HARMONIC FUNCTIONS AND THEIR CONJUGATES ON HILBERT SPACE

BY

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Introduction. Let $1 < p < \infty$ and let H be a real separable Hilbert space. Denote by $y \to T_y$ the regular representation of the additive group of H acting on L^p (H, normal distribution). Let B be a one-one Hilbert-Schmidt operator on H and $e_1, e_2, e_3, \ldots$ be an orthonormal basis for H. Let n_t denote the normal distribution with variance parameter $t/2$, let the function $n_t(z) = (\pi t)^{-1/2} \exp[-z^2/t]$, and let the function $N_t(z) = z n_t(z)$. Set

$$(1) \qquad\qquad P_z(f) = \int_0^\infty H_t(f) N_t(z)\, dt/t,$$

where

$$(2) \qquad\qquad H_t(f) = \int_H T_y f\, dn_t \circ B^{-1}(y)$$

for f in $L^p(H)$. $P_z(f)$ is the Poisson integral of f. The conjugate Poisson integrals of f are the

$$(3) \qquad\qquad Q_z^{(j)}(f) = \lim_{R \to \infty} \int_0^R \left[\int T_{By} f(y, e_j)\, dn_t(y) \right] n_t(z)\, dt/t.$$

[At this point we can not specify the domain of integration for the inner integral.] $Q_z^{(j)}(f)$ is a type of singular integral operator and is bounded on $L^p(H)$. We shall show that $Q_z^{(j)}$ is the composition of the Poisson integral with a singular integral operator of the type discussed in [1].

Let A_j denote the infinitesimal generator of the semigroup T_{tBe_j}, $t > 0$. The A_j will play the role of derivatives. $P_z(f)$ is infinitely differentiable for each $z > 0$ and we have an integral formula for its derivatives. For $p = 2$, $(\partial^2/\partial z^2) P_z(f) + \sum_{j=1}^\infty A_j^2 P_z(f) = 0$ and, conversely, if $U(z)$ is a twice differentiable function from $(0, \infty)$ to $L^2(H)$ such that $(\partial^2/\partial z^2) U(z) + \sum_{j=1}^\infty A_j^2 U(z) = 0$ and $\|U(z)\|_2$ is bounded in $z > 0$, there is a unique f in $L^2(H)$ such that $U(z) = P_z(f)$.

For f in $L^2(H)$, set $U(z) = P_z(f)$, $v_j(z) = -Q_z^{(j)}(f)$, $V(z) = (v_1(z), v_2(z), v_3(z), \ldots)$, and $F(z) = (U(z), V(z))$. Then $|F(z)|$ is in $L^2(H)$ for $z \geq 0$ and the Cauchy-Riemann equations relate the components of F. If $|F(z)| > 0$ almost everywhere for $z > 0$, the sum $(\partial^2/\partial z^2)|F(z)| + \sum_{j=1}^\infty A_j^2 |F(z)|$ converges in measure and the sum is positive

almost everywhere. If $G(z)=((\partial/\partial z)U(z), A_1v_1(z), A_2v_2(z), A_3v_3(z),\ldots)$ and if $|G(z)|>0$ almost everywhere for $z>0$, the sum $(\partial^2/\partial z^2)|G(z)|+\sum_{j=1}^{\infty}A_j^2|G(z)|$ converges in measure and is positive almost everywhere.

When H is a finite dimensional Euclidean space and B is the identity operator on H the Poisson integral and the conjugate Poisson integrals, given above, reduce to the classical finite dimensional forms of these operators. The results to be discussed in this paper correspond to known theorems on finite dimensional spaces; see Weiss and Stein [10] in this regard.

1. **Preliminaries.** To minimize the discussion of measure theory on Hilbert space we refer the reader to the papers [3] and [5] of L. Gross and [8] and [9] of I. E. Segal.

DEFINITION (SEGAL). A weak distribution on a real Hilbert space, H, is an equivalence class of linear maps, F, from the conjugate space H^* of H to real valued measurable functions on a probability space (depending on F). Two such maps, F and F', are equivalent if for any finite set of vectors $y_1,\ldots,y_k$ in H^*, $F(y_1),\ldots,F(y_k)$ and $F'(y_1),\ldots,F'(y_k)$ have the same joint distribution in k-space. A weak distribution is continuous if a representative is a continuous linear map (the range space has the topology of convergence in measure).

In what follows we shall be most interested in the normal distribution with variance parameter $c>0$. This distribution is uniquely determined by the following properties: (1) for any y in H^*, $n(y)$ is normally distributed with mean zero and variance $c\|y\|^2$; (2) n maps orthogonal vectors to independent random variables. The normal distribution is continuous. There is an essentially unique (up to expectation preserving isomorphism) probability space (S, Σ, μ) and a continuous linear map, F, from H^* to the real valued measurable functions on (S, Σ, μ) such that F is a representative of the normal distribution. Σ has no proper sub-σ-field with respect to which all of the $F(y)$, y in H^*, are measurable. The measurable functions on H are the measurable functions on (S, Σ, μ). $L^p(H, n_c)=L^p(S, \Sigma, \mu)$. When the variance parameter is 1, we denote $n=n_1$ and $L^p(H)=L^p(H, n)$. The expectation, $E(f)$, of a measurable function f is $E(f)=\int_S f\,d\mu$.

A function $f(x)$ on the points of H is a tame function if there is a Baire function g on a finite dimensional Euclidean space, E_k, and orthonormal vectors, $h_1,\ldots,h_k$ in H^* such that $f(x)=g((x, h_1),\ldots,(x, h_k))$. The span of the $h_i, i=1,\ldots,k$ in H is called the base space of f. If F is a representative of the normal distribution and $f(x)=g((x, h_1),\ldots,(x, h_k))$ is a tame function $\tilde{f}(s)=g(F(h_1)(s),\ldots,F(h_k)(s))$ is a measurable function on H. The expectation of $\tilde{f}$ is

$$E(\tilde{f}) = (2\pi c)^{-k/2}\int_{E_k} g(t)\exp\left[-\frac{\|t\|^2}{2c}\right]dt,$$

where k is the dimension of the base space of f. This equality holds in the sense that if either side exists and is finite then so does the other and the two are equal.

Several very useful representatives of the normal distribution are known. Of

these the one in which we shall be most interested is the mapping studied by Gross (in [5]) from H^* to Borel measurable functions on an abstract Wiener space. We adopt the notation and terminology of [5]. Let B be a one-one Hilbert-Schmidt operator on a real separable Hilbert space H. Then $\|Bx\| = |x|_1$ is a measurable norm on H. Let H_B denote the completion of H in this norm. Let S denote the σ-field generated by the closed subsets of H_B. The normal distribution n_c induces a Borel probability measure N_c on H_B such that the extension of the identity map on H_B^* ($\subset H^*$), regarded as a densely defined map on H^* to measurable functions on (H_B, S, N_c), to H^* is a representative of the normal distribution on H. Continuous functions, f, on H_B are measurable functions on H and if g denotes the restriction of f to H and if $\mathscr{F}$ denotes the directed set (ordered by inclusion of the ranges) of finite dimensional projections on H, the net $\{\tilde{g}(Qx) \mid Q \in \mathscr{F}\}$ of measurable tame functions converges in measure to f as Q tends strongly to the identity through $\mathscr{F}$.

Let N_c be as above. We may regard B as an isometry from H_B to H. Hence $N_c \circ B^{-1}$ is a Borel measure on H. This measure is usually denoted by $n_c \circ B^{-1}$. See [4] for a discussion of these measures. If f is a bounded continuous function from H to a Banach space E, $\int_H f(x)\, dn_c \circ B^{-1}(x) = \int_{H_B} f(By)\, dN_c(y) = E(f \circ B)^{\sim}$.

If f, g, and fg are absolutely integrable tame functions on H, $(fg)^{\sim} = \tilde{f}\tilde{g}$, $(af+g)^{\sim} = a\tilde{f} + \tilde{g}$ for constants a, and if $f \leqq g$ on H, $\tilde{f} \leqq \tilde{g}$ almost everywhere. We shall use these properties often. Gross [3] has studied other functions on the points of H which give rise to measurable functions on H. If C is a Hilbert-Schmidt operator on H, $\|Cx\|$ determines a measurable function $\|C(\cdot)\|^{\sim}$ on H by

$$\|C(\cdot)\|^{\sim} = \lim \{\|CQx\|^{\sim} \mid Q \in \mathscr{F}\}$$

where $\mathscr{F}$ is as above and Q tends strongly to the identity through $\mathscr{F}$. If f is a bounded continuous function on H, $f(Cx)$ determines a measurable function on H. Such elementary functions as $\|x\|$, $\exp[-\|x\|^2]$, and $\exp[i\|x\|]$ fail to determine measurable functions on H. We refer to [3] for further discussion of these points.

