SOME TAMENESS CONDITIONS INVOLVING SINGULAR DISKS[1]

BY
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Introduction. A familiar sort of lemma in the study of E^3 is the following:

LEMMA. *Let D and F be two disks in E^3 with $\partial D \cap F = D \cap \partial F = \varnothing$, and let U be a neighborhood of F° in E^3. Then there is a disk D' in E^3 such that $\partial D' = \partial D$, $D' \subset D \cup U$, and $O(D', F) \subset U$ where $O(D', F)$ is $D' \backslash F$ minus the component containing ∂D.* ($D' \backslash F$ means $D' \backslash D' \cap F$.)

Theorem 4 generalizes this lemma, allowing E to be a singular disk with its "interior" disjoint from its "boundary". It is necessary to redefine $O(D', E)$, and this is done in §2; the new definition is motivated by Lemma 5A.

Applications of Theorem 4 to the study of 2-spheres in E^3 are given in §6. Burgess has shown (Theorem 7 in [6]) that a 2-sphere S in E^3 is tame from the interior (i.e., $S \cup \operatorname{int} S$ is a 3-cell) if it is "locally spanned" by disks in the interior; Theorem 6 partially extends this result, letting the spanning disks be singular but imposing a condition on their boundaries. Corollary 6A notes that S is then tame from the interior if "small loops in S can be shrunk to points in small subsets of the interior." Corollary 6B answers a question raised by Bing [5, §5].

1. **Notation and terminology.** We use the letter d to denote the Euclidean metric for Euclidean 3-space E^3, and let $\rho(f, g) = \sup_{x \in A} d(f(x), g(x))$ for any two maps f and g of a space A into E^3. A map f of a subspace of E^3 into E^3 is a *δ-map* if $\rho(f, I) < \delta$, where I is the identity map.

An *n-manifold* N is a separable metric space such that each point $p \in N$ has an n-cell neighborhood in N. $N^\circ = \{p \in N : p$ has a neighborhood in N homeomorphic to $E^n\}$, and $\partial N = N \backslash N^\circ$. N is an *n-manifold-with-boundary* if $\partial N \neq \varnothing$. A *Euclidean neighborhood* of a point $p \in N$ is an n-cell neighborhood U together with a linear structure on U. If S is a connected $(n-1)$-manifold in N which separates N, and V is a component of $N \backslash S$, then *S is tame from V* if $S \cup V$ is an n-manifold. All 2-manifolds and 3-manifolds are assumed to be triangulated [2, Theorem 6], and we use the same symbol for both the manifold and its triangulation.

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Two subsets X and Y of an n-manifold N are in *relative general position* if, for each point $p \in N$, there is a Euclidean neighborhood U of p and triangulations T_X and T_Y of $X \cap U$ and $Y \cap U$ such that

(i) each simplex of T_X and T_Y is a simplex in U,

(ii) dimension $(|T_X^i| \cap |T_Y^j|) \leqq i+j-n$.

A map $f: X \to N$ is *in general position* if, for each point $p \in N$, there is a Euclidean neighborhood U of p and a triangulation T of $f^{-1}(U)$ such that

(i) for each simplex $\sigma \in T$, $f(\sigma)$ is a simplex in U,

(ii) for any two distinct simplices $\sigma_1 \in T^i$ and $\sigma_2 \in T^j$,

$$\text{dimension } (f(\sigma_1^\circ) \cap f(\sigma_2^\circ)) \leqq i+j-n.$$

If X and Y are two triangulated spaces, then $X \oplus Y$ denotes the disjoint union of both the spaces X and Y and their triangulations.

A *Dehn disk* D in E^3 is the image of a real disk Δ under a map $f: \Delta \to E^3$ such that, for some subdisk $\Delta_1 \subset \Delta^\circ$, $f(\Delta_1) \cap f(\Delta \backslash \Delta_1) = \varnothing$ and $f|_{\Delta\backslash\Delta_1}$ is piecewise linear and 1-1. The *singularities of* f are the points of Δ in the closure of $\{x \in \Delta : f^{-1}f(x) \neq x\}$, and the *singular points of* D are the images under f of these singularities. $\partial D = f(\partial \Delta)$.

If S is a 2-sphere in E^3, then int S and ext S are, respectively, the bounded and unbounded components of $E^3 \backslash S$. *Sierpinski curve* and *inaccessible point* are as defined in [5].

2. **Algebraic separation.** Let N be a simply-connected n-manifold, $n \leqq 3$. An $(n-1)$-polyhedron K is an *algebraic separator of* N if $K \cap N^\circ$ can be given a triangulation in which each $(n-2)$-simplex is the face of an even number of $(n-1)$-simplices.

