# SELF-EQUIVALENCES OF $S^n \times S^k$(1)

BY
JEROME LEVINE

**Introduction.** Topology is primarily concerned with classifying various (topological) objects with respect to various equivalence relations. Given two equivalent objects, a secondary problem is then to classify the group (under composition) of equivalences between these objects.

We will deal with the following three categories:

$\mathscr{H}$: Topological spaces and homotopy classes of maps,

$\mathscr{P}$: Piecewise-linear manifolds and piecewise-linear maps,

$\mathscr{D}$: Smooth manifolds and smooth maps.

The equivalences (i.e. invertible maps) in these categories are homotopy equivalences, piecewise-linear homeomorphisms, and diffeomorphisms, respectively. We will be concerned with the group of *self*-equivalences of an object, with respect to some suitable relation between equivalences. The most interesting relation, aside from the equality relation, is *isotopy*. Two equivalences are isotopic if they can be joined by a continuous family of equivalences. We will work with the ostensibly weaker relation of *concordance* or *weak isotopy*. Two equivalences $f_0, f_1\colon X \to Y$ are *concordant* if there is an equivalence $F\colon I\times X \to I\times Y$ such that $F(t, x) = (t, f_t(x))$ for $t=0$, 1 and $x \in X$ (in $\mathscr{D}$, some elaboration is needed if $X$ has a boundary). In $\mathscr{H}$, both isotopy and concordance are equality.

The simplest nontrivial object in any of the categories is the $n$-sphere $S^n$. The group of concordance classes of self-equivalences is largely determined. In the category $\mathscr{H}$ and $\mathscr{P}$, it is $Z_2$, generated by a reflection. This follows, in $\mathscr{P}$, from the Alexander process (see [6]). In $\mathscr{D}$, the group is a semidirect product $\Gamma_{n+1} \times Z_2$, where $\Gamma_{n+1}$ is the subgroup of concordance classes of orientation-preserving diffeomorphisms of $S^n$; the generator of $Z_2$ acts on $\Gamma_{n+1}$ by inversion. $\Gamma_{n+1}$ has been extensively studied by Kervaire and Milnor [12].

In this work we will study the group of concordance classes of self-equivalences of $S^n \times S^k$. In §1, the induced homological automorphisms are examined; it then suffices to study a certain subgroup $A^{n,k}$ containing all the homologically trivial self-equivalences. In §2, a group theoretic structure for $A^{n,k}$ is established, in terms of certain subgroups of $A^{n,k}$ and group actions involving these subgroups. In §3, the subgroups are identified with certain groups of knots, in the categories $\mathscr{P}$ and $\mathscr{D}$, and with the homotopy groups of the space of self-equivalences of a sphere, in

Received by the editors May 2, 1968.

(1) Research supported in part by NSF Grant no. 6868. The author is a Sloan Foundation Fellow.

the category $\mathscr{H}$. In §4, the group actions are investigated. We conclude, in §5, with a number of consequences. For example, in a metastable range of dimensions, we see that there are no unfamiliar self-equivalences of $S^n \times S^k$. Another application is the classification of a certain family of objects in each of our categories, up to equivalence. A special case gives an example of a smooth closed manifold tangential homotopy equivalent, but not piecewise-linearly homeomorphic, to $S^6 \times S^2$.

Of perhaps, independent interest, we obtain an extension of Whitney's technique for removing intersections of submanifolds to the case where one of the manifolds is two-dimensional (see Lemma 3.6).

It is interesting to compare our results with the general obstruction theory developed in [23]. We also refer the reader to [10] for another treatment of the special case of homotopy equivalences of $S^n \times S^n$. Related results have been obtained by Morlet (Comptes Rendus, 1968) and Cerf (Proceedings International Congress of Mathematicians, Moscow 1966), and Lashof-Shaneson [35].

## 1. **Automorphisms of homology.**

1.1. $\mathscr{A}$ will denote any of the categories $\mathscr{H}$, $\mathscr{P}$ or $\mathscr{D}$. The group of concordance classes of self-equivalences of $S^n \times S^k$ in the category $\mathscr{H}$, $\mathscr{P}$ or $\mathscr{D}$, will be denoted $\bar{H}^{n,k}$, $\bar{P}^{n,k}$ or $\bar{D}^{n,k}-\bar{A}^{n,k}$ will denote any of them. We have natural homomorphisms:

$$\bar{D}^{n,k} \xrightarrow{\mu_1} \bar{P}^{n,k} \xrightarrow{\mu_2} \bar{H}^{n,k}.$$

$\mu_1$ is defined by the process of approximating a diffeomorphism by a piecewise-linear homeomorphism [27]; $\mu_2$ is defined by considering a piecewise-linear homeomorphism as merely a homotopy equivalence.

1.2. Let Auto $H^*(S^n \times S^k)$ be the group of graded ring automorphisms of $H^*(S^n \times S^k)$. If $n>k$, it is isomorphic to $Z_2+Z_2$. If $n=k$, we can obviously identify Auto $H^*(S^n \times S^n)$ with a subgroup of the group of $2\times 2$-unimodular matrices GL $(2, Z)$, by associating to such an automorphism its matrix representative in $H^n(S^n \times S^n)$ with respect to the natural basis (see [10]). It follows easily from the commutativity of $H^*(S^n \times S^n)$, that Auto $H^*(S^n \times S^n) = \mathrm{GL}\,(2, Z)$, if $n$ is odd, but consists only of the eight matrices:

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$$

if $n$ is even.

Let $\Phi\colon \bar{A}^{n,k} \to$ Auto $H^*(S^n \times S^k)$ be the obvious homomorphism.

PROPOSITION. *If $n>k$, or $n=k=1$, 3 or 7, or $n=k$ is even, then $\Phi$ is onto. If $n=k$ is odd, but $\neq 1$, 3 or 7, then* Image $\Phi$ *is the subgroup of* GL $(2, Z)$ *consisting of matrices*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

*where* $ab \equiv cd \equiv 0 \bmod 2$.

**Proof.** If $n>k$ or $n=k$ is even, this is easy to verify. If $n=k$ is odd, but $\neq 1$, 3 or 7, this is proven in [24, Lemma 5]. The proof of [24, Lemma 5] can be simplified to also prove the case $n=k=1$, 3 or 7 as follows. It is proved in [13, Appendix B] that GL $(2, Z)$ is generated by

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

But the corresponding automorphisms of $H^n(S^n \times S^n)$ are induced by diffeomorphisms; one for

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is defined by $(x, y) \mapsto (x, \phi(x)\cdot y)$, where $\phi\colon S^n \to SO_{n+1}$ is a map such that $p \circ \phi$ has degree $+1$ if $p\colon SO_{n+1} \to S^n$ is the usual evaluation map.

1.3. It now seems reasonable to consider the subgroup, Kernel $\Phi$, of $\bar{A}^{n,k}$. But our final results are simpler to state if we, instead consider the somewhat larger subgroup defined by orientation-preserving self-equivalences $f$ such that $f|x_0 \times S^k$ is homotopic to the inclusion, for any $x_0 \in S^n$. We denote this subgroup by $A^{n,k}$. Since $n \geqq k$, Kernel $\Phi \subset A^{n,k}$; if $n>k$, clearly Kernel $\Phi = A^{n,k}$. If $n=k$ is even, it follows from (1.2) that Kernel $\Phi = A^{n,n}$. But if $n=k$ is odd, $A^{n,n}$/Kernel $\Phi$ is infinite cyclic; in fact by (1.2) $\Phi(A^{n,n})$ is the subgroup of GL $(2, Z)$ consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix},$$

where $a$ is any integer, if $n=1$, 3 or 7, or $a$ is even if $n$ is otherwise odd.

Notice that $A^{n,n}$ is not normal in $\bar{A}^{n,n}$, if $n$ is odd, because $\Phi(A^{n,n})$ is not normal in GL $(2, Z)$. In this case, we leave it to the reader to deduce explicit results on Kernel $\Phi$ from the results to be obtained concerning $A^{n,k}$.

## 2. A decomposition of $A^{n,k}$.

2.1. Recall the notion of *semidirect product* of groups. If $G$ is a group with subgroups $G_0$ and $G_2$ satisfying:

(i) $G_0G_2 = G$, $G_0 \cap G_2 = \{1\}$,

(ii) $G_0$ is normal in $G$,

then the group structure of $G$ is entirely determined by the left action of $G_2$ on $G_0$, $\phi\colon G_2 \to \text{Auto } G_0$, defined by $\phi(g)\cdot g_0 = gg_0g^{-1}$. Conversely, given groups $G_0$ and $G_2$ and such an action $\hat{\phi}$, the Cartesian product $G_0 \times G_2$ is given a group structure $G$ by the multiplication:

$$(g_0, g_2)\cdot(g_0', g_2') = (g_0\cdot(\hat{\phi}(g_2)\cdot g_0'), g_2\cdot g_2').$$

2.2. Suppose, in addition, that $G_0$ is abelian and has a direct sum splitting $G_0 = G_1 \oplus \gamma$. Also, suppose that the action $\hat{\phi}$ is trivial on $\gamma$, i.e. $\hat{\phi}(g)|\gamma$ is the inclusion, for every $g \in G_2$.

PROPOSITION. *There is a unique left action, i.e. homomorphism* $\phi: G_2 \to \text{Auto } G_1$ *and function* $\tau: G_2 \to \text{Hom }(G_1, \gamma)$ *satisfying*:

(i) $\hat{\phi}(g)\cdot g' = \phi(g)\cdot g' + \tau(g)\cdot g'$, *for* $g \in G_2$, $g' \in G_1$,

(ii) $\tau(g_1 g_2) = \tau(g_1)\phi(g_2) + \tau(g_2)$, *for* $g_1, g_2 \in G_2$.

**Proof.** $\phi$ and $\tau$ are defined by (i), but it must be checked that $\phi(g)$ is an automorphism, $\phi$ is a left action and that $\tau(g)$ is a homomorphism. These are straightforward computations, as is the verification of (ii).

It is a simple matter to reverse this procedure. Given abelian groups $G_1$ and $\gamma$, and a group $G_2$ with an action $\phi$ of $G_2$ on $G_1$ and a function $\tau$ satisfying (ii), one can define by (i) an action $\hat{\phi}$ of $G_2$ on $G_1 \oplus \gamma$ and use $\hat{\phi}$ to construct the semidirect product. We will denote the resulting group by:

$$(G_1 \oplus \gamma) \times_{\phi,\tau} G_2.$$

2.3. Let $A = A^{n,k}$ be as defined in 1.3. Define subgroups $A_0$, $A_1$, $A_2$ and $\alpha$ of $A$ to consist of those elements represented by $f: S^n \times S^k \to S^n \times S^k$ satisfying:

($A_0$) $f | D^n_+ \times S^k = \text{inclusion}$,

($A_1$) $f$ extends to a self-equivalence of $S^n \times D^{k+1}$,

($A_2$) $f$ extends to a self-equivalence of $D^{n+1} \times S^k$,

($\alpha$) For some $(n+k)$-disk $D \subset S^n \times S^k, f(D) \subset D$ and $f | (S^n \times S^k - D) = \text{inclusion}$.

