ON A COEFFICIENT PROBLEM IN UNIVALENT FUNCTIONS

BY

DEAN PHELPS(1)

Introduction. Let S denote the family of functions which are regular and univalent in the unit disc and which possess a power series expansion about the origin of form

(1) $$f(z) = z+A_2z^2+A_3z^3+\cdots.$$

The coefficient problem for univalent functions proposed by Bieberbach is to determine the precise region, V_n, in $2n-2$ dimensional euclidean space occupied by points $(A_2,\ldots,A_n)$ where the A_j's appear in (1) for some $f\in S$. Bieberbach [1] determined V_2, and Schaeffer and Spencer [2] determined V_3. Earlier Peschl [3] determined V_3 in the special case in which A_2 and A_3 are real, denoting the region $E_S^{(3)}$. Using a slight modification of Peschl's notation we determine the region $E(4, S)$ in this paper. We also adopt the following notation: D denotes the unit disc centered at 0, E denotes $\{z : |z|>1\}\cup\{\infty\}$, R denotes $\{f: 1/f(1/z)\in S\}$, $E(n, S)$ denotes V_n when the A_j's are real for $j=2,\ldots,n$. The statements $(A_2,\ldots,A_n)$ belongs to $f\in S$ and $f\in S$ belongs to $(A_2,\ldots,A_n)$ will mean that $(A_2,\ldots,A_n)\in E(n, S)$ and the A_j's appear in (1) for f.

Implicit in results concerning V_n in [2] are the following two propositions about $E(n, S)$.

PROPOSITION 1. *$E(n, S)$ is a bounded closed set, the closure of a domain, and is homeomorphic to the closed $n-1$ dimensional full sphere.*

PROPOSITION 2. *The following statements are equivalent:*

(i) *$(A_2,\ldots,A_n)$ is an interior point of $E(n, S)$.*

(ii) *There is a bounded function in S belonging to the point $(A_2,\ldots,A_n)$.*

Proofs of these propositions follow directly from the proofs in [2] upon obvious modifications.

The determination of the functions belonging to boundary points of $E(4, S)$ using the General Coefficient Theorem (GCT) leads to consideration of certain quadratic differentials on the Riemann sphere. We refer to [4] for definitions and terminology associated with the GCT and to [5] for the form of the GCT used here.

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Since the GCT in [4] is phrased in terms of local uniformizing parameters which represent poles of the quadratic differentials as the point at infinity, we consider functions of the family R and their expansions about infinity of form

$$f(z) = z+C_0+C_1/z+C_2/z^2+\cdots. \tag{2}$$

The coefficients in (1) and (2) are related in

$$C_0 = -A_2, \qquad C_1 = A_2^2-A_3, \qquad C_2 = -A_2^3+2A_2A_3-A_4. \tag{3}$$

Clearly A_2, A_3, and A_4 are real iff C_0, C_1, and C_2 are real. The functions from R to be considered are identified in

PROPOSITION 3. *Let t_1 and t_2 be real parameters with $-4 \leqq t_1 \leqq 4$ and $-\infty < t_2 < \infty$. Then corresponding to the quadratic differentials*

$$\begin{aligned} Q_1(w, t_1, t_2)\, dw^2 &= [(w-t_1)(w-t_2)/w]\, dw^2, && t_2 \geqq \max(0, t_1),\\ Q_2(w, t_1, t_2)\, dw^2 &= -[(w-t_1)(w-t_2)/w]\, dw^2, && t_2 \leqq \min(0, t_1),\\ Q_3(w, t_1)\, dw^2 &= [(w-t_1)/w]\, dw^2 \end{aligned}$$

on the sphere, there are families $F_1(t_1, t_2)$, $F_2(t_1, t_2)$, and $F_3(t_1)$(2) *of functions such that $F_j \subset R$ and each $f \in F_j$ maps E conformally onto a domain G admissible with respect to Q_j, and G is bounded as follows:*

(a) *If $t_1 \in [-4, 4]$, and $t_2 > \max(0, t_1)$ or $t_2 < \min(0, t_1)$ or t_2 does not occur, then G is bounded by the segment from 0 to t_1 plus two slits of equal length along trajectories of Q_j having an endpoint at t_1, the slits including t_1.*

(b) *If $t_1 \in [-4, 4]-\{0\}$ and $t_2 = t_1$, then G is bounded by the segment from 0 to t_1 plus three slits along trajectories of Q_j with an endpoint at t_1. One of the slits lies along the real axis while the other two slits are of equal length, possibly zero, and all slits include t_1.*

(c) *If $t_1 \in [-4, 4]$ and $t_2 = 0$, then G is bounded by the segment from 0 to t_1 plus a slit from 0 to a point on the real axis on the opposite side of the origin from t_1, plus two slits of equal length on the trajectories of Q_j with an endpoint at t_1, the latter point included.*

