tau} = \rho_s(c_{\sigma,\tau})$.

Proof. For each $\sigma \in \Gamma$, we have $\pi \circ \rho_s^\sigma = (\rho \circ f)^\sigma = A_\sigma (\rho \circ f) A_\sigma^{-1} = \pi(\bar{A}_\sigma \rho_s \bar{A}_\sigma^{-1})$. So $\rho_s^\sigma(g) = e(g) \bar{A}_\sigma \rho_s(g) \bar{A}_\sigma^{-1}$ where $e(g) = +1$ or -1. But, since G_1 is connected, $e(g)$ is always 1 and so $\rho_s^\sigma(g) = \bar{A}_\sigma \rho_s(g) \bar{A}_\sigma^{-1}$ for all $g \in G_1$. Using this fact, the lemma follows immediately.

Before stating Theorem I.1, we recall a few definitions about quadratic spaces (W, S) defined over K. Assume that $n = \dim W$ and that in diagonal form S is diag $(a_1, \ldots, a_n)$ where $a_i \in K^*$ (the multiplicative group of nonzero elements in K). Then one puts $\Delta(S) = (-1)^{n(n-1)/2} a_1 \cdots a_n \bmod (K^*)^2$. The invariant $c(S)$ is the cohomology class of a certain 2-cocycle of Γ in $\bar{K}^*$ and is defined in the proof of Theorem I.1. It can be shown [4] that the invariants dim, Δ, and c are enough to determine S if K is a nonarchimedean local field.

THEOREM I.1. *Let* G_1 *and* G *be simply connected algebraic groups defined over* K (char $K = 0$) *and assume that there is a* $\bar{K}$*-isomorphism* $f: G_1 \to G$ *such that* $f^{-\sigma} \circ f = I_{g_\sigma}$ *for each* $\sigma \in \Gamma$. *Define elements* $c_{\sigma,\tau} \in Z(G_1)$ *by setting* $c_{\sigma,\tau} = g_{\sigma,\tau}^{-1} g_\sigma^\tau g_\tau$. *Let* (V, ρ, B) *be an orthogonal representation of* G *defined over* K *and assume that* *each* $(\rho \circ f)(c_{\sigma,\tau})$ *is* 1. *Then there is an orthogonal representation* (V_1, ρ_1, B_1) *of* G_1 *defined over K such that* $\rho_1 \sim \rho \circ f$ *and* B_1 *is related to B as follows*: dim $V_1=$dim V, $\Delta(B_1)=\Delta(B)$, *and* $c(B_1)=c(B)(\rho_s(c_{\sigma,\tau}))$ *where* ρ_s: $G_1 \to$ Spin (B) *and* $\pi \circ \rho_s=\rho \circ f$.

Proof. As before, we set $A_\sigma=(\rho \circ f)(g_\sigma^{-1})$ and $\bar{A}_\sigma=\rho_s(g_\sigma^{-1})$. Since $A_\sigma^\tau A_\tau=A_{\sigma\tau}$, there is an element $X \in GL(V)$ such that $A_\sigma=X^{-\sigma}X$. Using X, we set $\rho_1=X(\rho \circ f)X^{-1}$ and $B_1={}^tX^{-1}BX^{-1}$. It is easy to check that ρ_1 is defined over K and that the image of G_1 under ρ_1 preserves B_1 which is also defined over K. Also, since $A_\sigma \in SO(V, B)$, $(\det X)^\sigma(\det X)^{-1}=1$ for all $\sigma \in \Gamma$ and so $(\det X) \in K^*$. Hence, $\Delta(B_1)=\Delta(B)$.

Finally, it is necessary to compute $c(B_1)$. To do this, we look at the Clifford algebra $C(B)$ of B. (If dim $V \equiv 1$ (2), we really need $C^+(B)$, the set of even elements of $C(B)$, but we write $C(B)$ to avoid some notational clumsiness.) Let h: $C(B) \to M(t, \bar{K})$ be an isomorphism of $C(B)$ onto a total matrix algebra. For each $\sigma \in \Gamma$, there is $Y_\sigma \in GL(t, \bar{K})$ such that $h^\sigma(x)=Y_\sigma h(x)Y_\sigma^{-1}$ for all $x \in C(B)$. The system $\{Y_\sigma\}$ satisfies the relation $Y_\sigma^\tau Y_\tau=b_{\sigma,\tau}Y_{\sigma,\tau}$ with $b_{\sigma,\tau} \in \bar{K}^*$ and, by definition, the cohomology class of the 2-cocycle $b_{\sigma,\tau}$ is $c(B)$.

The map X^{-1}: $(V_1, B_1) \to (V, B)$ is a quadratic space isomorphism and induces a mapping X^{-1}: $C(B_1) \to C(B)$. (In the following when we write X^{-1}, we shall always mean the mapping of the Clifford algebras.) The composite map $H=h \circ X^{-1}$ gives an isomorphism of $C(B_1)$ with a total matrix algebra. We now determine the corresponding 2-cocycle. For each $\sigma \in \Gamma$, $H^\sigma \circ H^{-1}=I_{N_\sigma}$ where $N_\sigma=Y_\sigma h(\bar{A}_\sigma)$. From this it follows that $N_\sigma^\tau N_\tau=b_{\sigma,\tau}\rho_s(c_{\sigma,\tau})N_{\sigma\tau}$ and our theorem is proved.