PROPOSITION. $\alpha = 0$ *in* $\mathscr{H}$ *and* $\mathscr{P}$, *while* $\alpha \approx \Gamma^{n+k+1}$ *in* $\mathscr{D}$.

**Proof.** In $\mathscr{H}$ and $\mathscr{P}$, any two self-equivalences of a disk, which agree on the boundary, are concordant, by the Alexander process in $\mathscr{P}$ (see [6]). In $\mathscr{D}$, $\Gamma^{n+k+1}$ is precisely the group of concordance classes of self-equivalences of the $(n+k)$-disk which are the identity on the boundary; thus there is an epimorphism:

$$\eta: \Gamma^{n+k+1} \to \alpha.$$

If $f$ represents $\eta(\sigma)$, for any $\sigma \in \Gamma^{n+k+1}$, then the smooth manifold $S^n \times D^{k+1} \cup_f D^{n+1} \times S^k$ is a topological $(n+k+1)$-sphere representing $\sigma$. This observation implies that $\eta$ is injective.

Notice that $\alpha$ is abelian.

2.4 THEOREM. $A = (A_1 \oplus \alpha) \times_{\phi,\tau} A_2$, *for suitable* $\phi$ *and* $\tau$, *assuming* $n \geqq 3$ *if* $\mathscr{A} = \mathscr{P}$ *or* $\mathscr{D}$. *Moreover* $A_1$ *and* $A_2$ *are abelian, and, if* $\mathscr{A} = \mathscr{P}$ *or* $\mathscr{D}$, $A_0 = A_1 \oplus \alpha$.

2.5. We first deal with $\mathscr{A} = \mathscr{H}$. Notice that a homotopy equivalence $f: S^n \times S^k \to S^n \times S^k$ represents an element of $H_i$ if and only if $p_i \circ f \simeq p_i$, where $p_i$ is projection on the $i$th factor ($i = 1$ or 2). It follows immediately that $H_1 \cap H_2 = \{1\}$.

We now show $H_2 H_1 = H$. Suppose $f$ represents an element of $H$ and $f_i = p_i \circ f$ are the coordinate functions. By definition of $H$, the map $y \mapsto f_2(x, y)$ has degree $+1$, for any $x \in S^n$. Therefore the map $g$ defined by $g(x, y) = (x, f_2(x, y))$ represents an element of $H$ and, by the above paragraph, of $H_1$. Now let $h: S^n \times S^k \to S^n \times S^k$ be a map satisfying $h \circ g \simeq f$. It follows that $f_2$ is homotopic to the map $(x, y)$

$\mapsto h_2(x, f_2(x, y))$, where $h_2 = p_2 \circ h$. Since we can consider $f_2$ and $h_2$ as representing elements of $\pi_n(G_{k+1}) - G_{k+1}$ is the space of maps $S^k \to S^k$ of degree $+1$—we can interpret this fact as saying that $h_2$ represents the zero element of $\pi_n(G_{k+1})$. But this means $h_2 \simeq p_2$, which implies that $h$ represents an element of $H_2$.

We now show $H_1$ is normal in $H$. Let $g$ represent an element of $H_1$—we may assume $p_1 \circ g = p_1$. If $g_2 = p_2 \circ g$, then the map $y \mapsto g_2(x, y)$ has degree $+1$, for any $x \in S^n$. Thus we may assume $g_2 | D^n_+ \times S^k = p_2 | D^n_+ \times S^k$ and, therefore, $g | D^n_+ \times S^k$ $=$ inclusion. Now suppose $f$ represents any element of $H$. By definition of $H$ we may assume that $f | D^n_- \times S^k =$ inclusion. We must show that a map $h$, satisfying $f \circ g \simeq h \circ f$, represents an element of $H_1$, i.e. $p_1 \circ h \simeq p_1$. Notice that $p_1 \circ f \circ g = p_1 \circ f$, since $g | D^n_+ \times S^k =$ inclusion, $p_1 \circ g = p_1$, and $f | D^n_- \times S^k =$ inclusion. Now we observe $p_1 \circ h \circ f \simeq p_1 \circ f \circ g \simeq p_1 \circ f$; since $f$ is a homotopy equivalence, we have $p_1 \circ h \simeq p_1$.

Finally, we show $H_1$ and $H_2$ are abelian. As pointed out above, any representative $g$ of an element of $H_1$ may be assumed to satisfy $p_1 \circ g = p_1$ and $g | D^n_+ \times S^k =$ inclusion. But it follows that another such $g'$ may be assumed to satisfy $p_1 \circ g' = p_1$ and $g' | D^n_- \times S^k =$ inclusion. Now we observe that $g \circ g' = g' \circ g$. Similarly for $H_2$.

2.6. We now assume $\mathscr{A} = \mathscr{P}$ or $\mathscr{D}$ and $n \geqq k$, $n \geqq 3$.

Lemma. *Every element of $A$ can be represented by $f$ satisfying $f(D^n_+ \times S^k) = D^n_+ \times S^k$.*

**Proof.** By definition of $A$, $f | x_0 \times S^k$ is homotopic to the inclusion. Choose $x_1 \in S^n - x_0$; then we may isotopically deform $f$ so that $f(x_0 \times S^k)$ is disjoint from $x_1 \times S^k$. If $n > k$, this follows from general position; if $n = k$, we use the technique of Whitney [28] (see [25] or 3.6 below for the piecewise-linear case) since $n \geqq 3$. Note that the intersection number of $f(x_0 \times S^k)$ and $x_1 \times S^k$ is 0.

We may now move $f(x_0 \times S^k)$ inside $D^n_+ \times S^k$, by another isotopy. In fact we may even move $f(D^n_+ \times S^k)$ inside $D^n_+ \times S^k$, if $x_0 \in D^n_+$. For $n + k \geqq 6$, it follows from the $h$-cobordism theorem [21] if $k > 1$, or the $s$-cobordism theorem [17] if $k = 1$, that $\mathrm{Cl}\,(D^n_+ \times S^k - f(D^n_+ \times S^k))$ is equivalent to $I \times S^{n-1} \times S^k$. Using this equivalence, there is an obvious isotopy to make $f(D^n_+ \times S^k) = D^n_+ \times S^k$.

If $n + k \leqq 5$ (and $n \geqq 3$), we may assume $f | x_0 \times S^k =$ inclusion—by general position if $k = 1$, or [29], [30] if $k = 2$. Then $D^n_+ \times S^k$ and $f(D^n_+ \times S^k)$ are both tubular or regular neighborhoods of $f(x_0 \times S^k)$; therefore, by the tubular or regular neighborhood theorem, an ambient isotopy of $S^n \times S^k$ will carry $D^n_+ \times S^k$ onto $f(D^n_+ \times S^k)$.

2.7 Lemma. *Suppose $f$ represents an element of $A_2$ ($A_1$) and satisfies $f | D^n_+ \times S^k = $ inclusion ($f | S^n \times D^k_+ =$ inclusion). Then $f$ is concordant to the identity.*

**Proof.** Suppose $f | D^n_+ \times S^k =$ inclusion and $f$ extends to a self-equivalence $F$ of $D^{n+1} \times S^k$. We may assume that $F$ is the "product extension" in a neighborhood of $S^n \times S^k$ i.e.

$$F(tx, y) = (tf_1(x, y), f_2(x, y))$$

for $(x, y) \in S^n \times S^k$ and $t$ near 1. Therefore $F|V=$inclusion, where $V$ is a neighborhood of $D^n_+ \times S^k$. If $D_0^{n+1}$ is the disk of radius 1/2 in $D^{n+1}$, then, by an isotopy, we may move $D_0^{n+1} \times S^k$ inside $V$. So we may assume $F|D_0^{n+1} \times S^k=$inclusion. But now $F|(D^{n+1}-D_0^{n+1}) \times S^k$ defines a concordance between $f$ and the identity.

A similar argument works for $A_1$.

2.8 Lemma. *Any element of $A_1$ ($A_2$) admits a representative $f$ satisfying $f|D^n_+ \times S^k$ $=$inclusion ($f|S^n \times D^k_+ =$inclusion).*

**Proof.** Given $f$ representing an element of $A_1$, we will construct a self-equivalence $g$, concordant to the identity, which agrees with $f$ on a neighborhood of $x_0 \times S^k$. After expanding this neighborhood, by an isotopy, to contain $D^n_+ \times S^k$ we may assume $g$ agrees with $f$ on $D^n_+ \times S^k$. Then $g^{-1} \circ f$ is concordant to $f$ and satisfies the condition of the lemma.

Let $F$ be a self-equivalence of $S^n \times D^{k+1}$ extending $f$. Given $x_0 \in S^n$, we may assume that $f|U$ is the inclusion, for some neighborhood $U$ of $(x_0, 0)$ in $S^n \times D^{k+1}$. We would now like to alter $F$ by an isotopy so that $F(x_0 \times D^{k+1})$ meets $S^n \times 0$ only at $(x_0, 0)$.

If $k=1$, it follows from general position for $n \geqq 4$ and [29] or [30] for $n=3$, that $f|x_0 \times D^{k+1}$ is isotopic to the inclusion. If $k \geqq 2$, we can apply Whitney's technique to remove the undesired intersections, since $S^n \times 0$ and $F(x_0 \times D^{k+1})$ have intersection number $\pm 1$ and an intersection of precisely this sign already occurs at $(x_0, 0)$.

It now follows that for sufficiently small disk neighborhoods $V_1$ of $x_0$ (in $S^n$) and $V_2$ of 0 (in $D^{k+1}$), $F(V_1 \times D^{k+1})$ meets $S^n \times V_2$ only at $F(V_1 \times V_2)=V_1 \times V_2$, and $F|V_1 \times V_2=$inclusion.

We now begin to define a self-equivalence $G$ of $S^n \times D^{k+1}$ by $G|S^n \times V_2=$inclusion, and $G|V_1 \times D^{k+1}=F|V_1 \times D^{k+1}$. It follows from our considerations that this defines an imbedding $G_0\colon S^n \times V_2 \cup V_1 \times D^{k+1} \to S^n \times D^{k+1}$. If $k=1$, we have shown that $F(x_0 \times D^{k+1})=x_0 \times D^{k+1}$ and, therefore, we may assume $G_0$ maps onto $S^n \times V_2 \cup V_1 \times D^{k+1}$. The extension to a self-equivalence of $S^n \times D^{k+1}$ is, then, formally equivalent to the extension of a self-equivalence of $S^{n-1} \times I \times S^k \cup D^n \times 0 \times S^k$ to one of $D^n \times I \times S^k$. But this is always possible.