Proof. If $t_2 > \max(0, t_1)$ or $t_2 < \min(0, t_1)$ or t_2 does not appear, and if $t_1 \neq 0$, then Q_j has a simple zero at t_1. Let G_1 be the simply connected domain on the sphere bounded by the segment from 0 to t_1 and two slits of equal length, L, along the other two trajectories of Q_j with endpoint at t_1. By the Riemann Mapping Theorem there is a conformal mapping, f, of E onto G_1 with expansion about infinity of form

$$f(z) = d(L)z+d_0+d_1/z+d_2/z^2+\cdots, \qquad d(L) > 0.$$

For $L=0$, G_1 is bounded by the segment from 0 to t_1 hence f reduces to

$$f(z) = (|t_1|/4)(z+2 \operatorname{sgn} t_1 + 1/z)$$

(2) Differentials and families are often written briefly as Q_j and F_j, $j=1, 2, 3$, in what follows.

by uniqueness in the Riemann Mapping Theorem. If δ denotes the diameter of the complement of G_1, then the diameter theorem for functions from the family Σ gives $2d(L) \leqq \delta \leqq 4d(L)$. Schwarz's Lemma implies that $d(L)$ increases with L. By the above bounds on $d(L)$, $\sup_L d(L) \geqq 1$. Also $d(L)$ is a continuous function of L as a result of Carathéodory's theorem on variable regions [7, Theorem 2.1, p. 343]. Thus as L ranges from 0 to ∞, $d(L)$ increases continuously from $|t_1|/4$ through 1 so for some value of L, $d(L)=1$. The corresponding function is in R and since $d(L)$ increases with L, this function is the only member of $F_j(t_1, t_2)$. Analogous reasoning gives the result if $t_1=0$ and $t_2 \neq 0$. If $t_1=t_2\neq 0$, Q_1 and Q_2 have double zeros at t_1 and four trajectories with limiting endpoints at t_1. Let Δ be a domain on the sphere bounded by the segment from 0 to t_1 plus two slits of equal length L on the two trajectories of Q_j, $j=1, 2$, not lying along the real axis. As above, for some choice of L the function mapping E conformally onto Δ has $d(L)=1$ in its expansion of form (4) about ∞, and $d(L)$ decreases as L decreases. Fixing L so that $d(L)<1$, we can increase $d(L)$ by introducing a slit of length L_1 along the real axis on the trajectory of Q_j from t_1 to ∞. Then writing $d(L, L_1)$ instead of $d(L)$, we have that for some choice of L_1, $d(L, L_1)=1$ and the corresponding function is in R. Q.E.D.

Note that for each choice of $t_1 \in (-4, 4)-\{0\}$ with $t_2=t_1$, $F_j(t_1, t_2)$ is a one parameter family of functions in which L can be chosen as the parameter. Note also that if $t_1=\pm 4$ in any of the above cases, or when $t_1=t_2$, if $L=0$, the corresponding function is one of the Koebe functions $k_1(z)=z+2+1/z$ or $k_2(z)=z-2+1/z$.

Construction of the mappings of Proposition 3 and expressions for the coefficients C_0, C_1, and C_2 of expansion (2) proceeds as follows. The upper half w-plane is mapped into the ζ-plane by

$$\zeta_w = \int_0^w [Q_j(w)]^{1/2}\, dw$$

where the branch of $[Q_j(w)]^{1/2}$ is the one taking large positive values for w large and positive. The domain $E \cap \operatorname{Im} z>0$ is mapped onto the upper half of the W-plane by $W=z+2 \operatorname{sgn} t_1+1/z$ where we make the agreement that $\operatorname{sgn} t_1=1$ if $t_1=0$. Next the upper half W-plane is mapped into the ζ-plane so that the image of the former coincides with the image of the upper half w-plane under ζ_w with the exception that a horizontal segment is appended to the boundary of the latter image at the point $\zeta_w(t_1)$. The mappings are to be conformal on their domains so that the composed mapping from z to the w-plane is conformal from $E \cap \operatorname{Im} z>0$ onto the set $\operatorname{Im} w>0$ minus a slit on a trajectory of Q_j emanating from t_1. Reflection then extends the composed mapping to a conformal mapping of E onto a domain bounded as described in (a), (b), or (c) of Proposition 3.

In the case of Q_1 we have

$$\zeta_w = \int_0^w [(w-t_1)(w-t_2)/w]^{1/2}\, dw$$

and the mapping from the W-plane into the ζ-plane is given by

$$\zeta_W = \int_0^W (W-\alpha)(W-\beta)^{1/2}W^{-1/2}(W-4\operatorname{sgn} t_1)^{-1/2}\,dW+T$$

where α, β, and T are real parameters with α between 0 and 4 sgn t_1,

$$\beta \geqq \max(0, 4 \operatorname{sgn} t_1),$$

$T=-2(-t_1)^{3/2}/3-32/3-4\alpha$ with T appearing only if $t_2=0$. Boundaries in the ζ-plane are matched by the conditions:

$$\text{(5)}\quad \begin{array}{lll} \zeta_w(t_1) = \zeta_W(4), & \zeta_w(t_2) = \zeta_W(\beta) & \text{if } 0 \leqq t_1 < t_2, \\ \zeta_w(t_1) = \zeta_W(-4), & \zeta_w(t_2) = \zeta_W(\beta) & \text{if } -4 \leqq t_1 < t_2, \\ \zeta_w(t_1) \geqq \zeta_W(\alpha), & \zeta_w(t_1) \leqq \zeta_W(4) & \text{if } 0 \leqq t_1 = t_2, \\ \zeta_w(t_1) = \zeta_W(-4), & \zeta_W(0) \geqq 0 & \text{if } -4 \leqq t_1 < 0 = t_2. \end{array}$$

