INFINITE PRIMES AND UNIQUE FACTORIZATION IN A PRINCIPAL RIGHT IDEAL DOMAIN(1)

BY

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1. **Introduction.** Throughout this paper all rings are assumed to be integral domains with unity. If R is an integral domain R^* will denote the set of nonzero elements of R. An integral domain in which the sum and intersection of any two principal right ideals is principal whenever the intersection is nonzero is called a weak Bézout domain. In particular, if we do not require that the intersection be nonzero then R is called a right Bézout domain (see [2]). If R is a weak Bézout domain and $a \in R^*$ then the set $[aR, R]$ of all principal right ideals of R that contain aR is a sublattice of the lattice of all right ideals of R. In this case dim a is defined to be the dimension of the lattice $[aR, R]$, i.e., dim a is the length of the longest chain in $[aR, R]$. We let R' denote the set of finite dimensional elements of R. If each right ideal of an integral domain R is a principal right ideal then R is called a PRI (principal right ideal) domain. Thus each PRI domain is a weak Bézout domain.

The known types of unique factorization that occur in a PRI domain R deal with the members of R'. For example, it is well known [4, p. 34] that each nonunit z of R' can be factored into primes: $z = p_1 \cdots p_n$, and if $z = q_1 \cdots q_m$ is another such factorization, then $n = m$ and there is a permutation Π on $\{1, 2, \ldots, n\}$ such that p_i and $q_{\Pi(i)}$ are similar, $i = 1, 2, \ldots, n$. There is another type of unique factorization that occurs in R' which is described by R. E. Johnson in [6]. An element $a \in R'$ is called simple if $[aR, R]$ has the property that $[aR, R] = [aR, B] \cup [B, R]$ implies $B = aR$ or $B = R$. Johnson proves that each element z in R' can be factored into simple elements: $z = a_1 \cdots a_n$, and no subproduct $a_i \cdots a_j$, $i < j$, of z is simple. Any other factorization of z into simple elements of this type must have the form $z = (a_1 u_1)(u_1^{-1} a_2 u_2) \cdots (u_{n-1}^{-1} a_n)$ where $u_1, \ldots, u_{n-1}$ are units in R.

The type of unique factorization that we describe in the present paper concerns all of the nonzero elements of a PRI domain R. In §2 we develop some general results (in particular Theorem 2) for weak Bézout domains that satisfy the ascending chain condition for principal right ideals. In §3 we define inf$^{(\alpha)}$ primes for each non-limit ordinal α. Inf$^{(0)}$ primes are the usual primes, and inf$^{(\alpha)}$ primes have infinite dimension if $\alpha \neq 0$. The unique factorization theorem (Theorem 3) that follows

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states that each nonzero element a of a PRI domain R can be factored as $a=z_{\alpha_1} \cdots z_{\alpha_n}$ where z_{α_i} is a uniquely determined product of $\inf^{(\alpha_i)}$ primes. The factorization of a is unique in the sense that if $a=y_{\beta_1}\cdots y_{\beta_m}$ is another such factorization of a then $n=m$, $\alpha_i=\beta_i$ $(i=1, 2, \ldots, n)$, and there are units $u_1, \ldots, u_{n-1}$ in R such that $z_{\alpha_1}=y_{\alpha_1}u_1$, $z_{\alpha_n}=u_{n-1}^{-1}y_{\alpha_n}$, and $z_{\alpha_i}=u_{i-1}^{-1}y_{\alpha_i}u_i$ $(i\neq 1, n)$.

The important applications of Theorem 3 obviously occur in PRI domains R such that $R^*\neq R'$. Until recently few examples of such were available. Such examples however do occur, for instance, in P. M. Cohn [3]. In [5] A. V. Jategaonkar describes the method of skew polynomial extensions. Using these methods it is possible to construct PRI domains with $\inf^{(\alpha)}$ primes, given an arbitrary ordinal α. We include this discussion in §4.

2. **Right quotient monoids.** Suppose R is an integral domain and $\varnothing\neq S\subset R^*$. We call S a *right quotient monoid* in R if S satisfies the following *right quotient conditons*:

(1) $ab\in S$ iff a and $b\in S$, where $a, b\in R$.

(2) $a\in R$, $b\in S$ implies there exists $\bar{a}\in R$, $\bar{b}\in S$ such that $a\bar{b}=b\bar{a}$.

In this case S contains the group of units of R. For, $a\in S$ for some element a of R. From $1a\in S$ we obtain $1\in S$ by condition (1). Also, if u is a unit in R then $uu^{-1}=1\in S$ and therefore $u\in S$.