Now assume $k \geqq 2$. Let

$$X = S^n \times D^{k+1} - \text{Image } G_0, \qquad X_0 = \text{Cl}\,(S^n \times S^k - G_0(V_1 \times S^k)),$$
$$X_1 = G_0\,(\text{Cl}\,(S^n - V_1) \times \partial V_2), \quad \text{and} \quad Y = G_0(\partial V_1 \times \text{Cl}\,(D^{k+1} - V_2)).$$

Then $X$ is an $h$-cobordism from $X_0$ to $X_1$, extending the $h$-cobordism $Y$ from $\partial X_0$ to $\partial X_1$. If we choose $V_2$ to be a concentric disk in $D^{k+1}$, then $G_0$ defines an obvious equivalence of $Y$ with $\partial V_1 \times \partial V_2 \times I$. According to the $h$-cobordism theorem ($\dim X \geqq 6$ and $X$, $X_0$, $X_1$ are 1-connected) this extends to an equivalence of $X$ with $\text{Cl}\,(S^n - V_1) \times \partial V_2 \times I$. This equivalence can be used to extend $G_0$ to the desired self-equivalence $G$, since $\text{Cl}\,(D^{k+1}) - V_2$ is equivalent to $\partial V_2 \times I$.

Now define $g=G|S^n\times S^k$. Since $G|S^n\times V_2=$inclusion, $G|S^n\times \mathrm{Cl}\,(D^{k+1}-V_2)$ defines a concordance from $g$ to the identity. Also $g|V_1\times S^k=F|V_1\times S^k=f|V_1\times S^k$. Thus $g$ is as desired at the beginning of the proof.

When $f$ represents an element of $A_2$, the argument is easier. Let $F$ be a self-equivalence of $D^{n+1}\times S^k$ extending $f$. Then $F|0\times S^k$ is isotopic to the inclusion (by general position if $n>k$, and [29] or [30] if $n=k$). Furthermore, since $F$ is orientation-preserving, we may assume that $F|D$ is the inclusion for any $(n+k+1)$-disk $D \subset$ interior $(D^{n+1}\times S^k)$. If $V$ is a concentric disk neighborhood of 0 in $D^{n+1}$, we may assume by the tubular or regular neighborhood theorem that $F(V\times S^k)=V\times S^k$. If $D$ contains $V\times D^k_+$, then $F|\mathrm{Cl}\,(D^{n+1}-V)\times S^k$ defines a concordance from $f$ to a self-equivalence with the desired property.

2.9. We are now ready to prove Theorem 2.4 for $\mathscr{P}$ or $\mathscr{D}$.

$A=A_0A_2$. Suppose $f$ represents an element of $A$. By Lemma 2.6 we may assume $f(D^n_+\times S^k)=D^n_+\times S^k$. Now extend $f|D^n_+\times S^k$ to a self-equivalence $G$ of $D^{n+1}\times S^k$; the restriction $g$ to $S^n\times S^k$ represents an element of $A_2$. Clearly $f\circ g^{-1}$ represents an element of $A_0$, and $f=(f\circ g^{-1})\circ g$ is the desired factorization of $f$.

$A_0\cap A_2=\{1\}$. This is implied by Lemma 2.7.

*$A_0$ is normal in $A$.* Let $f$ represent an element of $A$ and $g$ an element of $A_0$. Then we may assume $f(D^n_+\times S^k)=D^n_+\times S^k$, by Lemma 2.6, and $g|D^n_+\times S^k=$inclusion, by definition. But then $f\circ g\circ f^{-1}|D^n_+\times S^k=$inclusion, and, therefore, represents an element of $A_0$.

$\alpha\subset A_0$. This follows from the definitions by choosing $D$ disjoint from $D^n_+\times S^k$.

$A_1\subset A_0$. This follows from Lemma 2.8.

$A_1\cap\alpha=\{1\}$. This follows from Lemma 2.7 by choosing $D$ disjoint from $S^n\times D^k_+$.

$A_0=A_1\alpha$. Suppose $f$ is a self-equivalence of $S^n\times S^k$ satisfying $f|D^n_+\times S^k=$inclusion. We can extend $f$ to a self-equivalence $F_0$ of a neighborhood $U$ of $S^n\times S^k\cup x_0\times D^{k+1}$ in $S^n\times D^{k+1}$ by the "product extension" (see proof Lemma 2.7) near $S^n\times S^k$ and the identity near $x_0\times D^{k+1}$. The complement of $U$ in $S^n\times D^{k+1}$ is an $(n+k+1)$-disk $D$. It is well known that, after changing $F_0$ on a disk in $\partial D$, $F_0|\partial D$ may be extended to a self-equivalence of $D$. But such a change in $F_0$ can be effected by changing $f$ on a disk in $S^n\times S^k$.

This argument shows that a representative of any element of $A_0$, after being changed on a disk, extends to a self-equivalence of $S^n\times D^{k+1}$.

*$\alpha\subset$ center of $A$.* Any orientation-preserving self-equivalence of $S^n\times S^k$ may be assumed to leave a disk $D$ fixed. It therefore commutes with any representative of an element of $\alpha$, which is fixed outside of $D$.

*$A_0$ and $A_2$ are abelian.* If $f$, $g$ represent elements of $A_0$ ($A_2$), then, by definition (Lemma 2.8), we may assume that $f|D^n_+\times S^k$ and $g|D^n_-\times S^k(f|S^n\times D^k_+$ and $g|S^n\times D^k_-)$ are inclusions.

Then $f$ and $g$ commute.

This completes the proof of Theorem 2.4.

2.10. Notice that the homomorphisms:

$$\bar{D}^{n,k} \xrightarrow{\mu_1} \bar{P}^{n,k} \xrightarrow{\mu_2} \bar{H}^{n,k}$$

defined in 1.1 preserve the subgroups $A$, $A_0$, $A_1$ and $A_2$, and, therefore, the action $\phi$.

**3. Determination of the groups $A_1$ and $A_2$.**

3.1. Let $G_p$ denote the space of maps $S^{p-1} \to S^{p-1}$ of degree $+1$. We write $H(m, p) = \pi_m(G_{p+1})$. In the categories $\mathscr{P}$ and $\mathscr{D}$, let $A(m, p)$ be the group of concordance classes of framed imbeddings $S^m \to S^{m+p+1}$ i.e. imbeddings $S^m \times R^{p+1} \to S^{m+p+1}$. Two such $f_0, f_1$ are concordant if there is an imbedding $F\colon I \times S^m \times R^{p+1} \to I \times S^{m+p+1}$ such that $F(t, x) = (t, f_t(x))$, for $t = 0$, 1 and $x \in S^m \times R^{p+1}$.

Observe that $D(m, p) = FC_m^{p+1}$, in the notation of [7]. Furthermore, one can show, using the arguments in [8, §3], that $P(m, p) \approx F\Gamma_m^{p+1}$, the group of concordance classes of smooth framed submanifolds $\Sigma$ of $R^{m+p+1}$, where $\Sigma$ is piecewise-smoothly homeomorphic to $S^m$. These groups are extensively studied in [7] and [16]. In particular there are monomorphisms:

$$D(m, p) \xrightarrow{\mu_1'} P(m, p) \xrightarrow{\mu_2'} H(m, p).$$

$\mu_1'$ is defined by the passage from a framed imbedding to the framed submanifold determined by its image; $\mu_2'$ is defined in [16]. Recall from [7], [16] the exact sequences:

(a) $$\cdots \longrightarrow \Gamma_{m+1} \longrightarrow D(m, p) \xrightarrow{\mu_1'} P(m, p) \xrightarrow{\theta} \Gamma_m \longrightarrow \cdots$$

(b) $$\cdots \longrightarrow P_{m+1} \xrightarrow{\partial} P(m, p) \xrightarrow{\mu_2'} H(m, p) \xrightarrow{\omega} P_m \longrightarrow \cdots.$$

$P_m$ is defined to be 0 for $m$ odd, $Z$ for $m = 0 \bmod 4$, and $Z_2$ for $m = 2 \bmod 4$. Sequences (a) and (b) are valid for $p \geqq 2$, $m \geqq 1$ (see [16], [7]).

3.2. We construct homomorphisms:

$$\lambda_1\colon A_1^{n,k} \to A(n, k), \qquad \lambda_2\colon A_2^{n,k} \to A(k, n).$$

$\mathscr{A} = \mathscr{H}$. An element of $H_1$ is represented by a map of the form $(x, y) \mapsto (x, g(x, y))$, where $g\colon S^n \times S^k \to S^k$ represents an element of $\pi_n(G_{k+1}) = H(n, k)$. This induces the homomorphism $\lambda_1$ and, in a similar fashion, $\lambda_2$ is defined.

$\mathscr{A} = \mathscr{P}$ *or* $\mathscr{D}$. Let $f$ represent an element $\xi \in A_1^{n,k}$. Then $(x, t) \to (f(x), t)$ defines a self-equivalence $\hat{f}$ of $S^n \times S^k \times R$. Choose an orientation-preserving imbedding $R^k \subset S^k$ and notice the standard imbeddings

$$S^n \times S^k \times R \subset S^n \times R^{k+1} = S^n \times R \times R^k \subset R^{n+1} \times R^k = R^{n+k+1}$$

defined by $(x, y, t) \to (x, e^t y)$, $(x, t, y) \to (e^t x, y)$. Then the composite imbedding:

$$S^n \times R^{k+1} = S^n \times R^k \times R \subset S^n \times S^k \times R \xrightarrow{\hat{f}} S^n \times S^k \times R \subset S^n \times R^{k+1} \subset R^{n+k+1}$$

represents the element $\lambda_1(\xi) \in A(n, k)$. $\lambda_1$ is clearly well defined. $\lambda_2$ is defined similarly.

An alternative description of $\lambda_1(\xi)$ is as follows. Let $F$ be an extension of $f$ to a self-equivalence of $S^n \times D^{k+1}$. The standard equivalence of $R^{k+1}$ with the interior of $D^{k+1}$ induces an imbedding $S^n \times R^{k+1} \subset S^n \times D^{k+1}$. Now, the composition:

$$S^n \times R^{k+1} \subset S^n \times D^{k+1} \xrightarrow{F} S^n \times D^{k+1} \subset S^n \times R^{k+1} \subset R^{n+k+1}$$

represents $\lambda_1(\xi)$. This seems to depend upon $F$, but its equivalence to the first definition is easily verified. A similar description holds for $\lambda_2$.