Expanding the integrands in the expressions for ζ_w and ζ_W, choosing the earlier mentioned branches of the root functions, we integrate termwise, insert a trial expansion of form (2) into the resulting expression for ζ_w, express ζ_W in terms of z and equate coefficients of like powers of z giving

$$\text{(6)}\quad \begin{aligned} C_0 &= g_1+C+r-\beta-2\alpha, \\ C_1 &= -g_2-C_0^2/4+C_0g_1/2+2/3+B_1+(C/3)(C+(3/2)(r-\beta-2\alpha)), \\ C_2 &= -g_1g_2/6+g_2C_0/2-C_0C_1/2+C_0^3/24+C_1g_1/2-C_0^2g_1/8+B_2 \\ &\qquad -CB_1/2+C/3 \end{aligned}$$

where $g_1=t_1+t_2$, $g_2=(t_1-t_2)^2/4$, $C=2\operatorname{sgn} t_1$, $r=2C$,

$$B_2 = (\beta+r)^3/24-r\beta^2/12-r^3/4+\alpha r^2/4-\alpha\beta^2/12-r\alpha\beta/6$$

and

$$B_1 = (\beta+r)^2/4-r^2+r\alpha-\alpha\beta.$$

In the case of Q_2 the same expressions for C_0, C_1, and C_2 result but parameter ranges are changed so that $t_2 \leqq \min(t_1, 0)$, $-4 \leqq t_1 \leqq 4$, $\beta \leqq \min(4 \operatorname{sgn} t_1, 0)$ and α is between 0 and 4 sgn t_1.

For Q_3, C_0 and C_1 were determined in [6, p. 170] and found there to be

$$\begin{aligned} C_0 &= (t_1/2)[1-\ln(|t_1|/4)], \\ C_1 &= (t_1^2/8)[1-2\ln(|t_1|/4)]-1 \end{aligned}$$

with $C_0=0$ and $C_1=-1$ for $t_1=0$. Use of the explicit mapping [6, Equation (8), p. 170](3) gives

$$C_2 = (t_1^3/32)(1-2\ln(|t_1|/4)-2\ln^2(|t_1|/4))-t_1/2.$$

Now define F to be $\bigcup [F_1(t_1, t_2) \cup F_2(t_1, t_2) \cup F_3(t_1)]$ where the outer union is over $t_1 \in [-4, 4]$ and t_2 restricted as described in Proposition 3. Then the family F

(3) In [6, Equation (8), p. 170] a factor τ multiplying the log term is missing.

gives the complete collection of extremal functions for $E(4, S)$ in the following sense.

PROPOSITION 4. $(A_2, A_3, A_4) \in \partial E(4, S)$ *iff* (A_2, A_3, A_4) *belongs to* g *and* $f(z) = 1/g(1/z) \in F$.

Proof. Let $f \in F$. From the description in Proposition 3 of the boundary of the image of E under a member of F it follows that the range of f never excludes a neighborhood of the origin. Thus $1/f(1/z) \in S$ is unbounded. If the point (A_2, A_3, A_4) belonging to $1/f(1/z)$ is an interior point of $E(4, S)$, then by Proposition 2 there is a bounded function g belonging to the point. Applying the GCT to the Riemann sphere with the image of E under f as the admissible domain, the function $(1/g) \circ (1/f^{-1})$ as the admissible function, and the differential Q_j associated with the family F_j to which f belongs, we find the fundamental inequality to be a zero equality. Since Q_j has a pole of order 4 or 5 at ∞, the equality statement in the GCT gives $1/g(1/z) \equiv f(z)$, a contradiction since $1/f(1/z)$ is unbounded while $g(z)$ is bounded. Thus (A_2, A_3, A_4) is a boundary point of $E(4, S)$. To complete the proof of Proposition 4 we introduce a topology on the function family F and show that the resulting space is topologically the two sphere. Let the topology on F be given by the metric $\operatorname{dist}(f, g) = \sup |f(2e^{i\theta}) - g(2e^{i\theta})|$, with the sup taken over $\theta \in [0, 2\pi]$. We map the set $F - \{k_1(z), k_2(z)\}$ into the plane as follows:

(a) If $-4 < t_1 < 4$ and $t_2 > \max(t_1, 0)$ or $t_2 < \min(t_1, 0)$, the single member of $F_j(t_1, t_2)$ is mapped onto the plane point $(t_1, [4 + \operatorname{Arctan} |t_2|] \operatorname{sgn} t_2)$ for $j = 1, 2$.

