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ASYMPTOTIC BEHAVIOR OF PRODUCTS $C^p = C + \cdots + C$ IN LOCALLY COMPACT ABELIAN GROUPS

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1. **Summary.** Let G be a locally compact group and $|C|$ the left Haar measure of an open relatively compact set $C \subset G$. We consider the asymptotic behavior of powers $\{C^p : p = 1, 2, \ldots\}$. The behavior of $|C^p|$ as $p \to \infty$ is of interest in a variety of problems; one of particular interest to the authors is the relationship between "polynomial growth" of $|C^p|$ and the existence of left invariant means on G (see [5, §3.6], and [4]). On the other hand, interesting connections have been found between the growth of $|C^p|$ in discrete solvable groups and the properties of fundamental groups for Riemannian manifolds with negative curvature (see [11], [8], [9]). We shall mention some of the results drawn from [11] below, but will not pursue applications to differential geometry.

In §§2 and 3 we shall review some elementary estimates giving upper bounds for the growth of $|C^p|$ in abelian and nilpotent groups, and will outline the relevance of these results to the study of invariant means. Then we shall review some recent efforts to derive upper and lower bounds for $|C^p|$ in solvable groups.

The main point of this article is to give sharp asymptotic estimates of $|C^p|$ for G a locally compact *abelian* group. Our main result is:

THEOREM 1.1. *If G is a locally compact abelian group and C an open relatively compact set, then there is a constant $A > 0$ and an integer $k \geqq 0$ such that $|C^p| = Ap^k + O(p^{k-1} \log p)$ as $p \to \infty$. In a connected group, k is independent of C; in general it is an invariant associated with the smallest closed subgroup $H \subset G$ such that C lies within a coset of H.*

COROLLARY 1.2. *If G is a locally compact abelian group then the following condition*

$$\text{(SA)} \qquad \lim \left\{ \frac{|C^{p+1}|}{|C^p|} : p = 1, 2, \ldots \right\} = 1$$

is satisfied for any nonempty open relatively compact set $C \subset G$.

Our derivation of these results will exhibit much information about the geometric behavior of C^p in terms of the structural features of G. One of the main steps is the analysis of the combinatory problem which arises when G is discrete (so $C = \{g_1, \ldots, g_n\}$ is a finite nonempty set); here $|C^p|$ is just cardinality and we show that $|C^p| = Ap^k + O(p^{k-1})$ where k is the rank $(0 \leqq k \leqq |C| - 1)$ of the subgroup H

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generated by the differences $\{g_2-g_1, \ldots, g_n-g_1\}$(1). In another direction we must repair some gaps (2) in a note by Kawada [7] which elegantly describes the behavior of C^p when G is a vector group:

THEOREM 1.3. *Let C be an open relatively compact set in a vector group $G=\boldsymbol{R}^N$. If for $\delta>0$ we define $C_\delta=\{x \in G : x \in C, \text{dist}(x, \text{bdry } C)>\delta\}$, then there exists a $p(\delta)$ such that $(\text{conv } C)^p \supset C^p \supset (\text{conv } (C_\delta))^p$ for all $p \geqq p(\delta)$. Here conv (X) stands for the convex hull of a set $X \subset \boldsymbol{R}^N$.*

Thus C^p behaves like $(\text{conv } C)^p$ as $p \to \infty$. Next, we show that if G is any *compact* group (possibly nonabelian) there exists an open subgroup $G_0 \subset G$, an element $x_0 \in G$, and an integer p_0 such that $C^p=(x_0)^{p-p_0}G_0$ for all $p \geqq p_0$. These results are combined, using structure theory of abelian locally compact groups, to describe the behavior of C^p in general abelian groups(3).

2. **Notations.** All our groups are locally compact Hausdorff (possibly discrete, and sometimes noncommutative), left Haar measure of a set $X \subset G$ is written $|X|$, and we leave the reader to deduce from context the group in which X lives when there are several different Haar measures about—this will cause little difficulty in our discussion. We write $\boldsymbol{Z}$ for the integers, $\boldsymbol{R}$ for the reals, so $\boldsymbol{R}^N$ is a typical vector group and $\boldsymbol{Z}^N$ its lattices of integral points. For a set E in a vector group $\boldsymbol{R}^N$ we usually denote its convex hull (not the *closed* convex hull) by conv (E). We will invariably refer to the usual Euclidean metric in $\boldsymbol{R}^N$. If X, Y are sets we write $X \sim Y$ for their difference and $X \Delta Y=(X\sim Y) \cup (Y \sim X)$. We shall also write $U^+(X, \delta) =\{x : \text{dist}(x, X)<\delta\}$ and $U^-(X, \delta)=\{x : x \in X \text{ and dist}(x, \text{bdry } X)>\delta\}$ for $\delta>0$ and X in a vector space; if x is a point in a vector space we write $U(x, \delta)$ for the open ball of radius $\delta>0$ about x and abbreviate $U(0, 1)=U$ and $U(x, \delta)=x+\delta U$. We often write the "δ-retract" $U^-(X, \delta)$ as X_δ for brevity; notice that this set X_δ may be empty if δ is too large.

Finally, if G is an abelian group and $C \subset G$, we write

$$C^p = C+\cdots+C = \{c_1+\cdots+c_p : c_i \in C\} \quad \text{and} \quad pC = \{pc = c^p : c \in C\}.$$

Obviously $pC \subset C^p$, but these sets are usually quite different. In general the sets

(1) For any $1 \leqq l \leqq n$ the differences $\{g_1-g_l, \ldots, g_n-g_l\}$ generate the same subgroup $H \subset G$ and H is precisely the smallest subgroup of G such that C lies within a coset of H, as described in Theorem 1.1.

(2) One of the two major gaps in Kawada's note occurs in his proof of Theorem 1.3 (in [7], lines 2–5(t), p. 226, are wrong; for one thing they overlook the need to have $n(\delta)$ independent of y_0 in the terminology defined there).

(3) In [7] Kawada treats general connected abelian G, i.e. $G=\boldsymbol{R}^N \times K$ where K is connected compact abelian. However, a serious gap occurs in this discussion when he asserts that eventually some power of a given open set $C \subset G$ will have the form $C^p=A \times K$ (a union of K-cosets). This would allow us to use Theorem 1.3 to study general connected abelian G; unfortunately, there are easy counterexamples to this assertion. We repair this gap in §6 of this article.

C^p, pC are well defined even if G is nonabelian; on the other hand, if $G=\mathbf{R}^N$ then the notation pC is consistent with the scaling operation $C \to \lambda C=\{\lambda c : c \in C\}$, which is defined for $\lambda \in \mathbf{R}$ and is an automorphism of $\mathbf{R}^N$ if $\lambda \neq 0$.

3. **Growth properties.** Let us define

$$W(p, C) = \{x_1^{i_1} \cdots x_n^{i_n} : 0 \leqq n \leqq p, i_k = \pm 1, x_k \in C\},$$

the set of all words of length at most p formed from elements of C. Evidently $W(p+1, C) \supset W(p, C) \supset C^p$ and in fact $W(p, C)=\tilde{C}^p$ where $\tilde{C}=C \cup C^{-1} \cup \{e\}$, e being the unit. Let C be open and relatively compact. Then we say C has:

(i) *At most polynomial growth* $\Leftrightarrow$ there exist constant $A>0$ and integer $k \geqq 0$ such that $|C^p| \leqq Ap^k$, all p.

(ii) *At least polynomial growth* $\Leftrightarrow$ there exist constant $A>0$ and integer $k \geqq 0$ such that $|C^p| \geqq Ap^k$, all p.

(iii) *Exponential growth* $\Leftrightarrow$ there exist constants $A>0$, $\rho>1$ such that $|C^p| \geqq A\rho^p$, all p.

We say that G has one of these modes of growth if every set C which generates G (in the sense that $\bigcup_{p=1}^{\infty} W(p, C)=G$) has the property.

It seems natural to ask for upper and lower bounds on the growth of C^p for a generating set $C \subset G$. One of the main results of [11] gives such estimates for discrete nilpotent groups.

THEOREM 3.1. *Let G be a discrete finitely generated nilpotent group with derived series $G=G_0 \supset G_1 \supset \cdots \supset G_s \supset G_{s+1}=\{e\}$, where $G_{k+1}=[G, G_k]$. Then each group G_k/G_{k+1} is finitely generated abelian, say $G_k/G_{k+1} \cong A_k \times \mathbf{Z}^{n_k}$ with A_k a finite abelian group. Define invariants*:

$$E_1(G) = \sum_{k=0}^{s} (k+1)n_k, \qquad E_2(G) = \sum_{k=0}^{s} 2^k n_k.$$

Then if C is any set which generates G, there exist constants $\alpha, \beta>0$ such that

$$\alpha p^{E_1(G)} \leqq |W(p, C)| \leqq \beta p^{E_2(G)}, \qquad p = 1, 2, \ldots.$$

Somewhat more complicated estimates are given in [11] for finitely generated solvable groups which have nilpotent subgroups of finite index; in finitely generated solvable groups which do not have such a subgroup, every set of generators is shown to have exponential growth. Incidentally, the following estimates are obtained for the growth of a *minimal* set of generators C in $\mathbf{Z}^N$: there exist constants $0<\alpha<\beta$ such that[4] $\alpha p^N \leqq |W(p, C)| \leqq \beta p^N$. Kawada's theorem (Theorem 1.3) seems to be the first attempt in the literature to describe the asymptotic behavior of C^p in nondiscrete groups.

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[4] If $D=C \cup C^{-1} \cup \{e\}$ this estimate is not sharp enough to give the result we assert in Corollary 1.2; it only gives: $1 \leqq \liminf \{|D^{p+1}|/|D^p|\} \leqq \limsup \{|D^{p+1}|/|D^p|\} \leqq \beta/\alpha$.