That $\lambda_1$ is a homomorphism follows by choosing representative self-equivalences which restrict to the inclusion on $D^n_+ \times S^k$ or $D^n_- \times S^k$, by Lemma 2.8. The framed imbeddings constructed are then standard on $D^n_+$ or $D^n_-$. The composition of the self-equivalences then clearly yields the sum of the framed imbeddings.

The functions $\mu_i$ and $\mu_i'$ are related by the $\lambda_i$ according to the commutative diagram:

$$\begin{array}{ccccc} D_1 & \xrightarrow{\mu_1} & P_1 & \xrightarrow{\mu_2} & H_1 \\ \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_1} \\ D(n, k) & \xrightarrow{\mu_1'} & P(n, k) & \xrightarrow{\mu_2'} & H(n, k). \end{array}$$

Similarly for $\lambda_2$.

3.3. The fibration $G_{n+1} \to S^n$ defined by evaluation at a point of $S^n$ induces a homomorphism $H(k, n) = \pi_k(G_{n+1}) \to \pi_k(S^n)$. We define:

$$\varepsilon: A(k, n) \to \pi_k(S^n)$$

by composing this with the necessary $\mu_i'$, and write $A_0(k, n) = \text{Kernel } \varepsilon$. Notice that $A_0(k, n) = A(k, n)$ unless $k = n$ is odd (see (1.2)).

THEOREM. (i) $P_1^{n,1} = 0$, $D_1^{n,1} \approx \theta^{n+1}/bP^{n+2}$ *(see* [12]) *for* $n \geqq 3$,

(ii) $\lambda_1$ *is an isomorphism onto* $A(n, k)$ *if either* $\mathscr{A} = \mathscr{H}$ *or* $n \geqq 3$ *and* $k \geqq 2$,

(iii) $\lambda_2$ *is an isomorphism onto* $A_0(k, n)$ *if either* $\mathscr{A} = \mathscr{H}$ *or* $n \geqq 3$.

Compare [1, §5] for $k = 1$.

3.4. The definition of $A$ (see 1.3) implies that Image $\lambda_2 \subset A_0(k, n)$. The statements of the theorem for $\mathscr{A} = \mathscr{H}$ follow immediately from the previous (2.5) observation that elements of e.g. $H_1'$ are represented by maps of the form $(x, y) \mapsto (x, g(x, y))$, where $g$ represents an element of $\pi_n(G_{k+1})$. The correspondence, $\lambda_1$, so defined, is clearly injective. Since any map $S^n \times S^k \to S^k$ extends to a map $S^n \times D^{k+1} \to D^{k+1}$, by radial extension, it follows that $\lambda_1$ is surjective. For $\lambda_2$ the same arguments work except that the definition of $A$ (see above) restricts Image $\lambda_2$ to $A_0(k, n)$.

3.5. We now restrict ourselves to $\mathscr{A} = \mathscr{P}$ or $\mathscr{D}$. The surjectivity of $\lambda_1$ and $\lambda_2$ is seen as follows. Given a framed imbedding of $S^n$ in $S^{n+k+1}$, we consider the restriction

$F: S^n \times D^{k+1} \to S^{n+k+1}$. By an argument in [16, §3.5] which uses the $h$-cobordism theorem—since $k \geqq 2$, $n \geqq 3$—we may assume that $F(S^n \times D^{k+1}) = S^n \times D^{k+1}$. It then follows from our second description of $\lambda_1$ that $\lambda_1$ is onto. A similar argument works for $\lambda_2$, if $k > 1$. If $k = 1$, notice that $A(1, n) \approx Z_2$, and its generator is the image, under $\lambda_2$, of the element of $A_2^{n,1}$ represented by $(x, y) \mapsto (\phi(y) \cdot x, y)$, where $\phi: S^1 \to SO_{n+1}$ is essential.

3.6. Before continuing, we digress briefly to prove a slight extension of Whitney's theorem [28] on removing intersections of submanifolds. This will be needed so that we can apply Whitney's technique in all the dimensions needed. The result is stated for both the smooth and piecewise-linear category.

LEMMA. *Suppose $M$, $N$ are oriented connected locally flat submanifolds of the oriented simply-connected manifold $V$ where* $\dim M + \dim N = \dim V$, $\dim N \geqq 2$ $\dim M > 2$, *and $M$ meets $N$ in general position. If the intersection number of $M$ and $N$ is zero and, for* $\dim N = 2$, $\Pi_1(V - M)$ *is abelian, there is an isotopy of $V$, stationary on $\partial V$, which separates $M$ and $N$.*

**Proof.** We describe the necessary modifications in Whitney's proof [28] (see also [20])—placing ourselves in the differential situation. Recall that the intersections of $M$ and $N$ are removed by a sequence of local isotopies, each of which removes a pair of oppositely oriented intersections.

The construction of a local isotopy is begun by joining the points by arcs $a_1$ in $M$ and $a_2$ in $N$ and then constructing vector fields $v_i$ along $a_i$ such that $v_1$ is normal to $M$ and inwardly tangent to $a_2$ at $a_1 \cap a_2$—similarly for $v_2$. If we push $a_i$ slightly along $v_i$, we obtain new arcs $a_i'$ in $V$ which, we may assume, again meet at two points. Then $a_1' \cup a_2'$ contains a loop $l$ in $V - (M \cup N)$. If $M$ and $N$ both have dimension $> 2$, it follows by general position that $l$ is null-homotopic in $V - (M \cup N)$ and bounds an imbedded 2-disk $D$. If dimension $N = 2$, it is not necessarily true that $l$ is null-homotopic in $V - (M \cup N)$. Now notice that $H_1(V - M)$ is generated by the element $\alpha$ represented by any small loop linking $M$ once. But $l$ may be modified, e.g. by adding twists to $v_1$ or letting $a_2$ wind around one of its endpoints extra times, to add any multiple of $\alpha$ to its homology class. Thus we may assume $l$ is null-homologous, and so null-homotopic, in $V - M$. Since $\dim M > 2$, a general position argument allows us to complete the construction of $D$.

The next step (see [20]) is to construct a normal frame $w_1, \ldots, w_n$ to $D$, so that $w_1, \ldots, w_k$ is tangent to $N$ along $a_2$ and normal to $M$ along $a_1$, where $\dim V = n + 2$ and $\dim N = k + 1$. This is begun by constructing $w_1, \ldots, w_k$ along $a_1 \cup a_2$ and extending over $D$. The obstruction to the extension is an element of $\pi_1(V_{n,k})$, where $V_{n,k}$ is the Stiefel manifold of $k$-frames in $n$-space; since $n - k = (\dim M) - 1 > 1$, $\pi_1(V_{n,k}) = 0$.

The remainder of the proof is identical to Whitney's. For the piecewise-linear case, we can construct the arcs $a_i$ as above and then smooth $M$ and $N$ in neighborhoods of $a_1$ and $a_2$, respectively, to enable us to apply the procedures of Whitney with the above modifications—see [3] for a description of the necessary smoothing. Alternatively, we could modify the arguments of [25] to obtain the piecewise-linear result.

3.7. It is illuminating to examine two simple examples toward understanding the necessity of the hypotheses of Lemma 3.6.

EXAMPLE 1. Let $K_1, K_2 \subset S^3$ be trivial knots with linking number zero, and $D_1, D_2 \subset D^4$ disks bounded by $K_1, K_2$ respectively. Then the intersection number of $D_1$ and $D_2$ is zero, but, if $K_1$ and $K_2$ are nontrivially linked we cannot remove the intersections of $D_1$ and $D_2$. Note that, we may choose $D_1, D_2$ so that $\pi_1(D^4 - D_i)$ is abelian.

EXAMPLE 2. Let $K \subset S^{n+2}$ be an imbedded $n$-sphere such that $\pi_1(S^{n+2}-K)$ is not abelian. Let $C \subset S^{n+2}-K$ be an imbedded circle representing a nonzero element of the commutator subgroup of $\pi_1(S^{n+2}-K)$. If $n>1$, $C$ is unknotted. Let $V=I\times S^{n+2}$, $M=I\times K$, and $N$ an imbedded 2-disk in $V$ bounded by $0\times C$. The intersection number of $M$ and $N$ is zero, but the intersections cannot be removed since $C$ is essential in $S^{n+2}-K$.

3.8. We now return to the proof of Theorem 3.3. It remains to prove the injectivity of $\lambda_1$ and $\lambda_2$, and assertion (i). We first consider $A_1^{n,k}$. Suppose $F_0$, $F_1$ are self-equivalences of $S^n\times D^{k+1}$, and the compositions:

$$S^n\times R^{k+1} \subset S^n\times D^{k+1} \xrightarrow{F_i} S^n\times D^{k+1} \subset S^n\times R^{k+1} \subset R^{n+k+1}$$

are concordant framed imbeddings. Let $C\colon I\times S^n\times R^{k+1}\to I\times S^{n+k+1}$ be the concordance.

Recall the imbedding $S^n\times R^{k+1}=S^n\times R^k\times R\subset R^{n+k+1}$ defined by $(x, y, t)\to(e^tx, y)$. If $S^{n+k+1}$ is represented as the one-point compactification of $R^{n+k+1}$, then the complement of this imbedding is a $k$-sphere $\Sigma$ which links $S^n\times 0$ once. Now the intersection number of $C(I\times S^n\times 0)$ with $I\times\Sigma$ is 0. *If* $k=1$, *we assume* $\pi_1(I\times S^{n+k+1}-C(I\times S^n\times 0))$ *is abelian.* Therefore we can apply Whitney's procedure, extended by Lemma 3.6, to change $C$ so that $C(I\times S^n\times 0)\subset I\times S^n\times R^{k+1}$.

We can then move $C(I\times S^n\times 0)$ into an arbitrarily small neighborhood of $I\times S^n\times 0$. If $D_0^{k+1}$ is a disk of small radius, we may assume $C(I\times S^n\times D_0^{k+1})\subset I\times S^n\times D^{k+1}$. The annular region $\mathrm{Cl}\,(I\times S^n\times D^{k+1}-C(I\times S^n\times D_0^{k+1}))$ is an $h$-cobordism between $I\times S^n\times S^k$ and $C(I\times S^n\times\partial D_0^{k+1})$, and $F_0$, $F_1$ define an equivalence of

$$\dot I\times S^n\times(S^k\times I) = \dot I\times S^n\times \mathrm{Cl}\,(D^{k+1}-D_0^{k+1})$$

with $\mathrm{Cl}\,(I\times S^n\times D^{k+1}-C(I\times S^n\times D_0^{k+1}))$. By the $h$-cobordism theorem, for $k>1$, and the $s$-cobordism theorem [17] if $k=1$ (note that, if $k=1$ and $C$ as above, $\pi_1(I\times S^n\times D^{k+1}-C(I\times S^n\times D_0^{k+1}))\approx Z$), $F_0$, $F_1$ and $C|I\times S^n\times D_0^{k+1}$ extend to a self-equivalence of $I\times S^n\times D^{k+1}$. Thus $F_0$ and $F_1$ are concordant, and we have proved (ii) when $k\geqq 2$.