Suppose that K is an algebraic separator of N. Any arc $A \subset N$ in general position relative to K hits K at a finite number $\|A \cap K\|$ of points, and standard counting arguments show that:

PROPOSITION 2A. *If $A \subset N$ and $B \subset N$ are polygonal arcs in general position relative to K, and A, B have the same endpoints, then $\|A \cap K\| = \|B \cap K\|$(mod 2). In particular, if $\|A \cap K\|$ is odd then the endpoints of A are separated in N by K.*

Suppose that D is a disk and $K \subset D^\circ$ is an algebraic separator of D. It follows from Proposition 2A that we can define a map $\phi_{D\backslash K}$ on $D \backslash K$ by setting $\phi_{D\backslash K}(x) = \|A \cap K\|$ (mod 2), where A is any arc from ∂D to x in general position relative to K. We let $O(D, K) = \{x \in D \backslash K : \phi_{D\backslash K}(x) = 1\}$.

Now suppose that Δ is a disk, M a 3-manifold, and $f: \Delta \to M$ a map such that $f|_{\Delta^\circ}$ is locally piecewise linear and in general position. Let $D \subset M$ be a polyhedral disk in general position relative to $f(\Delta^\circ)$, such that $\partial D \cap f(\Delta) = D \cap f(\partial \Delta) = \varnothing$.

PROPOSITION 2B. *$f^{-1}(D) = J_1 \cup \cdots \cup J_s$, where the J_i are disjoint simple closed curves. $f(J_i)$ and $f(\bigcup J_i) = D \cap f(\Delta)$ are algebraic separators of D, and $O(D, f(\bigcup J_i)) \subset \bigcup O(D, f(J_i))$.*

Proof. To check that $O(D, f(\bigcup J_i)) \subset \bigcup O(D, f(J_i))$, just note that, for any polygonal arc A in general position relative to $f(\bigcup J_i)$, $\|A \cap f(\bigcup J_i)\| = \|A \cap f(J_1)\| + \cdots + \|A \cap f(J_s)\|$. The other statements follow from the general position of $f|_{\Delta^\circ}$, and of D relative to $f(\Delta^\circ)$.

3. **Induction lemma.**

LEMMA 3. *Let M be a 3-manifold-with-boundary, D and Δ disks. Let $f: \Delta \to M$ be a simplicial map in general position, U an open neighborhood of $f(\Delta)$ in M.*

Suppose $i: D \to M$ is a simplicial embedding such that $i(D)$ is in general position relative to $f(\Delta)$, $i(D) \cap \partial M = i(\partial D)$, and $i(\partial D) \cap U = i(D) \cap f(\partial\Delta) = \varnothing$.

If $O(i(D), f(J)) \not\subset U$ for some simple closed curve $J \subset f^{-1}i(D)$, then there is a polyhedral disk D' in M such that:

(3.1) $\partial D' = i(\partial D)$,

(3.2) $D' \subset i(D) \cup U$,

(3.3) $(i(D)\backslash U)\backslash(D'\backslash U) \neq \varnothing$.

Proof. Our proof will be analogous to those of Papakyriakopoulos [10] and Stallings [11], but where they dealt with maps of disks, we will be working with the map $i \oplus f: D \oplus \Delta \to M$ defined by $i \oplus f|_D = i$, $i \oplus f|_\Delta = f$. To measure the singularity of this map we use the complex $S(i \oplus f)$ defined by Stallings in his proof of [11, Lemma 3]; for completeness, we reproduce the definition here.

For any simplicial map γ of a complex X into a complex Y, a simplicial map $\gamma \times \gamma: X \times X \to Y \times Y$ can be constructed, where $X \times X$ and $Y \times Y$ are the cartesian products of complexes as defined in [7, p. 67]. We define $S(\gamma)$ to be the inverse image under $\gamma \times \gamma$ of the diagonal of $Y \times Y$; since this diagonal is a subcomplex of $Y \times Y$, it follows that $S(\gamma)$ is a subcomplex of $X \times X$. The useful property of $S(\gamma)$ is that, if $\Pi: Y \to Z$ is a simplicial map into some complex Z, then $S(\gamma) \subset S(\Pi\gamma)$, and $S(\gamma) = S(\Pi\gamma)$ if and only if Π is 1-1.

We will induct on the number $\mathscr{H}(i, f)$ of simplices in $S(i \oplus f)$; assume that $O(i(D), f(J)) \not\subset U$ for some simple closed curve $J \subset f^{-1}i(D)$.

To simplify notation, we will identify D with $i(D)$ from this point on in the proof of Lemma 3.

Through standard combinatorial techniques and the ideas Stallings uses in proving Lemma 3 of [11], one can show:

PROPOSITION 3A. *There is a regular neighborhood N of $D \cap f(\Delta)$ in M, a closed neighborhood V of $D \cap f(\Delta)$ in D, and a piecewise linear embedding $h: D \times [-1, 1] \to N$ such that*

(i) *$h(x \times 0) = x$ and $h(x \times \pm 1) \in \partial N$ for all $x \in D\backslash V$,*

(ii) *$N \subset U \cup h((D\backslash V) \times [-1, 1])$ and $f(\Delta) \cap h((D\backslash V) \times [-1, 1]) = \varnothing$,*

(iii) *the maps $i: D \to N$ and $f: \Delta \to N$ are simplicial.*

The proof of the lemma splits into two parts, depending on whether or not N is simply connected.