(b) If $0 \leqq t_1 < 4$ and $t_1 = t_2$, then the proper one of conditions (5) gives $t_1 \leqq \alpha \leqq 4/3 + t_1^{3/2}/3$ and as noted earlier $F_1(t_1, t_1)$ is a one parameter family of functions with parameter L defined in Proposition 3. As α increases from t_1 to $4/3 + t_1^{3/2}/3$, L increases from 0 to its maximum. Then using α instead of L as parameter and calling the corresponding member of $F_1(t_1, t_1)$ f_α, we map f_α onto $(t_1 + 2, \alpha)$ for $t_1 < \alpha \leqq 4/3 + t_1^{3/2}/3$.

(c) If $-4 < t_1 < 0$ and $t_2 = 0$, then from conditions (5), $-4 \leqq \alpha \leqq -8/3 - (-t_1)^{3/2}/6$. With α as parameter map f_α in $F_1(t_1, 0)$ onto the point $(t_1 + 2, \alpha + 4)$.

(d) If $0 \leqq t_1 < 4$ and t_2 and $\alpha - 4$ are not simultaneously zero, then the boundary matching conditions for Q_2 analogous to (5) give $8/3 + t_1^{3/2}/6 \leqq \alpha \leqq 4$. Map f_α in $F_2(t_1, 0)$ onto the point $(t_1 - 2, \alpha - 4)$ for α in the above range, $\alpha \neq 4$.

(e) If $-4 < t_1 < 0$ and $t_1 = t_2$, the conditions analogous to (5) give $-4/3 - (-t_1)^{3/2}/3 \leqq \alpha \leqq t_1$. Map f_α in $F_2(t_1, t_1)$ onto the point $(t_1 - 2, \alpha)$.

(f) Map the single member of $F_3(t_1)$ onto the point $(t_1, 4 + \pi/2)$ for $-4 < t_1 < 4$.

Denote the two disjoint plane sets described in (a) by D_1 and D_2 where $t_2 > 0$ in D_1, and sets described in (b) through (f) by D_3 through D_7. The mapping just described is one-to-one from $F - \{k_1(z), k_2(z)\}$ into the plane. The functions $k_1(z)$ and $k_2(z)$ correspond to those points on the boundaries of the D_j's for which $t_1 = \pm 4$ in any of the D_j's; $\alpha = t_1$ in D_3 and D_6; $\alpha = -4$ and $t_1 = 0$ in D_4; and $\alpha = 4$, $t_1 = 0$ in D_5. This statement is easily verified by substituting the appropriate values for the parameters in equations (6) and in later expressions for the C_j's in the case of F_3. In each case it will be seen that $C_0 = \pm 2$.

Denote the set of boundary points of the D_j's corresponding to k_1 and k_2 by K and $\bigcup (D_j) \cup K$, $j=1,\ldots,7$, by H. We now introduce a topology on H under which the correspondence just described from F into H becomes a homeomorphism. Using Cl for plane closure we form the free union $\mathrm{Cl}\,(D_2)+\mathrm{Cl}\,(D_1 \cup D_7)+\mathrm{Cl}\,(D_3 \cup D_4 \cup D_5 \cup D_6)$ and denote it P. Certain points in this space are identified.

$(t_1, 4+\operatorname{Arctan} t_1)$ in ∂D_1 is identified with $(t_1+2, 4/3+t_1^{3/2}/3)$ in ∂D_3 for $0 \leqq t_1 \leqq 4$.

$(t_1, 4)$ in ∂D_1 is identified with $(t_1+2, +4/3-(-t_1)^{3/2}/6)$ in ∂D_4 for $-4 \leqq t_1 \leqq 0$.

$(t_1, -4)$ in ∂D_2 is identified with $(t_1-2, -4/3+t_1^{3/2}/6)$ in ∂D_5 for $0 \leqq t_1 \leqq 4$.

$(t_1, -4-\operatorname{Arctan}(-t_1))$ in ∂D_2 is identified with $(t_1-2, -4/3-(-t_1)^{3/2}/3)$ in ∂D_6 for $-4 \leqq t_1 \leqq 0$.