The proof that $\lambda_2$ is injective is the same, except that special consideration when $n=1$ is not required. On the other hand, when $k=1$, it is necessary to point out that $\pi_1(I\times S^{n+k+1}-I\times\Sigma)\approx Z$ in order to apply Lemma 3.6.

3.9. We now prove (i). Let $F_0$ be a self-equivalence of $S^n\times D^2$ and $\tilde{F}_0$ the imbedding

$$S^n\times D^2 \xrightarrow{F_0} S^n\times D^2 \subset S^n\times R^2 \subset R^{n+2} \subset S^{n+2};$$

notice that $S^{n+2}-\tilde{F}_0(S^n\times 0)$ is a homotopy circle. It is proved in [15] or [33] for $n\geqq 4$, and [32] for $n=3$, that any collared imbedded $n$-sphere in the $(n+2)$-sphere whose complement is a homotopy circle bounds a collared $(n+1)$-disk. From this it follows that, for some orientation-preserving self-equivalence $h$ of $S^n$ we can define a self-equivalence $F_1=h\times 1$ of $S^n\times D^2$ such that the framed imbeddings:

$$S^n\times R^2 \longrightarrow S^n\times D^2 \xrightarrow{F_i} S^n\times D^2 \subset S^n\times R^2 \subset R^{n+2}$$

are isotopic. Now we can apply the argument in (3.8) to prove that $F_0$ and $F_1$ are concordant, since $C$ can be chosen to be an *isotopy* from which it follows that $\pi_1(I\times S^{n+2}-C(I\times S^n\times 0))$ is abelian.

In $\mathscr{P}$ all orientation-preserving self-equivalences of $S^n$ are isotopic; thus we have shown that $P_1^{n,1}=0$ for $n\geqq 3$.

3.10. We can define a homomorphism $\theta\colon \Gamma^{n+1}\to D_1^{n,1}$ by associating to a self-equivalence $h$ of $S^n$, the self-equivalence $h\times 1$ of $S^n\times S^1$. The argument of (3.9) shows $\theta$ is onto. Since $\Gamma^4=0$ ([5]), this proves $D_1^{3,1}=0$; if $n\geqq 4$, $\theta^{n+1}\approx\Gamma^{n+1}$. We will show that Kernel $\theta=bP^{n+2}$, completing the proof of (i).

If $h$ represents an element of Kernel $\theta$, then the imbedding $i\circ h$, where $i\colon S^n \to S^{n+2}$ is the standard inclusion, is concordant to $i$. By [31, Theorem III. 3], this proves $h$ represents an element of $bP^{n+2}$. Thus Kernel $\theta\subset bP^{n+2}$.

Conversely if $h$ represents an element of $bP^{n+2}$, let $C'$ be a concordance between $i$ and $i\circ h$. $C'$ also defines a concordance between the framed imbedding

$$S^n\times R^2 \xrightarrow{h\times 1} S^n\times R^2 \subset S^{n+2}$$

and some framing of $i$. But since $\pi_n(SO_2)=0$ $(n\geqq 3)$, this is isotopic to $S^n\times R^2\subset S^{n+2}$. Now, if we knew that $\pi_1(I\times S^{n+2}-C'(I\times S^n))$ were abelian, we could use the argument of (3.8) to prove that $h\times 1$ (on $S^n\times S^1$) is concordant to the identity (and so $bP^{n+2}\subset\text{Ker }\theta$).

3.11. It remains to construct $C'$ as desired. Recall the argument of [31, p. 262]. Let $\Sigma^{n+1}=D^{n+1}\cup_h D^{n+1}$, representing an element of $bP^{n+2}$; then $\Sigma^{n+1}$ imbeds smoothly in $S^{n+3}$. In fact, if $V^{n+2}$ is a $[(n+1)/2]$-connected parallelizable manifold bounded by $\Sigma^{n+1}$, we can imbed $V^{n+2}$ in $S^{n+3}$. Notice also that, for an imbedding of $\Sigma^{n+1}$ in $S^{n+3}$ constructed this way, $\pi_1(S^{n+3}-\Sigma^{n+1})$ is abelian (see e.g. [15]). Let $B_1$, $B_2$ be disjoint $(n+3)$-balls in $S^{n+3}$ with $B_i\cap\Sigma^{n+1}=(n+1)$-ball. There is a diffeomorphism $S^{n+3}-(B_1\cup B_2)\approx I\times S^{n+2}$ which carries $\Sigma^{n+1}-(B_1\cup B_2)$

onto the image of a concordance $C'$ between $i \circ h$ and $i$ (see [31, p. 262]). Then $\pi_1(I \times S^{n+2} - C'(I \times S^n)) \approx \pi_1(S^{n+3} - \Sigma^{n+1})$ which is abelian.

This completes the proof of Theorem 3.3.

## 4. **Group structure of $A^{n,k}$.**

4.1. We now study the action $\phi$, and the function $\tau$, which occur in the description of $A^{n,k}$ given by Theorem 2.4. By Theorem 3.3, $\phi$ corresponds to an action of $A_0(k, n)$ on $A(n, k)$, also called $\phi$, and $\tau$ corresponds to a bilinear pairing $D(n, k) \otimes D_0(k, n) \to \Gamma^{n+k+1}$, also called $\tau$.

We first study $\phi$ in the category $\mathscr{H}$. The fibration $G_{m+1} \to S^m$ (see 3.3) has fiber $F_m$, the space of base-point preserving maps $S^m \to S^m$ of degree $+1$. There is a canonical isomorphism $\pi_p(F_m) \approx \pi_{p+m}(S^m)$ (see [26, p. 465]). Note $F_m$ is not the usual loop space; in the notation of [26], $F_m = F_\iota^m(S^m, x)$. There are homomorphisms $\varepsilon: \pi_p(G_{m+1}) \to \pi_p(S^m)$ and $\nu: \pi_{p+m}(S^m) \to \pi_p(G_{m+1})$ induced by the fibration and inclusion of the fiber.

4.2 Proposition. *If $\xi \in \pi_{n+k}(S^n)$ and $\beta \in \pi_n(G_{k+1})$, then $\phi(\nu(\xi)) \cdot \beta = \beta - \nu(\varepsilon(\beta) \circ \xi)$. Note $H_0(k, n) = \nu(\pi_{n+k}(S^n))$.*

**Proof.** Consider representatives $f: S^n \times S^k \to S^n$ of $\nu(-\xi)$, and $g: S^n \times S^k \to S^k$ of $\beta$. Let $\rho: (S^n, D^n_-) \to (S^n, x_0)$ be a map of degree $+1$; we may assume $f(x, y) = \rho(x)$ for $x \in D^n_+$ or $y \in D^k_+$.

Consider a new map $f'$: $S^n \times S^k \to S^n$ defined by:

$$\begin{aligned} f'(x, y) &= f(x, y) \quad \text{if } x \in D^n_- \\ &= x_0 \quad\quad\;\; \text{if } x \in D^n_+. \end{aligned}$$

Recall from [26, p. 465] that $\nu$ is defined as a composition:

$$\pi_{m+p}(S^m) \xrightarrow{\nu'} \pi_{m+p}(F^0_m) \xrightarrow{\nu''} \pi_{m+p}(F_m),$$

where $F^0_m$ is the space of base-point preserving maps $S^{m-1} \to S^{m-1}$ of degree 0, $\nu'$ is the usual isomorphism and $\nu''$ is induced by "adding" a fixed map of degree $+1$. Then $f'$ represents $\nu'(-\xi)$.

Now it follows that $-\nu'(\varepsilon(\beta) \circ \xi) = \nu'(\varepsilon(\beta) \circ (-\xi))$ is represented by a map $h': S^n \times S^k \to S^k$ defined by $h'(x, y) = g(f'(x, y), y_0)$. Notice that $h'(x, y) = g(x_0, y_0)$ if $y \in D^k_+$; we may assume $g(x_0, y_0) = y_0$. Let $\gamma$: $(S^k, D^k_-) \to (S^k, y_0)$ be a map of degree $+1$; then $-\nu(\varepsilon(\beta) \circ \xi)$ is represented by $h$: $S^n \times S^k \to S^k$ defined by:

$$\begin{aligned} h(x, y) &= h'(x, y) \quad \text{if } y \in D^k_- \\ &= \gamma(y) \quad\quad\;\; \text{if } y \in D^k_+. \end{aligned}$$

Now $f'(x, y) = x_0$, if $y \in D^k_+$, and, therefore, the map $(D^k_+, S^{k-1}) \to (S^k, y_0)$, defined by $y \mapsto g(f'(x, y), \gamma(y))$, is independent of $x$. Let $\bar{\gamma}$: $(S^k, D^k_-) \to (S^k, y_0)$ be the extension of degree $+1$. Since, in the definition of $h$, $\gamma$ was arbitrary, replace $\gamma$ with $\bar{\gamma}$. Then it may be checked that:

$$h(x, y) = g(f'(x, y), \gamma(y)) \quad \text{for } x \in S^n,\ y \in S^k.$$

Since $\gamma$ is homotopic to the identity, we can change the definition of $h$, without altering its homotopy class, to: $h(x, y)=g(f'(x, y), y)$ for $x \in S^n$, $y \in S^k$.

Now $\beta$ is also represented by the map $(x, y) \mapsto g(\rho(x), y)$. Therefore, by the definition of addition, $\beta-\nu(\varepsilon(\beta) \circ \xi)$ is represented by the map $l: S^n \times S^k \to S^k$ defined by:

$$l(x, y) = g(f'(x, g(\rho(x), y)), g(\rho(x), y)).$$

If $g$ is chosen to satisfy $g(x_0, y)=y$, for $y \in S^k$, then it may be checked that $l(x, y)=g(f(x, y), y)$. But this is also a representative of $\phi(\nu(\xi)) \cdot \beta$.

4.3. Proposition 4.2, together with Theorems 2.4 and 3.3, completely determine the group $H^{n,k}$ in terms of standard homotopy groups. Notice also that Proposition 4.2 gives us a great deal of information about $\phi$ in the categories $\mathscr{P}$ and $\mathscr{D}$ (see 2.10).