2. **Harmonic functions.** We shall be interested in functions $U: (0, \infty) \to L^p(H)$, $1 < p < \infty$, which are twice strongly differentiable. Let $e_1, e_2, e_3, \ldots$, be an orthonormal basis for the real separable Hilbert space H. Let B be a one-one Hilbert-Schmidt operator on H and let $y \to T_y$ denote the regular representation of the additive group of H acting on $L^p(H)$. $\{T_y\}$ is a strongly continuous group of isometries on $L^p(H)$; if f is a bounded tame function $(T_y f)(x) = f(x-y)D_p(x, y)$ where $D_p(x, y) = \exp[(x, y)/p - \|y\|^2/2p]$. See [1]. Let A_j denote the infinitesimal generator of the semigroup T_{tBe_j}, $t > 0$. For any h in H, let A_h be the infinitesimal generator of the semigroup T_{tBh}, $t > 0$. Each A_h is a closed densely defined linear operator on $L^p(H)$. We say that f in $L^p(H)$ is differentiable if f is in the domain of A_h for each h in H. f is twice differentiable if $A_h f$ exists for $h \in H$ and if f and $A_h f$ are in the domain of A_k for all h and k in H. f is n-times differentiable if when $h_1, \ldots, h_i$ are vectors in H and $\alpha = (\alpha_1, \ldots, \alpha_i)$ is a multi-index of order $|\alpha| = \sum_{j=1}^{i} \alpha_j \leqq n-1$

then $A^\alpha f = A_{h_1}^{\alpha 1} \cdots A_{h,i}^{\alpha i} f$ exists and is in the domain of A_h for any h in H. f is infinitely differentiable if it is n-times differentiable for every n.

Let $U(z)$ be a twice strongly differentiable function from $(0, \infty)$ to $L^p(H)$ such that $U(z)$ is twice differentiable as a function in $L^p(H)$ for each $z > 0$. Such a function will be said to be harmonic if for $z > 0$ $U_{zz}(z) + \sum_{j=1}^\infty A_j^2 U(z) = 0$ where the sum converges in $L^p(H)$. In this section we shall characterize the harmonic functions $U: (0, \infty) \to L^2(H)$ for which $\|U(z)\|_2$ is a bounded function of z for $z > 0$.

Let n_t denote the normal distribution with variance parameter $t/2$, and let $n_t \circ B^{-1}$ denote the Gaussian measure with representative $F \circ B^*$ when F is the representative of n_t, see [4]. For f in $L^p(H)$ set $H_t(f) = \int_H T_y f \, dn_t \circ B^{-1}(y)$. Then $\|H_t(f)\|_p \leq \|f\|_p$ for $t \geq 0$. By taking Fourier transforms one sees that $(n_t \circ B^{-1}) * (n_s \circ B^{-1}) = (n_{t+s} \circ B^{-1})$, so that $H_t(H_s(f)) = H_{t+s}(f)$. Since

$$H_t(f) = \int_H T_{t^{1/2}y} f \, dn_1 \circ B^{-1}(y),$$

it follows from the strong continuity of the T_y and from the dominated convergence theorem that $\lim_{t \to 0^+} H_t(f) = f$. Thus H_t is a strongly continuous contraction semigroup on $L^p(H)$.

Set

$$n_t(z) = (\pi t)^{-1/2} \exp[-z^2/t], \qquad N_t(z) = z n_t(z),$$

and

$$P_z(f) = \int_0^\infty H_t(f) N_t(z) \, dt/t.$$

Since $\int_0^\infty N_t(z) \, dt/t = 1$, $\|P_z(f)\|_p \leq \|f\|_p$ for $z > 0$.

PROPOSITION 1. P_z is a strongly continuous contraction semigroup on $L^p(H)$. There is a Borel probability measure p_z on H such that $P_z(f) = \int_H T_y f \, dp_z(y)$.

Proof. The integral $P_z(f) = \int_0^\infty H_t(f) N_t(z) \, dt/t$ converges absolutely in $L^p(H)$ and $\|P_z(f)\|_p \leq \|f\|_p$ as above.

$$\|P_z(f) - f\|_p \leq \int_0^\infty \|H_t(f) - f\|_p N_t(z) \, dt/t$$

since $\int_0^\infty N_t(z) \, dt/t = 1$. For $\varepsilon > 0$ there is a $\delta > 0$ such that $\|H_t(f) - f\|_p < \varepsilon$ when $0 < t < \delta$. Thus

$$\|P_z(f) - f\|_p \leq \varepsilon + \int_\delta^\infty \|H_t(f) - f\|_p N_t(z) \, dt/t.$$

$N_t(z)/t \to 0$ as $z \downarrow 0$ and for $t \geq \delta$, $N_t(z)/t \leq Kt^{-3/2}$. By the dominated convergence theorem, $\lim \sup_{z \downarrow 0} \|P_z(f) - f\|_p \leq \varepsilon$ for every $\varepsilon > 0$, and $\lim_{z \downarrow 0} P_z(f) = f$.

The set of Borel measurable functions f on H such that $f(t^{1/2}x)$ is Borel measurable on $(0, \infty) \times H$ is closed under pointwise limits and includes the continuous

functions and hence consists of all Borel functions. Let $\chi_E(y)$ denote the characteristic function of a Borel set E in H and set $\mu_t(E) = \int_H \chi_E(y) \, dn_t \circ B^{-1}(y)$. Then $\mu_t(E) = \int_H \chi_E(t^{1/2}y) \, dn_1 \circ B^{-1}(y)$ is a bounded measurable function of t. Set $p_z(E) = \int_0^\infty \mu_t(E)N_t(z) \, dt/t$. By the bounded convergence theorem, $p_z(E)$ is a countably additive Borel measure on H. For simple functions, F, from H to $L^p(H)$,

$$\int_H F(y) \, dp_z(y) = \int_0^\infty \int_H F(y) \, dn_t \circ B^{-1}(y)N_t(z) \, dt/t.$$

Since for each bounded continuous function F from H to $L^p(H)$, there is a bounded sequence of simple functions converging pointwise to $F(y)$, the above equality of integrals holds for all bounded continuous functions by the bounded convergence theorem. In particular, $P_z(f) = \int_H T_y f \, dp_z(y)$.

To show that $P_z \circ P_w = P_{z+w}$ we show that $p_z * p_w = p_{z+w}$ since $P_z(P_w(f)) = \int_H T_y f \, d(p_z * p_w)(y)$. To prove this identity we shall calculate the Fourier transform, $\hat{p}_z$, of measure p_z. We shall see that $\hat{p}_z(y) = \exp[-z\|B^*y\|]$ and hence $(p_z * p_w)^\wedge(y) = \exp[-(z+w)\|B^*y\|]$. Thus $p_z * p_w = p_{z+w}$ by uniqueness of the Fourier transforms of Borel measures [4].

It follows from the above considerations on the measure p_z that

$$\hat{p}_z(u) = \int_0^\infty \int_H \exp[i(u, y)] \, dn_t \circ B^{-1}(y)N_t(z) \, dt/t$$

$$= \int_0^\infty \exp[-t/4\|B^*u\|^2]N_t(z) \, dt/t.$$

Set $a = 2^{-1}\|B^*u\|$ and $F(a, z) = \int_0^\infty \exp[-ta^2]N_t(z) \, dt/t$ for $z > 0$ and set $F(a, 0) = 1$. $F(a, z)$ is continuous on $z \geq 0$ for each $a \geq 0$. If $s = t^{-1}z^2$,

$$F(a, z) = \pi^{-1/2} \int_0^\infty \exp\left[-\frac{(za)^2}{s} - s\right] ds/s^{1/2}$$

and by a theorem on uniform convergence

$$F_2(a, z) = (-2za^2)\pi^{-1/2} \int_0^\infty \exp\left[-\frac{(za)^2}{s} - s\right] ds/s^{3/2}$$

for $z > 0$ and $a > 0$. Set $t = s^{-1}(za)^2$, to get $F_2(a, z) = -2aF(a, z)$ and $F(a, z) = F(a, 0) \exp[-2az]$. Thus $\hat{p}_z(u) = \exp[-z\|B^*u\|]$. This concludes the proof.