If S is a right quotient monoid in R then the set $K=RS^{-1}=\{rs^{-1}\mid r\in R, s\in S\}$ can be made into a ring in the usual way (see Bourbaki [1, p. 162])(3). It is easy to prove that K is an integral domain with the property that $s\in R$ is a unit in K iff $s\in S$. Further, if A is a right ideal of R then $AS^{-1}=\{as^{-1}\mid a\in A, s\in S\}$ is a right ideal of K, and if R is a PRI domain then so is K. The easy proofs of these facts are omitted.

LEMMA 1. *Let R be an integral domain and let S be a right quotient monoid in R. Let A, B be right ideals of R. Then*

(1) $(A\cap B)S^{-1}=AS^{-1}\cap BS^{-1}$.

(2) $(A+B)S^{-1}=AS^{-1}+BS^{-1}$.

Proof. Clearly $(A\cap B)S^{-1}\subset AS^{-1}\cap BS^{-1}$. Now suppose $as_1^{-1}=bs_2^{-1}\in AS^{-1}\cap BS^{-1}$ $(s_1, s_2\in S$ and $a, b\in R)$. Because S is a right quotient monoid we can choose $\bar{s}_1, \bar{s}_2\in S$ such that $s_1\bar{s}_2=s_2\bar{s}_1$. It follows that $a\bar{s}_2=b\bar{s}_1\in A\cap B$. Also, $as_1^{-1}=a\bar{s}_2(s_2\bar{s}_1)^{-1}\in(A\cap B)S^{-1}$. This proves $AS^{-1}\cap BS^{-1}\subset(A\cap B)S^{-1}$ and (1) is established.

Now it is obvious that $(A+B)S^{-1}\subset AS^{-1}+BS^{-1}$. To show the reverse inclusion let $as_1^{-1}+bs_2^{-1}\in AS^{-1}+BS^{-1}$ $(s_1, s_2\in S$ and $a, b\in R)$. Choose $\bar{s}_1, \bar{s}_2\in S$ such that $s_1\bar{s}_2=s_2\bar{s}_1$. Then $as_1^{-1}+bs_2^{-1}=(a\bar{s}_2+b\bar{s}_1)(s_1\bar{s}_2)^{-1}$. Hence $as_1^{-1}+bs_2^{-1}\in(A+B)S^{-1}$. This proves (2). Q.E.D.

(3) For this purpose condition (1) in the definition of right quotient monoid is usually replaced by the weaker condition that S be multiplicatively closed.

COROLLARY. *If R is a weak Bézout or a right Bézout domain, then so is $K=RS^{-1}$*

We recall from [2] that if a, $\bar{a} \in R$, then a and $\bar{a}$ are similar ($a \sim \bar{a}$ or $a \sim_R \bar{a}$) if $R/aR \cong R/\bar{a}R$ as right R-modules, and this is true iff there exists $b \in R$ such that $aR+bR=R$ and $aR \cap bR=b\bar{a}R$. It is also shown in [2] that the definition of similarity is left-right symmetric.

LEMMA 2. *Let S be a right quotient monoid in an integral domain R and let $K=RS^{-1}$. If a, $\bar{a} \in R$ and $a \sim_R \bar{a}$ then $a \sim_K \bar{a}$. Further, if $a \sim_R \bar{a}$ and $a \in S$ then $\bar{a} \in S$.*

Proof. Let $a, \bar{a} \in R$ with $a \sim_R \bar{a}$. Then $aR+bR=R$ and $aR \cap bR=b\bar{a}R$ for some $b \in R$. Therefore $aK+bK=K$ and $aK \cap bK=b\bar{a}K$ by Lemma 1. Hence $a \sim_K \bar{a}$. If in addition $a \in S$ then a is a unit in K. Since $a \sim_K \bar{a}$ it follows that $\bar{a}$ is a unit in K. Hence $\bar{a} \in S$. Q.E.D.

LEMMA 3. *Suppose R is a right Bézout domain and $\varnothing \neq S \subset R^*$ such that $ab \in S$ iff a and $b \in S$. Then S is a right quotient monoid in R iff elements similar to members of S belong to S, that is, iff $a \in S$, $\bar{a} \in R$, and $a \sim \bar{a}$ implies $\bar{a} \in S$.*

Proof. Assume the hypotheses. If S is a right quotient monoid in R then elements similar to members of S belong to S by Lemma 2. Conversely assume that S has the property that $a \in S$, $\bar{a} \in R$, and $a \sim \bar{a}$ implies $\bar{a} \in S$. To show that S is a right quotient monoid in R we need only establish condition (2) of the definition. Accordingly let $a \in S$ and $b \in R$. Because R is a right Bézout domain we can choose d, $m \in R$ such that $aR+bR=dR$ and $aR \cap bR=mR$. Then $a=da'$, $b=db'$, and $m=a\bar{b}=b\bar{a}$ for some a', b', $\bar{a}$, $\bar{b} \in R$. It follows that $a'R+b'R=R$ and $a'R \cap b'R =b'\bar{a}R$. Therefore $a' \sim \bar{a}$. Now $a \in S$ implies $a' \in S$. Hence $\bar{a} \in S$ by our assumption. This shows that S is a right quotient monoid in R. Q.E.D.

