A VARIATIONAL METHOD FOR FUNCTIONS OF BOUNDED BOUNDARY ROTATION

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Introduction. In this paper we present a variational method for functions of bounded boundary rotation and solve certain general extremal problems for these functions. The variational method is based upon a general method of G. M. Goluzin [1], and has previously been used by the present author to solve extremal problems for several classes of univalent functions ([7] and [8]). Once the variation formulas are derived, the details of solving the extremal problems are very similar to those in [7] and [8].

Theorem 1 here has been obtained independently by Professor J. Pfaltzgraff, who used essentially the same methods. I wish to thank Professor Pfaltzgraff for providing me with his unpublished work.

Those functions of bounded boundary rotation which are also *univalent* form a subclass which has recently been studied by M. M. Schiffer and O. Tammi [9]. They use a different (though closely related) variational method. We shall compare their results with ours.

1. **Preliminaries.** Let V_k denote the class of normalized analytic functions $f(z)=z+a_2z^2+\cdots$ in the open unit disc $D=\{z : |z|<1\}$ which satisfy the condition $f'(z)\neq 0$ for all z in D, and which map D onto a domain with boundary rotation bounded by $k\pi$.

V. Paatero [6] has shown that $f(z) \in V_k$ iff

$$f'(z) = \exp\left(\int_0^{2\pi} -\log(1-ze^{-i\theta})\, d\psi(\theta)\right) \tag{1}$$

where $\psi(\theta)$ is a real-valued function of bounded variation with

$$\int_0^{2\pi} d\psi(\theta) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\psi(\theta)| \leqq k.$$

The geometric interpretation is as follows (see [5]): Let $f(z)$ map D onto a domain G. If G is a schlicht domain with a continuously differentiable boundary curve, let $\pi\psi(\theta)$ denote the angle of the tangent vector at the point $f(e^{i\theta})$ to the boundary curve with respect to the positive real axis. The boundary rotation of G is equal to $\pi \int_0^{2\pi} |d\psi(\theta)|$. If G does not have a sufficiently smooth boundary curve,

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the boundary rotation is defined by a limiting process. Finally, if G is a non-schlicht domain without interior branch points, the boundary rotation is similarly defined.

We define S_k to be the subclass of V_k consisting of the univalent functions in V_k.

2. **Univalent functions.** Clearly, $f(z) \in V_2$ iff $f(z)$ is a normalized univalent function mapping D onto a convex domain. Furthermore, V. Paatero [6] has shown that for $2 \leqq k \leqq 4$ the classes V_k consist entirely of univalent functions. We strengthen this assertion by showing:

PROPOSITION 1. *For $2 \leqq k \leqq 4$, the classes V_k consist entirely of close-to-convex functions.*

Proof. The proof of this proposition consists of stating an appropriate definition of close-to-convex functions. We choose the geometric interpretation originally given by Kaplan [3, p. 177]. Let $f(z)$ be analytic in D and continuously differentiable for $z = e^{i\theta}$, $0 \leqq \theta \leqq 2\pi$. $f(z)$ is close-to-convex if the angle of the tangent vector to the boundary curve of the image of D under $f(z)$ with respect to the positive real axis either increases as θ increases or else decreases in such a manner that it never drops to a value π radians below a previous value. If $f(e^{i\theta})$ is not sufficiently smooth, we appeal to the suitable limiting process (see also [2, p. 391]).

Now, if $f(z) \in V_k$, $2 \leqq k \leqq 4$, it clearly satisfies the above conditions and is close-to-convex. Since every close-to-convex function is univalent, see [3], the observation of Paatero is contained here.

3. **Variations for V_k.** G. M. Goluzin [1] derived variational formulas for classes E_g of analytic functions in D defined by the condition: $f(z) \in E_g$ iff

$$f(z) = \int_0^{2\pi} g(z, \theta)\, d\psi(\theta) \tag{2}$$

where $g(z, \theta)$ is a given function, analytic in D for $0 \leqq \theta \leqq 2\pi$, and $\psi(\theta)$ runs through all nondecreasing functions on $0 \leqq \theta \leqq 2\pi$ subject to the condition $\int_0^{2\pi} d\psi(\theta) = A$.

Goluzin's variations consist of appropriately varying the function $\psi(\theta)$ so as to preserve its monotonicity and total variation.