If C and D are generating sets (open, relatively compact) in G and if one set has upper or lower polynomial bounds on the growth of $W(p, C)$ then the other set has similar limits on its growth. The following lemmas are obtained by trivial modifications of the results given in [11] for discrete groups.

LEMMA 3.2. *Let C be an open relatively compact set in G.*

(i) *Suppose $r \geqq 0$, $q \geqq 0$ are integers such that $|C^p| \geqq \alpha(p-r)^q$ for some $\alpha > 0$. Then there exists another constant $\alpha' > 0$ such that $|C^p| \geqq \alpha' p^q$, all p.*

(ii) *Suppose $l > 0$, $q \geqq 0$ are integers such that $|C^{lp}| \geqq \alpha(lp)^q$ all p, for some constant $\alpha > 0$. Then there exists an $\alpha'' > 0$ such that $|C^p| \geqq \alpha'' p^q$, all p.*

LEMMA 3.3. *Let C and D be relatively compact open generating sets for G. Suppose C admits polynomial growth estimates $\alpha p^m \leqq |W(p, C)| \leqq \beta p^n$ where $0 \leqq m \leqq n$ are integers and $0 < \alpha \leqq \beta$. Then there are constants $0 < \alpha_1 \leqq \beta_1$ such that $\alpha_1 p^m \leqq |W(p, D)| \leqq \beta_1 p^n$, all p.*

In a discrete group exponential growth is the worst that can happen since $|C^p| \leqq |C|^p$ with equality holding, for example, if $C = \{a, b\}$ where $G = F_2$ is the free group on these two elements. Since F_2 contains as a subgroup the free group on n generators, it is clear we may invent (nonminimal) generating sets $D \subset F_2$ with $|D^p| \sim n^p$, so generating sets in such a group do not have comparable growth rates of the sort described in Lemma 3.3(5). We have noted that in the category of finitely generated discrete solvable groups there is a dichotomy between exponential and polynomial growth of generating sets. An interesting question is whether a generating set in some other type of group can exhibit an intermediate mode of growth, say $|C^p| \sim e^p/\log p$, or whether in a nondiscrete group $|C^p|$ can exhibit worse than exponential growth(6). It would be very interesting to know if the above dichotomy persists for, say, connected solvable groups. In the Lie group $G = A(1, \boldsymbol{R})$ of all affine transformations of $\boldsymbol{R}$, any neighborhood C of the unit has $|C^p|$ growing at least exponentially; in fact, $A(1, \boldsymbol{R})$ is well known to be nonunimodular and if $\delta = \sup\{\Delta(x) : x \in C\}$, where Δ is the modular function, then $\delta > 1$ and

$$|C^p| \geqq |C^{p-1}|\delta \geqq \cdots \geqq |C|\delta^{p-1}.$$

The theorem on discrete solvable G presented in [11] seems to be one of the best growth results at present. It is not known how sharp these estimates are, even in nilpotent groups; as we noted above, the property $|C^{p+1}|/|C^p| \to 1$, which we shall prove for abelian groups, does not follow from the estimates in [11], even using the special estimates set down there for the group $G = \boldsymbol{Z}^N$. We conjecture

(5) It is interesting to compare $|C^p|$ with $|D^p|$ where $D = \{a, b, a^{-1}, b^{-1}\}$ in F_2. One easily shows that $|D^p| \geqq 4 \cdot 3^{p-1} \geqq 3^p$, while $|C^p| = 2^p$. Since $|W(p, C)| \geqq |D^p|$, the growth rates of $|C^p|$ and $|W(p, C)|$ are not comparable.

(6) If $G = \boldsymbol{R}^2$ let B be the union of the x-axis and y-axis. Then $|B| = 0$, $|B^2| = +\infty$. Approximating B with open relatively compact sets C we can arrange $|C| = 1$ and $|C^2| > 10$ so $|C^2| > |C|^2$.

that this property holds for all nilpotent groups, but a proof seems very difficult even for discrete groups. It is not hard (see [5, §3.6]) to prove that

$$\liminf \{|C^{p+1}|/|C^p|\} = 1$$

for any open relatively compact generating set in a nilpotent locally compact group. Notice that $|C^{p+1}|/|C^p| \geqq 1$ for all p; moreover, if we have $|C^{p+1}|/|C^p| \to 1$ for every C, this implies $|W(p+1, C)|/|W(p, C)| \to 1$ on taking $C \cup C^{-1} \cup \{e\}$ in place of C.

4. **Growth properties and invariant means.** If G is a locally compact group and $L^\infty(G)$ the essentially bounded measurable functions equipped with the usual ess. sup norm $\|f\|_\infty$, a *mean* on L^∞ is any continuous linear functional $m \in (L^\infty)^*$ such that

(i) $m(\bar{f}) = \overline{m(f)}$,

(ii) $m(f) \geqq 0$ if $f \geqq 0$ locally a.e.,

(iii) $m(1) = 1$ where 1 is the constant function on G,

and m is *left invariant* (m a LIM) if $m({}_xf) = m(f)$ all $x \in G$, where ${}_xf(t) = f(x^{-1}t)$. We say G is *amenable* if L^∞ admits at least one LIM; for a number of equivalent definitions of amenability, (see [5, §2.2]).

Amenability is tied up with certain geometric properties of G via the recent discovery (see [5, §3.6]) that amenability is equivalent to being able to construct sets $U \subset G$ which are "large" in the sense that they are moved only slightly by a prescribed compact set $K \subset G$ of left translations.

THEOREM 4.1. *The following are equivalent for any locally compact group G.*

(i) *G is amenable.*

(ii) *If $\varepsilon > 0$ and $K \subset G$ is compact, then there is a compact set $U \subset G$ with $0 < |U| < \infty$ such that*

(FC) $$|xU \,\Delta\, U|/|U| < \varepsilon, \qquad \textit{all } x \in K.$$

(iii) *If $\varepsilon > 0$ and $K \subset G$ is compact, then there is a compact set $U \subset G$ with $0 < |U| < \infty$ and*

(A) $$|KU \,\Delta\, U|/|U| < \varepsilon.$$

Condition (A) is formally stronger than (FC). This theorem is proved in [4]. If U satisfies (FC) for a pair (K, ε) then $\varphi_U = (1/|U|)\chi_U$, the normalized characteristic function of the measurable set U, is a unit vector in $L^1(G)$, $\varphi_U \geqq 0$, and φ_U is moved at most a distance ε by any left translation $x \in K$: i.e. we have $\|{}_x(\varphi_U) - \varphi_U\|_1 < \varepsilon$, all $x \in K$.

If G has the geometric property (FC) (or (A)) it is easy to use it to produce invariant means on L^∞, proving (FC) $\Rightarrow$ (amenable)(7); if we make the set of

(7) It is the converse which is the hard part in [4].

pairs $J=\{(K, \varepsilon)\}$ a directed set in the obvious way, then for $j=(K, \varepsilon) \in J$ take any set U_j satisfying (FC) for (K, ε) and set $\varphi_j=\varphi_{U_j}$. Then, regarding $\{\varphi_j\}$ as a net of means in $L^1 \subset (L^\infty)^*$, it is trivial to show that every weak-* limit point of $\{\varphi_j\} \subset (L^\infty)^*$ is a left invariant mean. Such points must exist due to the easily verified weak-* compactness of the convex set of all means in $(L^\infty)^*$.

If G is amenable (and let us note that all solvable groups are amenable) it is natural to ask how we may *systematically* construct large sets in G. A natural conjecture is that if C is an open relatively compact neighborhood of the unit which generates G then the sets C^p eventually (resp. frequently) satisfy condition (FC) for any fixed pair (K, ε). This is more or less what Kawada was trying to prove for $\mathbf{R}^N$ in [7]. As we now show, this conjecture sometimes fails. For other approaches to the problem of constructing large sets, see [6].

DEFINITION 4.2. A locally compact group is *strongly amenable* if, for every open relatively compact neighborhood C of the unit which is symmetric ($C^{-1}=C$), we have $|C^{p+1}|/|C^p| \to 1$.

THEOREM 4.3. *In any locally compact group G, (strongly amenable) $\Rightarrow$ (amenable), and every strongly amenable group is necessarily unimodular.*

Proof. Let $\varepsilon>0$, $K \subset G$ compact. Let C be any symmetric neighborhood of the unit large enough to give $C \supset KK \cup KK^{-1}$, so $C^{p+1} \supset KC^p \cup C^p$ for all p. For any U, $|KU \,\Delta\, U|=|KU \sim U|+|U \sim KU|$, and we have $|U \sim KU| \leqq |KU \sim U|$ since $|U \cap KU|+|U \sim KU|=|U| \leqq |KU|=|U \cap KU|+|KU \sim U|$. Thus

$$\frac{|KC^p \,\Delta\, C^p|}{|C^p|} \leqq 2\,\frac{|C^{p+1} \sim C^p|}{|C^p|} = 2\left(\frac{|C^{p+1}|}{|C^p|}-1\right) \to 0$$

and the sets C^p eventually satisfy (A) for the pair (K, ε).

If Δ is the modular function and G is not unimodular then there is an x_0 in G with $\Delta(x_0)>2$. If C is any small neighborhood of x_0 then $C^{p+1} \supset C^p x_0$, which implies that $|C^{p+1}| \geqq \sup\{\Delta|C\} \cdot |C^p| \geqq 2|C^p|$, all p, contradicting strong amenability. Q.E.D.

Our result, Theorem 1.2, shows that abelian groups are strongly amenable; their amenability is fairly trivial (see [5, Theorem 1.2.1]).

We digress for a moment to point out a few other connections between property (A) and the behavior of $|C^{p+1}|/|C^p|$; recall that

$$|C^{p+1}|/|C^p| \to 1 \Leftrightarrow \limsup\{|C^{p+1}|/|C^p|\} = 1$$

since $|C^{p+1}|/|C^p| \geqq 1$.