COROLLARY. *If R is a right Bézout domain then R^* is a right quotient monoid in R.*

THEOREM 1. *Let R be a weak Bézout domain satisfying the ascending chain condition for principal right ideals, and let S be a right quotient monoid in R. Each $z \in R^*$ can be written as $z=xs$ where $s \in S$ and x has no nonunit right factor in S. The factorization is unique in the sense that if $z=x_1s_1=x_2s_2$ are two such factorizations of z then there is a unit $u \in R$ such that $x_1=x_2u$ (and $us_1=s_2$).*

Proof. Let $z \in R^*$. Let C_S be the collection of submodules R/sR of R/zR such that $s \in S$. Note that $zR/zR \cong R/R$ is a member of C_S. We claim that C_S is closed under sums. For if aR/zR, $bR/zR \in C_S$, then $aR/zR \cong R/xR$, $bR/zR \cong R/yR$ where $z=ax=by$ and $x, y \in S$. Since R is a weak Bézout domain, $aR+bR=dR$, $aR \cap bR =mR$ for some d, $m \in R$. Choose a', b', $\bar{a}$, $\bar{b} \in R$ with $a=da'$, $b=db'$, $m=a\bar{b}=b\bar{a}$. Then $a'R+b'R=R$, $a'R \cap b'R=b'\bar{a}R$. Hence $a' \sim \bar{a}$. Now $ax=by \in mR$ implies

$by=mr$ for some $r \in R$. Hence $y=\bar{a}r$. This shows that $\bar{a} \in S$ because $y \in S$. Consequently $a' \in S$. Then $aR/zR+bR/zR=dR/zR=dR/da'xR \cong R/a'xR$ and $a'x \in S$. We conclude $aR/zR+bR/zR \in C_S$.

To prove the theorem observe that C_S has the ascending chain condition by hypothesis. Thus we may select a (not necessarily proper) maximal member xR/zR of C_S. Thus $xR/zR \cong R/sR$ where $z=xs$ and $s \in S$. Since xR/zR is maximal x has no nonunit right factor in S. Also, xR/zR is the unique maximal member of C_S because C_S is closed under sums. Therefore the factorization is unique. Q.E.D.

Let R be an integral domain and let $I=\{\alpha \mid 0 \leqq \alpha \leqq \alpha_0\}$ be an initial segment of ordinals. A collection $\{S_\alpha \mid \alpha \in I\}$ of right quotient monoids in R is called a *right quotient chain* in R if the following conditions hold:

(1) $S_\alpha \subsetneqq S_{\alpha+1}$ for each $\alpha \in I$, $\alpha \neq \alpha_0$.

(2) $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ if α is a limit ordinal.

For convenience we let S_{-1} denote the group of units of R. Then S_{-1} is contained in each S_α. Let $K_\alpha = R(S_\alpha)^{-1}$ if $\alpha=-1$ or if $\alpha \in I$. Then because of condition (1) $K_{\alpha-1} \subset K_\alpha$ for each $\alpha \in I$.

Theorem 2. *Let R be a weak Bézout domain satisfying the ascending chain condition for principal right ideals. Let $I=\{\alpha \mid 0 \leqq \alpha \leqq \alpha_0\}$ be an initial segment of ordinals and let $\{S_\alpha \mid \alpha \in I\}$ be a right quotient chain in R. Each $z \in R^*$ can be factored as $z=ra_{\alpha_1}\cdots a_{\alpha_n}$ where α_i are nonlimit ordinals such that $\alpha_0 \geqq \alpha_1 > \cdots > \alpha_n$, $a_{\alpha_i} \in S_{\alpha_i}$, a_{α_i} has no nonunit right factor in S_{α_i-1}, $r \in R$ and r has no nonunit right factor in S_{α_0}. The factorization is unique in the sense that if $z=sb_{\beta_1}\cdots b_{\beta_m}$ is another such factorization of z then $n=m$, $\alpha_i=\beta_i$ $(i=1, 2, \ldots, n)$, and there are units $u_0, u_1, \ldots, u_{n-1}$ in R such that $r=su_0$, $a_{\alpha_n}=u_{n-1}^{-1}b_{\alpha_n}$, and $a_{\alpha_i}=u_{i-1}^{-1}b_{\alpha_i}u_i$ $(i \neq 0, n)$.*