It is not difficult to reduce the general case where G_1 is not simply connected to the case above. For it is known that there are simply connected covering groups $(\bar{G}_1, p_1)$ and $(\bar{G}, p)$ of G_1 and G respectively which are defined over K. Then, it also can be shown that there is a $\bar{K}$-isomorphism $\bar{f}$: $\bar{G}_1 \to \bar{G}$ such that for each $\sigma \in \Gamma$, $\bar{f}^{-\sigma} \circ \bar{f}=I_{h_\sigma}$; here, h_σ is an element in $\bar{G}_1$ such that $p_1(h_\sigma)=g_\sigma$. In the statement of Theorem 1.1, G is replaced by $\bar{G}$, ρ by $\rho \circ p$, g_σ by h_σ, and so on.

1.3. **Problem 2.** In this section, we consider the case where $K\#$ is a quaternion division algebra (β, γ) and we begin by summarizing some results which can be found in [7, p. 235]. The algebra $K\#$ has a basis $(1, x_1, x_2, x_1x_2)$ over K such that $x_1^2=\beta$, $x_2^2=\gamma$, and $x_1x_2=-x_2x_1$. The elements β and γ are in K^* and we assume that the equation $\beta X^2+\gamma Y^2=1$ has no solution (X, Y) in K. An isomorphism M: $K\# \to M(2, \bar{K})$ is given by

$$M(Y_0+Y_1x_1+Y_2x_2+Y_3x_1x_2) = \begin{pmatrix} Y_0+Y_1\beta^{1/2} & \gamma(Y_2+Y_3\beta^{1/2}) \\ Y_2-Y_3\beta^{1/2} & Y_0-Y_1\beta^{1/2} \end{pmatrix}.$$

M is defined over $L=K(\beta^{1/2})$ and if we set Gal $(L/K)=\{1, \sigma\}$, then $M^\sigma(x) =M(n_\sigma^{-1}xn_\sigma)$ for all $x \in K\#$ where $n_\sigma=x_2$. There is a canonical involution $x \to \bar{x}$

of the first kind on $K\#$, namely, if $x = Y_0 + Y_1x_1 + Y_2x_2 + Y_3x_1x_2$, then $\bar{x} = Y_0 - Y_1x_1 - Y_2x_2 - Y_3x_1x_2$. Setting

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we see that $M(\bar{x}) = J^{-1}\,{}^tM(x)J$ for all $x \in K$. Furthermore, ${}^tJ = -J$.

Now we return to the situation in Problem 2 and assume that $K\#$ is a quaternion division algebra. If e_{ij} are matrix units in $K\#$, then, considering V_1e_{11} as a vector space over $\bar{K}$, there is a $\bar{K}$-isomorphism $f_1\colon V \to V_1e_{11}$ defined over L such that

$$R_{n_\sigma} = a_\sigma f_1 \circ A_\sigma^{-1} \circ f_1^{-\sigma} \tag{3}$$

where $R_{n_\sigma}\colon V_1e_{22} \to V_1e_{11}$ is given by $R_{n_\sigma}(v) = vn_\sigma$ for all $v \in V_1e_{22}$. The element a_σ is in $\bar{K}^*$ [7, p. 229].

Define B_{11} on V_1e_{11} so that f_1 is a quadratic space isomorphism and set $B_{ij}(v, w) = B_{11}(ve_{i1}, we_{j1})$ for all $v, w \in V_1$ and $i, j = 1, 2, 3, 4$. Then the form F is defined by the formula [7, p. 233]

$$JM(F(v, w)) = (B_{ij}(v, w)). \tag{4}$$

F is skew-hermitian if B is orthogonal.

LEMMA I.3. *In formula* (3), $a_\sigma^2 = -\gamma$.

Proof. First we show that $-\gamma B_{11}^\sigma(vn_\sigma^{-1}, yn_\sigma^{-1}) = B_{11}(v, w)$ for all $v, w \in V_1$. This is done by applying σ to (4) and remembering that F is defined over K, $M^\sigma \circ I_{n_\sigma} = M$, and $n_\sigma = \gamma e_{12} + e_{21}$.

Using this result we are able to prove the lemma. Again we use (3) and the fact that $A_\sigma \in SO(V, B)$. For choosing v to be K-rational in V_1, such that $B_{22}(v) = B_{22}(v, v) \neq 0$, we have: $B_{11}(R_{n_\sigma}(ve_{22})) = B_{11}(ve_{21}) = B_{22}(v)$. But also $a_\sigma^{-2}B_{11}(R_{n_\sigma}(ve_{22})) = B_{11}(f_1 \circ A_\sigma^{-1} \circ f_1^{-\sigma}(ve_{22})) = (B(f_1^{-1}(ve_{11})))^\sigma = (B_{11}(ve_{11}))^\sigma = B_{11}(vn_\sigma e_{11}n_\sigma^{-1})$ which by the first part of this lemma is just $(-\gamma)^{-1}B_{11}(vn_\sigma e_{11}) = (-\gamma)^{-1}B_{11}(ve_{21}) = (-\gamma)^{-1}B_{22}(v)$ and the lemma is complete.

Before stating Theorem I.2, we again review some fundamental definitions. For a skew-hermitian form F on a space V_1 over $K\#$, Tsukamoto [8] has determined a complete set of invariants when K is a nonarchimedean local field such that $[K^* : (K^*)^2] > 2$. The invariants are dim V_1 and $\delta(F)$. This last invariant is defined in the following way: let $\{v_1, \ldots, v_m\}$ be an orthogonal basis defined over K of V_1 over $K\#$. Since F is skew-hermitian, $F(v_i, v_i) = x_i = -\bar{x}_i$ for some $x_i \in K\#$. But $x_i^2 = a_i \in K^*$ and we set $\delta(F) = a_1 \cdots a_m \bmod (K^*)^2$.