Case I. N simply connected:

PROPOSITION 3B. *The components $S_1, \ldots, S_r$ of ∂N are spheres.*

Proof. See 7.2 in [10].

PROPOSITION 3C. *If p is a point of $O(D, f(J))\backslash U$, then $h(p \times -1)$ and $h(p \times 1)$ lie in different spheres S_i.*

Proof. If E is the disk in Δ bounded by J, then it follows from the general position of f, and of D relative to $f(\Delta)$, that $O(D, f(J)) \cup f(E) = K$ is an algebraic separator of N. $h(p \times [-1, 1])$ is a polyhedral arc in general position relative to K which hits K once; by Proposition 2A, $h(p \times -1)$ and $h(p \times 1)$ are separated in N by $K \subset N^\circ$, and must therefore lie in different components of ∂N.

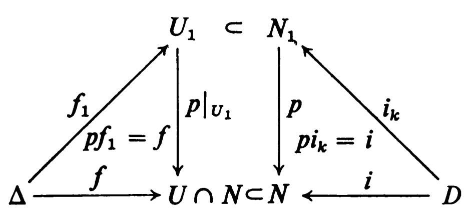
PROPOSITION 3D. *If p is a point of $O(D, f(J))\backslash U$, then ∂D bounds a polyhedral disk D' in $(D\backslash p) \cup U$.* (D' satisfies (3.1)–(3.3)).

Proof. Suppose ∂D lies in S_1. By Proposition 3C, S_1 does not contain both $h(p \times -1)$ and $h(p \times 1)$. ∂D misses $h(p \times \pm 1)$, so ∂D bounds a disk D_1 in S_1 missing $h(p \times \pm 1)$. For any $x \in D\backslash V$, D_1 contains at most one of the two points $h(x \times -1)$ and $h(x \times 1)$, since D is an algebraic separator of N separating them. Using the embedding of $D \times [-1, 1]$ in N given by Proposition 3A, we can therefore draw D_1 homeomorphically into $(D\backslash p) \cup U$.

Case II. N *not* simply connected:

Let (N_1, p) be the universal (simply connected) covering space for N. N_1 is a 3-manifold-with-boundary, and we triangulate N_1 so that $p: N_1 \to N$ is simplicial. Let $U_1 = p^{-1}(U \cap N)$.

Let $f_1: \Delta \to N_1$ be a lifting of f, and let $i_1, i_2, \ldots, i_k, \ldots : D \to N_1$ be the distinct liftings of i.



It is easy to check that

PROPOSITION 3E. *The hypotheses of Lemma* 3 *are satisfied by the substitution*:

for	*substitute*	*for*	*substitute*
M	N_1	U	U_1
Δ	Δ	D	D
f	f_1	i	*any* i_k

PROPOSITION 3F. *For any* i_k, $\mathscr{H}(i_k, f_1) < \mathscr{H}(i, f)$.

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} \pi_1(i_k(D) \cup f_1(\Delta)) & \xrightarrow{\psi_1} & \pi_1(N_1) = 0 \\ \Big\downarrow (p|_{i_k(D) \cup f_1(\Delta)})_* & & \Big\downarrow p_* \\ \pi_1(D \cup f(\Delta)) & \xrightarrow{\psi} & \pi_1(N) \neq 0, \end{array}$$

where ψ_1 and ψ are induced by inclusions. ψ is onto since N is a regular neighborhood of $D \cup f(\Delta)$.

Now, $S(i_k \oplus f_1) \subset S(p \circ (i_k \oplus f_1)) = S(i \oplus f)$. If $S(i_k \oplus f_1) = S(i \oplus f)$, then $p|i_k(D) \cup f_1(\Delta)$ is 1-1 and hence a homeomorphism, so $(p|i_k(D) \cup f_1(\Delta))_*$ is onto. But then

$$0 \neq \pi_1(N) = \psi(p|_{i_k(D) \cup f_1(\Delta)})_* \pi_1(i_k(D) \cup f_1(\Delta)) = p_* \psi_1 \pi_1(i_k(D) \cup f_1(\Delta)) = 0,$$

a contradiction. Thus, $S(i_k \oplus f_1)$ is properly contained in $S(i \oplus f)$, and $\mathscr{H}(i_k, f_1) < \mathscr{H}(i, f)$.

PROPOSITION 3G. *For some* K, $J \subset f_1^{-1} i_K(D)$ *and* $O(i_K(D), f_1(J)) \not\subset U_1$.