1. Let C be an open relatively compact set which is a symmetric neighborhood of the unit, and which generates G. Then

$$\limsup\{|C^{p+1}|/|C^p|\} = 1$$

$\Leftrightarrow$ the sets $\{C^p\}$ *eventually* satisfy condition (A) with respect to any pair (K, ε).

In fact, if (K, ε) is given there exists an integer p_0 such that $C^{p_0} \supset K \cup \{e\}$, so (using the elementary inequalities from the proof of 4.3)

$$\frac{|KC^p \,\Delta\, C^p|}{|C^p|} \leqq 2\frac{|KC^p \sim C^p|}{|C^p|} \leqq 2\left(\frac{|C^{p+p_0}|}{|C^p|}-1\right)$$
$$= 2\left(\frac{|C^{p+p_0}|}{|C^{p+p_0-1}|}\cdot\dots\cdot\frac{|C^{p+1}|}{|C^p|}-1\right) \to 0.$$

Conversely, suppose C is a neighborhood such that (A) holds eventually for the sets C^p and any pair (K, ε). Then consider the set $K=\bar{C}$; we get

$$0 \leftarrow \frac{|KC^p \,\Delta\, C^p|}{|C^p|} = \left(\frac{|\bar{C}C^p|}{|C^p|}-1\right) = \left(\frac{|C^{p+1}|}{|C^p|}-1\right)$$

since $\bar{C}C^p = C^{p+1}$ for $p \geqq 1$. Thus $|C^{p+1}|/|C^p| \to 1$ as required.

2. Assume that for every relatively compact symmetric neighborhood of the unit C we have

$$\liminf\{|C^{p+1}|/|C^p|\} = 1.$$

Let a compact set $K \subset G$ be given; then if we take any C with $C \supset KK \cup KK^{-1}$, the sets $\{C^p\}$ will *frequently* satisfy condition (A) with respect to the pair (K, ε), for any $\varepsilon > 0$.

This is seen by making trivial modifications in the proof of Theorem 4.3.

Example. Let us realize $G = A(1, \boldsymbol{R})$ as $\boldsymbol{R}^+ \times \boldsymbol{R}$ where $\boldsymbol{R}^+ = (0, +\infty)$, equipped with multiplication $(a, x)(b, y) = (ab, bx+y)$, which corresponds to the action $G \times \boldsymbol{R} \to \boldsymbol{R}$ via $(a, x)\colon t \to at+x$. Then G is solvable and nonunimodular, hence is amenable but not strongly amenable. Let $S=(a, 0)$ and $T=(0, b)$; then for suitable choices of $a, b>0$ the set $\{S, T\} = C \subset G$ generates a *free semigroup* in G—i.e. there are no identifications between distinct words built from *positive* powers of S, T—and in the subgroup H (regarded as discrete) generated by C we have cardinalities $|C^p| = |C|^p = 2^p$ and $|W(p, C)| \geqq |C^p| = 2^p$.

To verify that there are no identifications, note that a word $W_1 = S^{i_1}T^{j_1}\cdots S^{i_n}T^{j_n}$ $(i_1, j_n \geqq 0$; all other $i_k, j_k \geqq 1)$ acts on $\boldsymbol{R}$ via

$$t \to (a^{\sum_{k=1}^{n} j_k})t + b(i_n a^{\sum_{k=1}^{n-1} j_k} + i_{n-1}a^{\sum_{k=1}^{n-2} j_k} + \cdots + i_2 a + i_1).$$

Take $b=1$, $a>0$ so that no two distinct polynomials with integer coefficients have the same value at a. If word W_1 coincides as a group element with word $W_2 = S^{p_1}T^{q_1}\cdots S^{p_m}T^{q_m}$ then equality of the second bracketed terms forces $m=n$; $i_n = p_n, \dots, j_1 = q_1$ and so the words are the same.

The conjecture above fails in the discrete (unimodular) group H, even though H is solvable; in fact, the sets C^p never satisfy (A) or (FC) for the pair $K=C$, $\varepsilon = \frac{1}{4}$. The conjecture also fails in the (nonunimodular) connected Lie group G since $|C^p| \geqq |C|\delta^{p-1}$ where $\delta = \sup\{\Delta|C\}$ and C is any open relatively compact neighborhood of the unit.

Other growth properties and a conjecture. Another growth property has been mentioned in the literature on invariant means, namely the property discussed by Adelson-Welsky and Sreider [12]:

$$|C^p|^{1/p} \to 1 \quad \text{as } p \to \infty,$$

for any open relatively compact set (or neighborhood of the unit) in G. For any open set C it is obvious that

$$\limsup \{|C^p|^{1/p}\} \leqq \limsup \{|C^{p+1}|/|C^p|\}$$
$$\liminf \{|C^p|^{1/p}\} \geqq \liminf \{|C^{p+1}|/|C^p|\} \geqq 1,$$

so that the condition is formally weaker than the condition

$$|C^{p+1}|/|C^p| \to 1$$

we have been discussing. For example, it is easy to see that the growth estimates for discrete finitely generated nilpotent groups cited above from [11]:

$$\alpha p^{E_1} \leqq |C^p| \leqq \beta p^{E_2},$$

immediately yield the result that $|C^p|^{1/p} \to 1$, although they do not give

$$|C^{p+1}|/|C^p| \to 1.$$

If C is an open symmetric neighborhood of the unit which generates G, we have noted that the property: $|C^{p+1}|/|C^p| \to 1$ for C is equivalent to the property that the C^p eventually satisfy condition (A) with respect to any pair (K, ε). It is not so clear how the property $|C^p|^{1/p} \to 1$ is related to property (A), except that it is weaker than property (A).

Since we are dealing with open relatively compact sets C it does not seem unreasonable to expect that the shape of C^p will become more "regular" as p increases (at least in connected groups or Lie groups). We are led to the following conjecture:

Conjecture. If G is a (connected) locally compact group and C is an open relatively compact set, then

$$\lim \{|C^{p+1}|/|C^p|\} \textit{ exists},$$

although it may differ from 1.

Except for the abelian groups considered in this paper the conjecture is completely unresolved. This conjecture has some nice consequences.

1. To show that $|C^{p+1}|/|C^p| \to 1$, it suffices to show that $\liminf |C^{p+1}|/|C^p| = 1$. As the latter is known for connected nilpotent groups, it would follow that all nilpotent groups are strongly amenable; from the results in [11] it would follow that the only finitely generated discrete solvable groups with the strong amenability property are those with nilpotent subgroups of finite index, and it might be possible to arrive at similar results for connected solvable groups.

2. If the conjecture holds, then for an open relatively compact set C we have

$$\lim \{|C^p|^{1/p}\} = 1 \Leftrightarrow \lim \{|C^{p+1}|/|C^p|\} = 1.$$

This is immediate from the inequalities above.

5. **Behavior of C^p in vector groups.** In this section let $G=\mathbf{R}^N$ for some $N \geqq 1$. We prove results which improve the estimates of Kawada on the geometric behavior of C^p as $p \to \infty$, and at the same time correct errors in his proof. Clearly $pC \subset C^p$ for all $p=1, 2, \ldots$; if C is convex we actually have $pC=C^p$, for if $x \in C^p$ we may write it

$$x = x_1^{m_1} + \cdots + x_l^{m_l} = m_1 x_1 + \cdots + m_l x_l, \qquad l \leqq p,$$

where the $\{x_k\} \subset C$ are distinct, the $\{m_k\}$ are integers with $m_k \geqq 0$, and $\sum_k m_k = p$. Thus in $\mathbf{R}^N$:

$$x = p\left(\frac{1}{p}x\right) = p\left(\frac{m_1}{p}x_1 + \cdots + \frac{m_l}{p}x_l\right) = px^*$$

and obviously $x^* \in \operatorname{conv}(C) = C$ as required. We also have

$$(1) \qquad p(\operatorname{conv} C) = \operatorname{conv}(C^p) \quad \text{all } p = 1, 2, \ldots$$

for any subset of $\mathbf{R}^N$, for if $x \in \operatorname{conv} C$, say $x = \sum_i \alpha_i x_i$ with $x_i \in C$, $\alpha_i > 0$, and $\sum_i \alpha_i = 1$, then $px = \sum \alpha_i(px_i) \in \operatorname{conv}(C^p)$ since $px_i = x_i^p \in C^p$. On the other hand, if $x \in \operatorname{conv}(C^p)$ then $x = \sum_i \alpha_i(x_{i1} + \cdots + x_{ip}) = p(\sum_{i,j} (\alpha_i/p)x_{ij}) \in p(\operatorname{conv} C)$, as required. Notice that the mapping $x \to \lambda x$ is an automorphism of the additive group $\mathbf{R}^N$ for any $\lambda \neq 0$ in $\mathbf{R}$ and scales measures by a factor $|\lambda|^N$; this is fundamental to our discussion.