THEOREM I.2. *Let G_1 and G be semisimple algebraic groups defined over K* (char $K = 0$) *and assume that there is a $\bar{K}$-isomorphism $f_1\colon G_1 \to G$ such that $f^{-\sigma} \circ f = I_{g_\sigma}$ for each $\sigma \in \Gamma$. Let (V, ρ, B) be an orthogonal representation of G defined over K and let $(V_1/K\#, \rho_1, F)$ be a skew-hermitian representation of G defined over K where $K\#$ is a quaternion division algebra over K. Assume also that there is an* *absolutely irreducible representation* θ_1: End $(V_1/K\#) \to$ End (V) *defined over* $\bar{K}$ *such that* $\theta_1(\rho_1(g)) = (\rho \circ f)(g)$ *for each* $g \in G_1$. *Then the invariants of F are as follows*: dim $V_1 = \frac{1}{2}$ dim V *and* $\delta(F) = \Delta(B)$.

Proof. The dimension formula follows from the existence of f_1 in (3). To prove the relation on discriminants, let $\{v_1, \ldots, v_m\}$ be an orthogonal basis of F defined over K. Then $E = \{v_1e_{11}, \ldots, v_me_{11}, v_1e_{21}, \ldots, v_me_{21}\}$ is a basis for V_1e_{11} and $\delta(F) = (-1)^m \det(B_{11}, E)$. By this last term we mean the determinant of B_{11} in the basis E.

Let $\{x_1, \ldots, x_{2m}\}$ be a basis of V defined over K and let P be the matrix of $f^{-1}(E)$ with respect to $\{x_i\}$. Then $\delta(F) = (-1)^m \det(B, \{x_i\}) \cdot (\det P)^2$. Hence, $(\det P)^2 \in K^*$. If we can show that $\det P \in K^*$, we are done. Stated differently, it remains to be proved that $(\det P)^\sigma (\det P)^{-1} = 1$ where Gal $(L/K) = \{1, \sigma\}$.

To prove this statement, we compute determinants of both sides of (3). The matrix of $R_{n_\sigma^{-1}}(E)$ in the basis $E^\sigma = \{v_1e_{22}, \ldots, v_me_{22}, \gamma v_1e_{12}, \ldots, \gamma v_me_{12}\}$ is

$$\begin{pmatrix} 0 & 1_m \\ \gamma^{-1}1_m & 0 \end{pmatrix}$$

and has determinant $(-\gamma)^{-m}$. So, by (3), it follows that $(\det P)^\sigma (\det P)^{-1} = (-\gamma)^{-m} a_\sigma^{2m} = (-\gamma)^{-m}(-\gamma)^m$, by Lemma I.3, and we have proved the theorem.

1.4. **Steinberg groups.** In this brief section, we look at the results in this part from a slightly different viewpoint, namely that of Steinberg groups. A group G defined over K is called Steinberg if there is a Borel subgroup of G which is also defined over K. It is known that if G_1 is a connected semisimple group defined over K, then there is a Steinberg group G defined over K and a $\bar{K}$-isomorphism $f: G_1 \to G$ of inner type. In this case, the cohomology class of $c_{\sigma,\tau}$ is independent of f and is denoted by $\gamma_K(G_1)$. This last invariant has been studied by Satake [6], [7].

The division algebra associated with an irreducible representation of a Steinberg group is always trivial, i.e., is the underlying field [7, p. 241]. Hence, in terms of Steinberg groups, Theorems I.1 and I.2 say that to determine the form on a representation of G_1 it is enough to know the form on the corresponding representation of the Steinberg group G associated with G_1. Of course, for absolutely simple groups G_1, the associated Steinberg group G will always be the corresponding Chevalley group except possibly when G_1 is of type A_n, D_n, or E_6. In Part II, we shall study these three cases and show how orthogonal representations of Steinberg and Chevalley groups are related.

PART II

2.1. **The group G^*.** Throughout this section, let G be a semisimple Chevalley group defined over K (char $K = 0$) and let T be a maximal split torus in G defined over K. Denote by $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the corresponding fundamental root system.

The automorphism group of G is the semidirect product of a finite group Θ and the inner automorphisms of G. We choose Θ in such a way that for each $\theta \in \Theta$, θ is defined over K, $\theta(T)=T$, and $\theta(\Delta)=\Delta$. We define an algebraic group G^* to be $G \cdot \Theta$, the semidirect product of G and Θ where group multiplication is given in the following way: $(g_1\theta_1)(g_2, \theta_2)=(g_1\theta_1(g_2), \theta_1\theta_2)$. In what follows, we consider G as a subgroup of G^*. By our choice of Θ, both are algebraic groups defined over K.

LEMMA II.1. *Let $\rho: G \to GL(V)$ be an absolutely irreducible representation of G defined over K. Then there exists a representation $\rho^*: G^* \to GL(V)$ defined over $\bar{K}$ such that $\rho^* \mid G=\rho$ if and only if there is a homomorphism $\theta \to A_\theta$ of Θ to $GL(V)$ such that $\rho(\theta(g))=A_\theta \rho(g) A_\theta^{-1}$ for all $g \in G$.*

Proof. If ρ^* exists, set $\rho^*(1, \theta)=A_\theta$. Then $\rho^*[(1, \theta)(g, 1)(1, \theta^{-1})]=A_\theta\rho(g)A_\theta^{-1}$ and is also $\rho^*((\theta(g), 1))=\rho(\theta(g))$.

Conversely, if such A_θ exist, define $\rho^*(g, \theta)=\rho(g)A_\theta$. It is easy to check that ρ^* becomes a homomorphism and so the lemma is proved.

COROLLARY. *Assume that Θ is a cyclic group generated by θ. Then ρ^* exists if and only if $\rho \circ \theta \sim \rho$.*

Proof. Assume that $\theta^r=1$ and $\rho \circ \theta=A_\theta \rho A_\theta^{-1}$. It is easy to see that $A_\theta^r=a1$ for some $a \in \bar{K}^*$ and modifying A_θ we can assume $A_\theta^r=1$. This completes the proof.