Proof. To prove existence of the factorization let $z \in R^*$. If z has no nonunit right factor in S_{α_0} then we are finished. Otherwise by Theorem 1 $z=rs_0$ for some nonunit $s_0 \in S_{\alpha_0}$ and r has no nonunit right factor in S_{α_0}. Let α_1 be the least ordinal such that $s_0 \in S_{\alpha_1}$. Clearly α_1 is not a limit ordinal and $\alpha_0 \geqq \alpha_1$. It follows by Theorem 1 that $s_0=a_{\alpha_1}s_1$ for some element $s_1 \in S_{\alpha_1-1}$ and a_{α_1} has no nonunit right factor in S_{α_1-1}. Clearly $a_{\alpha_1} \in S_{\alpha_1}$ because $s_0 \in S_{\alpha_1}$. If s_1 is not a unit let α_2 be the least ordinal such that $s_1 \in S_{\alpha_2}$. Then $\alpha_1 > \alpha_2$ and α_2 is not a limit ordinal. Another application of Theorem 1 yields $s_1=a_{\alpha_2}s_2$ where $s_2 \in S_{\alpha_2-1}$ and a_{α_2} has no nonunit right factor in S_{α_2-1}. Clearly $a_{\alpha_2} \in S_{\alpha_2}$ because $s_1 \in S_{\alpha_2}$. If s_2 is not a unit we may repeat the argument. Now this process cannot continue indefinitely since we would obtain an infinite sequence $\alpha_1 > \alpha_2 > \cdots$ contradicting the well ordering of the ordinals. Thus the process stops, say, with the integer n. That is, a_{α_n} has no nonunit right factor in S_{α_n-1} and s_n is a unit. This establishes the existence of the factorization.

To prove uniqueness suppose $z=ra_{\alpha_1}\cdots a_{\alpha_n}=sb_{\beta_1}\cdots b_{\beta_m}$ are two factorizations of z of the type stated in the theorem. Then Theorem 1 applies and yields $r=su_0$ for some unit $u_0 \in R$. Therefore $a_{\alpha_1}\cdots a_{\alpha_n}=u_0^{-1}b_{\beta_1}\cdots b_{\beta_m}$. Evidently $\alpha_1=\beta_1$. Again Theorem 1 applies and yields $a_{\alpha_1}=u_0^{-1}b_{\beta_1}u_1$ for some unit $u_1 \in R$. Cancelling this factor we obtain $a_{\alpha_2}\cdots a_{\alpha_n}=u_1^{-1}b_{\beta_2}\cdots b_{\beta_m}$. Uniqueness now follows by induction. Q.E.D.

3. **Unique factorization and infinite primes.** In this section we shall construct a natural set $\{R^{(\alpha)} \mid \alpha\in I\}$ which is a right quotient chain in a right Bézout domain R. We shall then apply Theorem 2 to this right quotient chain. We begin by characterizing the peculiar factors that appear in Theorem 2.

Let $I=\{\alpha \mid 0\leqq\alpha\leqq\alpha_0\}$ be an initial segment of ordinals and let $\{S_\alpha \mid \alpha\in I\}$ be a right quotient chain in an integral domain R. If α is a nonlimit ordinal in I then $x\in S_\alpha$ is called an *α-prime* if xR is maximal in $\{xR \mid x\in S_\alpha\backslash S_{\alpha-1}\}$.

LEMMA 4. *Let $I=\{\alpha \mid 0\leqq\alpha\leqq\alpha_0\}$ be an initial segment of ordinals and let $\{S_\alpha \mid \alpha\in I\}$ be a right quotient chain in a* PRI *domain R. If α is a nonlimit ordinal in I and x is an α-prime then x is prime in $K_{\alpha-1}$.*

Proof. Assume the hypotheses and let x be an α-prime. Suppose $xK_{\alpha-1}\subsetneqq yK_{\alpha-1}\subset K_{\alpha-1}$. Then $xR\subset xK_{\alpha-1}\cap R\subsetneqq yK_{\alpha-1}\cap R\subset R$. Let $\bar{y}\in R$ be such that $\bar{y}R=yK_{\alpha-1}\cap R$. Then $xR\subsetneqq\bar{y}R$. The definition of α-prime implies that $\bar{y}\in S_{\alpha-1}$ and therefore $\bar{y}$ is a unit in $K_{\alpha-1}$ and $\bar{y}K_{\alpha-1}=K_{\alpha-1}$. Now $\bar{y}K_{\alpha-1}=yK_{\alpha-1}$ and so $yK_{\alpha-1}=K_{\alpha-1}$. This shows that x is prime in $K_{\alpha-1}$. Q.E.D.