Proof. $J \subset f^{-1}(D) = (pf_1)^{-1}(D) = f_1^{-1}(p^{-1}(D)) = f_1^{-1}(\bigcup i_k(D)) = \bigcup f_1^{-1} i_k(D)$; since the disks $i_k(D)$ are disjoint, $J \subset f_1^{-1} i_K(D)$ for some K. $pi_K = i$ is a homeomorphism, so $p(O(i_K(D), f_1(J)) = O(D, f(J))$; if $O(i_K(D), f_1(J)) \subset U_1$, then $O(D, f(J)) \subset p(U_1) = U \cap N$, a contradiction to our assumption that $O(D, f(J)) \not\subset U$.

PROPOSITION 3H. *There is a polyhedral disk* D' *in* M *satisfying* (3.1)–(3.3).

Proof. Let $D_1 = i_K(D)$, where K is given by Proposition 3G. By our induction, there is a polyhedral disk D_1' in N_1 such that:

(i) $\partial D_1' = \partial D_1$,

(ii) $D_1' \subset D_1 \cup U_1$,

(iii) $(D_1 \backslash U_1) \backslash (D_1' \backslash U_1) \neq \varnothing$.

Since $p|_{D_1}$ is a homeomorphism, the singularities of $p\colon D_1' \to M$ all lie in U_1. $p(U_1) \cap p(\partial D_1') \subset U \cap \partial D = \varnothing$, so we can apply Dehn's lemma [9, Theorem IV.3] to get a polyhedral disk D' in M such that:

(iv) $\partial D' = p(\partial D_1')$,

(v) $D' \subset p(D_1') \cup U$.

It is easy to check that (i)–(v) imply that D' satisfies (3.1)–(3.3).

4. Using a singular disk to "cut back" a real disk.

THEOREM 4. *Let* U_0 *be an open subset of* E^3, Δ_0 *a disk, and* $f_0\colon \Delta_0 \to E^3$ *a map such that* $f_0(\Delta_0) \cap U_0 = f_0(\Delta^\circ_0)$ *and* $f_0|_{\Delta^\circ_0}\colon \Delta^\circ_0 \to E^3$ *is locally piecewise linear and in general position.*

Suppose that $D \subset E^3$ is a polyhedral disk such that $D \cap f_0(\partial\Delta_0) = \partial D \cap f_0(\Delta_0) = \varnothing$. Then there is a polyhedral disk D' in E^3 such that

(4.1) $\partial D' = \partial D$,

(4.2) $D' \subset D \cup U_0$,

(4.3) *D' is in general position relative to $f_0(\Delta_0^\circ)$,*

(4.4) $O(D', D' \cap f_0(\Delta_0)) \subset U_0$.

Proof. We may assume that $\overline{U}_0$ is locally polyhedral mod $f_0(\partial\Delta_0)$, and that $U_0 \cap \partial D = \varnothing$. For any disk D' satisfying (4.1) and (4.2), let $\mathscr{H}(D')$ be the number of components of $D' \backslash U_0$; $\mathscr{H}(D')$ is finite because $\overline{U}_0$ is polyhedral near D'.

D satisfies (4.1) and (4.2); we will induct on $\mathscr{H}(D)$. By adjusting D within U_0, if necessary, we may assume that D is in general position relative to $f_0(\Delta_0^\circ)$; if $O(D, D \cap f_0(\Delta_0)) \subset U_0$, as is the case when $\mathscr{H}(D) = 1$, then we have nothing to prove. Suppose that $O(D, D \cap f_0(\Delta_0)) \not\subset U_0$.

Since $(D \cup U_0) \cap f_0(\partial\Delta_0) = \varnothing$, we can choose a disk $\Delta \subset \Delta_0^\circ$ such that $f = f_0|_\Delta : \Delta \to E^3$ is piecewise linear and in general position, and

$$D \cap f_0(\Delta_0) \subset f(\Delta^\circ) \backslash f(\partial\Delta).$$

Using standard Euclidean-space techniques, together with the fact that

$$f_0(\Delta_0^\circ) \cap f_0(\partial\Delta_0) = \varnothing,$$

one can show:

Proposition 4A. *There is a 3-manifold-with-boundary $M \subset E^3$ such that*

(i) $D \cup f(\Delta) \subset M$,

(ii) $D \cap \partial M = \partial D$,

(iii) $M \cap f_0(\partial\Delta_0) = \varnothing$.

Furthermore, M, D, and Δ may be triangulated so that the hypotheses of Lemma 3 are satisfied by M, D, Δ, f, $U = U_0 \cap M$, and the natural injection $i: D \to M$.

Since $O(D, D \cap f_0(\Delta_0)) \not\subset U_0$, we have also $O(D, D \cap f(\Delta)) \not\subset U$. By Proposition 2B, $O(D, D \cap f(J)) \not\subset U$ for some simple closed curve $J \subset f^{-1}(D)$. Lemma 3 then gives us a polyhedral disk D' such that

(4.1) $\partial D' = \partial D$,

(4.2) $D' \subset D \cup U \subset D \cup U_0$,

(3.3) $(D \backslash U) \backslash (D' \backslash U) \neq \varnothing$.