2.2. **The groups A_n, D_n, and E_6.** In this section, we shall take a closer look at the group G^* when G is a Chevalley group of type A_n, D_n, or E_6. In particular, let (V, ρ, B) be an orthogonal representation of G defined over K with highest weight λ. We shall give conditions on λ in order that $\rho^*: G^* \to GL(V)$ exists; furthermore, in each case we shall show that ρ^* can be chosen to be defined over K and $\rho^*: G^* \to O(V, B)$.

LEMMA II.2. *Let G be a Chevalley group of type A_n defined over K* (char $K=0$) *and let (V, ρ, B) be an orthogonal representation of G defined over K. Then $\rho^*: G^* \to O(V, B)$ exists and is defined over K. Furthermore, if* $\dim V \equiv 1$ (2), *ρ^* can be chosen so that $\rho^*: G^* \to SO(V, B)$.*

Proof. For easy reference, the proof is divided into small sections.

(i) The group Θ is of order 2 and is generated by θ where $\theta(\alpha_r)=\alpha_{n-r+1}$. If $\lambda=\sum_{r=1}^n m_r\alpha_r$ with $m_r \in \boldsymbol{Q}$, $m_r \geq 0$, then $\rho \circ \theta \sim \rho$ if and only if $m_r=m_{n-r+1}$. But it is known [3, p. 196] that all orthogonal representations of A_n have this property and also that each $m_r \in \boldsymbol{Z}$. Since ρ and $\rho \circ \theta$ are both defined over K, there is an $A \in GL(V, K)$ such that $A\rho(g)=\rho(\theta(g))A$. Let x be a K-rational highest weight vector in V. Since $\theta\lambda=\lambda$, it is easy to see that Ax is also a K-rational highest weight vector. Hence, $Ax=ax$ for some $a \in K^*$ and $A^2=a^21$. Set $A_\theta=a^{-1}A$; then $A_\theta \in GL(V, K)$, $A_\theta\rho(g)=\rho(\theta(g))A_\theta$ for all $g \in G$, and $A_\theta^2=1$. If $\dim V \equiv 1$ (2), we may assume that $\det A_\theta=1$, multiplying A_θ by -1 if necessary. We also note that $A_\theta x=ex$ where $e^2=1$. Next, we shall show that A_θ is in $O(V, B)$.

(ii) Let $W=N(T)/T$ be the Weyl group of G. It is known that there is an element w in W such that $w(\Delta)=-\Delta$, i.e. $w(\alpha_r)=-\alpha_{n-r+1}$. Choose a representative g in $N(T)$ for w, i.e. $w=gT$. The element $\theta(g)$ is also in $N(T)$ and it is easy to see that $I_{\theta(g)}=\theta\circ I_g\circ\theta=I_g$ on T. (It is enough to check that the induced mappings on Δ agree.) Hence, there is a t in T such that $\theta(g)=gt$. Applying θ again to this equation we get

$$t\theta(t) = 1. \tag{5}$$

(iii) Next, we show that $B(x, \rho(g)x)\neq 0$. If x_1 and x_2 are weight vectors in V corresponding to weights λ_1 and λ_2, respectively, then for t in T, $B(x_1, x_2)=B(\rho(t)x_1, \rho(t)x_2)=\lambda_1(t)\lambda_2(t)B(x_1, x_2)$. So $B(x_1, x_2)=0$ except possibly when the character $\lambda_1+\lambda_2$ is 0. (We use additive notation on the character module of T.) In the case above, the highest weight space has dimension 1 and so if $\rho(g)x$ has weight $-\lambda$, then we are done with (iii). But this follows from the facts that $g\in N(T)$ and $I_g(\lambda)=-\lambda$.

Since ${}^tA_\theta BA_\theta$ is also invariant under $\rho(G)$, there is $a_\theta\in K^*$ such that ${}^tA_\theta BA_\theta=a_\theta B$. In particular $0\neq a_\theta B(x, \rho(g)x)=B(A_\theta x, A_\theta\rho(g)x)=B(A_\theta x, \rho(\theta(g))A_\theta x)=B(x, \rho(gt)x)=\lambda(t)B(x, \rho(g)x)$. Hence, $a_\theta=\lambda(t)$. The map $\theta\to a_\theta$ is a homomorphism and so $a_\theta^2=1$, i.e. $\lambda(t)^2=1$, a result which can also be seen by applying λ to (5).

(iv) Finally, we show that $\lambda(t)=1$. If $n\equiv 0$ (2), this follows immediately. For by (5), $(\alpha_r+\alpha_{n-r+1})(t)=1$; but λ is an integral combination of such terms. If $n\equiv 1$ (2), then it is enough to show that $\alpha_r(t)=1$ where $r=\frac{1}{2}(n+1)$. We saw that ${}^tA_\theta BA_\theta=\lambda(t)B$. In particular, if dim $V\equiv 1$ (2), then $\lambda(t)=1$ (as can be seen by taking determinants). But for $n\equiv 1$ (2), the representation with highest weight $\lambda=\alpha_1+\alpha_2+\cdots+\alpha_n$ is orthogonal and has dimension $n(n+2)$ which is odd. Hence, $\lambda(t)=\alpha_r(t)=1$ and the lemma is proved.

We have proved this lemma in such generality so that the proof will apply in the cases D_n and E_6. We indicate below the way in which this happens.