LEMMA 5. *Let R be a weak Bézout domain, let I be an initial segment of ordinals, and let $\{S_\alpha \mid \alpha\in I\}$ be a right quotient chain in R. If $x_1,\ldots,x_k$ are α-primes, then $x_1\cdots x_k$ has no nonunit right factor that belongs to $S_{\alpha-1}$.*

Proof. The proof is by induction on k. The lemma is true if $k=1$ by the definition of α-prime. Assume k is an integer greater than 1 and the lemma holds for positive integers less than k. Suppose $x_1\cdots x_k=ab$ with $b\in S_{\alpha-1}$, $a\in R$ and x_i are α-primes. We shall show that b must be a unit. If $aR\subset x_1R$ then $a=x_1s$, $s\in R$. Hence $x_2\cdots x_k=sb$. It follows by induction that b must be a unit. Suppose on the other hand that $aR\not\subset x_1R$. Then since $x_1R\cap aR\neq 0$ and R is a weak Bézout domain it follows that $x_1R+aR=dR$ and $x_1R\cap aR=mR$ for some d, $m\in R$. Choose x', a', $\bar{a}$, $\bar{x}_1\in R$ such that $x_1=dx'$, $a=da'$, and $m=x_1\bar{a}=a\bar{x}_1$. Then $x'R+a'R=R$ and $x'R\cap a'R=a'\bar{x}_1R$. Consequently $x'\sim\bar{x}_1$. Now $x_1\cdots x_k=x_1\bar{a}z$ for some $z\in R$. Therefore $ab=x_1\cdots x_k=x_1\bar{a}z$ and so $b=\bar{x}_1z$. Hence $z\in S_{\alpha-1}$ since $b\in S_{\alpha-1}$. It follows from $x_2\cdots x_k=\bar{a}z$ and by induction that z is a unit. Consequently b is a right associate of $\bar{x}_1$. Therefore $x'\sim\bar{x}_1$ yields $x'\sim b$ and hence $x'\in S_{\alpha-1}$ by Lemma 2. Now $d\in S_{\alpha-1}$ because $x_1R\subsetneqq dR$ and x_1 is an α-prime. Therefore $x_1=dx'\in S_{\alpha-1}$. However this contradicts the fact that x_1 is an α-prime. Q.E.D.

Whenever R is a weak Bézout domain satisfying the ascending chain condition for principal right ideals the converse to Lemma 5 holds as follows.

LEMMA 6. *Let R be a weak Bézout domain satisfying the ascending chain condition for principal right ideals. Let I be an initial segment of ordinals and let $\{S_\alpha \mid \alpha\in I\}$ be a right quotient chain in R. Let α be a nonlimit ordinal in I, and let $a\in S_\alpha\cap(K_{\alpha-1})'$. If a has no nonunit right factor that belongs to $S_{\alpha-1}$ then a is a product of α-primes.*

Proof. Assume the hypotheses. Since $a \in S_\alpha \backslash S_{\alpha-1}$ we may choose x_1 (by the ascending chain condition for principal right ideals) such that x_1R is maximal in $\{xR \mid aR \subset xR \text{ and } x \in S_\alpha \backslash S_{\alpha-1}\}$. Then x_1 is an α-prime and $a = x_1 s_1$ for some $s_1 \in R$. Clearly $s_1 \in S_\alpha$ and if s_1 is not a unit then $s_1 \in S_\alpha \backslash S_{\alpha-1}$ because of the assumption on a. We repeat the argument and obtain $s_1 = x_2 s_2$ where x_2 is an α-prime and $s_2 \in S_\alpha$. If this process does not terminate we obtain, for each positive integer i, $s_i = x_{i+1} s_{i+1}$ where x_{i+1} is an α-prime and hence a nonunit in $K_{\alpha-1}$. Let $r_i = x_1 \cdots x_i$. Then $a = r_i s_i$ and $r_{i+1} = r_i x_{i+1}$ for each positive integer i. Therefore $aK_{\alpha-1} \subsetneqq \cdots \subsetneqq r_{i+1}K_{\alpha-1} \subsetneqq r_i K_{\alpha-1} \subsetneqq \cdots \subsetneqq r_1 K_{\alpha-1}$. This contradicts the hypothesis that $a \in (K_{\alpha-1})'$. Therefore the process terminates, say, with the integer k. Thus $a = x_1 \cdots x_k s_k$ and s_k is a unit in R. Q.E.D.

Each right Bézout domain contains a natural right quotient monoid as follows.

Lemma 7. *Let R be a right Bézout domain. Then R' is a right quotient monoid in R.*

Proof. Clearly $\varnothing \neq R' \subset R^*$. Suppose $a, b \in R^*$. Then $aR/abR \cong R/bR$ and therefore $\dim ab = \dim a + \dim b$. Hence $ab \in R'$ iff $a, b \in R'$. Also if $a \in R'$ and $\bar{a} \in R$ with $a \sim \bar{a}$ then $aR + bR = R$ and $aR \cap bR = b\bar{a}R$ for some $b \in R$. Therefore $[aR, R] = [aR, aR + bR] \cong [aR \cap bR, bR] = [b\bar{a}R, bR] \cong [\bar{a}R, R]$ as lattices. Thus $\dim a = \dim \bar{a}$ and so $\bar{a} \in R'$. It follows by Lemma 3 that R' is a right quotient monoid in R. Q.E.D.