For a given function $f(z) \in E_g$ having the representation (2) Goluzin derived the following variational formulas:

$$f_*(z) = f(z) + \lambda \int_{\theta_1}^{\theta_2} \frac{\partial g(z, \theta)}{\partial \theta} |\psi(\theta) - c|\, d\theta \tag{3}$$

where θ_1 and θ_2 are arbitrary but fixed with $0 \leqq \theta_1 < \theta_2 \leqq 2\pi$, λ varies in the interval $-1 \leqq \lambda \leqq 1$, and c is real number independent of λ and θ.

$$f_{**}(z) = f(z) + \lambda(g(z, \theta_1) - g(z, \theta_2)) \tag{4}$$

where θ_1 and θ_2 with $0 \leqq \theta_1 < \theta_2 \leqq 2\pi$ are *jump points* for $\psi(\theta)$, and λ varies in the interval $-a < \lambda < a$, where $a > 0$ is sufficiently small.

To apply these variational formulas to V_k, we express the function $\psi(\theta)$ appearing in (1) as the difference of two increasing functions, $\psi(\theta)=\alpha(\theta)-\beta(\theta)$. Since

$$\int_0^{2\pi} d\psi(\theta) = \int_0^{2\pi} d\alpha(\theta)-\int_0^{2\pi} d\beta(\theta) = 2$$

and

$$\int_0^{2\pi} |d\psi(\theta)| = \int_0^{2\pi} d\alpha(\theta)+\int_0^{2\pi} d\beta(\theta) \leqq k,$$

we conclude that

$$\int_0^{2\pi} d\alpha(\theta) \leqq \frac{k}{2}+1 \tag{5}$$

and

$$\int_0^{2\pi} d\beta(\theta) \leqq \frac{k}{2}-1. \tag{6}$$

The representation (1) for V_k can now be rewritten as

$$f'(z) = \exp\left(\int_0^{2\pi} \log(1-ze^{-i\theta})\, d\beta(\theta)-\int_0^{2\pi} \log(1-ze^{-i\theta})\, d\alpha(\theta)\right) \tag{7}$$

where $\alpha(\theta)$ and $\beta(\theta)$ are nondecreasing functions on $[0, 2\pi]$ satisfying (5) and (6) respectively.

(7) can be written as

$$f'(z) = \exp(H(z)-G(z))$$

where

$$H(z) = \int_0^{2\pi} \log(1-ze^{-i\theta})\, d\beta(\theta)$$

and

$$G(z) = \int_0^{2\pi} \log(1-ze^{-i\theta})\, d\alpha(\theta).$$

As $\alpha(\theta)$ and $\beta(\theta)$ range over all nondecreasing functions satisfying (5) and (6) respectively, we obtain two classes of the form E_g described above. We apply the Goluzin formulas to these classes.

Varying $H(z)$ we obtain for V_k,

$$f_*'(z) = \exp(H_*(z)-G(z)) = \exp\left(H(z)-G(z)+\lambda\int_{\theta_1}^{\theta_2} \frac{ize^{-i\theta}}{1-ze^{-i\theta}}\, |\beta(\theta)-c|\, d\theta\right)$$

$$= f'(z)\cdot\exp\left(\lambda\int_{\theta_1}^{\theta_2} \frac{ize^{-i\theta}}{1-ze^{-i\theta}}\, |\beta(\theta)-c|\, d\theta\right).$$

Expanding the exponential up to first order terms in λ we have,

$$f_*'(z) = f'(z)+\lambda\int_{\theta_1}^{\theta_2} f'(z)\frac{ize^{-i\theta}}{1-ze^{-i\theta}}\, |\beta(\theta)-c|\, d\theta+O(\lambda^2). \tag{8}$$

Here, $O(\lambda^2)$ is an analytic function of z, and the error is uniform in each disc $|z| \leqq r < 1$.

Varying $G(z)$ we obtain the same formula with $\beta(\theta)$ replaced by $\alpha(\theta)$.

If θ_1 and θ_2 are jump points for $\alpha(\theta)$ or $\beta(\theta)$ we have the following variational formula for V_k:

$$(9) \qquad f'_{**}(z) = f'(z) + \lambda f'(z)(\log(1 - ze^{-i\theta_1}) - \log(1 - ze^{-i\theta_2})) + O(\lambda^2).$$

It should be remarked at this point that the formulas (8) and (9) are identical in form to the variational formulas derived in [8] for convex functions of order α. It is for this reason that, as mentioned in the introduction, the details to the solution of extremal problems for V_k will be very similar to those in [8].

4. A general extremal problem.

The solution of extremal problems for V_k is equivalent to finding the functions $\alpha(\theta)$ and $\beta(\theta)$ which appear in the representation (7) for the extremal function. We begin with a general extremal problem for V_k.