LEMMA 11.3. *Let G be a Chevalley group of type D_n $(n\neq 4)$ defined over K* (char $K=0$) *and let (V, ρ, B) be an orthogonal representation of G defined over K with highest weight $\lambda=\sum_{r=1}^n m_r\alpha_r$. Then $\rho^*: G^*\to O(V, B)$ exists and is defined over K if and only if $m_n=m_{n-1}$. Furthermore, if* dim $V\equiv 1$ (2), *ρ^* can be chosen so that $\rho^*: G^*\to SO(V, B)$.*

Proof. We take $G=SO(2n)$, the special orthogonal group on a $2n$-dimensional vector space W defined over K. Let $\{e_1, \ldots, e_{2n}\}$ be a K-rational basis of weight vectors where e_i has weight λ_i and e_{n+i} has weight $-\lambda_i$ for $i=1, \ldots, n$. A fundamental root system $\{\alpha_1, \ldots, \alpha_n\}$ is given by $\alpha_1=\lambda_1-\lambda_2, \ldots, \alpha_{n-1}=\lambda_{n-1}-\lambda_n$, and $\alpha_n=\lambda_{n-1}+\lambda_n$. Define a linear transformation $J\in O(2n)$ by $Je_r=e_r$, $r\neq n, 2n$, $Je_n=e_{2n}$, and $Je_{2n}=e_n$. Then det $(J)=-1$.

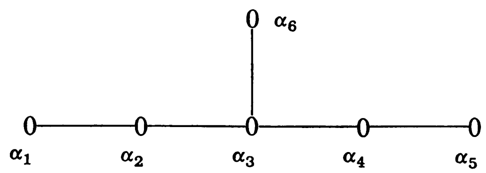
(i) The group Θ is of order 2 and is generated by θ where $\theta(\alpha_{n-1})=\alpha_n$. If $\lambda=\sum_{r=1}^n m_r\alpha_r$ with $m_r\in\mathbf{Q}$, $m_r\geq 0$, then $\rho\circ\theta\sim\rho$ if and only if $m_n=m_{n-1}$. It is easy to see that $\theta=I_J$. Hence, G^* may be identified with $O(2n)$.

(ii) The element $w=gT$ is given in the following way: if $n\equiv 1$ (2), $ge_r=e_{r+n}$ for $r=1,\ldots,n-1$, $ge_n=e_n$, $ge_{2n}=e_{2n}$, and $g^2=1$. If $n\equiv 0$ (2), $ge_r=e_{r+n}$ for $r=1,\ldots,n$ and $g^2=1$. In either case, $\theta(g)=JgJ=g$ and so $t=1$. The lemma now follows immediately.

The case D_4 is complicated by the fact that $\Theta=S_3$, the symmetric group on 3 elements. We postpone our study of it, looking first at E_6.

LEMMA 11.4. *Let G be a Chevalley group of type E_6 defined over K* (char $K=0$) *and let (V,ρ,B) be an orthogonal representation of G defined over K. Then ρ^*: $G^*\to O(V,B)$ exists and is defined over K. Furthermore, if* dim $V\equiv 1$ (2), *ρ^* can be chosen so that $\rho^*: G^*\to SO(V,B)$.*

Proof. The group G has the following Dynkin diagram:



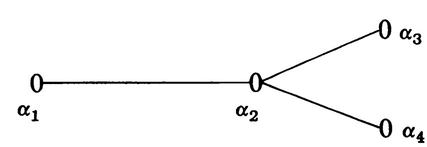
(i) The group Θ is of order 2 and is generated by θ where $\theta(\alpha_1)=\alpha_5$, $\theta(\alpha_2)=\alpha_4$, $\theta(\alpha_3)=\alpha_3$, and $\theta(\alpha_6)=\alpha_6$. If $\lambda=\sum_{r=1}^{6} m_r\alpha_r$ with $m_r\in \mathbf{Q}$, $m_r\geqq 0$, then $\rho\circ\theta\sim\rho$ if and only if $m_1=m_5$ and $m_2=m_4$. But it is known [3, p. 202] that all orthogonal representations of E_6 have this property and also that each $m_r\in\mathbf{Z}$.

(ii) The element w is given by: $w(\alpha_1)=-\alpha_5$, $w(\alpha_2)=-\alpha_4$, $w(\alpha_3)=-\alpha_3$, and $w(\alpha_6)=-\alpha_6$.

(iv) We know that λ is an integral combination of $\alpha_1+\alpha_5$, $\alpha_2+\alpha_4$, α_3, and α_6. From (5), it follows that $(\alpha_1+\alpha_5)(t)=1$, $(\alpha_2+\alpha_4)(t)=1$ and $\alpha_3(t)^2=\alpha_6(t)^2=1$. Hence, it only remains to be shown that $\alpha_3(t)=\alpha_6(t)=1$. The representation with highest weight $\lambda=2(\alpha_1+\alpha_5)+4(\alpha_2+\alpha_4)+6\alpha_3+3\alpha_6$ is orthogonal and has odd dimension. But then $\lambda(t)=\alpha_6(t)=1$. Similarly, the representation with highest weight $\lambda=5[(\alpha_1+\alpha_5)+2(\alpha_2+\alpha_4)+3\alpha_3+2\alpha_6]$ is orthogonal and has odd dimension. Hence, $\alpha_3(t)=1$ and the proof of the lemma is completed.