LEMMA 5.1. *Let A be any bounded set in $\mathbf{R}^N$ of diameter $D = \operatorname{diam}(A)$. Then every point of $\operatorname{conv}(A^p)$ is within distance $2N \operatorname{diam}(A)$ of some point in A^p; i.e., we have $\operatorname{conv}(A^p) \subset A^p + U(2N \operatorname{diam}(A))$ for $p=1, 2, \ldots$, where $U(\delta)$ is the ball of radius $\delta > 0$ in $\mathbf{R}^N$.*

Proof. The result is independent of translations of A, so we may assume $0 \in A$. We recall a theorem due to Carathéodory: if X is a set in $\mathbf{R}^N$ and $x \in \operatorname{conv}(X)$, we may always write x as a convex sum of at most $N+1$ points in X (see [2, Theorem 18]). Apply this to the set $\operatorname{conv}(A^p) = p \cdot (\operatorname{conv} A)$ to express a point $x \in \operatorname{conv}(A^p)$ as a sum

$$x = p\left(\sum_{i=1}^{N+1} \lambda_i a_i\right), \qquad \lambda_i \geqq 0, \qquad \sum_{i=1}^{N+1} \lambda_i = 1, \qquad a_i \in A.$$

We may assume $0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_{N+1}$ and define $d = p - \sum_{i=1}^N [p\lambda_i] > 0$, where $[\lambda]$ is the largest integer less than or equal to $\lambda \in \mathbf{R}$. Consider the point $x^* \in A^p$:

$$x^* = \sum_{i=1}^{N} [p\lambda_i] a_i + d a_{N+1}.$$

Then $\|a\| \leqq \operatorname{diam}(A)$ for $a \in A$, since $0 \in A$, and

$$\|x-x^*\| = \left\| \sum_{i=1}^{N} (p\lambda_i - [p\lambda_i])a_i + (p\lambda_{N+1}-d)a_{N+1} \right\|$$
$$\leqq \sum_{i=1}^{N} \|a_i\| + N\|a_{N+1}\| \leqq 2N \operatorname{diam}(A). \qquad \text{Q.E.D.}$$

THEOREM 5.2. *Let C be any bounded subset of $\boldsymbol{R}^N$ such that some power of C has interior. Then there is a vector $t \in \boldsymbol{R}^N$ and an integer $p_0 \geqq 1$ such that: $C^p \supset t + (p-p_0)(\operatorname{conv} C)$ for all $p \geqq p_0$. Clearly $p(\operatorname{conv} C) \supset C^p$, all p.*

Proof. Suppose C^{p_1} has interior and let $D = \operatorname{diam}(C)$. For some $\delta > 0$, C^{p_1} includes some ball: $C^{p_1} \supset t^* + U(\delta)$ where $t^* \in \boldsymbol{R}^N$. Then C^{p_2} includes the ball $t + U(2ND)$ if we let $t = ([2ND/\delta]+1)t^*$ and $p_2 = ([2ND/\delta]+1)p_1$. Thus $p \geqq p_2 \Rightarrow$

$$(t+U(2ND)) + C^{p-p_2} \subset C^{p_2} + C^{p-p_2} = C^p;$$

but by 5.1 (taking $A = C$)

$$\begin{aligned} C^p &\supset t+(C^{p-p_2}+U(2ND)) \supset t+\operatorname{conv}(C^{p-p_2}) \\ &= t+(p-p_2)(\operatorname{conv} C) \quad \text{all } p \geqq p_2. \end{aligned} \tag{2}$$ Q.E.D.

Kawada's Theorem is an easy consequence of 5.2.

COROLLARY 5.3. *If C is open relatively compact set (or if C has some power with interior), and if $\delta > 0$ is given, then there exists a $p(\delta)$ such that*

$$p(\operatorname{conv} C) \supset C^p \supset p(\operatorname{conv}(C_\delta))$$

for all $p \geqq p(\delta)$.

Note. If C^p has interior for some p, $\operatorname{int}(\operatorname{conv} C)$ is dense in $\operatorname{conv} C$ since $p(\operatorname{conv} C) = (\operatorname{conv} C)^p \supset C^p$, (see [2, p. 11]).

Proof. For all large p (say $p \geqq p_1$) we have t such that

$$p(\operatorname{conv} C) \supset C^p \supset t + (p-p_1)\operatorname{conv} C;$$

on scaling by $1/p$ we have

$$\operatorname{conv} C \supset \frac{1}{p} C^p \supset \frac{1}{p} t + \left(1 - \frac{p_1}{p}\right) \operatorname{conv} C.$$

Now $\operatorname{conv}(\overline{C}_{\delta/2})$ is a compact subset of $\operatorname{int}(\operatorname{conv} C)$ so there is a $p_2 \geqq p_1$ such that

$$(1-p_1/p)\operatorname{conv} C \supset \operatorname{conv}(C_{\delta/2}) \quad \text{all } p \geqq p_2,$$

and there is a $p_3 \geqq p_2$ such that $\|(1/p)t\| < \delta/2$ for $p \geqq p_3$. Taking $p_0 = p_3$ we have

$$\operatorname{conv}(C_\delta) - \frac{1}{p} t \subset \operatorname{conv}(C_{\delta/2}) \subset \frac{1}{p}(p-p_1)\operatorname{conv} C \quad \text{all } p \geqq p_0$$

so that $p \geqq p_0 \Rightarrow$

$$\operatorname{conv}(C_\delta) \subset \frac{1}{p} t + \frac{1}{p}(p-p_1)\operatorname{conv} C \subset \frac{1}{p} C^p. \qquad \text{Q.E.D.}$$

We now derive the measure theoretic estimates in Theorems 1.1 and 1.2. However, note that if $C\subset \boldsymbol{R}^N$ is relatively compact, the C^p may fail to be Borel sets (or even μ-measurable where μ is Haar measure); we systematically circumvent this difficulty in our measure theoretic estimates by considering, for $X\subset \boldsymbol{R}^N$,

$$|X|^+ = \inf\{|U| : U \supset X,\ U \text{ open}\},$$
$$|X|^- = \sup\{|U| : U \subset X,\ U \text{ open}\}.$$

Here $|X|^+$ is the usual outer measure, so $|X|^+=|X|$ if X is μ-measurable; $|X|^-$ is not a measure, but $|X|^-=|X|$ if X is *open*. Note that $|X|^+\geqq|X|^-$.

THEOREM 5.4. *Let $C\subset \boldsymbol{R}^N$ be relatively compact and open* [*having some power with interior*]. *Then*

(i) $\dfrac{|C^{p+1}|}{|C^p|} = 1+O(1/p)\qquad\left[\dfrac{|C^{p+1}|^+}{|C^p|^-} = 1+O(1/p)\right]$,

(ii) $\dfrac{|p \operatorname{conv} C|}{|C^p|} = 1+O(1/p)\qquad\left[\dfrac{|p \operatorname{conv} C|}{|C^p|^\pm} = 1+O(1/p)\right]$,

(iii) $|C^p| = p^N|\operatorname{conv} C|+O(p^{N-1})\qquad[|C^p|^\pm = p^N|\operatorname{conv} C|+O(p^{N-1})]$.

Proof. We have $C^p\subset(\operatorname{conv} C)^p=p(\operatorname{conv} C)$ all p, so if some C^p has interior, so must conv C; thus conv C is at least μ-measurable (if not a Borel set) with $|\operatorname{int}(\operatorname{conv} C)|=|\operatorname{conv} C|=|(\operatorname{conv} C)|^-$ by standard results on convex sets with interior (see [2, Theorem 3, ff.]). Let p_0 and $t\in \boldsymbol{R}^N$ be as in 5.2. Then we have for $p\geqq p_0$:

$$0 \leqq \frac{|C^{p+1}|^+}{|C^p|^-}-1 \leqq \frac{|(p+1)\operatorname{conv} C|}{|(p-p_0)\operatorname{conv} C|}-1$$
$$= \left(\frac{p+1}{p-p_0}\right)^N-1 = O(1/p).$$

Furthermore, since $p \operatorname{conv} C\supset C^p \Rightarrow |p \operatorname{conv} C|\geqq|C^p|^+$, we have (for $p\geqq p_0$):

$$1 \geqq \frac{|C^p|^+}{|p\operatorname{conv} C|} \geqq \frac{|C^p|^-}{|p\operatorname{conv} C|} = \frac{|C^p|^-}{|t+(p-p_0)\operatorname{conv} C|}\cdot\frac{|(p-p_0)\operatorname{conv} C|}{|p\operatorname{conv} C|}$$
$$\geqq \frac{|(p-p_0)\operatorname{conv} C|}{|p\operatorname{conv} C|} = \left(\frac{p-p_0}{p}\right)^N = 1+O(1/p).$$

Finally, we note (iii) follows from (ii) and $|p \operatorname{conv} C|=|\operatorname{conv} C|p^N$. Q.E.D.

6. **Behavior of powers in compact groups and certain extensions.** In this section we consider compact groups K (not necessarily abelian). We assume Haar measure normalized so $|G|=1$.

THEOREM 6.1. *Let K be any compact group and $C\subset K$ a subset such that C^p has interior for large p. Then there exist open subgroup $K_0\subset K$, an element $k_0\in K$, and an integer p_0, such that*

$$C^p = (k_0)^{p-p_0}K_0$$

all $p \geqq p_0$. *Thus we have* $|C^p| = |K_0| = 1/m$ *for all large* p, *where* m *is the index of* K_0 *in* K, *and* $|C^p| \sim \text{const.}$ *as* $p \to \infty$.

Proof. We need a lemma which may have independent interest.

Lemma 6.2. *Let* G *be any locally compact group,* $U \subset G$ *a subset satisfying*:
(i) $U^2 \subset U$ *and*
(ii) $0 < |\text{int}(U)| < \infty$. *Then* U *is a compact open subgroup of* G.

Note. Condition (ii) is necessary; otherwise consider $U = (0, +\infty) \subset \boldsymbol{R}$.

Proof (6.2). We first show U has compact closure: by (ii) there is an open set $V \subset U$ and we may take V to have compact closure. If U does not have compact closure we cannot cover U by finitely many left translates of VV^{-1}; therefore we may inductively define a sequence of points $\{x_i\} \subset U$ such that $x_j \notin \bigcup_{i<j} x_i VV^{-1}$ for $j = 1, 2, \ldots$. But this readily implies that the sets $\{x_i V\}$ are disjoint, and since they are open and included in U (by (i)) we contradict (ii).