These identifications are homeomorphisms between certain boundary continua on the D_j's. Further each of $\mathrm{Cl}\,(D_1 \cup D_7)$, $\mathrm{Cl}\,(D_2)$, and $\mathrm{Cl}\,(D_3 \cup D_4 \cup D_5 \cup D_6)$ is homeomorphic to the closed two disc, $\mathrm{Cl}\,(D)$. Calling the equivalence relation given by the above identifications R_1, we have that the space P/R_1 is also homeomorphic to the closed two disc since identifications were made along boundary continua in a manner preserving simple connectivity. Using $[x, y]$ to denote the equivalence class containing the plane point (x, y), we now map P/R_1 onto a rectangle in the plane with sides parallel to the coordinate axes so that images of the points $[t_1, 4+\pi/2]$ form the upper boundary and images of the points $[t_1, -4-\pi/2]$ form the lower boundary in such a way that images of points having the same first coordinate also have the same first coordinate. Note that the vertical sides of the rectangle correspond to K while the lower horizontal side does not correspond to any member of F. If g denotes the mapping of P/R_1 onto the rectangle, we remove the additional boundary points by identifying the upper and lower horizontal sides of the rectangle under the equivalence relation R_2 defined by

$$g([t_1, 4+\pi/2]) \sim g([t_1, -4-\pi/2]), \qquad -4 < t_1 < 4.$$

Since identified points have the same first coordinate in the rectangle, the quotient space $(P/R_1)/R_2$ is just $S^1 \times I$ where S^1 is the circle and I is a nondegenerate closed interval, say $[a, b]$. The sets $A=\{(x, a) : a \in S^1\}$ and $B=\{(x, b) : x \in S^1\}$ represent the functions $k_1(z)$ and $k_2(z)$ while every other point (x, y) in $S^1 \times I$ is the unique representative of a point of the set $F-\{k_1(z), k_2(z)\}$. We identify all points in A, and identify all points in B calling the equivalence relation R_3. Then $(S^1 \times I)/R_3$ is the suspension of S^1 and hence is homeomorphic to S^2. Thus the topology given to H is the quotient of plane topology under the equivalence relations R_1 followed by R_2 and R_3, and the resulting topological space is homeomorphic to S^2. To show that the one-to-one correspondence from F onto H is a homeomorphism, we consider its inverse, call it h. Points of H are equivalence classes of plane points with the following structures: (i) classes with a single member, (ii) classes containing pairs of identified boundary points, (iii) two classes each containing a continuum of points representing $k_1(z)$ and $k_2(z)$ respectively. Because of the topologies on H and F it suffices to deal with sequences in any discussion of continuity. Let $\{[x_n, y_n]\}$ be a sequence of points of H which converges to $[x, y]$, the latter being in H by compactness. The class $[x, y]$ contains plane points as enumerated in (i) through (iii) above. Suppose first that (x, y) is the only member of $[x, y]$. Then for all but finitely many n, $[x_n, y_n]$ has only one member since (x, y) is an interior point of D_j, $j=1, \dots, 6$, and the topology there is essentially plane topology. Thus $x_n \to x$ and $y_n \to y$. If $(x, y) \in D_1 \cup D_2$ we can conclude that $t_1^{(n)} \to t_1$ and $t_2^{(n)} \to t_2$ where $t_1^{(n)}$ and $t_2^{(n)}$ are the distinct zeros of the quadratic differentials

$$Q_j^{(n)}(w)\, dw^2 = \pm [(w-t_1^{(n)})(w-t_2^{(n)})/w]\, dw^2.$$

The corresponding sequence of functions in F is $\{f_n\}$ where f_n maps E conformally onto a domain G_n admissible with respect to $Q_j^{(n)}$ as described in Proposition 3. We show that $h([x_n, y_n]) \to h([x, y])$ as $n \to \infty$ by considering cases. Consider first the case in which $[x, y] \in D_1$. Let $h([x, y])=f$. To show that $f_n \to f$ in the topology of F it is enough to show that $f_n \to f$ uniformly on compact subsets of E. The latter will follow by Carathéodory's theorem on variable regions [7, Theorem 2.1, p. 343] if it is first shown that $\{G_n\}$ converges to its kernel with respect to ∞, G_∞, and that $G_\infty = G$, the image of E under f. The boundary of G as described in Proposition 3 is the segment from 0 to t_1 plus two symmetric slits of equal length along the other two trajectories of $[(w-t_1)(w-t_2)/w]\, dw^2$ with endpoint at t_1. Extend these slits along the trajectories until they meet the circle $|w|=4$ and denote the so augmented boundary of G by B. Then it is asserted that the boundary of G_∞ is contained in B. First note that the complement of G_∞ contains the segment from 0 to t_1 along the real axis and is symmetric in the real axis. If ∂G_∞ is not contained in B, take $w_0 \in \partial G_\infty$ at distance $\delta > 0$ from B. With no loss of generality we may assume that w_0 is in the half plane $\operatorname{Im} w \geqq 0$. Suppose first that $w_0 \neq t_2$. Let $N(w_0)$ be a neighborhood of w_0 of radius less than $\delta/2$ and chosen so that 0, t_1, and t_2 are not in $N(w_0)$. Since $w_0 \in \partial G_\infty$ there is a sequence $\{n(k)\}$ of integers such that $\partial G_{n(k)}$ intersects $\operatorname{Cl}(N(w_0))$ and the zeros of $Q_1^{(n(k))}$ are exterior to $\operatorname{Cl}(N(w_0))$ for $k=1, 2, \dots$. If we restrict consideration to $\operatorname{Im} w > 0$ where the functions