4.4. We can define suspension homomorphisms:

$$\sigma_1: A_1^{n,k} \to A_1^{n,k+1}, \qquad \sigma_2: A_2^{n,k} \to A_2^{n+1,k}.$$

Suppose $f$ represents an element $\xi \in A_1^{n,k}$ and $F$ is an extension of $f$ to a self-equivalence of $S^n \times D^{k+1}$. If we choose equivalences $\varepsilon_+ : D_+^{k+1} \to D^{k+1}$ and $\varepsilon_- : D_-^{k+1} \to D^{k+1}$ extending the inclusion $S^k \to D^{k+1}$, then $F$ determines a self-equivalence $\hat{f}$ of $S^n \times S^{k+1}$, defined as follows:

$$\begin{aligned} \hat{f}(x, y) &= F(x, \varepsilon_+(y)) \quad \text{if } y \in D_+^{k+1}, \\ &= F(x, \varepsilon_-(y)) \quad \text{if } y \in D_-^{k+1}. \end{aligned}$$

The concordance class of $\hat{f}$ depends only on that of $f$. For certainly it depends only on the concordance class of $F$. But $F$ is isotopic to $F'$, where $F'$ is a "product extension" of $f$ (see 2.7) in a neighborhood of $S^n \times S^k$. If we use $F'$, instead of $F$, in the construction of $\hat{f}$, we see that $\hat{f}$ is determined by $f$ in $S^n \times U$, where $U$ is some neighborhood of $S^k$ in $S^{k+1}$. It now follows from Lemma 2.7 that $\hat{f}$ is determined by $f$; notice that $\hat{f}$ extends to a self-equivalence of $S^n \times D^{k+2}$. We define $\sigma_1(\xi)$ to be the concordance class of $\hat{f}$. Clearly $\sigma_1$ is a homomorphism. $\sigma_2$ is defined similarly.

Recall the suspension homomorphisms $\sigma: A(m, p) \to A(m, p+1)$ (see [7], [16]). The following diagrams are clearly commutative:

$$\begin{array}{ccc} A_1^{n,k} & \xrightarrow{\sigma_1} & A_1^{n,k+1} \\ \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_1} \\ A(n, k) & \xrightarrow{\sigma} & A(n, k+1) \end{array} \qquad \begin{array}{ccc} A_2^{n,k} & \xrightarrow{\sigma_2} & A_2^{n+1,k} \\ \downarrow{\scriptstyle \lambda_2} & & \downarrow{\scriptstyle \lambda_2} \\ A(k, n) & \longrightarrow & A(k, n+1). \end{array}$$

4.5 Proposition. *For any* $\xi \in A_2$, $\phi(\xi)$ *acts trivially on* Image $\sigma_1$.

**Proof.** If $\mathscr{A}=\mathscr{H}$, this follows immediately from Proposition 4.2, since $\varepsilon \circ \sigma=0$. We assume $\mathscr{A}=\mathscr{P}$ or $\mathscr{D}$.

Let $g$ represent $\xi \in A_2^{n,k}$ and $f$ represent $\beta \in \sigma_1(A_1^{n,k-1})$. We may, therefore, assume that $f(S^n \times D_+^k) = S^n \times D_+^k$. By Lemma 2.8, we may also assume that $g|S^n \times D_+^k$ is the inclusion. Now $\phi(\xi)\cdot\beta$ is represented by $g \circ f \circ g^{-1} = f'$. But clearly $f'|S^n \times D_+^k = f|S^n \times D_+^k$; then Lemma 2.7 implies $f$ and $f'$ are concordant.

4.6. We now restrict ourselves to $\mathscr{A} = \mathscr{P}$. Referring to the exact sequence 3.1(b), the remarks of 4.3 tell us that the action $\phi$ of $P_0(k, n)$ on $P(n, k)$ is determined, by Proposition 4.2, modulo the image of $\partial: P_{n+1} \to P(n, k)$. Hence, there remains indeterminacy in $\phi$ only if $n$ is odd. If $n \equiv 3 \bmod 4$, then $P_{n+1} = Z$ and $\partial$ is a monomorphism (see [16]). Therefore $\mu_2'|\text{Torsion } P(n, k)$ is injective and the indeterminacy will be resolved by: (except when $n = k$).

PROPOSITION. *For any $\xi \in P_0(k, n)$ and $\beta \in P(n, k)$, $\phi(\xi)\cdot\beta - \beta$ is a torsion element of $P(n, k)$, unless $n = k$ is odd.*

**Proof.** Suppose $H(n, k) = \pi_n(G_{k+1})$ is finite, i.e. $n \neq 2k-1$ for $k$ even. Then $r\beta \in \partial P_{n+1}$, for some positive integer $r$, and $r(\phi(\xi)\cdot\beta) = \phi(\xi)\cdot r\beta$, by Proposition 4.5 ($\partial P_{n+1} \subset \text{Image } \sigma_1$).

Suppose $n = 2k-1$, $k$ even. By Proposition 4.2, $\phi(\xi)\cdot\beta - \beta \in P_0(n, k)$ and it follows that $\mu_2'(\phi(\xi)\cdot\beta - \beta) \in H_0(n, k)$, which is finite (since $\pi_{n+k}(S^k)$ is finite). Therefore $r(\phi(\xi)\cdot\beta - \beta) \in \partial P_{n+1}$, for some positive integer $r$. We will now prove the formula:

$$\phi(\xi)^s\cdot\beta' - \beta' = s(\phi(\xi)\cdot\beta' - \beta') = rs(\phi(\xi)\cdot\beta - \beta)$$

where $\beta' = r\beta$, for any positive integer $s$. Since $P_0(k, n)$ is finite ($k$ is even, and so $P_{k+1} = 0$), this will show that $\phi(\xi)\cdot\beta - \beta$ is torsion.

For $s = 1$, the formula is obvious. Suppose it is true for $s$. Applying $\phi(\xi)$ to both sides, and observing from Proposition 4.5 that $\phi(\xi)$ acts trivially on the right-hand side, we obtain:

$$\phi(\xi)^{s+1}\cdot\beta' - \phi(\xi)\cdot\beta' = s(\phi(\xi)\cdot\beta' - \beta').$$

Adding $\phi(\xi)\cdot\beta' - \beta'$ to both sides yields the desired formula for $s+1$.

4.7 REMARK. One can consider the stable suspension:

$$P(m, p) = F\Gamma_m^{p+1} \xrightarrow{\hat{\sigma}} F\Gamma_m \approx \pi_m(PL)$$

where $PL = \lim_{q\to\infty} PL_q$ and $PL_q$ is the semi-simplicial group of germs of piecewise-linear homeomorphisms of $R^n$ onto itself (see e.g. [8]). It should not be difficult to verify the formula

$$\hat{\sigma}(\phi(\xi)\cdot\beta - \beta) = \pm\hat{\sigma}(\xi)\circ\varepsilon(\beta) \quad \text{for } \xi \in P_0(k, n),\ \beta \in P(n, k).$$

Since $\hat{\sigma}|\partial P_{n+1}$ is injective, unless, perhaps, $n = 2^e - 3$ for some integer $e$, (see [16], [4] and [2]), this would resolve the indeterminacy quite often. Notice, also, the implication that Proposition 4.6 is false when $n = k$ is odd.

4.8. Finally we comment briefly on the pairing $\tau: D_1^{n,k} \otimes D_2^{n,k} \to \Gamma^{n+k+1}$. Recall the pairing

$$\pi_n(SO_{k+1}) \otimes \pi_k(SO_n) \to \Gamma^{n+k+1}$$

studied by Milnor in [18]. Also notice that there are homomorphisms:

$$\pi_n(SO_{k+1}) \to D_1^{n,k} \quad \text{and} \quad \pi_k(SO_n) \to D_2^{n,k}$$

defined by associating to a smooth map $f: S^n \to SO_{k+1}$, the diffeomorphism $(x, y) \mapsto (x, f(x) \cdot y)$—similarly for $\pi_k(SO_n)$, except that we must suspend $SO_n \subset SO_{n+1}$ first. It is then clear that these homomorphisms make the pairing of Milnor a special case of $\tau$.

In [14] a generalization of Milnor's pairing is also studied. By considerations of the above type, the construction of [14, §7] is also a special case of $\tau$. Furthermore the results of [14] can be generalized, with little effort, to give analogous results about $\tau$. We will not, at this time, present any details.

## 5. **Examples and applications.**

5.1. We begin by making some specific comparisons of self-equivalences of $S^n \times S^k$ in the three categories, which amounts to considering the homomorphisms $\mu_1$ and $\mu_2$ defined in (1.1). We concentrate on $\mu_2$, and leave a similar study of $\mu_1$ to the reader. By (3.2) and (3.3), it suffices to consider $\mu_2'$ in sequence 3.1(b).

The homomorphism $\omega: H(m, p) \to P_m$ is defined, using the Thom construction, by taking the index or Kervaire invariant of a closed framed $m$-dimensional submanifold of $S^m \times S^p$. According to the Index Theorem and results of Brown-Peterson-Browder ([2], [4]), $\omega = 0$ unless $m = 2^e - 2$, for some $e \geqq 2$. On the other hand if $e = 3$, 4 or 5 and $p$ is large, then $\omega \neq 0$ (see [12], [2]). More specifically, $\omega \neq 0$ when $e = 3$ and $p \geqq 4$, or $e = 4$ and $p \geqq 8$ and $\omega = 0$ when $e = 3$ and $p \leqq 3$, or $e = 4$ and $p \leqq 6$. These facts follow from the observation that the suspension $\pi_m(G_{p+1}) \to \pi_m(G)$ is onto (for $\omega \neq 0$) or zero (for $\omega = 0$) in the stated cases.

Now it follows immediately from 3.1(b) that $\mu_2': P(m-1, p) \to H(m-1, p)$ is not injective when $\omega = 0$ and $P_m \neq 0$ and $\mu_2': P(m, p) \to H(m, p)$ is not surjective when $\omega = 0$. These phenomena imply, therefore:

1. The existence of piecewise-linear homeomorphisms which are homotopic but not concordant on $S^m \times S^p$ when $m$ (or $p$) is odd but not of the form $2^e - 3$—see [1] for $m \equiv 3 \bmod 4$—or when $m = 5$ and $p = 2$ or 3, or $m = 13$ and $2 \leqq p \leqq 6$. The lowest dimensional example is $S^5 \times S^2$.

2. The existence of a homotopy equivalence not homotopic to a piecewise-linear homeomorphism on $S^m \times S^p$ when $m = 6$, $p \geqq 4$ or $m = 14$, $p \geqq 8$ or $m = 30$, $p$ large.

In fact, it follows from work of Sullivan ([23] and [34]) that "piecewise-linear" may be replaced by "topological" in these examples, since $H^*(S^m \times S^p)$ contains no 2-torsion.