Let R be a right Bézout domain. We construct, by transfinite induction, a natural chain $\{R^{(\alpha)} \mid \alpha \text{ is an ordinal}\}$ of right quotient monoids in R as follows.

Let $R^{(0)} = R'$. Let α be an ordinal greater than zero and assume $R^{(\beta)}$ has been defined and is a right quotient monoid in R whenever $\beta < \alpha$, and let $K_\beta = R(R^{(\beta)})^{-1}$. Then K_β is a right Bézout domain (Corollary to Lemma 1) and hence K'_β is a right quotient monoid in K_β by Lemma 7. We define $R^{(\alpha)}$ by

$$R^{(\alpha)} = \bigcup_{\beta < \alpha} R^{(\beta)} \text{ if } \alpha \text{ is a limit ordinal,}$$

$$R^{(\alpha)} = (K_{\alpha-1})' \cap R \text{ if } \alpha \text{ is not a limit ordinal.}$$

To show that the induction is valid we must show that $R^{(\alpha)}$ is a right quotient monoid in R. If α is a limit ordinal the proof is obvious. Assume that α is not a limit ordinal. Clearly $\varnothing \neq R^{(\alpha)} \subset R^*$. Also $ab \in R^{(\alpha)}$ iff $a, b \in R^{(\alpha)}$ because $(K_{\alpha-1})'$ has this property. Now if $a \in R^{(\alpha)}$, $\bar{a} \in R$ and $a \sim_R \bar{a}$ then $a \sim_{K_{\alpha-1}} \bar{a}$ by Lemma 2. It follows (as in the proof of Lemma 7) that $\dim_{K_{\alpha-1}} a = \dim_{K_{\alpha-1}} \bar{a}$. Hence $\bar{a} \in (K_{\alpha-1})'$ since $a \in (K_{\alpha-1})'$. It follows that $\bar{a} \in R^{(\alpha)}$. The hypotheses of Lemma 3 are satisfied and therefore $R^{(\alpha)}$ is a right quotient monoid in R. Q.E.D.

If α, β are ordinals such that $\alpha \leqq \beta$, then $R^{(\alpha)} \subset R^{(\beta)} \subset R$. Also $R^{(\alpha)} = R^{(\alpha+1)}$ for some ordinal α. For if $R^{(\alpha)} \neq R^{(\alpha+1)}$ for each ordinal α then $\text{card}\,(R^{(\alpha)}) \geqq \text{card}\,(\alpha)$ for each ordinal α. Choosing β such that $\text{card}\,(\beta) > \text{card}\,(R)$ we obtain $\text{card}\,(\beta) > \text{card}\,(R) \geqq \text{card}\,(R^{(\beta)})$, a contradiction. We let α_0 denote the least ordinal such that $R^{(\alpha_0)} = R^{(\alpha_0+1)}$, and we call $\{R^{(\alpha)} \mid 0 \leqq \alpha \leqq \alpha_0\}$ the *right D-chain* (Dimension chain) in R. In this situation $R^{(-1)}$ will denote the group of units of R.

Evidently the right D-chain in a right Bézout domain R is a right qotient chain in R. If R is a PRI domain then the right D-chain has the following additional property.

LEMMA 8. *Let R be a* PRI *domain and let $\{R^{(\alpha)} \mid 0 \leqq \alpha \leqq \alpha_0\}$ be the right D-chain in R. Then $R^{(\alpha_0)} = R^*$.*

Proof. Suppose $R^* \neq R^{(\alpha_0)}$. Then by the maximum condition in the PRI domain R we may choose x such that xR is maximal in $\{xR \mid x \in R^* \backslash R^{(\alpha_0)}\}$. The proof of Lemma 4 can be used to show that x is prime in K_{α_0}. In particular $x \in (K_{\alpha_0})' \cap R = R^{(\alpha_0+1)}$. This contradicts $R^{(\alpha_0+1)} = R^{(\alpha_0)}$. Therefore $R^* = R^{(\alpha_0)}$. Q.E.D.

Let R be a PRI domain and let $\{R^{(\alpha)} \mid 0 \leqq \alpha \leqq \alpha_0\}$ be the right D-chain in R. We shall call the α-primes in R $\inf^{(\alpha)}$ *primes*. If α is a nonlimit ordinal such that $\alpha \leqq \alpha_0$ we let $Z^{(\alpha)}$ be the set of (finite) products of $\inf^{(\alpha)}$ primes.