LEMMA 11.5. *Let G be a Chevalley group of type D_4 defined over K ($K=Q_{\mathfrak{p}}$) and let (V,ρ,B) be an orthogonal representation of G defined over K with highest weight $\lambda=m_1\alpha_1+m_2\alpha_2+m_3\alpha_3+m_4\alpha_4$. Then $\rho^*: G^*\to O(V,B)$ exists and is defined over K if and only if $m_1=m_3=m_4$. Furthermore, if* dim $V\equiv 1$ (2), *ρ^* can be chosen so that $\rho^*: G^*\to SO(V,B)$.*

Proof. The group G has the following Dynkin diagram:



(i) The group Θ is of order 6 and is the symmetric group on $\{\alpha_1, \alpha_3, \alpha_4\}$. We distinguish two elements θ and ψ in Θ. The element θ has order 2 and is defined by $\theta(\alpha_3)=\alpha_4$ and the element ψ, having order 3, is defined by $\psi(\alpha_1)=\alpha_3$, $\psi(\alpha_3)=\alpha_4$, and $\psi(\alpha_4)=\alpha_1$. If $\lambda=m_1\alpha_1+m_2\alpha_2+m_3\alpha_3+m_4\alpha_4$, it follows that a necessary condition for $\rho^*: G^* \to GL(V)$ to exist is that $m_1=m_3=m_4$. We show now that these equalities are also sufficient. For let x be a K-rational highest weight vector of ρ. Then, as in the proof for A_n, there are elements A_θ, $A_\psi \in GL(V, K)$ such that $A_\theta^2=A_\psi^3=1$, $A_\theta\rho(g)=\rho(\theta(g))A_\theta$ and $A_\psi\rho(g)=\rho(\psi(g))A_\psi$ for all $g \in G$, $A_\theta x=x$, and $A_\psi x=x$. The defining relations for S_3 are $\theta^2=\psi^3=1$ and $\theta\psi\theta=\psi^2$. Hence, we need to show that $A_\theta A_\psi A_\theta=A_\psi^2$. But since

$$\rho(\psi^2(g)) = A_\psi^2\rho(g)A_\psi^{-2} = (A_\theta A_\psi A_\theta)\rho(g)(A_\theta A_\psi A_\theta)^{-1}$$

it follows that there exists $a \in K^*$ such that $A_\psi^2=aA_\theta A_\psi A_\theta$. Applying both sides to x, we see that $a=1$ and this part of the lemma is proved. It should be noticed, also, that we can assume det $A_\theta=1$ if dim $V\equiv 1$ (2).

As in Lemma II.3, it can be shown that $A_\theta \in O(V, B)$. Therefore, if we can show that A_ψ is in $O(V, B)$, the proof will be complete. As a matter of fact, since the mapping $\psi \to A_\psi$ gives a homomorphism of the group of order 3 generated by ψ, if $A_\psi \in O(V, B)$, then $A_\psi \in SO(V, B)$.

We know that ${}^tA_\psi BA_\psi=a_\psi B$ where $a_\psi \in K^*$ and $a_\psi^3=1$. But since G is a Chevalley group, we may assume that $K=\mathbf{Q}$ and then a_ψ must be 1. This completes the proof of the lemma.

To conclude this section, we prove a result about the Clifford algebra $C(B)$ of B which will be useful when we return to Problem 1. As above, the set of even elements in $C(B)$ will be denoted by $C^+(B)$.

LEMMA II.6. *Let G be a Chevalley group of type A_n, D_n, or E_6 defined over K* (char $K=0$) *and let $\theta \in \Theta$ be an element of order* 2. *Let (V, ρ, B) be an orthogonal representation of G defined over K and assume that $\rho^*: G^* \to O(V, B)$ exists and is defined over K. Then there is an element $\bar{A}_\theta$ in $C^+(B)$ if* det $A_\theta=1$ *or in $C(B)$ if* det $A_\theta=-1$ *satisfying the following conditions*:

(i) $\bar{A}_\theta x\bar{A}_\theta^{-1}=A_\theta x$ *for all* $x \in V$.

(ii) $A_\theta(\text{Spin}(B))A_\theta^{-1}=\text{Spin}(B)$.

Proof. Since $A_\theta=\rho^*(\theta)$ is defined over K, $A_\theta^2=1$, and $A_\theta \in O(V, B)$, the spaces $V^+=\{x \in A_\theta \mid x=x\}$ and $V^-=\{x \in V \mid A_\theta x=-x\}$ are defined over K, span V, and are perpendicular. Let $\{e_1, \dots, e_r\}$ and $\{e_{r+1}, \dots, e_n\}$ be orthogonal bases of V^+ and V^-, respectively, which are defined over K.

If det $A_\theta=1$ (i.e., $n-r\equiv 0$ (2)), we set $\bar{A}_\theta=e_{r+1}\cdots e_n \in C^+(B)$. If det $A_\theta=-1$ (i.e., $n\equiv 0$ (2) and $n-r\equiv 1$ (2)), we set $\bar{A}_\theta=e_1\cdots e_r \in C(B)$. In both cases it is easy to see that $\bar{A}_\theta$ has the desired properties and so the lemma is proved.

COROLLARY 1. *Let ρ_s: $G \to$* Spin (B) *be such that $\pi \circ \rho_s=\rho$ where π is the natural mapping from* Spin (B) *onto $SO(V, B)$. Then $\rho_s(\theta(g))=\bar{A}_\theta\rho(s)\bar{A}_\theta^{-1}$ for all $g \in G$.*

COROLLARY 2. *If* $\det A_\theta = 1$, *then* $\bar{A}_\theta^2 = \Delta^-$ *where* Δ^- *is the discriminant of* B *restricted to* $V^- = \{x \in V \mid A_\theta x = -x\}$. *If* $\det A_\theta = -1$, *then* $\bar{A}_\theta^2 = \Delta^+$ *where* Δ^+ *is the discriminant of* B *restricted to* $V^+ = \{x \in V \mid A_\theta x = x\}$.