REMARK. The operator P_z is the Poisson integral of f. It follows from the fact that $\hat{p}_z(u) = \exp[-z\|B^*u\|]$ that if H is n-dimensional Euclidean space and B is the identity operator, then

$$dp_z(y) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} z(\|y\|^2 + z^2)^{-(n+1)/2} \, dy,$$

which is the Poisson kernel in E_n.

For f in $L^p(H)$, $T_y f$ is a bounded uniformly continuous function from H to $L^p(H)$. If B is a one-one Hilbert-Schmidt operator on H, $T_{By}f$ is uniformly continuous in

the norm $\|Bx\|$ and $T_{By}f$ extends uniquely to a continuous function from H_B, the completion of H in the norm $\|Bx\|$. In the next proposition we shall use the Wiener space representative for the normal distribution n_t, the normal distribution with variance parameter $t/2$.

The Wiener space representative, F, of the normal distribution n_t may depend on t since for y in H^*, $F(y)(s)$ is defined only almost everywhere. It is known [6], however, that F may be chosen so that if $h_n \in H_B^*$ and $h_n \to h$ in H^* then $F(h_n)$ converges to $F(h)$ in $L^2(H, n_t)$ for all $t > 0$. We shall assume in what follows that we are working with this representative for the normal distribution n_t.

Denote by $y \to S_y$ the regular representation of the additive group of H acting on $L^1(H, n_t)$. For h in H, let C_h denote the infinitesimal generator of the semigroup S_{uh}, $u > 0$.

PROPOSITION 2. *Let $h_1, \ldots, h_n$ be in H, let $A_h^\alpha = A_{h_1}^{\alpha_1} \cdots A_{h_n}^{\alpha_n}$, and let $C_h^\alpha = C_{h_1}^{\alpha_1} \cdots C_{h_n}^{\alpha_n}$. Then*

$$A_h^\alpha H_t(f) = \int_{H_B} T_{By} f C_h^\alpha(1)(y) \, dn_t(y)$$

and

$$A_h^\alpha P_z(f) = \int_0^\infty A_h^\alpha H_t(f) N_t(z) \, dt/t.$$

$P_z(f)$ is an infinitely differentiable function of z for $z > 0$ and

$$\frac{d^n}{dz^n} P_z(f) = \int_0^\infty H_t(f) \frac{d^n}{dz^n} N_t(z) \, dt/t.$$

Proof. For $t > 0$ and h in H,

$$A_h H_t(f) = \lim_{s \downarrow 0} s^{-1}(T_{sBh} H_t(f) - H_t(f))$$

$$= \lim_{s \downarrow 0} H_t(s^{-1}(T_{sBh} f - f))$$

$$= \lim_{s \downarrow 0} s^{-1} \int_H (T_{B(y+sh)} f - T_{By} f) \, dn_t(y).$$

Let $g \in L^{p'}(H)$, $p' = p/(p-1)$ and $\langle \, , \, \rangle$ denote the dual pairing between L^p and $L^{p'}$. Set $\phi(y) = \langle T_{By} f, g \rangle$. By Theorem 3 of [9],

$$\int_{H_B} \phi(y + sh) \, dn_t(y) = \int_{H_B} \phi(y) D_{t/2}(y, sh) \, dn_t(y)$$

where

$$D_p(x, y) = \exp\left[\frac{(x, y)}{p} - \frac{\|y\|^2}{2p} \right].$$

Thus

$$A_h H_t(f) = \lim_{s \downarrow 0} s^{-1} \int_{H_B} T_{By} f [D_{t/2}(y, sh) - 1] \, dn_t(y) \cdot s^{-1}[D_{t/2}(y - sh) - 1].$$

$$s^{-1}[D_{t/2}(y, sh) - 1] = s^{-1}[S_{sh}(1)(y) - 1]$$

converges to $C_h(1)$ in $L^1(H, n_t)$ as $s \downarrow 0$ and

$$|s^{-1}(S_{sh}(1)(y) - 1)| \leq \text{const. exp} \, [2/t\,|(h, y)|](|(h, y)| + 1)$$

for $s \leq 1$, so that by the dominated convergence theorem,

$$A_h H_t(f) = \int_{H_B} T_{By} f C_h(1)(y) \, dn_t(y).$$

Proceed by induction. Let $\alpha = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$ and $\beta = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_n)$.

$$A_h^\beta H_t(f) = \lim_{s \downarrow 0} s^{-1} \int_{H_B} (T_{B(y + sh_i)} f - T_{By} f) C_h(1)(y) \, dn_t(y).$$

Let $g \in L^{p'}(H)$ and set $\phi(y) = \langle T_{By} f, g \rangle$.

$$\left\langle s^{-1} \int_{H_B} (T_{B(y + sh_i)} f - T_{By} f) C_h^\alpha(1)(y) \, dn_t(y), g \right\rangle$$

$$= s^{-1} \int_{H_B} (\phi(y + sh_i) - \phi(y)) C_h^\alpha(1)(y) \, dn_t(y).$$

If ϕ were a bounded tame function, this last integral would be

$$\int_{H_B} \phi(y) s^{-1} [S_{sh_i} C_h^\alpha(1)(y) - C_h^\alpha(1)(y)] \, dn_t(y).$$

Since the bounded tame functions are weakly dense in $L^\infty(H, n_t)$, this equality holds for all bounded n_t-measurable functions ϕ. Thus

$$A_h^\beta H_t(f) = \lim_{s \downarrow 0} s^{-1} \int_{H_B} T_{By} f [S_{sh_i} C_h^\alpha(1)(y) - C_h^\alpha(1)(y)] \, dn_t(y).$$

Each $C_h^\alpha(1)(y)$ is a polynomial and $s^{-1}[S_{sh_i} C_h^\alpha(1)(y) - C_h^\alpha(1)(y)]$, the difference quotient, is dominated in absolute value by an n_t-integrable tame function which is independent of s for sufficiently small s. Thus the dominated convergence theorem assures that $A_h^\beta H_t(f)$ exists in $L^p(H)$ and that the desired equation holds.

Similarly one shows that the difference quotients $s^{-1}(T_{sBh_i} A_h^\alpha H_t(f) - A_h^\alpha H_t(f))$ are dominated in p-norm by a polynomial in $t^{-1/2}$. Hence the differentiability of $P_z(f)$ follows from the dominated convergence theorem; $A_h^\alpha P_z(f)$ has the desired form.

Each of the difference quotients $h^{-1}[N_t^{(k)}(z + h) - N_t^{(k)}(z)]$ is dominated in absolute value by a function of the form $p(z, t^{-1/2}) \exp [-z^2/t]$ where $p(z, t^{-1/2})$ is a polynomial in each variable which is independent of h for sufficiently small h. Differentiability of $P_z(f)$ in z now follows from the dominated convergence theorem.

We now turn to the characterization of the bounded harmonic functions $U: (0, \infty) \to L^2(H)$. Let $p(x)$ be a polynomial on H. Set

$$W(p)(y) = \int_H p(2^{1/2} x + iy) \, dn(x).$$

$W(p)$ is the Wiener transform of $p(x)$ and W extends to a unitary operator on $L^2(H)$ (see [8]). If $f \in L^2(H)$, $W(T_y f) = \exp[iF(y)/2]W(f)$ where F is the representative of the normal distribution with variance parameter 1. Hence if $f \in L^2(H)$ is in the domain of A_j, $W(A_j f) = [iF(Be_j)/2]W(f)$. Thus if $U: (0, \infty) \to L^2(H)$ is harmonic; i.e. $U_{zz}(z) + \sum_{j=1}^{\infty} A_j^2 U(z) = 0$ for $z > 0$, then

$$W(U(z))_{zz} - \frac{(\|B^*(\cdot)\|^{\sim})^2}{4} W(U(z)) = 0$$

where $\|B^*(\cdot)\|^{\sim}$ is the measurable function determined by $\|B^* u\|$. In particular if $f \in L^2(H)$ the Poisson integral, $P_z(f)$, of f is harmonic. For as was shown in the proof of Proposition 1, the Fourier transform, $\hat{p}_z$, of the measure p_z is $\hat{p}_z(u)$ $= \exp[-z\|B^* u\|]$. Since $W(P_z(f)) = W(f) \int_H \exp[iF(y)/2] \, dp_z(y)$, we have that $W(P_z(f)) = W(f) \exp[-z/2\|B^*(\cdot)\|^{\sim}]$. Hence

$$W(P_z(f))_{zz} = \frac{(\|B^*(\cdot)\|^{\sim})^2}{4} W(P_z(f))$$

and $P_z(f)$ is harmonic.