To show that (3.3) implies $\mathscr{H}(D') < \mathscr{H}(D)$, we note

Proposition 4B. *$D' \backslash U = D' \backslash U_0$, $D \backslash U = D \backslash U_0$, and each component of $D' \backslash U$ is a component of $D \backslash U$.*

Proof. $D^* \backslash U = D^* \backslash (U_0 \cap M) = (D^* \backslash U_0) \cup (D^* \backslash M) = D^* \backslash U_0$, where D^* is either D' or D. That the components of $D' \backslash U$ are components of $D \backslash U$ follows from (4.1) and (4.2) above.

Remark. The proof of Theorem 4 shows that we can actually have D' satisfy

(4.4′) $O(D', D' \cap f_0(J)) \subset U_0$, for each simple closed curve $J \subset f_0^{-1}(D')$.

5. **Applying Theorem 4.** Throughout the remainder of the paper, Δ will represent a standard disk. Let M be a 3-manifold, S a 2-manifold in M, and $F \subset S$ a disk.

PROPOSITION 5A. *Let $\mathscr{G}$ be the class of all maps $g: \partial\Delta \to F^\circ$ which are piecewise linear into F and in general position. Then, for an arbitrary map $f: \partial\Delta \to F$, $\phi_{F\backslash f(\partial\Delta)} = \lim_{g\in\mathscr{G};\rho(f,g)\to 0} \phi_{F\backslash g(\partial\Delta)}$ exists on $F\backslash f(\partial\Delta)$, and $\phi_{F\backslash f(\partial\Delta)}(x) = \phi_{F\backslash g(\partial\Delta)}(x)$ for any map $g \in \mathscr{G}$ which is homotopic to f in $F\backslash x$.*

Proof. Both assertions follow from the easily demonstrated fact that if two maps g_1 and $g_2: \partial\Delta \to F$ are piecewise linear into F, in general position, and homotopic in $F\backslash x$, then $\phi_{F\backslash g_1(\partial\Delta)}(x) = \phi_{F\backslash g_2(\partial\Delta)}(x)$.

If V is a component of $M\backslash S$, then a *blister of F in V* is a map $f: \Delta \to F \cup V$ such that $f(\Delta) \cap S = f(\partial\Delta)$. We let $O(F,f) = \{x \in F\backslash f(\partial\Delta) : \phi_{F\backslash f(\partial\Delta)}(x) = 1\}$, and denote $f(\Delta) \cup O(F,f)$ by $(f)_F$.

LEMMA 5A. *Let S be a 2-sphere in E^3, $F \subset S$ a disk. Let f be a blister of F in* int S, *and B a 3-cell in E^3, such that $(f)_F \subset B^\circ$ and $f|_{\Delta^\circ}: \Delta^\circ \to E^3$ is locally piecewise linear and in general position.*

Suppose p is a point of $O(F,f)$, q is a point of int $S\backslash B$, *and $qp \subset E^3$ is a polygonal arc in general position relative to $f(\Delta^\circ)$ such that $qp\backslash p \subset$* int S. *Then $\|qp \cap f(\Delta)\|$ is odd.*

Proof. Pick a point r in ext $S\backslash B$. We can use Theorem 5.37 of [12] to extend qp to a polygonal arc $qpr \subset E^3$ such that pr misses $S\backslash O(F,f)$ and $f(\Delta)$.

Let $\delta = d(f(\partial\Delta), (E^3\backslash B) \cup qpr)$, and triangulate S so that F is a polyhedron in S. Using Bing's approximation theorem [1, Theorem 1], we can find a piecewise linear $\delta/2$-homeomorphism $h: S \to E^3$ such that

(1) $qpr \cap h(S) \subset h(O(F,f))$

$$(= \{x \in h(F)\backslash hf(\partial\Delta) : \phi_{h(F)\backslash hf(\partial\Delta)}(x) = 1\}),$$

(2) $q \in \text{int } h(S)$, $r \in \text{ext } h(S)$,

(3) $h(S)$ is in general position relative to $f(\Delta^\circ) \cup qpr$.

PROPOSITION 5B. *There is a polyhedron $K \subset B \subset E^3$ such that:*

(i) *K is an algebraic separator of E^3 in general position relative to qpr.*

(ii) $qpr \cap (f(\Delta) \cup h(S)) = qpr \cap K$.

Proof. Let $g: \partial\Delta \to E^3$ be a piecewise linear map in general position, such that:

(i) $g(\partial\Delta) \subset h(F^\circ)$,

(ii) g and $hf|_{\partial\Delta}$ are homotopic in $h(F)\backslash qpr$,

(iii) $\rho(g, hf|_{\partial\Delta}) < \delta/2$.

Let $\gamma: \Delta \to \Delta^\circ$ be a homeomorphism such that

(iv) $\rho(f\gamma, f) < \delta$,

(v) $f\gamma: \Delta \to E^3$ is piecewise linear and in general position, and $f\gamma(\Delta)$ is in general position relative to $h(S)$.