Furthermore some power of the set V must include the unit $e \in G$, for if $x \in V$ consider the neighborhood $V^{-1}x$ of the unit. Since $\{x^n : n = 1, 2, \ldots\} \subset U$ it must have a limit point $y \in \bar{U}$ and it follows easily that e is a limit point for $\{x^n\}$ (if $x^{n_i} \to y$ look at $x^{n_i - n_j}$ for $i > j$), therefore some power x^r ($r \geqq 2$) of x belongs to the neighborhood $V^{-1}x$ of e, which $\Rightarrow$ $x^r = z^{-1}x$ for some $z \in V$, which $\Rightarrow$ $e = zx^r x^{-1} = zx^{r-1} \in VV^{r-1} = V^r$. It now follows that U is open since $UV^r \subset U^{r+1} \subset U$, thus a neighborhood of each point lies within U. If w is any point in U then $w^m V^r \subset w^m U \subset wU$ for all $m = 1, 2, \ldots$, and since e is a limit point for $\{w^m : m = 1, 2, \ldots\}$ we frequently have $e \in w^m V^r$, which $\Rightarrow$ $e \in wU \Rightarrow w^{-1} \in U$ and U is inversion invariant. Hence U is an open subgroup in G. Q.E.D.

Corollary 6.3. *If* K *is a compact group and* $C \subset K$ *a subset such that* C^p *has interior for large* p, *then* $K_0 = \bigcup_{p=1}^{\infty} C^p$ *is an open subgroup of* K (*and therefore it is the subgroup of* K *generated by* C).

Proof. $U = K_0$ satisfies (i), (ii) of 6.2. Q.E.D.

Corollary 6.4. *If* K, C *are as above we have* $K_0 = \bigcup_{p=1}^{p_0} C^p$ *for some* $p_0 \geqq 1$; *furthermore there is some integer* $r \geqq 1$ *such that* $e \in \text{int}(C^{mr})$ *all* $m = 1, 2, \ldots$.

Proof. It is not true that e is *eventually* in the sets C^p, rather than recurrently as we assert here. Since some C^p has interior, some power of C^p is a neighborhood of the unit (as in proof of 6.2), say $e \in \text{int}(C^r)$. Thus for any $p = 1, 2, \ldots$ we have $C^p \subset \text{int}(C^p C^r)$, which $\Rightarrow$ $K_0 = \bigcup_{p=1}^{\infty} C^p \subset \bigcup_{p=1}^{\infty} \text{int}(C^p)$. As K_0 is compact, $K_0 = \bigcup_{p=1}^{p_0} \text{int}(C^p) = \bigcup_{p=1}^{p_0} C^p$ for some p_0. Q.E.D

To prove 6.1, take r so $e \in \text{int}(C^r)$; then $C^{(p+1)r} \supset \text{int}(C^{(p+1)r}) \supset C^{pr}(\text{int}\, C^r) \supset C^{pr}$. This implies that $K_0 = \bigcup_{p=1}^{\infty} C^{pr}$ satisfies the hypotheses of 6.2, and hence is an open/closed subgroup of K; as $\text{int}(C^{pr}) \nearrow K_0$ we obviously get $C^{pr} = K_0$ for all large p (say $p \geqq p_0$). The sequence of *open* sets $\{C^p : p \geqq rp_0\}$ may fail to be increasing by inclusion, but clearly we have $|C^{p+1}| \geqq |C^p|$, thus we get $|C^p| = |K_0|$ for all

$p \geqq p_0 r$. Take $p_1 = p_0 r$; then $C^{p_1} = K_0$ and $C^{p+1} = CC^{p_1} = CK_0$ consists of precisely one left coset of K_0, say $k_0 K_0$, for some $k_0 \in C$ [in fact for *any* $k_0 \in C$]. In general $C^{p_1+n+1} = CC^{p_1}C^n = k_0 C^{p_1+n} = (k_0)^{n+1} K_0$ for $n \geqq 0$. Q.E.D.

Now we consider groups H of the form $H = K \times G$ (direct product) where K is any compact group, G is any strongly amenable locally compact group (commutativity not required). We denote the projection homomorphism of H onto K, G by φ, π respectively. If $A \subset G$ is any subset, measurable or not, we define $|A|^+ = \inf\{|U| : U \supset A,\ U \text{ open}\}$ and $|A|^- = \sup\{|U| : U \subset A,\ U \text{ open}\}$, as we did in discussing 5.4. If $C \subset H$, we are interested in the cross-sections over points $y \in G : \varphi(\pi^{-1}(y) \cap C)$; we write $\mathscr{P} = \mathscr{P}(C)$ for the family of subsets of K which are cross sections of some power C^p $(p = 1, 2, \ldots)$ over some point in G. It is essential to notice that

$$(3) \qquad Q_1, Q_2 \in \mathscr{P} \Rightarrow \text{there is some } Q_3 \in \mathscr{P} \text{ such that } Q_3 \supset Q_1 Q_2.$$

In fact if $Q_i = \varphi(\pi^{-1}(y_i) \cap C^{m_i})$, then we obviously have

$$\varphi(\pi^{-1}(y_1 y_2) \cap C^{m_1+m_2}) \supset \varphi(\pi^{-1}(y_1) \cap C^{m_1}) \cdot \varphi(\pi^{-1}(y_2) \cap C^{m_2}).$$

THEOREM 6.5. *Let $H = K \times G$ as above. Let C be a relatively compact set in H such that some power has interior. Let $\gamma = \sup\{|\varphi(\pi^{-1}(y) \cap C^p)|^+ : p \geqq 1, y \in G\}$ $= \sup\{|Q|^+ : Q \in \mathscr{P}\}$. Then there exists an open/closed subgroup $K_0 \subset K$ and an integer $p_0 \geqq 1$ such that $\gamma = |K_0|$ and*

$$(4) \qquad \gamma|\pi(C^{p-p_0})|^+ \leqq |C^p|^- \leqq |C^p|^+ \leqq \gamma|\pi(C^{p+p_0})|^+$$

for all $p \geqq p_0$; K_0 is normal in the open/closed subgroup of K generated by $\varphi(C)$.

COROLLARY 6.6. *If $H = K \times G$ where K is compact and G is locally compact, then G strongly amenable $\Rightarrow$ H strongly amenable.*

Proof (6.6). From (4) we see that, for open relatively compact set C (so powers are measurable),

$$1 \leqq \frac{|C^{p+1}|}{|C^p|} \leqq \frac{|C^{p+1}|^+}{|C^p|^-} \leqq \frac{|\pi(C^{p+p_0+1})|^+}{|\pi(C^{p-p_0})|^+}$$
$$\leqq \prod_{i=0}^{2p_0} \frac{|\pi(C^{p-p_0+i+1})|^+}{|\pi(C^{p-p_0+i})|^-} \quad \text{all } p \geqq p_0.$$

Thus

$$1 \leqq \limsup \left\{\frac{|C^{p+1}|}{|C^p|}\right\} \leqq \left[\limsup \left\{\frac{|\pi(C^{p+1})|^+}{|\pi(C^p)|^-}\right\}\right]^{2p_0+1} = 1$$

since projection π is open and preserves relative compactness. Q.E.D.

Proof of 6.5. If C^{m_0} has interior, some cross-section $Q_0 \in \mathscr{P}$ has interior in K. But (3) shows there are sets $Q_p \in \mathscr{P}$ such that $Q_p \supset (Q_0)^p$ for $p = 1, 2, \ldots$, hence we may apply Theorem 6.1 (taking $C = Q_0$) to see there is an element of $\mathscr{P}$ which includes an open/closed subgroup of K. Let Q_1 be an element of $\mathscr{P}$ which includes an open/closed subgroup $K_0 \subset Q_1$ having maximum possible measure (measures of open/closed subgroups in K can only take values $\{1/m : m=1, 2, \ldots\}$, so the maximum value is achieved). We assert that

(i) Q_1 coincides with the subgroup K_0,

(ii) $|K_0| = \gamma$ (hence γ is achieved for some *measurable* element of $\mathscr{P}$).

In fact, the powers $Q_1 \subset Q_1^2 \cdots$ increase to fill an open/closed subgroup of K (by 6.4) in finitely many steps; thus all large powers Q_1^p (say $p \geqq m_1$) coincide with this open/closed subgroup. But $Q_1^{m_1} \supset Q_1 \supset K_0$ and $Q_1^{m_1}$ lies within some element $Q_2 \in \mathscr{P}$ by (3). This violates maximality of $|K_0|$ if $Q_1 \neq K_0$, and (i) is proved. Further, if $Q' \in \mathscr{P}$ has $|Q'|^+ > |K_0|$ and $Q'' \in \mathscr{P}$ is any element with interior, then $Q'Q''$ has interior; some $(Q'Q'')^m$ includes a neighborhood of the unit (6.4) and is included in some $Q \in \mathscr{P}$, and we have $|Q|^+ \geqq |(Q'Q'')^m| \geqq |Q'Q''|^+ \geqq |Q'|^+ > |K_0|$ due to left invariance of outer measure. But $Q \subset Q^2 \subset \cdots$ increases to fill an open/closed subgroup in finitely many steps (6.4), and $|Q^p|^+ \geqq |Q|^+ > |K_0|$ contradicts the maximality of $|K_0|$ since each Q^p lies in some element of $\mathscr{P}$; thus $|K_0| \geqq |Q'|^+$ all $Q' \in \mathscr{P}$, proving (ii). Choose integer r_0 and $y_0 \in \pi(C^{r_0})$ so that

$$K_0 = \varphi(\pi^{-1}(y_0) \cap C^{r_0});$$

i.e. $K_0 \times \{y_0\} \subset C^{r_0}$. We have noted that some power C^{m_0} has interior, say $C^{m_0} \supset U \times Y$ where U, Y are open in K, G respectively. Take $p_0 = m_0 + r_0$. Then $p \geqq p_0 \Rightarrow$

$$C^p = C^{r_0} C^{p-p_0} C^{m_0} \supset (K_0 \times \{y_0\}) C^{p-p_0} (U \times Y).$$

The set on the right is open and clearly has measure at least as large as

$$|K_0| \, |\pi(C^{p-p_0}) Y| \geqq \gamma |\pi(C^{p-p_0})|^+;$$

therefore, $\gamma |\pi(C^{p-p_0})|^+ \leqq |C^p|^-$ for $p \geqq p_0$. On the other hand, for any $p \geqq 1$ let $u \in U$ and $y \in Y$; then $C^p \subset (u^{-1}U \times y^{-1}Y)C^p$, an open set, and we have $|C^p|^+ \leqq |(u^{-1}U \times y^{-1}Y)C^p| = |(U \times Y)C^p|$. But we have $\pi((U \times Y)C^p) \subset \pi(C^{m_0+p})$ and for each $y \in (U \times Y)C^p$ the cross-section above y (an open set in K) can have measure at most γ. Thus Fubini's theorem gives

$$|C^p|^+ \leqq \gamma |\pi(C^{p+m_0})|^+ \leqq \gamma |\pi(C^{p+p_0})|^+$$

and our measure theoretic estimates are proved.