2.3. **Problem 1.** Having the above results in hand, we are now able to give solutions to Problems 1 and 2 if f is not of inner type. As we saw in §1.4, we have reduced Problem 1 to the case where G_1 is a Steinberg group of type A_n, D_n, or E_6 and G is the corresponding Chevalley group.

Let G be a semisimple Chevalley group defined over K and let Θ be chosen as above. Steinberg groups are just K-forms associated with continuous 1-cocycles in Θ. Indeed, let $\{\theta_\sigma\}$ be a continuous 1-cocycle in Θ, i.e., $\theta_\sigma\theta_\tau = \theta_{\sigma\tau}$ for all $\sigma, \tau \in \Gamma$ and let G_1 be the associated K-form. Let Δ_1 be a fundamental system in G_1 corresponding to Δ. Then $\Delta_1^\sigma = \Delta_1$ for all $\sigma \in \Gamma$ and using this it can be shown that G_1 is Steinberg. Furthermore, there is a finite extension K_0 of K over which G_1 is a Chevalley group. The elements $\sigma \in \mathrm{Gal}\,(K_0/K)$ correspond to $\theta_\sigma \in \Theta$ and if $\sigma \neq 1$, then $\theta_\sigma \neq 1$. This field K_0 is called the nuclear field of G_1 [5]. With the exception of D_4, K_0 is a quadratic extension of K. As we have seen, $\Theta = S_3$ is $G = D_4$ and $K = Q_{\mathfrak{p}}$. Hence, in this case, $[K_0/K]$ can be 2, 3, or 6. In stating the next theorem, we use the notation introduced in §2.1.

THEOREM II.1. *Let* G_1 *be a Steinberg group of type* A_n, D_n $(n \neq 4)$, *or* E_6 *defined over* K (char $K = 0$), *let* G *be the corresponding Chevalley group defined over* K, *and let* $f: G_1 \to G$ *be the isomorphism between* G_1 *and* G *so that* $f^\sigma \circ f^{-1} = \theta_\sigma \in \Theta$ *for all* $\sigma \in \Gamma$. *Assume that* (V, ρ, B) *is an orthogonal representation of* G *defined over* K *such that* $\rho^*: G^* \to O(V, B)$ *exists and is defined over* K. *Then there is an orthogonal representation* (V_1, ρ_1, B_1) *of* G_1 *defined over* K *such that* $\rho_1 \sim \rho \circ f$ *and* B_1 *is related to* B *as follows*:

(i) $\dim V_1 = \dim V$.

(ii) $\Delta(B_1) = \Delta(B)$ *if* $\det(\rho^*(\theta)) = 1$ *or* $\Delta(B_1) = \alpha\Delta(B)$ *if* $\det(\rho^*(\theta)) = -1$ *where* $\alpha \in K^*$ *is determined up to* $(K^*)^2$ *by the property that* $K_0 = K(\alpha^{1/2})$ *is the field fixed by* $\{\sigma \in \Gamma \mid \theta_\sigma = 1\}$.

(iii) $c(B_1) = c(B)(c_{\sigma,\tau})$ *where* $c_{\sigma,\tau} = 1$ *unless* $\theta_\sigma = \theta_\tau = \theta$. *Then* $c_{\sigma,\tau} = \Delta^-$ *if* $\det(\rho^*(\theta)) = 1$ *or* Δ^+ *if* $\det(\rho^*(\theta)) = -1$.

Proof. This proof is a slight generalization of that for Theorem I.1. For $\sigma \in \Gamma$, $(\rho \circ f)^\sigma = \rho \circ \theta_\sigma \circ f = A_\sigma(\rho \circ f)A_\sigma^{-1}$ where $A_\sigma = \rho^*(\theta_\sigma)$. Hence, since ρ^* is defined over K, $A_\sigma^\tau A_\tau = \rho^*(\theta_\sigma\theta_\tau) = \rho^*(\theta_{\sigma\tau}) = A_{\sigma\tau}$. There is an element $X \in GL(V)$ such that $A_\sigma = X^{-\sigma}X$ for all $\sigma \in \Gamma$. We put $\rho_1 = X(\rho \circ f)X^{-1}$ and $B_1 = {}^tX^{-1}BX^{-1}$. Then it is immediate that $\rho_1 \sim \rho \circ f$, ρ_1 and B_1 are defined over K, ρ_1 preserves B_1, and $\dim V_1 = \dim V$. Since $(\det X)^\sigma(\det X)^{-1} = \det A_\sigma = +1$ or -1, the result on $\Delta(B_1)$ follows.

Finally, let $h: C(B) \to M(t, \bar{K})$ be an isomorphism of $C(B)$ onto a total matrix algebra. (Again, if $\dim V \equiv 1$ (2), we should write $C^+(B)$, but since nothing would change in the proof below, we do not distinguish these cases.) For $\sigma \in \Gamma$, there is $Y_\sigma \in GL(t, \bar{K})$ such that $h^\sigma(x) = Y_\sigma h(x) Y_\sigma^{-1}$ for all $x \in C(B)$. The system $\{Y_\sigma\}$ satisfies $Y_\sigma^\tau Y_\tau = b_{\sigma,\tau} Y_{\sigma\tau}$ with $b_{\sigma,\tau} \in \bar{K}^*$ and $c(B) = (b_{\sigma,\tau})$.

Next, we use Lemma II.6. For setting $H = h \circ X^{-1}$ we have an isomorphism of $C(B_1)$ onto $M(t, \bar{K})$. For $\sigma \in \Gamma$, $H^\sigma \circ H^{-1} = I_{N_\sigma}$ where $N_\sigma = Y_\sigma h(\bar{A}_\sigma)$. Then $N_\sigma^\tau N_\tau = b_{\sigma,\tau} c_{\sigma,\tau} N_{\sigma\tau}$. The elements $c_{\sigma,\tau}$ in $\bar{K}^*$ are defined by $\bar{A}_\sigma \bar{A}_\tau = c_{\sigma,\tau} \bar{A}_{\sigma\tau}$ and (iii) follows on applying Corollary 2 of Lemma II.6. Hence, the theorem is proved.