5.2. We will now show that, in a certain metastable range of dimensions, every self-equivalence of $S^n \times S^k$ (in the categories $\mathscr{P}$ and $\mathscr{D}$) is concordant to a composition of familiar ones.

Consider the following types of *diffeomorphisms* $f$ of $S^n \times S^k$.

(a) $f(x, y) = (\rho(y) \cdot x, y)$, where $\rho: S^k \to O_{n+1}$ is a smooth map,

(b) $f(x, y) = (x, \rho(x) \cdot y)$, where $\rho: S^n \to O_{k+1}$ is a smooth map and,

(c) $f$ is the identity outside an $(n+k)$-disk.

PROPOSITION. *If $n < 2k-1$, $n \geqq 3$, then every diffeomorphism of $S^n \times S^k$ is concordant to a composition of diffeomorphisms of types* (a), (b) *and* (c).

**Proof.** First notice that every element of $\bar{D}^{n,k}/D^{n,k}$ is representable as desired. This follows immediately from §1 and the diffeomorphisms considered in [24, Lemma 5]. We can, therefore, consider only $D^{n,k} = (D_1^{n,k} \oplus \alpha) \times_{\phi,\tau} D_2^{n,k}$ (see Theorem 2.4).

By definition, every element of $\alpha$ is represented by a diffeomorphism of type (c). Consider the homomorphisms:

$$\pi_n(SO_{k+1}) \to D_1^{n,k} \quad \text{and} \quad \pi_k(SO_n) \to D_2^{n,k}$$

defined in 4.8. It is only necessary to show these are surjective. But they correspond to the homomorphism $\pi_m(SO_{p+1}) \to D(m, p+1) = FC_m^{p+1}$ defined by twisting the framing on a representative of the zero element in $FC_m^{p+1}$ (see [7, §5.9]). There is an exact sequence:

$$\cdots \to \pi_m(SO_{p+1}) \to FC_m^{p+1} \to C_m^{p+1} \to \cdots$$

where $C_m^{p+1}$ is the group of concordance classes of smooth imbeddings $S^m \to S^{m+p+1}$. But if $m < 2p-1$, this group is zero (see e.g. [7, §6.6]), and we have the desired surjectivity in the asserted range.

5.3. A similar result is true in the category $\mathscr{P}$. We define two types of piecewise-linear homeomorphisms of $S^n \times S^k$:

(d) Let $\alpha$ be a smoothing of $S^n$ such that $S_\alpha^n \times S^k$ is diffeomorphic to $S^n \times S^k$. Then consider the composition

$$S^n \times S^k \xrightarrow{1} S_\alpha^n \times S^k \xrightarrow{h} S^n \times S^k$$

where 1 is the identity and $h$ is a piecewise-linear approximation to a diffeomorphism.

(e) Similarly, using a smoothing of $S^k$.

PROPOSITION. *Every piecewise-linear homeomorphism of $S^n \times S^k$ onto itself is concordant to a composition of types* (d) *and* (e).

**Proof.** If $n+k \leqq 4$, this follows from the obstruction theory of Munkres [19], and the fact that $\Gamma_m = 0$ for $n \leqq 4$ [5]. We will show that type (d) generates the cokernel of $\mu_1: D_1 \to P_1$ and type (e) generates the cokernel of $\mu_1: D_2 \to P_2$. Referring to the sequence 3.1 (a), for $n = m$ and $k = p$, $S_\alpha^n$ represents an element of the image $\theta: F\Gamma_m^{k+1} \to \Gamma_n$ if and only if $S_\alpha^n \times D^{k+1}$ is diffeomorphic to $S^n \times D^{k+1}$—by the argument in 3.5. If $\xi \in F\Gamma_n^{k+1}$ is represented by a submanifold $\Sigma$ of $S^{n+k+1}$, with some framing, then $\theta(\xi)$ is represented by $\Sigma$. It follows easily then that if $\alpha$ is a smoothing of $S^n$ derived from an element $\xi$ in the image of $\theta$, we can use $\alpha$ to construct a homeomorphism of type (d) and the image under $\theta$ (using the isomorphism $\lambda_1$) of the element of $P_1^{n,k}$ it represents is $\xi$. Similarly for $\mu_2$.

5.4. We now give an application which depends on the actual group structure of $A^{n,k}$. If $f$ is a self-equivalence of $S^n \times S^k$, we can use $f$ to identify the boundaries of two copies of $D^{n+1} \times S^k$. The resulting space $X_f$ is an object in the category $\mathscr{A}$. Moreover its equivalence class in $\mathscr{A}$ is easily seen to depend only upon the concordance class of $f$. Thus we have defined for $\alpha \in \bar{A}^{n,k}$ an equivalence class $X_\alpha$ of objects in $\mathscr{A}$.

Let $\bar{A}_2^{n,k} \subset \bar{A}^{n,k}$ be the subgroup of self-equivalences which extend to self-equivalences of $D^{n+1} \times S^k$. If $n > k$, we see easily that $\bar{A}_2/A_2 \to \bar{A}/A \approx Z_2 + Z_2$ is an isomorphism.

LEMMA. *If $n > k$ and $\alpha, \alpha' \in \bar{A}^{n,k}$, then $X_\alpha = X_{\alpha'}$ if and only if $\alpha' = \beta_1 \alpha \beta_2$ for some $\beta_i \in \bar{A}_2^{n,k}$.*

**Proof.** Suppose $f, f'$ represent $\alpha, \alpha'$, respectively, and assume the existence of $\beta_i$, represented by $g_i$. Then $f'$ is concordant to $g_1 \circ f \circ g_2$. Let $\bar{g}_i$ be an extension of $g_i$ to a self-equivalence of $D^{n+1} \times S^k$; by using $\bar{g}_2$ on one copy of $D^{n+1} \times S^k$ and $\bar{g}_1^{-1}$ (or a homotopy inverse, in $\mathscr{H}$) on the other, we can define an equivalence $X_{f'} \to X_f$.

5.5. For the converse, we first deal with the category $\mathscr{H}$. Let $g = p_2 \circ f$, $g' = p_2 \circ f'$, where $p_2 \colon S^n \times S^k \to S^k$ is projection on the second factor. Then $X_\alpha$ (or $X_{\alpha'}$) can be constructed by using $g$ (or $g'$) to attach $D^{n+1} \times S^k$ to $S^k$. It will suffice to construct self-equivalences $h$ of the pair $(D^{n+1} \times S^k, S^n \times S^k)$, and $h'$ of $S^k$, such that $g' \circ h \simeq h' \circ g$ on $S^n \times S^k$.

Suppose $h'' \colon X_\alpha \to X_{\alpha'}$ is a homotopy equivalence. We may assume $h''(S^k) \subset S^k$; since $n > k$, $h''$ induces a self-equivalence $h'$ of $S^k$. Let $q \colon (D^{n+1} \times S^k, S^k) \to (X_\alpha, S^k)$ be the identification map—similarly define $q'$ from $X_{\alpha'}$. We want to complete the commutative diagram:

$$\begin{array}{ccc} (D^{n+1} \times S^k, S^k) & \xrightarrow{q} & (X_\alpha, S^k) \\ h \downarrow & & \downarrow h'' \\ (D^{n+1} \times S^k, S^k) & \xrightarrow{q'} & (X_{\alpha'}, S^k). \end{array}$$

Let $E$ be the space of paths in $X_\alpha$, which start in $0 \times S^k$. Let $p \colon E \to X_{\alpha'}$ be the fibration defined by evaluating at the endpoint and $E_0 = p^{-1}(S^k)$; note that the fiber $F$ is $(n-1)$-connected. Now $h'' \circ q$ and $q'$ both lift to maps $s$ and $s''$, respectively, $(D^{n+1} \times S^k, S^n \times S^k) \to (E, E_0)$, since the only obstruction lies in $\pi_{k-1}(F) = 0$.

By the homotopy excision theorem [22, p. 484], $q_* \colon \pi_i(D^{n+1} \times S^k, S^n \times S^k) \to \pi_i(X_\alpha, S^k)$, $q'_* \colon \pi_i(D^{n+1} \times S^k, S^n \times S^k) \to \pi_i(X_{\alpha'}, S^k)$ are isomorphisms for $i \leqq 2n-1$ and epimorphisms for $i = 2n$. The same is, therefore, true of $s_*$ and $s'_*$. Since $s, s' \colon D^{n+1} \times S^k \to E$ are homotopy equivalences, the maps $S^n \times S^k \to E_0$, induced by $s$ and $s'$, are $(2n-1)$-connected.

We wish to complete the commutative diagram:

$$\begin{array}{ccc}
(D^{n+1}\times S^k,\ S^n\times S^k) & & \\
 & \searrow^{s} & \\
\Big\downarrow{\scriptstyle h} & & (E,\ E_0) \\
 & \nearrow_{s'} & \\
(D^{n+1}\times S^k,\ S^n\times S^k) & &
\end{array}$$

If we replace $E$ by the mapping cylinder of $s'$, we may assume $s'$ is an inclusion. Since $s'(D^{n+1}\times S^k)$ is, then, a (weak) deformation retract of $E$, it is only necessary to deform $s$ so that $s(S^n\times S^k)\subset s'(S^n\times S^k)$. But $(E_0, s'(S^n\times S^k))$ is $(2n-1)$-connected and $n+k\leqq 2n-1$.

5.6. To complete the proof of Lemma 5.4, we now consider $\mathscr{A}=\mathscr{P}$ or $\mathscr{D}$. Suppose $f, f'$ represent $\alpha, \alpha' \in \bar{A}^{n,k}$, and an equivalence $h\colon X_f \to X_{f'}$ exists. Since $n>k$, $\pi_k(X_{f'})$ is infinite cyclic and $h|0\times S^k$ represents a generator. After an isotopy, we may assume $h(0\times S^k)=0\times S^k$ (by general position). By the tubular or regular neighborhood theorem, we may assume $h(D^{n+1}\times S^k)=D^{n+1}\times S^k$. Therefore $h$ determines two self-equivalences $h_1$, $h_2$ of $D^{n+1}\times S^k$—one for each copy used in constructing $X_f$ and $X_{f'}$. Since $h_1$ and $h_2$ determine an equivalence $X_f\to X_{f'}$, it follows that $h_2\circ f$ is equal to $f'\circ h_1$ on $S^n\times S^k$. We now let $\beta_i$ be the self-equivalence of $S^n\times S^k$ determined by restricting $h_i$.

5.7. Now $\bar{A}_2/A_2\approx \bar{A}/A\approx Z_2+Z_2$ (recall $n>k$) and is generated by $\rho$, $\rho'\in\bar{A}_2$, where if $r'$ and $r$ are reflections of $S^n$ and $S^k$, respectively, we can take $r'\times 1$ and $1\times r$ as representatives of $\rho'$ and $\rho$, respectively.