For the normality of K_0 we may assume $\varphi(C)$ generates K, so $K = \bigcup_{p=1}^{\infty} \varphi(C)^p$ by 6.3. Let $k \in K$, then $k \in \varphi(C)^p = \varphi(C^p)$ for some p, which $\Rightarrow$ k belongs to some cross-section of C^p. Now $K_0 \in \mathscr{P}$, so (3) $\Rightarrow kK_0 \subset P'$ for some $P' \in \mathscr{P}$; likewise there is a $P'' \in \mathscr{P}$ with $P'' \supset k^{-1}K_0kK_0 \supset K_0$. But the middle set is a union of left cosets of K_0 and if it includes more than one coset we violate $|K_0| = \gamma$; thus we have $k^{-1}K_0k \subset k^{-1}K_0kK_0 = K_0$, all $k \in K$, and K_0 is normal. Q.E.D.

In the special case $G = \boldsymbol{R}^N$ these considerations extend to give a clear geometric description of the C^p.

THEOREM 6.7. *Let $H=K\times V$, where K is compact and $V=\boldsymbol{R}^N$, a vector group. Let C be a relatively compact open subset in H. Let $E=\pi(C)$ and let $\delta>0$ be given. Then there exists an integer n_0, an open/closed subgroup $K_0\subset K$, and a point $k_0\in K$ such that*

$$\phi(\pi^{-1}(y)\cap C^p) = (k_0)^{p-n_0}K_0 \quad \text{all } y\in p \operatorname{conv}(E_\delta),$$

for $p\geqq n_0$.

Proof. We have $\pi(C^p)=(\pi(C))^p=E^p$ and have seen (5.3) that $p(\operatorname{conv} E)\supset E^p \supset p(\operatorname{conv}(E_\delta))$ for large p, say $p\geqq p_1$. We continue the discussion in proof of (6.5), defining K_0 and $r_0, m_0, p_0=r_0+m_0$ as above. We have seen that $K_0\times\{y_0\}\subset C^{r_0}$, so $y_0\in E^{r_0}$; furthermore we have

$$y_0+E^{p-r_0}\supset p\operatorname{conv}(E_\delta) \tag{5}$$

for all large p, say $p\geqq p_3\geqq \max\{p_0, p_1\}$ [(5) holds $\Leftrightarrow$

$$\operatorname{conv}(E_\delta)-\frac{1}{p}y_0 \subset \frac{1}{p}E^{p-r_0} = \frac{p-r_0}{p}\left(\frac{1}{p-r_0}E^{p-r_0}\right);$$

but the latter includes $(p-r_0)/p\cdot\operatorname{conv}(E_{\delta/2})$ for large p].

Fix $p\geqq p_3$ and let y vary within $p\operatorname{conv}(E_\delta)$. We first show that the cross-section of C^p over each y consists of precisely one coset of the subgroup K_0; then we will show this coset remains constant as y runs through $p\operatorname{conv}(E_\delta)$. As $y\in y_0+E^{p-r_0}$ we may write $y=y_0+y_1$, so

$$\begin{aligned}\varphi(\pi^{-1}(y)\cap C^p) \supset \varphi(\pi^{-1}(y_1)\cap C^{p-r_0})&\varphi(\pi^{-1}(y_0)\cap C^{r_0})\\ &= \varphi(\pi^{-1}(y_1)\cap C^{p-r_0})K_0.\end{aligned}$$

The right side is nonempty, hence includes at least one left coset of K_0; but if the left side meets more than one left coset of K_0, we violate $|K_0|=\gamma$ for the cross-section $\varphi(\pi^{-1}(y+y_0)\cap C^{p+r_0})\supset\varphi(\pi^{-1}(y)\cap C^p)\cdot K_0$, so each cross-section is a single left coset. The cross-sections of C^p vary "lower semicontinuously" over E^p, i.e., if $\varphi(\pi^{-1}(y)\cap C^p)$ includes some coset kK_0, then so do the cross-sections $\varphi(\pi^{-1}(y')\cap C^p)$ for y' near y in E^p. This follows easily since C^p is open and the coset kK_0 compact. Therefore, if $p\geqq p_3$ the cross-sections of C^p must remain constant on the *connected* set $p\operatorname{conv}(E_\delta)\subset E^p$.

We may assume $\varphi(C)$ generates K, so K_0 is normal in K. Write X_p for the coset of K_0 giving the cross-section of C^p over $p\operatorname{conv}(E_\delta)$. If $m, n\geqq p_3$ we have

$$X_{m+n} = X_m X_n. \tag{6}$$

In fact if $y\in(m+n)\operatorname{conv}(E_\delta)=m\operatorname{conv}E_\delta+n\operatorname{conv}E_\delta$ we can split $y=y'+y''$, and we have $X_{m+n}\supset\varphi(\pi^{-1}(y)\cap C^{m+n})\supset\varphi(\pi^{-1}(y')\cap C^m)\cdot\varphi(\pi^{-1}(y'')\cap C^n)=X_mX_n$; but the left side is a single coset of K_0, so we have equality. As K/K_0 is finite there is a $p_4=mp_3$ $(m=1,2,\dots)$ such that $X_{p_4}=(X_{p_3})\cdots(X_{p_3})=(kK_0)^m=K_0$. Write

$X_{p_4+1}=k_0K_0$ where $k_0 \in K$; then we use induction to show $X_{2p_4+m}=(k_0)^mK_0$ for all $m \geqq 0$, proving our theorem if we set $n_0=2p_4$. In fact (6) shows: $X_{2p_4}=K_0^2=K_0$ ($m=0$) and $X_{2p_4+1}=X_{p_4+1}X_{p_4}=k_0K_0$ ($m=1$), and by induction $X_{2p_4+m+1}=X_{p_4+1}X_{p_4+m}=k_0K_0X_{p_4+m}=k_0X_{p_4}X_{p_4+m}=k_0X_{2p_4+m}=(k_0)^{m+1}K_0$. Q.E.D.

The analog of the measure theoretic estimate in Theorem 1.1 can be stated for general locally compact groups G as follows:

(7) For every relatively compact set $C \subset G$ such that some power has interior, there is an integer $k \geqq 0$ and a constant A such that $|C^p|^{\pm}=Ap^k+O(p^{k-1}\log p)$. We have seen this for $G \cong \boldsymbol{R}^N$ (with $k=N$ depending only on G) in 5.4; we will eventually see that every abelian G has this property, and in most cases (e.g., if G is compactly generated) the parameter k depends only on G. For the present we note:

Corollary 6.8. *Let $H=K\times G$ with K compact and G a locally compact group satisfying* (7). *Let $C \subset H$ be relatively compact such that some power of C has interior. Then*

$$|C^p|^{\pm} = \gamma Ap^k+O(p^{k-1})$$

where γ is defined as in 6.5 *and A, k are the constants associated with the set $E=\pi(C)\subset G$ in* (7). *If k depends only on G in* (7), *the same is true for H.*

Proof. Here $E=\pi(C)$ is relatively compact and $E^p=\pi(C^p)$ has interior for large p. Thus $|E^p|^{\pm}=Ap^k+O(p^{k-1})$. Taking γ, p_0 as in 6.5 we have $p \geqq p_0 \Rightarrow$

$$\gamma|E^{p+p_0}|^{+} \geqq |C^p|^{+} \geqq |C^p|^{-} \geqq \gamma|E^{p-p_0}|^{-}.$$

Now apply (7) to E^p and note that $(p-p_0)^k=p^k+O(p^{k-1})$ as $p \to \infty$. Q.E.D.

7. **Behavior of powers in discrete groups and groups $\boldsymbol{R}^m \times \boldsymbol{Z}^n$.** If H is any locally compact abelian group and C a relatively compact subset such that C^p has interior for large p, then the group $H' \subset H$ generated by C is open/closed and is compactly generated; thus the study of powers C^p in arbitrary H immediately reduces to the corresponding problem for compactly generated H. But groups of the form $G=\boldsymbol{R}^m \times \boldsymbol{Z}^n$ are the fundamental building blocks of such groups—in fact we can always write $H=K\times \boldsymbol{R}^m \times \boldsymbol{Z}^n$, with K compact, for any such H. The considerations of §6 reduce the study of powers C^p in H to those in $\boldsymbol{R}^m \times \boldsymbol{Z}^n$, which we take up here.

The case of discrete torsion free groups ($m=0$) is essentially a problem concerning the geometric and simple analytic properties of lattices (discrete subgroups) in Euclidean space, while the case $m>0$ requires a rather different (and complicated) discussion of convex sets in Euclidean spaces, linear families of convex sets, etc. Thus we single out the case of discrete finitely generated torsion free G ($G \cong \boldsymbol{Z}^n$) for our starting point. We need a number of special results on convex sets in Euclidean spaces which do not seem to be set forth in the literature; these are discussed separately in an Appendix.