THEOREM 1. *If f is in $L^2(H)$ and if $P_z(f)$ is its Poisson integral, $P_z(f)$ is harmonic and is bounded as a function of z, $z > 0$, to $L^2(H)$. Conversely, if $U: (0, \infty) \to L^2(H)$ is a harmonic function such that $\|U(z)\|_2$ is a bounded function of z, there is a unique function f in $L^2(H)$ such that $U(z) = P_z(f)$.*

Proof. The facts that $P_z(f)$ is bounded in z and is harmonic were established above.

Let $\|U(z)\|_2 \leq C < \infty$ for $z > 0$ and let U be harmonic. Let $z_1, z_2, z_3, \ldots$ be a sequence of positive real numbers which converges to zero. By virtue of the boundedness of the sequence $\{U(z_i)\}$ in $L^2(H)$ and by virtue of the weak compactness of a closed ball in $L^2(H)$ there is a subsequence, again denoted by $\{U(z_i)\}$, and a function f in $L^2(H)$ such that $\{U(z_i)\}$ converges weakly to f.

Set $\phi(z) = U(z + z_i) - P_z(U(z_i))$. Then $\phi(0) = 0$ and

$$\begin{aligned} W(\phi(z))_{zz} &= W(U(z + z_i))_{zz} - W(P_z(U(z_i)))_{zz} \\ &= \frac{(\|B^*(\cdot)\|^{\sim})^2}{4} [W(U(z + z_i)) - W(P_z(U(z_i)))] \\ &= \frac{(\|B^*(\cdot)\|^{\sim})^2}{4} W(\phi(z)). \end{aligned}$$

Set $a = 2^{-1}\|B^*(\cdot)\|^{\sim}$ and $\psi(z) = W(\phi(z))$. We have $\psi''(z) = a^2 \psi(z)$ or $\psi(z) = A \exp(az) + B \exp(-az)$. Since $\psi(0) = 0$, $B = -A$ and $\psi(z) = 2A \, \text{Sinh}(az)$; A is a measurable function. Since $a > 0$ and $z > 0$, $|2A \, \text{Sinh}(az)| \geq 2az|A|$. Since $\|U(z)\|_2 \leq C$, $\|\psi(z)\|_2$ $= \|\phi(z)\|_2 \leq 2C$; but $z\|A\|B^*(\cdot)\|^{\sim}\|_2 \leq \|\psi(z)\|_2$ and the left side of this inequality is unbounded in z unless $A = 0$ almost everywhere. Hence $\phi(z) = 0$ for $z \geq 0$ and $U(z + z_i) = P_z(U(z_i))$.

If g is in $L^2(H)$,

$$\langle g, P_z(U(z_i))\rangle = E(gP_z(U(z_i))) = \int_H \langle g, T_y U(z_i)\rangle \, dp_z(y) = \int_H \langle T_y^* g, U(z_i)\rangle \, dp_z(y).$$

Since $\{U(z_i)\}$ converges weakly to f as z_i tends to zero, $\langle g, P_z(U(z_i))\rangle$ converges to $\langle g, P_z(f)\rangle$ as z_i tends to zero by the bounded convergence theorem. Since $U(z+z_i) = P_z(U(z_i))$, $\langle g, U(z)\rangle = \langle g, P_z(f)\rangle$ for all g in $L^2(H)$. Hence $U(z) = P_z(f)$. The uniqueness of f follows by taking Wiener transforms and noting that the Fourier transform of p_z does not vanish.

REMARK 1. Let $1 < p < \infty$ and $f \in L^p(H)$ satisfy $\|T_y f - f\|_p \le K\|y\|$ where K is a finite positive constant and $y \to T_y$ is the regular representation of H on $L^p(H)$. We shall show that $P_z(f)$ is harmonic in a weak sense. Let $g \in L^{p'}(H)$, $p' = p/(p-1)$. Then $\phi(y) = \langle g, T_{By} f\rangle$ is a Lip 1 function on H_B. In the notation of [6] $p_t \phi(x) = \langle g, T_{Bx} H_t(f)\rangle = v(t, x)$. By Theorem 3 of [6], $v(t) = v(t, 0)$ is a boundedly and continuously differentiable function of t and

$$dv/dt = \tfrac{1}{4} \text{ trace } D^2 v(t, x)\big|_{x=0}$$

where D^2 is the second Fréchet derivative of v with respect to the second variable. (The change in the above constant is due to the fact that we have replaced Gross' parameter t by $t/2$.) $\int_0^\infty (dv/dt) N_t(z) \, dt/t$ exists and integration by parts shows that this integral is $-\tfrac{1}{4}\langle g, P_z(f)_{zz}\rangle$. By formula 9, p. 44 of [6],

$$((D^2 p_t \phi)(x)h_1, h_2) = 2/t \int_{H_B} \phi(x+y)\{2/t F(h_1)(y)F(h_2)(y) - (h_1, h_2)\} \, dn_t(y).$$

Thus

$$\text{trace } D^2 v(t, x)\big|_{x=0} = \sum_{j=1}^{\infty} 2/t \int_{H_B} \phi(y)\{2/t F(e_j)(y)^2 - 1\} \, dn_t(y) = \sum_{j=1}^{\infty} \langle g, A_j^2 H_t(f)\rangle$$

by Proposition 2. By inequality 38, p. 173 of [6], the trace class norm of $D^2 v(t, 0)$ is dominated by $Kt^{-1/2}$ where K is a positive constant. Hence $\sum_{j=1}^n |\langle g, A_j^2 H_t(f)\rangle|$ is dominated by $Kt^{-1/2}$ for all n. By the dominated convergence theorem we get that $P_z(f)_{zz} + \sum_{j=1}^\infty A_j^2 P_z(f)$ converges weakly to zero.

REMARK 2. An argument similar to that given in the proof of Theorem 1 shows that the bounded differentiable functions $V: (0, \infty) \to L^2(H)$ which satisfy $V'(z) = \sum_{j=1}^\infty A_j^2 V(z)$ for $z > 0$ must be of the form $V(t) = H_t(f)$ for some unique f in $L^2(H)$. H_t is as in equation (2).

3. **Conjugate harmonic functions.** Let B be the one-one Hilbert-Schmidt operator of §2 and let $e_1, e_2, e_3, \ldots$ be the orthonormal basis for the Hilbert space H. Let $n_t(z) = (\pi t)^{-1/2} \exp[-z^2/t]$ and let the representative of the normal distribution be the map F from H^* to Borel measurable functions on H_B; choose F so that it is independent of the variance parameter. For f in $L^p(H)$, $1 < p < \infty$, set

$$Q_z^{(j)}(f) = \lim_{R \to \infty} \int_0^R \left[\int_{H_B} T_{By} f F(e_j)(y) \, dn_t(y)\right] n_t(z) \, dt/t.$$

PROPOSITION 3. $Q_z^{(j)}$ *is a bounded operator on* $L^p(H)$ *for* $1 < p < \infty$.

Proof. Let e denote a generic element of the o.n. basis of H and let Q_z denote the conjugate Poisson integral corresponding to e.

$$\int_0^R \left[\iint_{H_B} T_{By} fF(e)(y) \, dn_t(y) \right] n_t(z) \, dt/t$$

$$= \pi^{-1/2} \int_0^R \left[\iint_{H_B} T_{t^{1/2}By} fF(e)(y) \, dn_1(y) \right] \exp\left[\frac{-z^2}{t} \right] dt/t;$$

for let $\phi(y) = \langle g, T_{By} f \rangle$ with g in $L^{p'}(H)$, $p' = p/(p-1)$. If $\phi(y)$ were a bounded tame function,

$$\int_{H_B} \phi(y) F(e)(y) \, dn_t(y) = t^{1/2} \int_{H_B} \phi(t^{1/2}y) F(e)(y) \, dn_1(y).$$

Since the bounded tame functions are weakly dense in $L^\infty(H)$, the desired equality holds for $\phi(y) = \langle g, T_{By} f \rangle$ and the first assertion holds.