THEOREM 1. *Let $\zeta \neq 0$ be a given point in D, and let $F(X_1, X_2, \ldots, X_{n+1})$ be analytic in a neighborhood of each point $F(f'(\zeta), f''(\zeta), \ldots, f^{(n)}(\zeta), \zeta), f(z) \in V_k$. Then the functional*

$$J(f') = \operatorname{Re} F(f'(\zeta), f''(\zeta), \ldots, f^{(n)}(\zeta), \zeta)$$

attains its maximum (or minimum) in V_k only for a function of the form

$$(10) \qquad f'(z) = \prod_{j=1}^{M} (1 - \varepsilon_j z)^{\beta_j} \Big/ \prod_{j=1}^{N} (1 - e_j z)^{\alpha_j}$$

where $M < n$, $N \leqq n$, $|\varepsilon_j| = |e_j| = 1$ and $\sum_{j=1}^{M} \beta_j \leqq k/2 - 1$ and $\sum_{j=1}^{N} \alpha_j \leqq k/2 + 1$.

Proof. The compactness of V_k assures the existence of a function $f(z) \in V_k$ which maximizes (or minimizes) $J(f')$. Let $f(z)$ with

$$f'(z) = \exp \int_0^{2\pi} \log(1 - ze^{-i\theta})(d\beta(\theta) - d\alpha(\theta))$$

be such a function. For the functions $f'_*(z)$ defined by (8) we have

$$(11) \quad J(f'_*) = J(f') + \lambda \int_{\theta_1}^{\theta_2} \operatorname{Re} \sum_{j=1}^{n} d_j [i\zeta f'(\zeta)(e^{i\theta} - \zeta)^{-1}]^{(j-1)} |\beta(\theta) - c| \, d\theta + O(\lambda^2)$$

where $d_j = \partial F(X_1, \ldots, X_{n+1}) / \partial X_j$ evaluated at the extreme point

$$(f'(\zeta), \ldots, f^{(n)}(\zeta), \zeta).$$

(Here, $f^{(0)}(z) = f(z)$.)

The extremal property of $f(z)$ implies that

$$(12) \qquad \int_{\theta_1}^{\theta_2} Q(\theta) |\beta(\theta) - c| \, d\theta = 0$$

where

$$Q(\theta) = \operatorname{Re} \sum_{j=1}^{n} d_j [i\zeta f'(\zeta)(e^{i\theta}-\zeta)^{-1}]^{(j-1)}.$$

V_k is a rotation invariant family, and a theorem of Kirwan [4] assures us that at least one $d_j \neq 0$.

If the continuous function $Q(\theta)$ does not vanish in the interval (θ_1, θ_2), (12) implies that $\beta(\theta)=c$ throughout that interval. Now, $Q(\theta)=0$ is equivalent to an equation of degree at most $2n$ in $e^{i\theta}$. We therefore conclude that $\psi(\theta)$ is a step function with at most $2n$ jump points and furthermore, these jumps can only occur at the zeros of $Q(\theta)$. Let these points be $\theta_1, \ldots, \theta_{2n}$, with $\theta_1 < \theta_2 < \cdots < \theta_{2n}$.

We now construct the functions $f'_{**}(z)$ defined by (9). For these functions we have

$$\begin{aligned} J(f'_{**}) &= J(f') \\ &\quad + \lambda \operatorname{Re} \sum_{j=1}^{n} d_j [f'(\zeta)(\log(1-\zeta e^{-i\theta_1}) - \log(1-\zeta e^{-i\theta_2})]^{(j-1)} + O(\lambda^2). \end{aligned} \tag{13}$$

The extremal property of $f'(z)$ implies that the coefficient of λ in (13) must vanish. Thus, the function

$$R(\theta) = \operatorname{Re} \sum_{j=1}^{n} d_j [f'(\zeta) \log(1-\zeta e^{-i\theta})]^{(j-1)}$$

has the same value at θ_1 and θ_2. The Mean Value Theorem implies that $R'(\theta)$ must vanish at some point θ', with $\theta_1 < \theta' < \theta_2$. But $R'(\theta) = Q(\theta)$ and this is a contradiction.

We conclude that $\beta(\theta)$ could not have jump points at each of the $2n$ roots of $Q(\theta)$, but at most at every other root. Thus, $\beta(\theta)$ is a step function with at most n jump points.

A similar procedure with $\beta(\theta)$ replaced by $\alpha(\theta)$ in (8) and (9) yields the conclusion that $\alpha(\theta)$ is a step function having at most n jump points. Evaluating (7) with these $\alpha(\theta)$ and $\beta(\theta)$ we arrive at (10) and the theorem is proved.

For the class V'_k of functions with boundary rotation *equal* to $k\pi$, the conclusion of Theorem 1 is strengthened to $\sum \alpha_j = k/2+1$ and $\sum \beta_j = k/2-1$. Furthermore, as we shall see in the next section, for certain specific extremal problems in V_k we also conclude $\sum \alpha_j = k/2+1$ and $\sum \beta_j = k/2-1$.