THEOREM 7.1. *Let G be finitely generated torsion free abelian ($G \cong \mathbf{Z}^n$ for some $n \geqq 0$) and let $C=\{g_1, \ldots, g_{N+1}\}$ be a finite subset. Let k be the rank of the subgroup in G generated by the differences $\{g_1-g_{N+1}, \ldots, g_N-g_{N+1}\}$, so $0 \leqq k \leqq N$. Then there is a constant A such that $|C^p|=Ap^k+O(p^{k-1})$ as $p \to \infty$.*

REMARK. See footnote (1). It is also helpful to notice that, for any $g \in G$, we have $|(C+g)^p|=|C^p+g^p|=|C^p|$ for $p=1, 2, \ldots$; thus we may as well assume that the unit 0 is in C, say $g_{N+1}=0$. Once this normalization is made, k is rank of the subgroup generated by $\{g_1, \ldots, g_N\}$. The case $k=0$ is trivial, for if our normalization is made it implies that $g_1=\cdots=g_{N+1}=0$, since G is torsion free, and we have $|C^p|=1$ all p. Thus we assume $1 \leqq k \leqq N$.

Proof. In G, C^p is the set of all elements of the form $\sum_{i=1}^{N+1} \alpha_i g_i$ for $(\alpha_i) \in \mathbf{Z}^{N+1}$ with $\alpha_i \geqq 0$, $\sum_{i=1}^{N+1} \alpha_i=p$; due to our normalization $g_{N+1}=0$, this set of elements is precisely $\{\sum_{i=1}^N \beta_i g : (\beta_i) \in \mathbf{Z}^N, \beta_i \geqq 0, \sum_{i=1}^N \beta_i \leqq p\}$ and $|C^p|$ is the number of distinct elements in this set. If we consider the N-dimensional simplex

$$S = \left\{(\lambda_i) \in \mathbf{R}^N : \lambda_i \geqq 0, \sum_i \lambda_i \leqq 1\right\},$$

then C^p is precisely $\{\sum_{i=1}^N \alpha_i g_i : (\alpha_i) \in pS \cap \mathbf{Z}^N\}$ and 7.1 is a special case of the following theorem, taking $C=\{g_1, \ldots, g_N\}$.

THEOREM 7.2. *Let G be a torsion free abelian group. Let $C=\{g_1, \ldots, g_N\}$ be any finite subset and let k be the rank $(0 \leqq k \leqq N)$ of the subgroup generated by the elements $\{g_i\}$. Let S be a compact convex set in $\mathbf{R}^N$ with interior and let*

$$\lambda S = \{\lambda x : x \in S\}$$

for all $\lambda > 0$. Finally, let $N(p)=N(S,p)$ be the number of distinct elements in $\{\sum_{i=1}^N \alpha_i g_i : (\alpha_i) \in pS \cap \mathbf{Z}^N\}$. Then

$$N(p) = Ap^k+O(p^{k-1}) \quad \text{as } k \to \infty,$$

where A is a constant depending on S, G which will be evaluated explicitly in the course of our proof.

Proof. One must develop several lemmas on lattices in $\mathbf{R}^N$ ($N \geqq 1$).

LEMMA 7.3. *Let S be a compact convex set with interior in $\mathbf{R}^N$ and let Γ be any lattice of rank N in $\mathbf{R}^N$ with $|\Gamma|$ the measure of a fixed fundamental region Σ for Γ (an invariant of Γ). Let $\eta_0 > 0$ be given. Then there is an integer p_0 and a constant B (independent of $0 \leqq \eta \leqq \eta_0$) such that:*

$$\left| |\Gamma \cap (pS)_\eta| - \frac{1}{|\Gamma|} |S| p^N \right| \leqq Bp^{N-1}$$

all $p \geqq p_0$, all $0 \leqq \eta \leqq \eta_0$.

Proof. Let $N^+(p)$ be the number of Γ-translates of Σ which meet pS and $N^-(p)$

the number of Γ-translates of Σ lying within $(pS)_{\eta_0}$. If $d_0 = \operatorname{diam}(\Sigma)$, we have $N^-(p) \leqq |\Gamma \cap (pS)_\eta| \leqq N^+(p)$ for $0 \leqq \eta < \eta_0$, all p, and clearly:

$$N^+(p) = \frac{1}{|\Gamma|} |\bigcup \{\gamma+\Sigma : (\gamma+\Sigma) \cap pS \neq \varnothing\}| \leqq \frac{1}{|\Gamma|} |\{x : \operatorname{dist}(x, pS) \leqq d_0\}|$$

$$\leqq \frac{1}{|\Gamma|} |\{x : \operatorname{dist}(x, pS) \leqq \eta_0 + d_0\}|,$$

$$N^-(p) = \frac{1}{|\Gamma|} |\bigcup \{\gamma+\Sigma : \gamma+\Sigma \subset (pS)_{\eta_0}\}|$$

$$\geqq \frac{1}{|\Gamma|} |\{x \in pS : \operatorname{dist}(x, \operatorname{bdry}(pS)) \geqq \eta_0 + d_0\}|$$

all p. Notice that $(1/|\Gamma|)|pS| = (1/|\Gamma|)|S|p^N$ lies between the right-hand bounds displayed here, as does $|\Gamma \cap (pS)_\eta|$, so for all $\eta < \eta_0$, all p, we have

$$\left| |\Gamma \cap (pS)_\eta| - \frac{1}{|\Gamma|} |S| p^N \right| \leqq \frac{1}{|\Gamma|} |\{x : \operatorname{dist}(x, \operatorname{bdry}(pS)) \leqq \eta_0 + d_0\}|.$$

It is not hard to see (we give a proof in A.2(ii) of the Appendix) that for all large p, say $p \geqq p_0$, the right side is dominated by Bp^{N-1}. Q.E.D.

Lemma 7.4. *Let* Λ *be a lattice in* $\boldsymbol{R}^N$ *of rank* N *and let* $\Lambda_0 \subset \Lambda$ *be any sublattice, say of rank* l. *Let* P *be the* $k = N - l$ *dimensional subspace in* $\boldsymbol{R}^N$ *consisting of all vectors in* $\boldsymbol{R}^N$ *orthogonal to the subset* Λ_0, *and let* Γ *be the orthogonal projection of* Λ *into* P. *Then* Γ *is a discrete subgroup* (*a lattice*) *in* $P \cong \boldsymbol{R}^k$ *of rank* k.

Proof. The rank of Λ is just the number of elements in a $\boldsymbol{Z}$-basis (Λ torsion free, finitely generated). We recall that no *discrete* subgroup $\Lambda \subset \boldsymbol{R}^N$ can have rank $(\Lambda) > N$. From this it is clear that rank $(\Lambda) =$ dimension of the vector space spanned by Λ. Let M be the space spanned by Λ_0, so $\dim M = l$; then $\Lambda \cap M$ is a discrete lattice in M so that $l \geqq \operatorname{rank}(\Lambda \cap M) \geqq \operatorname{rank}(\Lambda_0) = l$ and there is a $\boldsymbol{Z}$-basis $\{\lambda_1, \ldots, \lambda_l\} \subset \Lambda \cap M$ for $\Lambda \cap M \supset \Lambda_0$. By standard theorems on free $\boldsymbol{Z}$-modules, there are $k = N - l$ further elements $\{\lambda'_1, \ldots, \lambda'_n\}$, such that $\{\lambda_i\} \cup \{\lambda'_j\}$ form a $\boldsymbol{Z}$-basis for Λ. It is clear that Γ is a lattice of rank k in P which has for a $\boldsymbol{Z}$-basis the projected images of $\lambda'_1, \ldots, \lambda'_k$ in P. Q.E.D.

The following technical lemma is central to our asymptotic estimates for discrete groups.

Lemma 7.5. *Let* S *be a compact convex set with interior in* $\boldsymbol{R}^N$. *Let* Λ *be a lattice of rank* N *in* $\boldsymbol{R}^N$ *and* $\Lambda_0 \subset \Lambda$ *a sublattice of rank* $0 \leqq l \leqq N$. *Write* P *for the* $k = (N-l)$ *dimensional subspace orthogonal to* Λ_0, *and let* $\pi : \boldsymbol{R}^N \to P$ *be the orthogonal projection. Denote* $\Gamma = \pi(\Lambda)$ *and* $D = \pi(S)$. *We define a function* diam*: $D \to [0, +\infty)$ *by the formula*

$$\operatorname{diam}^*(x) = \sup\{r \geqq 0 : U(y, r) = \{z \in \boldsymbol{R}^N : \|y - z\| \leqq r\} \subset S \text{ for some } y \in \pi^{-1}(x)\},$$

so diam* (x) *is the radius of the largest closed ball contained in* S *which has center in the fiber* $\pi^{-1}(x)$. *Then*

(i) diam* *is continuous on* int (D).

Let r_{in} *be the insphere radius of* S (*see* [2, p. 76]).

(ii) *The constant* $\mu = \text{diam } S/r_{\text{in}}$ *is such that, for any* δ *with* $0<\delta<r_{\text{in}}$ *we have*

$$\text{diam}^*(x) > (1/\mu)\delta \quad \textit{for all } x \in D_\delta = \{x \in D : \text{dist}(x, \text{bdry } D) > \delta\}.$$

Proof. For (i), it is readily seen that $\pi(\text{int } S) = \text{int}(D)$ due to the convexity of S. Let $x_0 \in \text{int}(D)$, then $d_0 = \text{diam}^*(x_0) > 0$ and let $U(y_0, d_0)$, with $y_0 \in \pi^{-1}(x_0)$, be a closed ball of radius d_0 lying within S. If x_1 is any point of P such that $\|x_0 - x_1\| < d_0$, it is clear that $x_1 \in \pi(\text{int } S) = \text{int}(D)$ and $\text{diam}^*(x_1) \geqq d_0 - \|x_0 - x_1\|$ (consider the point $y_1 \in \pi^{-1}(x_1)$ closest to y_0). Therefore we see:

$$\|x_0 - x_1\| \geqq \text{diam}^*(x_0) - \text{diam}^*(x_1),$$

if $\|x_0 - x_1\| < \text{diam}^*(x_0)$, $x_0 \in \text{int}(D)$; but by symmetry:

$$\|x_1 - x_0\| \geqq \text{diam}^*(x_1) - \text{diam}^*(x_0),$$

if $\|x_1 - x_0\| < \text{diam}^*(x_1)$, $x_1 \in \text{int}(D)$. Hence if x_0 is fixed, we see that for all $x_1 \in \text{int}(D)$ with $\|x_1 - x_0\| < (\text{diam}^*(x_0)/2)$:

$$|\text{diam}^*(x_1) - \text{diam}^*(x_0)| \leqq \|x_1 - x_0\|,$$

which proves the continuity of diam*.