As a result of our care with δ, we can get a piecewise linear homotopy $G\colon \partial\Delta \times [0, 1] \to E^3$ such that

(vi) $G_0 = g$, $G_1 = f\gamma|_{\partial\Delta}$,

(vii) $G(\partial\Delta \times [0, 1]) \subset B\backslash qpr$,

(viii) $G(\partial\Delta \times (0, 1))$ is in general position relative to $h(S)$ and $f\gamma(\Delta)$.

It is simple to check that $K = O(h(F), g(\partial\Delta)) \cup G(\partial\Delta \times [0, 1]) \cup f\gamma(\Delta)$ satisfies the requirements.

Proposition 5C. *$\|qp \cap f(\Delta)\|$ is odd.*

Proof. K is contained in B, which does not separate q and r in E^3, so by Proposition 2A $\|qpr \cap K\|$ is even. $\|qpr \cap K\| = \|qpr \cap (f(\Delta) \cup h(S))\| = \|qpr \cap f(\Delta)\| + \|qpr \cap h(S)\|$, by condition (3) on h. $\|qpr \cap h(S)\|$ is odd since $h(S)$ is a manifold separating q and r in E^3, so $\|qpr \cap f(\Delta)\|$ is also odd.

Lemma 5B. *Let S be a 2-sphere in E^3, $F \subset S$ a disk. Let $f_1, \ldots, f_s$ be blisters of F in* int S, *and $B_1, \ldots, B_s$ 3-cells in E^3 such that $(f_i)_F \subset \mathring{B}_i$ for each i.*

Suppose D is a polyhedral disk in E^3 such that $\partial D \subset \operatorname{int} S\backslash \bigcup B_i$, $\operatorname{int} S \cup (F\backslash \bigcup f_i(\partial\Delta)) \cup D$ retracts to $\operatorname{int} S \cup (F\backslash \bigcup f_i(\partial\Delta))$, and $D \cap S \subset \bigcup O(F, f_i)$.

Then there is a disk D' in E^3 such that

(5.1) $\partial D' = \partial D$,

(5.2) $D' \subset D \cup (\bigcup B_i)$,

(5.3) $D' \subset \operatorname{int} S$.

Proof. Suppose that we have a polyhedral disk D_j in E^3 which satisfies the following conditions:

(1) $\partial D_j = \partial D$,

(2) $D_j \subset D \cup (\bigcup B_i)$,

(3) $\operatorname{int} S \cup (F\backslash \bigcup f_i(\partial\Delta)) \cup D_j$ retracts to $\operatorname{int} S \cup (F\backslash \bigcup f_i(\partial\Delta))$,

(4) $D_j \cap S \subset \bigcup_{j < i \leqq s} O(F, f_i)$.

We can choose $D_0 = D$, for example, and if we had D_s we could choose $D' = D_s$. For the proof of Lemma 5B it is, therefore, sufficient to produce D_{j+1}.

We may assume that, for each i, $f_i|_{\Delta^\circ}$ is locally piecewise linear and in general position. The hypotheses of Theorem 4 are then satisfied by the following substitutions:

for	substitute	for	substitute
Δ_0	Δ	U_0	$B_{j+1} \cap \operatorname{int} S$
f_0	f_{j+1}	D	D_j

There is, therefore, a polyhedral disk, which we shall call D_{j+1}, satisfying:

(4.1) $\partial D_{j+1} = \partial D_j$,

(4.2) $D_{j+1} \subset D_j \cup (B_{j+1} \cap \operatorname{int} S)$,

(4.3) D_{j+1} is in general position relative to $f_{j+1}(\Delta^\circ)$,

(4.4) $O(D_{j+1}, D_{j+1} \cap f_{j+1}(\Delta)) \subset B_{j+1} \cap \operatorname{int} S$.

From (4.1) and (4.2) it follows that D_{j+1} satisfies conditions (1)–(3); it remains to check (4).

PROPOSITION 5D. $D_{j+1} \cap S \subset \bigcup_{j+1<i\leqq s} O(F, f_i)$.

Proof. (4.2) implies that $D_{j+1} \cap S \subset \bigcup_{j<i\leqq s} O(F, f_i)$, so all we need check is $O(F, f_{j+1})$. Suppose that $D_{j+1} \cap O(F, f_{j+1}) \neq \varnothing$, and use (4.3) to choose a polygonal arc A with endpoints p and q, such that

(i) $A \subset D_{j+1}$,

(ii) $q \in \partial D_{j+1}$, $p \in O(F, f_{j+1})$,

(iii) A is in general position relative to $f_{j+1}(\Delta^\circ)$.

Using the facts that D_{j+1} satisfies (3) and int S is locally 0-connected [12, Theorem 5.35], we can get a polygonal arc A' with endpoints p' and q, such that

(iv) $A' \backslash p' \subset \operatorname{int} S$,

(v) $p' \in O(F, f_{j+1})$,

(vi) $A' \cap W = A \cap W$, for some neighborhood W of $f_{j+1}(\Delta^\circ)$ in int S.