Set $s^2 = t$ so that the above integral is

$$2\pi^{-1/2} \int_0^\rho \left[\iint_{H_B} T_{sBy} fF(e)(y) \, dn_1(y) \right] \exp\left[\frac{-z^2}{s^2} \right] ds/s, \quad \text{where } \rho = R^{1/2}.$$

Since $F(e)(y)$ is an odd function, this integral is

$$\int_{-\rho}^0 \left[\iint_{H_B} T_{sBy} fF(e)(y) \, dn_1(y) \right] \exp\left[\frac{-z^2}{s^2} \right] ds/s.$$

Thus

$$Q_z(f) = \pi^{-1/2} \lim_{R \to \infty} \int_{|t| \leq R} \left[\iint_{H_B} T_{tBy} fF(e)(y) \, dn_1(y) \right] \exp\left[\frac{-z^2}{t^2} \right] dt/t.$$

Let

$$Q_z^R(f) = \pi^{-1/2} \int_{|t| \leq R} \left[\iint_{H_B} T_{tBy} fF(e)(y) \, dn_1(y) \right] \exp\left[\frac{-z^2}{t^2} \right] dt/t.$$

Then

$$Q_z^R(f) = \pi^{-1/2} \int_{H_B} \left[\iint_{|t| \leq R} T_{tBy} \exp\left[\frac{-z^2}{t^2} \right] dt/t \right] F(e)(y) \, dn_1(y).$$

By Minkowski's integral inequality,

$$\|Q_z^R(f)\|_p \leq \int_{H_B} \left\| \int_{|t| \leq R} T_{tBy} f \exp\left[\frac{-z^2}{t^2} \right] dt/t \right\|_p |F(e)(y)| \, dn_1(y).$$

Set $s = t\|By\|$ so that

$$\int_{|t| \leq R} T_{tBy} f \exp\left[\frac{-z^2}{t^2} \right] dt/t$$

$$= \int_{|t| \leq R\|By\|} T_{t\omega} f \exp\left[\frac{-z^2\|By\|^2}{t^2} \right] dt/t \quad \text{where } \omega = By\|By\|^{-1}.$$

Let f be a bounded continuous tame function. Since the normal distribution on H is orthogonally invariant,

$$\left\| \int_{|t| \leq R\|By\|} T_{t\omega} f \exp\left[\frac{-z^2\|By\|^2}{t^2}\right] dt/t \right\|_p^p$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \int_{|s| \leq R\|By\|} f(x_1 - s, x_2, \ldots, x_n) \exp\left[\frac{x_1 s}{p} - \frac{s^2}{2p}\right] \right.$$

(*)
$$\left. \times \exp\left[\frac{-z^2\|By\|^2}{s^2}\right] ds/s \right|^p dn(x_1)\cdots dn(x_n)$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \int_{|s| \leq R\|By\|} f(x_1 - s, x_2, \ldots, x_n) \exp\left[\frac{-(x_1 - s)^2}{2p}\right] \right.$$

$$\left. \times \exp\left[\frac{-z^2\|By\|^2}{s^2}\right] ds/s \right|^p dx_1\, dn(x_2)\cdots dn(x_n).$$

Hence to prove that the $\|Q_z^R(f)\|_p$ are uniformly bounded, it suffices to show that

$$T_z^r(f) = \int_{|t| \leq r} f(x - t) \exp\left[\frac{-z^2}{t^2}\right] dt/t$$

is a uniformly bounded family of operators on $L^p((-\infty, \infty), dx)$, $1 < p < \infty$. Let f be a smooth rapidly decreasing function. Then the Fourier transform $\hat{T}_z^r(f)(u)$ of $T_z^r(f)$ is

$$2i\hat{f}(u) \int_0^r \sin(uy) \exp\left[\frac{-z^2}{y^2}\right] dy/y.$$

Let

$$\phi_r(u) = \int_0^r \sin(uy) \exp\left[\frac{-z^2}{y^2}\right] dy/y.$$

Then $\phi_r(u) = \int_0^{ru} \sin y \exp[-u^2 z^2/y^2]\, dy/y$ for $u \neq 0$ and $\phi_r(0) = 0$. Now,

$$\rho \int_{\rho/2}^{\rho} |\phi_r'(u)|^2\, du$$

can be estimated as follows.

$$\phi_r'(u) = \int_0^{ru} \exp\left[\frac{-u^2 z^2}{y^2}\right] \sin y \left[\frac{-2uz^2}{y^2}\right] dy/y + \exp\left[\frac{-z^2}{r^2}\right] \sin(ur)/u$$

and

$$\left(\int_{\rho/2}^{\rho} |\phi_r'(u)|^2\, du\right)^{1/2} \leq K \int_0^{\infty} \exp\left[\frac{-\rho^2 z^2}{4y^2}\right] \frac{z^2}{y^2} \left(\int_{\rho/2}^{\rho} u^2\, du\right)^{1/2} dy/y + K\rho^{-1/2} \leq K_1 \rho^{-1/2}.$$

Thus $\rho \int_{\rho/2}^{\rho} |\phi_r'(u)|^2\, du \leq K_1 < \infty$. By Hörmander's Theorem on L^p-multipliers [7], the T_z^r form a uniformly bounded family of operators on $L^p((-\infty, \infty), dx)$. Hence the Q_z^R form a bounded family on $L^p(H)$.

We shall prove that the Q_z^R converge strongly as R tends to infinity. It is sufficient to show that the T_z^r converge strongly as $r \to \infty$. Then two applications of the dominated convergence theorem show that the Q_z^R converge strongly.

$$T_z^r(f) - T_z^\rho(f) = \pm \int_{\rho \leq |y| \leq r} f(x-y) \exp\left[\frac{-z^2}{y^2}\right] dy/y.$$

Let f be continuous with compact support and let $\alpha, \beta > 1$ satisfy $1/p = 1/\alpha + 1/\beta - 1$. Apply Young's inequality to the above difference to get

$$\|T_z^r(f) - T_z^\rho(f)\|_p \leq \|f\|_\alpha \left(\int_{\rho \leq |y| \leq r} y^{-\beta} \, dy\right)^{1/\beta}.$$

The right side of the last inequality tends to zero as ρ, r tend to infinity. A density argument completes the proof of the strong convergence of the T_z^r.

For bounded continuous tame functions f on H

$$\left\|\int_{|t| \leq r} T_{tBy} f \exp\left[\frac{-z^2}{t^2}\right] dt/t\right\|_p \leq K\|f\|_p$$

where K is a constant which is independent of z, y, and r. Hence by the dominated convergence theorem,

$$\lim_{R,\rho \to \infty} \|Q_z^R(f) - Q_z^\rho(f)\|_p \leq \int_{H_B} \lim_{R,\rho \to \infty} \left\|\int_{\rho \leq |t| \leq R} T_{tBy} f \exp\left[\frac{-z^2}{t^2}\right] dt/t\right\|_p |F(e)(y)| dn_1(y).$$

Use the fact that f is tame to write $\|\int_{\rho \leq |t| \leq R} T_{tBy} f \exp[-z^2/t^2]\, dt/t\|_p$ as in (*). Since the $T_z^r(f)$ are uniformly bounded, we may apply the dominated convergence theorem to conclude that the $Q_z^R(f)$ converge in $L^p(H)$. A density argument completes the proof.

For f in $L^p(H)$, $1 < p < \infty$, the $Q_z^{(j)}(f)$ are the conjugate Poisson integrals of f.

If one applies the Wiener transform to the operator $Q_z^{(j)}$ on $L^2(H)$ one sees that

$$W(Q_z^{(j)}(f)) = i\frac{(B^*e_j, \cdot)^\sim}{\|B^*(\cdot)\|^\sim} \exp[-z/2\|B^*(\cdot)\|^\sim] W(f).$$

Hence, at least in $L^2(H)$, $Q_z^{(j)} = P_z \circ R_j$ where R_j is the singular integral operator

$$R_j(f) = \lim_{\rho \to \infty; \delta \downarrow 0} \int_\delta^\rho \left[\int_{H_B} T_{tBy} f F(e_j)(y) \, dn(y)\right] dt/t.$$

These operators have been studied in [1].