If R is a PRI domain and α is a nonlimit ordinal such that $\alpha \leqq \alpha_0$ then $x \in R$ is an $\inf^{(\alpha)}$ prime iff xR is maximal in $\{xR \mid x \in R \backslash R^{(\alpha-1)}\}$. For if xR is maximal in $\{xR \mid x \in R \backslash R^{(\alpha-1)}\}$ then the proof of Lemma 4 can be used to show that x is prime in $K_{\alpha-1}$. In particular $x \in (K_{\alpha-1})'$ and so $x \in R^{(\alpha)}$. Hence x is an $\inf^{(\alpha)}$ prime. As a consequence we note that $\inf^{(\alpha)}$ primes exist for each nonlimit ordinal $\alpha \leqq \alpha_0$ by the maximum condition in R. In fact for each $z \notin R^{(\alpha-1)}$, $zR \subset xR$ for some $\inf^{(\alpha)}$ prime x.

If R is a PRI domain we can combine Lemmas 5 and 6 into the following.

LEMMA 9. *Let R be a* PRI *domain and let $\{R^{(\alpha)} \mid 0 \leqq \alpha \leqq \alpha_0\}$ be the right D-chain in R. Let α be a nonlimit ordinal such that $\alpha \leqq \alpha_0$, and let $z \in R^{(\alpha)}$. Then z has no nonunit right factor in $R^{(\alpha-1)}$ iff $z \in Z^{(\alpha)}$,* i.e., *iff z is a product of* $\inf^{(\alpha)}$ *primes.*

Using Lemmas 8 and 9 we can state Theorem 2 for the present case as follows.

THEOREM 3. *Let R be a* PRI *domain and let $\{R^{(\alpha)} \mid 0 \leqq \alpha \leqq \alpha_0\}$ be the right D-chain in R. Each $a \in R^*$ can be written as $a = z_{\alpha_1} \cdots z_{\alpha_k} u$ where α_i are nonlimit ordinals such that $\alpha_0 \geqq \alpha_1 > \cdots > \alpha_k$ and $z_{\alpha_i} \in Z^{(\alpha_i)}$ and u is a unit in R. This factorization is unique in the sense that if $a = y_{\beta_1} \cdots y_{\beta_h} v$ is another such factorization of a then $h = k$, $\alpha_i = \beta_i$ $(i = 1, 2, \ldots, k)$, and there are units $u_1, \ldots, u_{k-1}$ in R such that $z_{\alpha_1} = y_{\alpha_1} u_1$, $z_{\alpha_k} = u_{k-1}^{-1} y_{\alpha_k}$, and $z_{\alpha_i} = u_{i-1}^{-1} y_{\alpha_i} u_i$ $(i \neq 1, k)$.*

We note that the factors that appear in Theorem 3 are themselves uniquely determined as follows. Let $\alpha \in [0, \alpha_0]$ be a nonlimit ordinal and let $z \in Z^{(\alpha)}$, z a nonunit. Then z is a product of $\inf^{(\alpha)}$ primes. Since an $\inf^{(\alpha)}$ prime is a prime in $K_{\alpha-1}$ (Lemma 4) the decomposition of z is unique up to similarity in $K_{\alpha-1}$. That is, if $z = x_1 \cdots x_n = y_1 \cdots y_m$ where x_i and y_i are $\inf^{(\alpha)}$ primes in R, then $n = m$ and there is a permutation Π on $\{1, 2, \ldots, n\}$ such that $x_i \sim_{K_{\alpha-1}} y_{\Pi(i)}$ $(i = 1, 2, \ldots, n)$.

4. **Example.** If L is a ring and σ is a monomorphism from L into L we shall denote by $H = L[x, \sigma]$ the ring of skew polynomials in an indeterminate x with coefficients in L (written on he right of x). Addition in H is the usual pointwise

addition and multiplication is determined by the associative and distributive laws and by the commutation rule $ax=xa^{\sigma}$ ($a\in L$). It is easy to prove that L is an integral domain iff H is an integral domain (see Ore [7]). Further, it is shown in [5] that H is a PRI domain iff L is a PRI domain and σ maps L^* into the group of units of L.

We turn our attention to skew polynomial extensions which are defined by A. V. Jategaonkar in [5]. Let $\bar{\alpha}$ be an ordinal, let $I=[0,\bar{\alpha}]$, and let L be an integral domain. Let H be a right twisted polynomial extension of L with $\{H_{(\alpha)}\mid \alpha\in I\}$ a chain of twisted subdomains from L to H. Thus for each nonlimit ordinal $\alpha\in I$ there exists a monomorphism $\rho_\alpha\colon H_{(\alpha-1)}\to H_{(\alpha-1)}$ and an indeterminate x_α such that

$$H = H_{(\bar{\alpha})},\ L = H_{(-1)},$$
$$H_{(\alpha)} = H_{(\alpha-1)}[x_\alpha,\rho_\alpha] \text{ if } \alpha\in I \text{ and } \alpha \text{ is not a limit ordinal,}$$
$$H_{(\alpha)} = \bigcup_{\beta<\alpha} H_{(\beta)} \text{ if } \alpha\in I \text{ and } \alpha \text{ is a limit ordinal.}$$