A' is in general position relative to $f_{j+1}(\Delta^\circ)$ since A is, so Lemma 5A tells us that $\|A' \cap f_{j+1}(\Delta)\|$ is odd. Since $\|A' \cap f_{j+1}(\Delta)\| = \|A \cap f_{j+1}(\Delta)\|$, this means that $p \in O(D_{j+1}, D_{j+1} \cap f_{j+1}(\Delta))$. According to (4.4), p then lies in int S; but we assumed that $p \in S$, which is a contradiction. Therefore, $D_{j+1} \cap O(F, f_{j+1}) = \varnothing$.

6. **2-spheres in E^3.** Let M be a 3-manifold, S a 2-manifold in M, and V a component of $M \backslash S$. *S satisfies condition* (1) *toward V at a point $p \in S$* if, for any neighborhood B of p in M, and any Cantor set C in S, there is a disk $F \subset S \cap B$ and a blister f of F in V such that $p \in O(F, f) \subset (f)_F \subset B$ and $f(\partial\Delta) \cap C = \varnothing$.

THEOREM 6. *Let S be a 2-sphere in E^3 which satisfies condition* (1) *toward its interior at every point. Then S is tame from the interior.*

Proof. Let $F \subset S$ be a disk, U a neighborhood of F in E^3, and $D \subset E^3$ a polyhedral disk with $\partial D \subset \operatorname{int} S$ and $D \cap S \subset F^\circ$.

PROPOSITION 6A. *To prove Theorem* 6, *it is sufficient to show that there is a disk D' in E^3 such that*

(i) $\partial D' = \partial D$,

(ii) $D' \subset D \cup U$,

(iii) $D' \subset \operatorname{int} S$.

Proof. As Hempel has noted (in the proof of [8, Theorem 1]), this is a consequence of Bing's proof of Theorem 1 in [3].

PROPOSITION 6B. *There is a Dehn disk D_0 in E^3 and a Cantor set C in S such that*

(i) $\partial D_0 = \partial D$,

(ii) $D_0 \subset D \cup U$, *with the singular points of D_0 contained in U,*

(iii) $D_0 \subset \operatorname{int} S \cap (F^\circ \cap C)$.

Proof. We can use the Tietze extension theorem to get a Dehn disk D_0' such that $\partial D_0' = \partial D$ and $D_0' \subset (D \cap \operatorname{int} S) \cup F^\circ$, with the singular points of D_0' contained in F. Theorem 2.1 of [5] then gives us D_0.

PROPOSITION 6C. *There are blisters $f_1, \ldots, f_s$ of F in* int S, 3*-cells $B_1, \ldots, B_s$ in $U \backslash \partial D$, and disjoint disks $G_1, \ldots, G_r$ in S, such that $(f_i)_F \subset B_i^\circ$ for each i, and $D_0 \cap S \subset \bigcup G_j^\circ \subset \bigcup G_j \subset (\bigcup O(F, f_i)) \backslash \bigcup f_i(\partial \Delta)$.*

Proof. For any point p in F°, we can choose a 3-cell neighborhood B in $U \backslash (\partial D \cup (S \backslash F))$. Let C be the Cantor set described in Proposition 6B; since S satisfies condition (1) toward int S at p, there is a disk $F_p \subset S \cap B_p \subset F$ and a blister f_p of F_p in int S such that $p \in O(F_p, f_p) = O(F, f_p) \subset (f_p)_F \subset B^\circ{}_p$ and $f_p(\partial \Delta) \cap C = \varnothing$. $D_0 \cap S$ is compact, so we can pick $p_1, \ldots, p_s \in F^\circ$ such that $D_0 \cap S \subset (\bigcup O(F, f_{p_i})) \backslash \bigcup f_{p_i}(\partial \Delta)$. We let $\{f_1, \ldots, f_s\} = \{f_{p_1}, \ldots, f_{p_s}\}$, $\{B_1, \ldots, B_s\} = \{B_{p_1}, \ldots, B_{p_s}\}$, and use the fact that $D_0 \cap S \subset C$ is 0-dimensional to choose the disks $G_1, \ldots, G_r$.

We can use the Tietze extension theorem to show:

PROPOSITION 6D. int $S \cup (\bigcup G_j)$ *is a retract of some neighborhood V of* int $S \cup (\bigcup G_j^\circ)$ *in E^3.*

PROPOSITION 6E. *There is a polyhedral disk D_1 in E^3 such that*

(i) $\partial D_1 = \partial D$,

(ii) $D_1 \subset D \cup U$,

(iii) *S, F, $\{f_1, \ldots, f_s\}$, $\{B_1, \ldots, B_s\}$, and $D = D_1$ satisfy the hypotheses of Lemma* 5B.