PROPOSITION 4. *For $1 < p < \infty$, $Q_z^{(j)} = P_z \circ R_j$ as an operator on $L^p(H)$.*

Proof. Let e denote a generic element of the basis $e_1, e_2, e_3, \ldots$ of H and let Q and R be the corresponding operators. Let F be the usual representative of the

normal distribution which maps to Borel measurable functions on the Wiener space H_B. Let S be a finite dimensional orthogonal projection on H. Set

$$Q_z^S(f) = \lim_{R \to \infty} \int_0^R \left[\int\!\!\int_{H_B} T_{SBy}fF(e)(y)\,dn_t(y) \right] n_t(z)\,dt/t,$$

$$P_z^S(f) = \int_0^\infty \left[\int\!\!\int_H T_{Sy}f\,dn_t \circ B^{-1}(y) \right] N_t(z)\,dt/t,$$

and

$$R^S(f) = \lim_{\delta \to 0; \rho \to \infty} \int_\delta^\rho \left[\int\!\!\int_{H_B} T_{tSBy}fF(e)(y)\,dn(y) \right] dt/t.$$

We may suppose that $f \geqq 0$. Each of these operators is bounded independently of S on $L^p(H)$.

Let $Q_z^{S\rho}(f) = \int_0^\rho \left[\int\!\!\int_{H_B} T_{SBy}fF(e)(y)\,dn_t(y) \right] n_t(z)\,dt/t$. Let f be a bounded continuous tame function on H and let V be the orthogonal projection onto the span of e and the base of f. Let $S > V$. We shall show that $Q_z^{S\rho}(f)$ converges uniformly in S to $Q_z^S(f)$ as $\rho \to \infty$.

$$\|Q_z^{S\rho}(f) - Q_z^{Sr}(f)\|_p \leqq K \left\| \int_r^\rho \left[\int\!\!\int_{H_B} T_{t^{1/2}SBy}fF(e)(y)\,dn_1(y) \right] \exp\left[\frac{-z^2}{t} \right] dt/t \right\|_p$$

$$\leqq K \int_{H_B} \left\| \int_r^\rho T_{t^{1/2}SBy}f \exp\left[\frac{-z^2}{t} \right] dt/t \right\|_p |F(e)(y)|\,dn_1(y)$$

by Fubini's theorem and Minkowski's inequality. Since $S > V$ and f is based in VH, $T_{t^{1/2}SBy}f = (T_{t^{1/2}VBy}f)D_\rho(\cdot, t^{1/2}(S-V)By)$. For each y in H_B the functions on the right are independent tame functions. The first is based in VH and the second is based in $(S-V)H$. The product is based in SH. We write the normal distribution on SH as a product and apply Minkowski's inequality to get

$$\left\| \int_r^\rho T_{t^{1/2}SBy}f \exp\left[\frac{-z^2}{t} \right] dt/t \right\|_p \leqq \left\| \int_r^\rho T_{t^{1/2}VBy}f \exp\left[\frac{-z^2}{t} \right] dt/t \right\|_p .$$

Set $u^2 = t$ and $\omega = VBy\|VBy\|^{-1}$, use the fact that f is tame and based on VH and the fact that the normal distribution is rotationally invariant. Let $1/p = 1/\alpha + 1/\beta - 1$, $\alpha, \beta > 1$ and apply Young's inequality. It follows that

$$\|Q_z^{S\rho}(f) - Q_z^{SR}(f)\|_p \leqq K\delta(\rho, R)\left(\int_{H_B} \|VBy\|^{(1-\alpha)/\alpha}|F(e)(y)|\,dn_1(y) \right)$$

where $\delta(\rho, R)$ tends to zero as $\rho, R \to \infty$ and where K is independent of S. We may choose α so that $\alpha < 2$. By Hölder's inequality the last integral is dominated by $K(\int_H \|VBy\|^{1-\alpha}\,dn_1(y))^{1/2}$. Let K_1 denote the kernel of VB. On $K_1^\perp$, VB is a one-one finite dimensional operator into H. So there is a constant C such that $\|y\| \leqq C\|VBy\|$

for y in K_1. Write the normal distribution on K_1^1 in polar coordinates; if V is sufficiently large, dim $(K_1) \geqq 2$, the last integral is finite. Thus as $r, \rho \to \infty$,

$$\| Q_z^{S\rho}(f) - Q_z^{Sr}(f) \|_p$$

converges to zero uniformly in S when f is a tame function and S is sufficiently large.

Thus if f is a bounded continuous tame function,

$$\| Q_z(f) - Q_z^S(f) \|_p \leqq \| Q_z(f) - Q_z^\rho(f) \|_p + \| Q_z^\rho(f) - Q_z^{S\rho}(f) \|_p + \| Q_z^{S\rho}(f) - Q_z^S(f) \|_p.$$

For $\varepsilon > 0$, there is a ρ_0 such that the first and third terms on the right are each $< \varepsilon/3$ when $\rho \geqq \rho_0$. Fix $\rho \geqq \rho_0$. By the strong continuity of the regular representation of H acting on $L^p(H)$ and by the bounded convergence theorem, the second term on the right converges to zero as S tends strongly to the identity through the directed set of finite dimensional projections on H. Hence $\lim_{S \to I} Q_z^S(f) = Q_z(f)$. Since the bounded continuous tame functions are dense in $L^p(H)$, Q_z is the strong limit of the net Q_z^S.

A similar argument shows that R is the strong limit of the net R^S [2]. The dominated convergence theorem implies that P_z is the strong limit of the P_z^S.

Let f be a bounded continuous tame function. Then $Q_z^S(f)$ and $P_z^S(R^S(f))$ are tame functions. By taking Wiener transforms (with variance parameter $p/2$) of each of these functions we see that $Q_z^S(f) = P_z^S(R^S(f))$. Since the Q_z^S, P_z^S, and R^S are uniformly bounded in S, we may take the limit in S on each side of this equality. This gives the conclusion of the proposition.

In what follows we shall consider only the case $p = 2$. For f in $L^2(H)$ we set $U(z) = P_z(f)$, $v_j(z) = -Q_z^{(j)}(f)$, $V(z) = (v_1, v_2, v_3, \ldots)$, and $F(z) = (U(z), V(z))$.

PROPOSITION 5. $(\partial/\partial z)U(z) + \sum_{j=1}^\infty A_j v_j(z)$ *converges to zero in* $L^2(H)$. *For* $i, j = 1, 2, 3, \ldots$ $A_j v_i = A_i v_j$, *and* $A_j U = (\partial/\partial z)v_j(z)$.

Proof. We shall first calculate the Wiener transform of v_j. By Proposition 4, $v_j(z) = -P_z(R_j(f))$. From §2 we know that $W(P_z(f)) = W(f) \exp [-z/2 \| B^*(\cdot) \|]$. From [1] we have that $W(R_j(g)) = W(g)[i(B^*(\cdot), e_j) \| B^*(\cdot) \|^{-1}]$. Thus $(\partial/\partial z)U(z) + \sum_{j=1}^N A_j v_j(z)$ has Wiener transform

$$-\frac{1}{2} \left[\| B^*(\cdot) \| - \| B^*(\cdot) \|^{-1} \left(\sum_{j=1}^N (B^*(\cdot), e_j)^2 \right) \right] \exp [-z/2 \| B^*(\cdot) \|] W(f).$$

The first term in this product converges to zero in measure. For each N, this quantity is dominated in absolute value by a constant multiple of $|W(f)|$. By the dominated convergence theorem $(\partial/\partial z)U(z) + \sum_{j=1}^\infty A_j v_j(z)$ converges to zero in $L^2(H)$.

The other assertions of the proposition follow by taking Wiener transforms.

The vector $F(z) = (U(z), V(z))$ has $|F(z)| = (U(z)^2 + \sum_{j=1}^\infty v_j(z)^2)^{1/2}$ in $L^2(H)$ for each z and $\| |F(z)| \|_2 = 2^{1/2} \| U(z) \|_2$. For

$$E \left(\sum_{j=m}^n v_j(z)^2 \right) = E \left(|W(U)|^2 \sum_{j=m}^n (B^*(\cdot), e_j)^2 \| B^*(\cdot) \|^{-2} \right).$$

$\sum_{j=m}^{n} \|B^*(\cdot)\|^{-2}(B^*(\cdot), e_j)^2$ converges boundedly to zero in measure as $n, m \to \infty$. The dominated convergence theorem implies that $E(\sum_{j=m}^{n} v_j(z)^2)$ converges to zero as $n, m \to \infty$. Hence $|F(z)|$ is in $L^2(H)$ for each $z > 0$. That $\| |F(z)| \|_2 = 2^{1/2}\|U(z)\|_2$ follows by taking Wiener transforms in

$$E(|F(z)|^2) = E(U(z)^2) + \sum_{j=1}^{\infty} E(v_j(z)^2)$$

$$= E(|W(U)|^2) + E\left(\sum_{j=1}^{\infty} (B^*(\cdot), e_j)^2 \|B^*(\cdot)\|^{-2}|W(U)|^2\right)$$

$$= 2E(|W(U)|^2) = 2E(U^2);$$

here the dominated convergence theorem is used as above to insure these equalities.