Suppose $\alpha, \alpha'\in A_0$ and $X_\alpha = X_{\alpha'}$. Then, by Lemma 5.4, $\alpha'=\beta_1\alpha\beta_2$ for some $\beta_i\in\bar{A}_2$. Projecting on $\bar{A}/A$ tells us that $\beta_1\beta_2\in A$. It follows that we may write $\alpha'=\beta(\beta_1\alpha\beta_2)\beta^{-1}$, where $\beta_i\in A_2$ and $\beta=1$, $\rho$, $\rho'$ or $\rho\rho'$. Now by projecting onto $A/A_0\approx A_2$, we find $\beta_1=\beta_2^{-1}$, since $\beta^{-1}\alpha'\beta\in A_0$.

We also point out that, if $\alpha\in A_0$, then $\rho'\alpha\rho'=-\alpha$. If $\mathscr{A}=\mathscr{H}$, this follows from the characterization of $H_0=H_1$ as $\pi_n(G_{k+1})$. If $\mathscr{A}=\mathscr{P}$ or $\mathscr{D}$, suppose $\alpha$ is represented by $f$ satisfying $f|D^n_-\times S^k=$ inclusion. We may extend $f|D^n_+\times S^k$ to a self-equivalence $g$ of $D^{n+1}\times S^k$ in such a way that $g|D^n_-\times S^k=(r'\times 1)\circ f\circ(r'\times 1)|D^n_-\times S^k$. It follows easily that $(r'\times 1)\circ f\circ (r'\times 1)\circ f=g|S^n\times S^k$, by checking on $D^n_+\times S^k$ and $D^n_-\times S^k$ separately. Thus $\rho'\alpha\rho'\alpha\in A_2\cap A_0=\{1\}$.

We can summarize the above observations in:

Proposition. *If $n>k$ and $\alpha, \alpha'\in A_0^{n,k}$, then $X_\alpha = X_{\alpha'}$ if and only if there exists $\beta\in A_2^{n,k}$ such that*:

$$\beta\alpha\beta^{-1}=\pm\alpha' \quad \textit{or} \quad \pm\rho\alpha'\rho^{-1}.$$

5.8. The objects $\{X_\alpha\}$, for $\alpha\in A_0$, include the $k$-sphere bundles over $S^{n+1}$. In the category $\mathscr{H}$, Proposition 5.7, in light of Proposition 4.2, agrees with the classification given by James and Whitehead [9].

5.9. As one application of Proposition 5.7 we show:

COROLLARY. *There exists a smooth closed manifold M, tangentially homotopy equivalent but not homeomorphic to* $S^n \times S^k$, *for* $n=6$ *and* $k=2$, 3 *or* $n=14$ *and* $2 \leqq k \leqq 6$.

**Proof.** In these dimensions $P_n \approx Z_2$, $\partial: P_n \to P(n-1, k)$ is nonzero and $\theta \circ \partial: P_n \to \Gamma_{n-1}$ is zero (see (3.1) and (5.1)). Let $\alpha \in P(n-1, k) \approx P_1^{n-1,k}$ be the nonzero element of $\partial(P_n)$; since $\theta(\alpha)=0$, let $\alpha=\mu_1(\alpha')$ (see 3.1(a)). Then $M=X_{\alpha'}$ is a smooth $(n+k)$-manifold. Because $\mu_2 \circ \mu_1(\alpha')=\mu_2(\alpha)=0$, $M$ is homotopy equivalent to $S^n \times S^k$ (Proposition 5.7). On the other hand $M$ is *not* piecewise-linearly homeomorphic to $S^n \times S^k$, by Proposition 5.7, since $\alpha$ is not conjugate to 0. Therefore, by Sullivan [34], $M$ is not homeomorphic to $S^n \times S^k$.

To complete the proof of the Corollary, we show that any homotopy equivalence $M \to S^n \times S^k$ is tangential. Since $\alpha'$ represents a diffeomorphism of $S^{n-1} \times S^k$ which extends to one of $S^{n-1} \times D^{k+1}$, we may use such an extension to attach two copies of $D^n \times D^{k+1}$ along $S^{n-1} \times D^{k+1}$, thereby obtained a smooth homology $n$-sphere bounded by $M$. This manifold, and, therefore, $M$, is stably parallelizable, since $\pi_{n-1}(SO)=0$. But $S^n \times S^k$ is also stably parallelizable and the Euler class of a manifold is a homotopy invariant. Therefore, any homotopy equivalence $M \to S^n \times S^k$ is tangential.

5.10. Proposition 5.8 can also be used to study the *inertia group* of a smooth manifold (see [14]).

COROLLARY. *Suppose* $\alpha \in D_1^{n,k}$, $\beta \in D_2^{n,k}$ *satisfy* $\phi(\beta)\cdot\alpha=\alpha$. *Then* $\tau(\beta)\cdot\alpha \in \Gamma^{n+k+1}$ *belongs to the inertia group of* $X_\alpha$.

**Proof.** We have $\beta\alpha\beta^{-1}=\phi(\beta)\cdot\alpha+\tau(\beta)\cdot\alpha=\alpha+\tau(\beta)\cdot\alpha$. If $\alpha'=\beta\alpha\beta^{-1}$, then $X_{\alpha'}=X_\alpha$. But it is easy to see (compare [1, Lemma 1]) that $X_{\alpha'}=X_\alpha \# \Sigma$, where $\Sigma$ represents $\tau(\beta)\cdot\alpha$.

For example, if $\alpha$ is the suspension of an element of $D_1^{n,k-1}$, the hypothesis is satisfied for any $\beta$, by Proposition 4.5 (compare [14, Theorem 1]).

## REFERENCES

1. W. Browder, *Diffeomorphisms of* 1*-connected manifolds*, Trans. Amer. Math. Soc. **128** (1967), 155–163.

2. ———, *The Kervaire invariant of framed manifolds and its generalization*, mimeographed notes, Princeton Univ., Princeton, N. J.

3. W. Browder and M. Hirsch, *Surgery on piecewise-linear manifolds*, Bull. Amer. Math. Soc. **72** (1966), 959–964.

4. E. Brown and F. Peterson, *The Kervaire invariant of* $(8k+2)$*-manifolds*, Bull. Amer. Math. Soc. **71** (1965), 190–193.

5. J. Cerf, *La nullité du groupe* $\Gamma_4$, Séminaire H. Cartan, École Normale Supérieure, Paris, 1962/1963.

6. V. Guggenheim, *Piece-wise linear isotopy of elements and spheres*, Proc. London Math. Soc. **3** (1953), 29–53.

7. A. Haefliger, *Differentiable embeddings of* $S^n$ *in* $S^{n+q}$ *for* $q>2$, Ann. of Math. (2) **83** (1966), 402–436.

8. A. Haefliger and C. T. C. Wall, *Piecewise linear bundles in the stable range*, Topology **4** (1965), 209–214.

9. I. James and J. H. C. Whitehead, *Homotopy theory of sphere bundles over spheres*. II, Proc. London Math. Soc. **5** (1955), 148–166.

10. P. Kahn, *Self-equivalences of* $(n-1)$*-connected* $2n$*-manifolds*, Bull. Amer. Math. Soc. **72** (1966), 562–566.

11. M. Kervaire, *On the Pontriagin classes of certain* $SO(n)$*-bundles over manifolds*, Amer. J. Math. **80** (1958), 632–638.

12. M. Kervaire and J. Milnor, *Groups of homotopy spheres*. I, Ann. of Math. (2) **77** (1963), 504–537.

13. A. G. Kurosh, *Theory of groups*. Vol. II, Chelsea, New York, 1955.

14. J. Levine, *Inertia groups of manifolds and diffeomorphisms of spheres*, mimeographed.

15. ———, *Unknotting spheres in codimension two*, Topology **4** (1965), 9–16.

16. ———, *A classification of differentiable knots*, Ann. of Math. **82** (1965), 15–50.

17. B. Mazur, *Relative neighborhoods and the theorems of Smale*, Ann. of Math. (2) **77** (1963), 232–249.

18. J. Milnor, *Differentiable structures on spheres*, Amer. J. Math. **81** (1959), 962–972.

19. J. Munkres, *Obstructions to the smoothing of piecewise differentiable homeomorphisms*, Ann. of Math. (2) **72** (1960), 521–524.

20. A. Shapiro, *Obstructions to the embedding of a complex in a Euclidean space*: I: *The first obstruction*, Ann. of Math. (2) **66** (1957), 256–269.

21. S. Smale, *On the structure of manifolds*, Amer. J. Math. **84** (1962), 387–399.

22. E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

23. D. Sullivan, *Triangulating and smoothing homotopy equivalences and homeomorphisms*, mimeographed notes, Princeton University, Princeton, N. J., 1967.

24. C. T. C. Wall, *Killing the middle homotopy groups of odd dimensional manifolds*, Trans. Amer. Math. Soc. **103** (1962), 421–433.

25. C. Weber, *L'élimination des points doubles dans le cas combinatoire*, Comment. Math. Helv. **41** (1966), 179–182.

26. G. Whitehead, *Products in homotopy groups*, Ann. of Math. (2) **47** (1946), 460–475.

27. J. H. C. Whitehead, *On* $C'$*-complexes*, Ann. of Math. (2) **41** (1940), 809–814.

28. H. Whitney, *The self-intersection of a smooth n-manifold in* $2n$*-space*, Ann. of Math. (2) **45** (1944), 220–246.

29. W. T. Wu, *On the isotopy of* $C^r$*-manifolds of dimension n in Euclidean* $(2n+1)$*-space*, Science Record N.S. **2** (1958), 271–275.

30. E. C. Zeeman, "Isotopies and knots in manifolds" in *Topology of* 3*-manifolds and related topics*, Prentice-Hall, Englewood Cliffs, N. J., 1962, pp. 187–193.

31. M. A. Kervaire, *Les noeuds de dimensions supérieures*, Bull. Soc. Math. France **93** (1965), 225–271.

32. J. L. Shaneson, *Embeddings of spheres in spheres of codimension two and h-cobordisms of* $S^1 \times S^3$, Bull. Amer. Math. Soc. **75** (1968), 972–973.

33. C. T. C. Wall, *Unknotting tori in codimension one and spheres in codimension two*, Proc. Cambridge Philos. Soc. **61** (1965), 659–664.

34. C. R. Rourke, *The hauptvermutung according to Sullivan*, mimeographed notes, Institute for Advanced Study, Princeton, N. J. 1967.

35. R. Lashof and J. Shaneson, *Classification of knots of codimension two*, Bull. Amer. Math. Soc. **75** (1969), 171–175.

Brandeis University,
Waltham, Massachusetts