$$\zeta_{n(k)}(w) = \int_0^w [Q_1^{(n(k))}(w)]^{1/2}\, dw \quad \text{and} \quad \zeta(w) = \int_0^w [Q_1(w)]^{1/2}\, dw$$

are continuous and one-to-one, we find that the trajectories of $Q_1^{(n(k))}$ from 0 to $t_1^{(n(k))}$ and from $t_1^{(n(k))}$ to ∞ are mapped by $\zeta_{n(k)}$ onto the negative real axis (recall the choice of root determination specified earlier). Thus there is a sequence $\{p_{n(k)}\}$ of points in $\operatorname{Cl}(N(w_0))$ such that $\operatorname{Im} \zeta_{n(k)}(p_{n(k)})=0$, $k=1, 2, \dots$. Choose a convergent subsequence $\{q_j\}$ of $\{p_{n(k)}\}$, and let $\{\zeta_j\}$ denote the corresponding subsequence of $\{\zeta_{n(k)}\}$. Then $\zeta_j \to \zeta$ uniformly on $\operatorname{Cl}(N(w_0))$. Thus if $q = \lim q_j$, we have $q \in \operatorname{Cl}(N(w_0))$ and ζ maps q onto the real axis which contradicts the one-to-one nature of ζ since $\operatorname{Cl}(N(w_0))$ is $\delta/2$ distance from the trajectories of Q_1 with limiting endpoint at t_1. If $w_0=t_2$, the above proof is valid upon replacing Cl $(N(w_0))$ by a half disc centered at w_0. Thus $\partial G_\infty \subseteq B$.

Note here that any subsequence of $\{G_n\}$ has kernel with respect to ∞ containing G_∞. Let $\{G_{n(k)}\}$ be a subsequence of domains, and $\{G_{m(k)}\}$ a sub-subsequence such that $\{f_{m(k)}\}$ converges uniformly inside E to the limit function $f^{\#}$. Then $f^{\#}$ maps E conformally onto the kernel of $\{G_{m(k)}\}$ with respect to ∞ which we denote by J. Then J is simply connected and $f^{\#}$ has expansion about ∞ of form (2). It now follows from Schwarz's Lemma that $f^{\#}$ and f are identical since f had expansion about ∞ of form (2) also and either $\partial G \subseteq \partial J$ or $\partial J \subseteq \partial G$. Similarly the kernel of any convergent subsequence $\{G_{n(k)}\}$ must be G, hence $G_\infty \subseteq G$ which implies $\partial G \subseteq \partial G_\infty$. If the last containment is proper, then for some point $p \in \partial G_\infty - \partial G$ there is a sequence $\{G_{n(k)}\}$ and a sequence of points $\{p_{n(k)}\}$ such that $p_{n(k)} \in \partial G_{n(k)}$ and $p_{n(k)} \to p$. Choosing a convergent subsequence $\{G_{m(k)}\}$ of $\{G_{n(k)}\}$ we have that $p_{m(k)} \to p$. But the kernel of $\{G_{m(k)}\}$ is G by above while $\partial[\text{Ker}\,\{G_{m(k)}\}]$ contains p, and this is a contradiction since $p \in G$. Thus $\partial G = \partial G_\infty$, so that $G = G_\infty$ since they are open, connected and nondisjoint. As remarked above $G_\infty \subseteq \text{Ker}\,\{G_{n(k)}\}$ and $\text{Ker}\,\{G_{n(k)}\} \subseteq G$ for any subsequence of $\{G_n\}$. Hence $\{G_n\}$ converges to G and so by Carathéodory's theorem $f_n \to f$ uniformly on compact subsets of E. Similarly if $(x, y) \in D_2$ we have $f_n \to f$ inside E. If (x, y) is an interior point of D_3, then $(x_n, y_n) = (t_1^{(n)}+2, \alpha^{(n)})$ and the coefficients C_0, C_1, and C_2 are given by (6), including proper superscripts, with $t_1^{(n)} = t_2^{(n)}$ and $\beta^{(n)}=4$. The coefficients are continuous functions of t_1 and α so that $t_1^{(n)} \to t_1$ and $\alpha^{(n)} \to \alpha$ imply that $C_j^{(n)} \to C_j$, $j=0, 1, 2$. Taking any convergent subsequence of $\{f_n\}$, the sequence of functions associated with (x_n, y_n), we have that the limit function, f^*, has C_0, C_1, and C_2 as constant term and coefficients of $1/z$ and $1/z^2$ in its expansion of form (2). This is also true of the function associated with (x, y), the limit of (x_n, y_n), since (x, y) is just (t_1+2, α). The fundamental inequality of GCT is a zero equality when applied to f and f^*. $Q_1(w, t_1, t_2)\, dw^2$ has a pole of order five at ∞ so $f \equiv f^*$. Thus every convergent subsequence of the normal family $\{f_n\}$ has limit f, so $f_n \to f$. If (x, y) is an interior point of D_j, $j=4, 5, 6$; then $C_0^{(n)}$, $C_1^{(n)}$, and $C_2^{(n)}$ are also given by (6) with proper values of the parameters and proper superscripts and the same argument as the one above for (x, y) in the interior of D_3 gives $f_n \to f$. Suppose now that $\{[x_n, y_n]\}$ has limit on the boundaries of the D_j's. First let the limit point correspond to the Koebe function k_1. Then it is possible that $\{(x_n, y_n)\}$ has subsequences in each D_j, $j \neq 6$, simultaneously. The subsequence in D_1 has terms of the form $(t_1^{(n)}, 4+\text{Arctan}\, t_2^{(n)})$ with $t_1^{(n)} \to 4$ but no requirement on $t_2^{(n)}$. To prove that the limit of the associated sequence of functions, $\{f_n\}$, is k_1 consider the sequence $\{G_n\}$ of domains which are images of E under the f_n. Since $t_1^{(n)} \to 4$, the boundary of the kernel of $\{G_n\}$ with respect to ∞ contains the segment of the real axis from 0 to 4. Taking any convergent subsequence $\{f_{n(k)}\}$ of $\{f_n\}$, we have that the boundary of the kernel of $\{G_{n(k)}\}$ also contains the segment from 0 to 4 because $t_1^{(n(k))} \to 4$. Thus the limit function of $\{f_{n(k)}\}$ must omit the value $w=4$. Hence by the Koebe