In what follows we shall denote $\partial/\partial z = A_0$ and the map $U(z) \to U(z+h)$ by $(T_h U)(z)$. Set $U(z) = v_0(z)$ and $A_j F(z) = (A_j v_0, A_j v_1, A_j v_2, \dots)$ for $j = 0, 1, 2, 3, \dots$. Then $|A_j F|$ and $|A_j^2 F|$ are in $L^2(H)$ and $\| |A_j F| \|_2 \leq \text{const} \|f\|_2$ and $\| |A_j^2 F| \|_2 \leq \text{const} \|f\|_2$.

THEOREM 2. *The notation is as established above. If $|F(z)| > 0$ almost everywhere for each $z > 0$, $A_j|F(z)| = (F(z), A_j F(z))|F(z)|^{-1}, j = 0, 1, 2, 3, \dots$. The second derivative $A_j^2|F(z)| = \lim_{t\to 0} t^{-1}[T_{tBe_j}A_j|F(z)| - A_j|F(z)|]$ exists in measure. The sum $\sum_{j=0}^{\infty} A_j^2|F(z)|$ converges in measure and the sum is positive almost everywhere.*

Proof. For brevity set $F(z) = F$ and $T_{tBe_j} = T_t$ for the computation of the derivatives.

Step I. We shall show that $\lim_{t\to 0} \|t^{-1}(T_t|F| - |F|) - |F|^{-1}(F, A_j F)\|_2 = 0$. Expand the integrand and consider

$$I_t = E[|F|^{-2}(F, A_j F)^2 - |F|^{-1}(F, A_j F)t^{-1}(T_t|F| - |F|)]$$

and

$$II_t = E[t^{-2}(T_t|F| - |F|)^2 - |F|^{-1}(F, A_j F)t^{-1}(T_t|F| - |F|)].$$

First note that $T_t|F| = |T_t F|$ almost everywhere. For if F were a vector with finitely many nonzero components and if each component were a bounded continuous tame function, this equality would hold. The set of all vectors with finitely many nonzero components each of which is a bounded continuous tame function is dense in the set of all square integrable functions from H to l_2. Hence the above equality holds.

$$I_t = E[|F|^{-1}(F, A_j F)(|F|^{-1}(F, A_j F) - t^{-1}(|T_t F|^2 - |F|^2)(|T_t F| + |F|)^{-1})].$$

$$|T_t F|^2 - |F|^2 = \sum_{i=0}^{\infty} [T_t v_i + v_i][t A_j v_i + o_i(t)]$$

where $o_i(t) = (T_t v_i - v_i) - t A_j v_i$. Let $o(t) = (o_0(t), o_1(t), o_2(t), \dots)$. Then

$$E(|t^{-1}o(t)|^2) = E\left(\sum_{i=0}^{\infty} [t^{-1}(\exp it/2F(Be_j) - 1) - i/2F(Be_j)]^2 |W(v_i)|^2\right)$$

tends to zero by the dominated convergence theorem (using the fact that v_i is a Poisson integral). Thus

$$I_t = E[|F|^{-2}(F, A_j F)^2 - (F, A_j F)|F|^{-1}(T_t F + F, A_j F)(|T_t F| + |F|)^{-1}]$$
$$+ E[(T_t F + F, t^{-1}o(t))(|T_t F| + |F|)^{-1}].$$

Each integrand tends to zero in measure as $t \to 0$. By the Schwarz inequality, the first integrand is dominated by an integrable function which is independent of t; hence the first integral converges to zero as $t \to 0$. The second integral is dominated in absolute value by $\|t^{-1}|o(t)|\|_2$ by the Schwarz inequality and this tends to zero as $t \to 0$. Thus $\lim_{t \to 0} I_t = 0$.

For II_t write

$$II_t = E[(|T_t F| + |F|)^{-2}(T_t F + F, A_j F)^2 - |F|^{-1}(F, A_j F)(T_t F + F, A_j F)(|T_t F| + |F|)^{-1}]$$
$$+ 2E[(|T_t F| + |F|)^{-2}(T_t F + F, A_j F)(T_t F + F, t^{-1}o(t))]$$
$$+ E[(T_t F + F, t^{-1}o(t))^2(|T_t F| + |F|)^{-2}]$$
$$- E[(|F|^{-1}(F, A_j F)(|T_t F| + |F|)^{-1}(T_t F + F, t^{-1}o(t))]$$

where $o(t)$ is as above. The integrand in the first integral converges to zero in measure as t tends to zero since T_t is strongly continuous, and this integrand is dominated by $2|A_j F|^2$. So the first integral tends to zero with t. The second integral is dominated in absolute value by $2E(|A_j F|t^{-1}|o(t)|) \leqq K\|t^{-1}|o(t)|\|_2$ which tends to zero with t. The third integral is dominated by $K\|t^{-1}|o(t)|\|_2^2$, as is the last integral, so that these tend to zero with t. We have calculated the first derivatives of $|F|$.

Step II. We show that $A_j^2|F| = |F|^{-1}|A_j F|^2 + |F|^{-1}(F, A_j^2 F) - |F|^{-3}(F, A_j F)^2$ and $t^{-1}(T_t A_j |F| - A_j |F|)$ converges in measure to $A_j^2|F|$ as t tends to zero. (This difference quotient does not converge in $L^2(H)$. It is convenient, however, to continue to denote the second partial by A_j^2.)

$A_j|F| = |F|^{-1}(F, A_j F)$ and $T_t A_j |F| = |T_t F|^{-1}(T_t F, T_t A_j F)$ almost everywhere. Consider $|F|T_t A_j |F|$. Write $|F|T_t A_j |F| - |F|A_j|F| = a_1(t) + a_2(t) + a_3(t)$ where $a_1(t) = |F| |T_t F|^{-1}(T_t F, T_t A_j F) - (T_t F, T_t A_j F) = |T_t F|^{-1}(T_t F, T_t A_j F)(|F| - |T_t F|)$. By Step I, $|F| - |T_t F| = -|F|^{-1}(F, A_j F)t + o(t)$ where $\|t^{-1}o(t)\|_2$ tends to zero as t tends to zero. So

$$a_1(t) = t[-|F|^{-1}(F, A_j F)]|T_t F|^{-1}(T_t F, T_t A_j F) + o(t)|T_t F|^{-1}(T_t F, T_t A_j F).$$

$$a_2(t) = (T_t F, T_t A_j F) - (F, T_t A_j F) = t(A_j F, T_t A_j F) + (o(t), T_t A_j F)$$

where $o(t)$ is as in Step I and $\|t^{-1}|o(t)|\|_2$ tends to zero with t.

$$a_3(t) = (F, T_t A_j F) - (F, A_j F) = t(F, A_j^2 F) + (F, o(t)).$$

Thus

$$\|t^{-1}|F|(T_t A_j |F| - A_j |F|) - |F|A_j^2|F|\|_1$$
$$\leqq \|t^{-1}a_1(t) + |F|^{-2}(F, A_j F)^2\|_1 + \|t^{-1}a_2(t) - |A_j F|^2\|_1 + \|t^{-1}a_3(t) - (F, A_j^2 F)\|_1$$
$$\leqq \| |F|^{-2}(F, A_j F)^2 - |F|^{-1}(F, A_j F)|T_t F|^{-1}(T_t F, T_t A_j F)\|_1$$
$$+ \|t^{-1}o(t)|T_t F|^{-1}(T_t F, T_t A_j F)\|_1 + \| |A_j F|^2 - (A_j F, T_t A_j F)\|_1$$
$$+ \|t^{-1}(A_j F, o(t))\|_1 + \|t^{-1}(F, o(t))\|_1.$$

By the Schwarz inequality the first term on the right is dominated by

$$K\||A_j|F| - T_t A_j|F|\|_2$$

which converges to zero with t because of the strong continuity of the T_t. The second term is dominated by $K\|t^{-1}o(t)\|_2$ and tends to zero with t. Similar arguments show that the other terms on the right tend to zero with t. This proves that $|F|[T_t A_j|F| - A_j|F|]/t$ converges in $L^1(H)$ to the desired quantity as t tends to zero. Since $|F| > 0$ almost everywhere for each $z > 0$, $t^{-1}[T_t A_j|F| - A_j|F|]$ converges in measure to $|F|^{-1}|A_j F|^2 + |F|^{-1}(F, A_j^2 F) - |F|^{-3}(F, A_j F)^2$.