REMARK. In §2.2, we saw that if $\rho \circ \theta \sim \rho$, then $\rho^*: G^* \to O(V, B)$ exists and is defined over K. Furthermore, if ρ_1 is a representation of G_1 defined over K and if ρ is the representation of G defined over K such that $\rho \sim \rho_1 \circ f^{-1}$, then ρ^* always exists since $\rho^\sigma = \rho$ implies $\rho_1 \circ f^{-1} \circ \theta_\sigma \sim \rho_1 \circ f^{-1}$. Therefore, Theorem II.1 is a complete reduction to the Chevalley case of the problem of finding invariant orthogonal forms on representations of Steinberg groups of type A_n, D_n $(n \neq 4)$, and E_6.

Groups of type D_4 present no new problems and we shall only outline the results.

(1) If $[K_0/K] = 2$, the situation is exactly as in Theorem II.1.

(2) If $[K_0/K] = 3$, let $\tau \in \text{Gal}(K_0/K)$ such that $\tau^3 = 1$. If $\rho^*: G^* \to O(V, B)$ exists, we have seen that $\det(A_\tau) = 1$. Furthermore, we may find $\bar{A}_\tau \in \text{Spin}(B)$ such that $\bar{A}_\tau^3 = 1$. So, $\dim V_1 = \dim V$, $\Delta(B_1) = \Delta(B)$, and $c(B_1) = c(B)$.

(3) The case $[K_0/K] = 6$ combines the results of (1) and (2). Indeed let $\sigma, \tau \in \text{Gal}(K_0/K)$ have orders 2 and 3 respectively and let θ, ψ be the corresponding elements in Θ. Then proceeding as in Theorem II.1, we get the following results: $\dim V_1 = \dim V$; $\Delta(B_1) = \Delta(B)$ if $\det \rho^*(\theta) = 1$ and otherwise $\Delta(B_1) = \alpha\Delta(B)$ where $\alpha \in K^*$ is such that $\sigma(\alpha^{1/2}) = -\alpha^{1/2}$. Finally $c(B_1) = c(B) \cdot$(2-cocycle). The elements of this 2-cocycle are given in the following table:

	1	σ	τ	τ^2	$\sigma\tau$	$\sigma\tau^2$
1	1	1	1	1	1	1
σ	1	δ	1	1	δ	δ
τ	1	1	1	1	1	1
τ^2	1	1	1	1	1	1
$\sigma\tau$	1	δ	1	1	δ	δ
$\sigma\tau^2$	1	δ	1	1	δ	δ

The element δ is Δ^+ or Δ^- depending on whether $\det(\rho^*(\theta))$ is -1 or $+1$.

REMARK. As in the remark above, we claim that we have reduced the case of Steinberg groups of type D_4 to that of Chevalley groups of type D_4. The verification is straightforward and we omit it.

2.4. **Problem 2.** Let G_1 be a connected group of type A_n, D_n, or E_6 defined over K (we do not assume that G_1 is a Steinberg group) and let G be the corresponding Chevalley group. We want to prove a theorem like Theorem I.2 under the assumption that G and G_1 are isomorphic only (i.e. we do not require that the isomorphism be of inner type). The important fact here is that if ρ^* exists, then $\rho^*: G^* \to O(V, B)$.

Let $f: G_1 \to G$ be the isomorphism. Then for $\sigma \in \Gamma$, $f^\sigma \circ f^{-1} = \theta_\sigma \circ I_{g_\sigma}$ for some $g_\sigma \in G$. If (V, ρ, B) is an orthogonal representation of G defined over K, then $(\rho \circ f)^\sigma = A_\sigma(\rho \circ f)A_\sigma^{-1}$ where $A_\sigma = \rho(g_\sigma)\rho^*(\theta_\sigma)$. Since $A_\sigma \in O(V, B)$, we may prove Lemma I.3 again. In the proof of Theorem I.2, the only change is in $\det(A_\sigma) = \det(\rho^*(\theta_\sigma))$ which may be -1.

THEOREM II.2. *Let G_1 be a connected algebraic group of type A_n, D_n, or E_6 defined over K* (char $K=0$), *let G be the corresponding Chevalley group defined over K, and let $f: G_1 \to G$ be an isomorphism between G_1 and G such that $f^\sigma \circ f^{-1} = \theta_\sigma \circ I_{g_\sigma}$ for all $\sigma \in \Gamma$. Assume that (V, ρ, B) is an orthogonal representation of G defined over K and assume that $\rho^*: G^* \to O(V, B)$ exists and is defined over K. Let $(V_1/K\#, \rho_1, F)$ be a skew-hermitian representation of G_1 defined over K where $K\# = (\beta, \gamma)$ is a quaternion division algebra over K. Set* Gal $(K(\beta^{1/2})/K) = \{1, \sigma\}$. *Assume also that there is an absolutely irreducible representation θ_1:* End $(V_1/K\#) \to$ End (V) *defined over $\bar{K}$ such that $\theta_1(\rho_1(g)) = (\rho \circ f)(g)$ for all $g \in G_1$. Then the forms F and B are related as follows*:

(i) $\dim V_1 = 1/2 \dim V$.

(ii) $\delta(F) = \Delta(B)$ *if* $\det(\rho^*(\theta_\sigma)) = 1$ *and* $\delta(F) = \beta\,\Delta(B)$ *if* $\det(\rho^*(\theta_\sigma)) = -1$.

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