In addition we assume that L is a skew field and $\rho_\alpha\colon H_{(\alpha-1)}\to L$ for each nonlimit ordinal $\alpha\in I$. Then H is a PRI domain [5]. Now each member of H is a polynomial in a finite number of indeterminates. Let S be the set of polynomials of H with nonzero constant terms in L. Then it can be shown [5] that S is a right quotient monoid in H, and $R=HS^{-1}$ is a local PRI domain with unique maximal ideal x_0R such that

$$(*)\qquad \begin{aligned} x_{\alpha+1}R &= \bigcap_{n=0}^{\infty} (x_\alpha)^n R \text{ if } \alpha\neq\bar{\alpha} \text{ and } \alpha \text{ is not a limit ordinal,}\\ x_{\alpha+1}R &= \bigcap_{\beta<\alpha} x_\beta R \text{ if } \alpha\neq\bar{\alpha} \text{ and } \alpha \text{ is a limit ordinal.}\end{aligned}$$

That $\alpha_0=\bar{\alpha}$ is a consequence of the next lemma.

LEMMA 10. *If* $\bar{\alpha}\neq\alpha\in I$ *or* $\alpha=-1$ *then* $R\backslash R^{(\alpha)}=x_{\alpha+1}R$.

Proof. The proof is by transfinite induction. Clearly $R\backslash R^{(-1)}=x_0R$ since x_0R is the unique maximal ideal of R. Let $\alpha\in I$, $\alpha\neq\bar{\alpha}$ and assume that $R\backslash R^{(\beta)}=x_{\beta+1}R$ if $\beta<\alpha$. If α is a limit ordinal then we obtain

$$R\backslash R^{(\alpha)} = R\Big\backslash\Big(\bigcup_{\beta<\alpha} R^{(\beta)}\Big) = \bigcap_{\beta<\alpha}(R\backslash R^{(\beta)}) = \bigcap_{\beta<\alpha} x_{\beta+1}R = x_{\alpha+1}R$$

by (*) and the induction hypothesis. Now assume that α is not a limit ordinal. Then from $R\backslash R^{(\alpha-1)}=x_\alpha R$ it follows that x_α is a prime in $K_{\alpha-1}$. Also from (*) we obtain $x_{\alpha+1}K_{\alpha-1}=\bigcap_{n=0}^{\infty}(x_\alpha)^nK_{\alpha-1}$ and therefore $\dim_{K_{\alpha-1}}(x_{\alpha+1})=\infty$. It follows that $x_{\alpha+1}\notin(K_{\alpha-1})'$. Therefore $x_{\alpha+1}\in R\backslash R^{(\alpha)}$ and $x_{\alpha+1}R\subset R\backslash R^{(\alpha)}$. To show the reverse inclusion let $f\in R\backslash R^{(\alpha)}$. Then $f\in R\backslash R^{(\alpha-1)}=x_\alpha R$. In fact $f\in\bigcap_{n=0}^{\infty}(x_\alpha)^nR$. Otherwise $f=(x_\alpha)^m g$ where m is the largest such integer. This implies $g\in R\backslash x_\alpha R$ $=R^{(\alpha-1)}$ and so g is a unit in $K_{\alpha-1}$. Since x_α is prime in $K_{\alpha-1}$ it follows that $\dim_{K_{\alpha-1}}$ $(f)<\infty$. Hence $f\in(K_{\alpha-1})'\cap R=R^{(\alpha)}$, a contradiction. Thus $f\in\bigcap_{n=0}^{\infty}(x_\alpha)^nR$, and so $f\in x_{\alpha+1}R$. Q.E.D.

COROLLARY. *For each nonlimit ordinal* $\alpha \leqq \alpha_0$, x_α *is the unique* (*up to right unit factor*) $\inf^{(\alpha)}$ *prime in* R.

Using the last corollary we may state Theorem 3 for the present case as follows.

THEOREM 4. *Let* R *be the ring of polynomials constructed in this section. Each nonzero element* $f \in R$ *can be written in the form* $f = x_{\alpha_1}^{n_1} \cdots x_{\alpha_k}^{n_k} u$ *where* n_i *are positive integers,* $\alpha_1 > \cdots > \alpha_k$ *and* u *is a unit in* R. *This expression is unique in the sense that if* $f = x_{\beta_1}^{m_1} \cdots x_{\beta_h}^{m_h} v$ *is another such factorization of* f, *then* $h = k$, $\alpha_i = \beta_i$ *and* $n_i = m_i$ $(i = 1, 2, \ldots, k)$.

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