Step III. $\sum_{j=0}^{\infty} A_j^2|F|$ converges in measure and the sum is positive almost everywhere.

Notice first that by the conclusion of Step II, $|F|A_j^2|F| \geq (F, A_j^2 F)$ since $|F|^{-2}(F, A_j F)^2 \leq |A_j F|^2$.

$\sum_{j=0}^{\infty} |A_j F|^2$ converges in $L^1(H)$. For

$$E\left(\sum_{j=n}^{m} |A_j F|^2\right) = \sum_{j=n}^{m} E(|A_j F|^2) = \sum_{j=n}^{m} E\left(\sum_{i=0}^{\infty} (A_j v_i)^2\right).$$

$$E((A_j v_i)^2) = \tfrac{1}{4} E((B^*(\cdot), e_j)^2 (B^*(\cdot), e_i)^2 \|B^*(\cdot)\|^{-2} \exp[-z\|B^*(\cdot)\|]|W(f)|^2)$$

$$\text{for } i, j \geq 1.$$

For $i = 0$,

$$E((A_j v_0)^2) = \tfrac{1}{4} E((B^*(\cdot), e_j)^2 \exp[-z\|B^*(\cdot)\|]|W(f)|^2).$$

By the dominated convergence theorem, $E(|A_j F|^2) = \tfrac{1}{2} E((B^*(\cdot), e_j)^2 |W(U)|^2)$. As $m, n \to \infty$, $\sum_{j=n}^{m} E(|A_j F|^2)$ converges to zero since

$$\sum_{j=n}^{m} (B^*(\cdot), e_j)^2 \exp[-z\|B^*(\cdot)\|]$$

converges boundedly to zero in measure.

Since $|F|^{-2}(F, A_j F)^2 \leq |A_j F|^2$, $\sum_{j=0}^{\infty} |F|^{-2}(F, A_j F)^2$ converges in $L^1(H)$.
$\sum_{j=0}^{\infty} (F, A_j^2 F)$ converges in $L^1(H)$. For

$$E\left(\left|\sum_{j=n}^{m} (F, A_j^2 F)\right|\right) = E\left(\left|\left(F, \sum_{j=n}^{m} A_j^2 F\right)\right|\right) \leq E\left(|F| \left|\sum_{j=n}^{m} A_j^2 F\right|\right)$$

$$\leq \left\|\,\||F|\|_2 \left|\sum_{j=n}^{m} A_j^2 F\right|\right\|_2.$$

$$\left\|\sum_{j=n}^{m} A_j^2 F\right\|_2^2 = E\left(\sum_{i=0}^{\infty} \left(\sum_{j=n}^{m} A_j^2 v_i\right)^2\right) \cdot E\left(\left(\sum_{j=n}^{m} A_j^2 v_i\right)^2\right) = E\left(\sum_{j,k=n}^{m} A_j^2 v_i A_k^2 v_i\right)$$

$$= \frac{1}{16} E\left(\sum_{j,k=n}^{m} (B^*(\cdot), e_j)^2 (B^*(\cdot), e_k)^2 (B^*(\cdot), e_i)^2 \|B^*(\cdot)\|^{-2} \exp[-z\|B^*(\cdot)\|]|W(f)|^2\right)$$

when $i \geq 1$ and

$$E\left(\left(\sum_{j=n}^{m} A_j^2 v_0\right)^2\right) = \frac{1}{16} E\left(\sum_{k,j=n}^{m} (B^*(\cdot), e_j)^2 (B^*(\cdot), e_k)^2 \exp[-z\|B^*(\cdot)\|]|W(f)|^2\right).$$

Thus

$$E\left(\left|\sum_{j=n}^{m} A_j^2 F\right|^2\right) = \frac{1}{8} E\left(\sum_{j,k=n}^{m} (B^*(\cdot), e_j)^2 (B^*(\cdot), e_k)^2 \exp\left[-z\|B^*(\cdot)\|\right]|W(f)|^2\right).$$

$$\sum_{j,k=n}^{m} (B^*(\cdot), e_j)^2 (B^*(\cdot), e_k)^2 \exp\left[-z\|B^*(\cdot)\|\right] \leq \|B^*(\cdot)\|^4 \exp\left[-z\|B^*(\cdot)\|\right]$$

and this sum converges boundedly and in measure to zero as n, m tend to infinity. Hence $\sum_{j=0}^{\infty} (F, A_j F)$ converges in $L^1(H)$.

Hence $\sum_{j=0}^{\infty} |F|A_j^2|F|$ converges in $L^1(H)$ and $\sum_{j=0}^{\infty} A_j^2|F|$ converges in measure.

Since $\sum_{j=0}^{\infty} |F|A_j^2|F| \geq \sum_{j=0}^{\infty} (F, A_j^2 F)$ and since each component v of F satisfies $\sum_{j=0}^{\infty} A_j^2 v = 0$, the right side of the last inequality converges to zero in $L^1(H)$. We know from above that $\sum_{j=0}^{\infty} (F, A_j^2 F)$ converges in $L^1(H)$. Consider

$$\sum_{j=0}^{n} (F, A_j^2 F) = \left(F, \sum_{j=0}^{n} A_j^2 F\right)$$

and

$$E\left(\left|F, \sum_{j=0}^{n} A_j^2 F\right|\right) \leq \|F\|_2 \left\|\sum_{j=0}^{n} A_j^2 F\right\|_2.$$

$$\left\|\sum_{j=0}^{n} A_j^2 F\right\|_2^2 = E\left(\sum_{i=0}^{\infty} \left(\sum_{j=0}^{n} (A_j^2 v_i)\right)^2\right).$$

Consider

$$E\left(\left(\sum_{j=0}^{n} A_j^2 v_i\right)^2\right) = E\left[\left(\frac{\|B^*(\cdot)\|^2}{4} - \sum_{j=1}^{m} \frac{(B^*(\cdot), e_j)^2}{4}\right)^2\right.$$

$$\left.\times \exp\left[-z\|B^*(\cdot)\|\right](B^*(\cdot), e_i)^2\|B^*(\cdot)\|^{-2}|W(f)|^2\right]$$

for $i \geq 1$ and

$$E\left(\left(\sum_{j=0}^{n} A_j^2 v_0\right)^2\right) = E\left[\left(\frac{\|B^*(\cdot)\|^2}{4} - \sum_{j=1}^{n} (B^*(\cdot), e_j)^2\right) \exp\left[-z\|B^*(\cdot)\|\right]|W(f)|^2\right].$$

By the dominated convergence theorem

$$\left\|\sum_{j=0}^{n} A_j^2 F\right\|_2^2 = 2E\left[\left(\frac{\|B^*(\cdot)\|^2}{4} - \sum_{j=1}^{n} \frac{(B^*(\cdot), e_j)^2}{4}\right)^2 \exp\left[-z\|B^*(\cdot)\|\right]|W(f)|^2\right].$$

$$\left(\frac{\|B^*(\cdot)\|^2}{4} - \sum_{j=1}^{n} \frac{(B^*(\cdot), e_j)^2}{4}\right)^2 \exp\left[-z\|B^*(\cdot)\|\right]$$

converges boundedly and in measure to zero. Hence $\||\sum_{j=0}^{n} A_j^2 F|\|_2^2$ converges to zero as n tends to infinity. Hence $\sum_{j=0}^{\infty} (F, A_j^2 F)$ converges to zero in $L^1(H)$ and $\sum_{j=0}^{\infty} A_j^2|F| \geq 0$ almost everywhere.

REMARK 1. If f is nonnegative and somewhere positive, $P_z(f)$ is positive almost everywhere and $|F|$ is positive almost everywhere.

REMARK 2. An argument similar to that given above shows that if $G(z)$ $=((\partial/\partial z)U, A_1v_1, A_2v_2, A_3v_3, \ldots)$ and if $|G(z)| > 0$ almost everywhere for $z > 0$, the sum $(\partial^2/\partial z^2)|G(z)| + \sum_{j=1}^{\infty} A_j^2|G(z)|$ converges in measure and is positive almost everywhere.

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