$\frac{1}{4}$-Theorem, this limit function is k_1. Since $\{f_n\}$ is a normal family it follows that $f_n \to k_1$. Similarly the subsequence of points in D_2 is such that the corresponding sequence of functions converges uniformly inside E to k_1. For the sequences in each of D_3, D_4, D_5, and D_7, the coefficient C_0 is a continuous function of t_1 and α, and an examination of (6) with proper values for the parameters shows that $C_0^{(n)} \to 2$ as (x_n, y_n) converges to a point representing k_1. Thus k_1 is the only function to which a subsequence of functions can converge, and since each subsequence is a normal family, k_1 is the limit of each subsequence. The situation is analogous for a sequence of points of H with limit point corresponding to the Koebe function k_2. Suppose now that $\{[x_n, y_n]\}$ converges to $[x, y]$ and that this equivalence class contains a point on the boundary of D_1 and a point on the boundary of D_3, neither corresponding to k_1. Assume also that $\{[x_n, y_n]\}$ consists of two subsequences of plane points $\{(x_{n(j)}, y_{n(j)})\}$ from D_1 and $\{(x^*_{n(j)}, y^*_{n(j)})\}$ from D_3. Then by the same arguments as above, the sequences of functions associated with the subsequences of plane points converge to functions f and f^*. The points of D_1 are of form $(t_1^{(n(j))}, 4+\operatorname{Arctan} t_2^{(n(j))})$ and their subsequence has limit $(t_1, 4+\operatorname{Arctan} t_2)$ on ∂D_1. The points of D_3 are of form $(t_1^{(n(j))}+2, \alpha^{(n(j))})$ and their subsequence has limit $(t_1+2, 4/3+t_1^{3/2}/3)$. f maps E onto a domain slit along trajectories of $Q_1(w, t_1, t_1)\,dw^2$ as described in part (b) of Proposition 3 with the slit on the trajectory from t_1 to ∞ having length zero. The last comment follows since any point to the right of t_1 on the real axis is in the kernel of $\{G_n\}$ because $t_1^{(n(j))} \to t_1$. Further f^* maps E onto the domain just described since generally the limit of a sequence of functions associated with points of D_3 maps E onto a domain bounded as described in part (b) of Proposition 3. In this case the length of the slit on the trajectory from t_1 to ∞ is $|-4t_1^{3/2}/3+4\alpha-16/3|$, and here $\alpha=4/3+t_1^{3/2}/3$, so the slit has length zero. The quadratic differential $Q_1(w, t_1, t_1)\,dw^2$ enters here as above and hence we can use Schwarz's Lemma to assert that $f \equiv f^*$. Suppose $\{[x_n, y_n]\}$ converges to a point $[x, y]$ which contains plane points $(t_1, \pi/2)$ on the boundary of D_1 and $(t_1, -\pi/2)$ on the boundary of D_2. Then $t_1^{(n)} \to t_1$ and $\{t_2^{(n)}\}$ consists of two subsequences, one with limit $+\infty$ and the other with limit $-\infty$. The quadratic differentials