5. **Distortion and rotation theorems.** We begin with the known distortion theorem for V_k [5].

COROLLARY 1. *Let* $f(z) \in V_k$. *Then,*

$$\frac{(1-|z|)^{k/2-1}}{(1+|z|)^{k/2+1}} \leqq |f'(z)| \leqq \frac{(1+|z|)^{k/2-1}}{(1-|z|)^{k/2+1}}.$$

These bounds are sharp and are attained by a function of the form (10) *with* $M=N=1$.

Proof. Set $J(f')=|f'(\zeta)|$ in Theorem 1.

COROLLARY 2. *Let $f(z)\in V_k$. Then for all $z\in D$,*

$$|\arg f'(z)| \leqq k\sin^{-1}|z|$$

where $\sin^{-1}0=0$.

This bound is sharp and is attained by a function of the form (10) *with* $M=N=1$.

Proof. Set $J(f')=\text{Re}\pm i\log f'(\zeta)$ in Theorem 1, and recall that

$$|\arg(1-re^{i\theta})| \leqq \sin^{-1}r.$$

It is interesting to observe that this is the same sharp bound on $|\arg f'(z)|$ obtained by Schiffer and Tammi [9, p. 143], for S_k.

If the functional $J(f)$ has the form $J(f)=\text{Re } F(\log f'(\zeta))$ we can strengthen the result of Theorem 1. In this case, the extremal function must be of the form

$$f'(z) = (1-\varepsilon z)^{k/2-1}/(1-ez)^{k/2+1}$$

with $|\varepsilon|=|e|=1$.

6. A general coefficient theorem.

THEOREM 2. *Let $F(X_2,\dots,X_n)$ be analytic in C^{n-1}. Then the functional*

$$C(f) = \text{Re } F(a_2,\dots,a_n)$$

where $f(z)=z+a_2z^2+\cdots$, attains its maximum (or minimum) in V_k only for a function of the form

$$f'(z) = \prod_{j=1}^{N}(1-\varepsilon_j z)^{\beta_j}\Big/\prod_{j=1}^{M}(1-e_j z)^{\alpha_j} \tag{14}$$

where $N\leqq n-1$, $M\leqq n-1$, $|\varepsilon_j|=|e_j|=1$, and $\sum_{j=1}^{N}\beta_j\leqq k/2-1$ and $\sum_{j=1}^{M}\alpha_j\leqq k/2+1$.

Proof. The proof is quite similar to the proof of Theorem 1 and we merely sketch it.

Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be extremal for the problem under consideration with

$$f'(z) = \exp\int_0^{2\pi}\log(1-ze^{-i\theta})(d\beta(\theta)-d\alpha(\theta)).$$

For the functions $f'_*(z)$ given by (8) we have

$$f_*(z) = z+\sum_{n=2}^{\infty}\left(a_n+\lambda\int_{\theta_1}^{\theta_2}\delta a_n|\beta(\theta)-c|\,d\theta\right)z^n+O(\lambda^2)$$

with

$$\delta a_n = \frac{i}{n}\sum_{j=0}^{n-2}(j+1)a_{j+1}\exp(-i(n-j-1)\theta),\quad (a_1 = 1).$$

Computing $C(f_*)$ and exploiting the extremal property of $f(z)$ we arrive at the condition

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \sum_{j=2}^{n} d_j\, \delta a_j |\beta(\theta)-c|\, d\theta = 0.$$

The continuous function $Q(\theta) = \operatorname{Re} \sum_{j=2}^{n} d_j\, \delta a_j$ has at most $2n-2$ roots with respect to θ in $[0, 2\pi]$. Thus, $\beta(\theta)$ is a step function with at most $2n-2$ jump points. Now, constructing $C(f_{**})$ we can prove, using the same reasoning as in the proof of Theorem 1, that $\beta(\theta)$ and $\alpha(\theta)$ are step functions having at most $n-1$ jump points, and the theorem is proved.

Setting $C(f) = |a_2|$, we conclude the known result (see e.g., [5]) that the function maximizing $|a_2|$ over V_k is a function of the form (14) with $N = M = 1$, $\beta_1 = k/2 - 1$, $\alpha_1 = k/2 + 1$.

For the problem of maximizing $|a_n|$ over S_k, Schiffer and Tammi [9, p. 137], have found the extremal function to be a mapping of D onto a polygon with N corners and $N \leqq 2(n-2)$. This agrees with the conclusion of Theorem 2. The expression for the extremal function in Theorem 2, (14), gives a bit more information about the structure of the extremal function.

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