Proof. The singular points of D_0 are contained in $U \cap V$, where V is as in Proposition 6D, so Dehn's Lemma gives us a polyhedral disk $D_1 \subset D_0 \cup (U \cap V)$ with $\partial D_1 = \partial D_0$. It is simple to check that D_1 meets the requirements.

If we apply Lemma 5B to D_1, we obtain a disk D' in E^3 such that

(5.1) $\partial D' = \partial D_1 = \partial D$,

(5.2) $D' \subset D_1 \cup (\bigcup B_i) \subset D \cup U$,

(5.3) $D' \subset \operatorname{int} S$.

D' satisfies conditions (i)–(iii) of Proposition 6A, and the proof is therefore complete.

Let M be a 3-manifold, S a 2-manifold in M, V a component of $M \backslash S$, and B a subset of M. A loop $f\colon \partial \Delta \to S$ *can be shrunk to a point through* $B \cap V$ if there is a homotopy $H_t\colon \partial \Delta \to M$ such that $H_0 = f$, H_1 is constant, and $H_t(\partial \Delta) \subset B \cap V$ for all $t > 0$. *S is* 1-LC *through V at a point $p \in S$* if, for any neighborhood B of p in M, there is a neighborhood B_1 of p in $B \cap S$ such that any loop $f\colon \partial \Delta \to B_1$ can be shrunk to a point through $B \cap V$.

COROLLARY 6A. *Let S be a* 2*-sphere in E^3 which is* 1-LC *through its interior at every point. Then S is tame from the interior.*

Proof. This follows from the observation:

PROPOSITION 6F. *Let B be a subset of E^3, and suppose that $i: \Delta \to S$ is an embedding such that $i(\partial\Delta)$ can be shrunk to a point through $B \cap$* int *S. Then there is a blister f of $i(\Delta)$ in* int *S such that $f(\partial\Delta) = i(\partial\Delta)$ and $f(\Delta) \subset B$, and we have $O(F, f) = i(\Delta^\circ)$, for any disk $F \subset S$ containing $i(\Delta)$.*

Let M be a 3-manifold, S a 2-manifold in M, and V a component of $M \setminus S$. A set X in S *can be deformed into* V if there is a homotopy $H_t: X \to M$ such that $H_0 = I$ and $H_t(X) \subset V$ for $t > 0$.

COROLLARY 6B. *Let S be a 2-sphere in E^3 such that every Sierpinski curve in S can be deformed into* int *S. Then S is tame from the interior.*

Proof. The following proposition is an adaptation of Theorem 14 in [6].

PROPOSITION 6G. *Let $E_1, E_2, \ldots, E_k, \ldots$ be a decreasing sequence of disks in S whose intersection is a point $p \in \bigcap E^\circ_k$, and suppose that $p \cup (\bigcup \partial E_k)$ can be deformed into* int *S.*

Then for any open set $B \subset E^3$ containing E_1, there is a blister f of E_1 in int *S such that $p \in O(E_1, f) \subset (f)_{E_1} \subset B$ and $f(\partial\Delta) = \partial E_K$ for some K.*

Proof. Let $H_t: p \cup (\bigcup \partial E_k) \to E^3$ be a homotopy such that $H_0 = I$ and $H_t(p \cup (\bigcup \partial E_k)) \subset \text{int } S$ for $t > 0$. We may assume that $H_t(p \cup (\bigcup \partial E_k)) \subset B$ for each t; let B' be an open 3-cell in $B \setminus S$ containing $H_1(p)$. $H_1(\partial E_K)$ lies in B' for large enough K, and can be shrunk to a point in B'. By Proposition 6F, there is then a blister f of E_K (and hence of E_1) in int S such that $p \in E_K^\circ = O(E_1, f) \subset (f)_{E_1} \subset B$ and $f(\partial\Delta) = \partial E_K$.

PROPOSITION 6H. *Let p be a point of S, C a Cantor set in S. Then p is an inaccessible point of some Sierpinski curve in S which misses C.*

Proof. We just construct such a Sierpinski curve, using the fact that C is 0-dimensional.

If p is an inaccessible point of a Sierpinski curve X in S, then there is a decreasing sequence of disks $E_1, E_2, \ldots, E_k, \ldots$ in S such that $\partial E_k \subset X$ and $\bigcap E_k = \bigcap E_k^\circ = p$. Therefore, Propositions 6G and 6H together imply that S satisfies condition (1) toward its interior at every point.

REMARKS. (1) The hypothesis of Corollary 6B requires that any Sierpinski curve in S can be *continuously* approximated from int S. For *any* 2-sphere S in E^3, any Sierpinski curve X in S, and any $\delta > 0$, it follows from Bing's side approximation theorem [4, Theorem 16] that there is a δ-homeomorphism $h: X \to \text{int } S$.

(2) Theorem 6 is stated for 2-spheres in E^3, but its proof is based on a local criterion for tameness [8, Condition A], so Theorem 1 of [6] can be used with Lemma 5B to extend our results to two-sided 2-manifolds in 3-manifolds.

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