$$[(w-t_1^{(n)})(w/t_2^{(n)}-1)/w]\,dw^2$$

have the same trajectory structure as the differentials $Q_1(w, t_1^{(n)}, t_2^{(n)})\,dw^2$ when $t_2^{(n)}$ is positive and $Q_2(w, t_1^{(n)}, t_2^{(n)})\,dw^2$ when $t_2^{(n)}$ is negative. As a subsequence of the sequence $\{t_2^{(n)}\}$ converges either to $+\infty$ or $-\infty$, the sequence of rational functions with terms $[(w-t_1^{(n)})(w/t_2^{(n)}-1)/w]$ converges uniformly on any compact neighborhood of w_0 not including 0 in the w-plane to $-[(w-t_1^{(n)})/w]$. We use this fact as before to prove that the sequence of domains, the images of E under the associated sequence of functions, converges to its kernel with respect to ∞, and that this kernel is bounded as described in part (c), Proposition 3. Then as before the associated sequence of functions in F converges to the function in F mapping E conformally onto the kernel of the above sequence of domains. The proofs that the convergence in H of sequences having limits at boundaries of other adjacent or attached sets D_j are carried out exactly as the proofs just completed. Thus the mapping h from H to F is one-to-one and onto, and continuous. H is compact hence h is a homeomorphism.

From (3) relating A_2, A_3, and A_4 to C_0, C_1, and C_2 we see that there is a natural correspondence from F to $\partial E(4, S)$ given by

$$f \to (-C_0, C_0^2 - C_1, 2C_0C_1 - C_2 - C_0^3).$$

The GCT can be used to show that this correspondence is one-to-one, and continuity follows because convergence of a sequence of functions in the topology of F implies convergence of their series coefficients in (2) to the corresponding coefficients of the limit function. Hence the mapping from F into $\partial E(4, S)$ is a homeomorphism. Finally, the image of F is all of $\partial E(4, S)$ for if this were not so, then because $\partial E(4, S)$ is homeomorphic to S^2, it would be possible to construct a homeomorphism of S^2 properly into itself in contradiction to the Jordan Separation Theorem. Thus the functions of F determine all of the boundary points of $E(4, S)$ and this completes the proof of Proposition 4.

It is interesting to note here that the coefficient domain $E(3, S)$ is readily found by considering the family F. Clearly $f \in F$ implies that $1/f(1/z)$ belongs to a point of $E(3, S)$. The functions belonging to boundary points of $E(3, S)$ are identified in

PROPOSITION 5. *Let $F^\#$ be the subset of F consisting of $F_3 = \bigcup F_3(t_1)$ with the union taken over $t_1 \in [-4, 4]$, and $F_2^\# = \bigcup F_2(t_1, 0)$ with the union taken over those t_1 in $[-4, 4]$ for which $-t_1 + T = 4$. Then $(A_2, A_3) \in \partial E(3, S)$ iff (A_2, A_3) belongs to $1/f(1/z)$ for some $f \in F^\#$.*

Proof. The proof that functions in $F^\#$ correspond only to boundary points of $E(3, S)$ is the same as the proof of the analogous assertion about F and $E(4, S)$ in Proposition 4. To see that $F^\#$ yields all of $\partial E(3, S)$ we recall that for $f \in F_3$ we have

$$\begin{aligned} A_2 &= -C_0 = -t_1[1 - \ln(|t_1|/4)]/2, \\ A_3 &= C_0^2 - C_1 = t_1^2[1 - \ln(|t_1|/4)]^2/4 - t_1^2[1 - 2\ln(|t_1|/4)]/8 + 1 \end{aligned} \tag{7}$$

for $t_1 \in (-4, 4)$ with $C_0 = 0$, $C_1 = -1$ for $t_1 = 0$. For $f \in F^\# - F_3$ we have

$$A_3 = A_2^2 - 1, \qquad -2 \leqq A_2 \leqq 2. \tag{8}$$

The pairs (A_2, A_3) satisfying (7) and (8) trace a simple closed curve hence the boundary of $E(3, S)$. Q.E.D.

For purposes of computation note that in those cases where $t_1 = t_2$ or $t_2 = 0$, equations (6) give the C_j's, and hence the A_j's, in terms of t_1 and α since β assumes fixed values with t_2 as above. Computations can then be made by choosing $t_1 \in [-4, 4]$ and α limited by the proper one of conditions (5) or its analog for Q_2. If $t_2 > \max(t_1, 0)$ or $t_2 < \min(t_1, 0)$ conditions (5) or their analog for Q_2 relate

t_1, t_2, β, and α in a way which can be put in terms of hypergeometric functions and computations appear to be very difficult to carry out. It is interesting to note however that only a relatively small portion of $\partial E(4, S)$ is associated with those cases where $t_2 \neq t_1$ or $t_2 \neq 0$. As a brief illustration of this the points $(A_2, A_3, A_4) = (-0.50563, 1.19559, -0.93709)$ and $(-0.50009, 1.07244, 1.08179)$ are on $\partial E(4, S)$ in the cross section where A_2 is approximately $-\frac{1}{2}$. These points lie on curves bounding the portion of $\partial E(4, S)$ associated with the conditions $t_2 > t_1 \geqq 0$. The point with smallest A_3 in this section is $(-0.50000, -0.75000, 0.87500)$ while the first mentioned point has the largest A_3 in this section. A table with points in representative cross sections of the coefficient body has been compiled.

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University of Florida,
Gainesville, Florida