For (ii), take any point $y_0 \in \text{int}(S)$ such that the closed ball $U(y_0, r_{\text{in}})$ lies within S. For all δ with $0 < \delta < r_{\text{in}}$, we obviously have (by definition of r_{in}) $x_0 = \pi(y_0) \in D_\delta = \{x \in P : x \in D, \text{dist}(x, \text{bdry}(D)) > \delta\}$. Consider any $0 < \delta < r_{\text{in}}$ and any point $x \neq x_0$ in D_δ (if $x = x_0$ then $\text{diam}^*(x_0) \geqq r_{\text{in}} > \delta \geqq \delta \cdot (r_{\text{in}}/\text{diam } S) = (\delta/\mu)$, so this is a trivial case). Let x' be the point where the ray from x_0 to x meets bdry (D); let y' be any point in $\pi^{-1}(x') \cap S$ and y the (unique) point in the segment $[y_0, y']$ with $\pi(y) = x$. The distance $\|y' - y\| \geqq \|x' - x\| > \delta$ since $x \in D_\delta$ implies

$$\text{dist}(x, \text{bdry}(D)) > \delta.$$

Consequently we have a convex representation

$$y = \alpha y_0 + (1-\alpha) y', \qquad \alpha = \frac{\|y'-y\|}{\|y'-y_0\|} \geqq \frac{\|x'-x\|}{\text{diam } S} > \frac{\delta}{\text{diam } S}.$$

But since S is convex we have $U(y, \alpha r_{\text{in}}) = \alpha U(y_0, r_{\text{in}}) + (1-\alpha)\{y'\} \subset S$, and since $\alpha r_{\text{in}} > r_{\text{in}}(\delta/\text{diam } S) = (\delta/\mu)$ our assertion is proved. Q.E.D.

We are now ready to prove 7.2. Consider lattice $\Lambda = \boldsymbol{Z}^N \subset \boldsymbol{R}^N$, let $C = \{g_1, \ldots, g_N\} \subset G$, and define the canonical homomorphism $\Phi: \boldsymbol{Z}^N \to G$

$$\Phi(\alpha_1, \ldots, \alpha_N) = \sum_{i=1}^{N} \alpha_i g_i. \tag{8}$$

The kernel $\Lambda_0 \subset \boldsymbol{Z}^N$ of this morphism is a sublattice, say with rank l $(0 \leqq l \leqq N)$. Since G is assumed to be torsion free one readily sees that the subspace M spanned by Λ_0 has $M \cap \boldsymbol{Z}^N = \Lambda_0$ (in general $M \cap \boldsymbol{Z}^N \supset \Lambda_0$); thus if $\{\lambda_1, \ldots, \lambda_l\} \subset \Lambda_0$ is a $\boldsymbol{Z}$-basis for Λ_0, so these vectors are an $\boldsymbol{R}$-basis for M, we may find additional vectors $\{\lambda_1', \ldots, \lambda_{N-l}'\} \subset \Lambda$ such that $\{\lambda_i\} \cup \{\lambda_j'\}$ is a $\boldsymbol{Z}$-basis for Λ. Therefore we see that the points $\{g_i^* = \Phi(\lambda_i') : i = 1, 2, \ldots, N-l\}$ are a $\boldsymbol{Z}$-basis for the group $G^* \subset G$ generated by $C = \{g_1, \ldots, g_N\}$, so that $N - l = \text{rank}\,(G^*)$, which we have denoted by k in our theorem, i.e. $k = N - l$.

If $k = 0$ then $l = N$, Ker $\Phi = \Lambda = \boldsymbol{Z}^N$, and we can only have $g_1 = \cdots = g_N = 0$ (unit in G); in this trivial case $N(p) = 1$ all p. Excluding this case we assume $1 \leqq k \leqq N$ hereafter. Note that if $k = N$ then Ker $\Phi = \{0\}$ (no relations exist among the $\{g_i\}$) and $N(p)$ is simply the cardinality of $|pS \cap \boldsymbol{Z}^N|$, which we have estimated in Lemma 7.3.

If $1 \leqq k \leqq N$ then clearly $N(p)$ is just the number of points in $pS \cap \boldsymbol{Z}^N$ which are incongruent (mod Λ_0)—i.e. the number of distinct cosets of Λ_0 which meet the set pS. However, this is just the number of points we obtain when we project $pS \cap \boldsymbol{Z}^N$ by π to the k-dimensional subspace P of vectors orthogonal to the set Λ_0 (this is because $M \cap \Lambda = \Lambda_0$). Let $\Gamma = \pi(\boldsymbol{Z}^N)$, which is a discrete subgroup (lattice) in P by Lemma 7.4, and let $D = \pi(S)$. Since Λ_0 spans the subspace M, there is some $d_0 > 0$ such that every point in M is within distance d_0 of some point in the lattice Λ_0; it follows immediately that, for any $x \in \Gamma \subset P$, every point in the coset $\pi^{-1}(x)$ of M lies within d_0 of some point in the lattice (a coset of Λ_0 in $\boldsymbol{Z}^N$) $\boldsymbol{Z}^N \cap \pi^{-1}(x)$.

Let $\mu = \text{diam}\,(S)/r_{\text{in}}(S)$ as in Lemma 7.5(ii), let $\eta = d_0\mu$, and for $p \geqq 1$ consider $(pD)_\eta = \{x \in pD : \text{dist}\,(x, \text{bdry}\,(pD)) > \eta\}$, which is readily identified as $p \cdot (D_{\eta/p})$. If $x \in \Gamma \cap (pD)_\eta$ then $\pi^{-1}(x)$ includes a coset of $\pi^{-1}(x) \cap \boldsymbol{Z}^N : \boldsymbol{Z}^N/\Lambda_0$; moreover Lemma 7.5(ii) applies (for all large p, say $p \geqq p_0$, we have $\eta = d_0$ diam $S/r_{\text{in}}(S) < r_{\text{in}}(pS) = p \cdot r_{\text{in}}(S)$, and may substitute $S \to pS$, $\delta \to \eta$ in 7.5(ii) for any $p \geqq p_0$) to show that $\pi^{-1}(x) \cap pS$ contains a closed ball centered on this fiber with radius at least $\eta/\mu = d_0$. Thus for $p \geqq p_0$ and $x \in \Gamma \cap (pD)_\eta$, $\pi^{-1}(x) \cap pS$ contains at least one point from $\pi^{-1}(x) \cap \boldsymbol{Z}^N$; this $\Rightarrow$ $\pi(pS \cap \boldsymbol{Z}^N) \supset (pD)_\eta \cap \Gamma$ for $p \geqq p_0$. But the inclusion $\pi(pS \cap \boldsymbol{Z}^N) \subset pD \cap \Gamma$ is trivial for all p. Now apply Lemma 7.3: for all large p, say $p \geqq p_1 \geqq p_0$, we have

$$\begin{aligned} \frac{1}{|\Gamma|}\,|D|p^k + Bp^{k-1} \geqq |pD \cap \Gamma| \geqq |\pi(pS \cap \boldsymbol{Z}^N)| &= N(p) \\ \geqq |(pD)_\eta \cap \Gamma| \geqq \frac{1}{|\Gamma|}\,&|D|p^k - Bp^{k-1}, \end{aligned}$$

and Theorem 7.2 is proved. Q.E.D.

Now we are ready to work out the asymptotic behavior of powers C^p in groups of the form $H = \boldsymbol{R}^m \times \boldsymbol{Z}^n$ where $m > 0$.

THEOREM 7.6. *Let $H = V \times G$ where V is a vector group of dimension $m > 0$ and G is a discrete torsion free abelian group. Let C be any open relatively compact set in H and let φ: $H \to G$ be the projection homomorphism, writing*

$$\varphi(C) = F = \{g_1, \ldots, g_{N+1}\}$$

(necessarily finite). Let k be the rank of the subgroup in G generated by

$$F' = \{g_1-g_{n+1}, \dots, g_N-g_{N+1}\}.$$

Then there exists a constant $A>0$ such that

$$|C^p| = Ap^{m+k}+O(p^{m+k-1}\log p) \quad \text{as } p\to\infty \tag{9}$$

(the constant A is constructively determined in the course of the proof).

Proof. Since $|(C+x)^p|=|C^p+x^p|=|C^p|$ we may normalize to assume C contains the unit of H, hence F includes the unit of G, say $g_{N+1}=0$ (note that the rank k is unchanged by translation). The case $m=0$ is precisely the discrete case just considered, in which we obtained an even better error estimate than in (9). Now C may be represented:

$$C = \bigcup\{E_i+g_i : 1 \leqq i \leqq N+1\}$$

where E_i are open, relatively compact in V—i.e. C is a finite union of open sets lying in distinct cosets of V in H.

We first prove the theorem assuming that the sets $\{E_i\}$ are *convex*; this case is easier to deal with as a consequence of the identity $E^p=pE$, valid for convex E. We shall then obtain the result for general $\{E_i\}$ via an approximation theorem analogous to Theorem 5.2.

Let $S=\{(\lambda_i)\in \boldsymbol{R}^N : \lambda_i\geqq 0, \sum_{i=1}^N \lambda_i\leqq 1\}$ be the unit simplex in $\boldsymbol{R}^N$. As in the discrete case, let Λ_0 be the kernel of the homomorphism $\Phi\colon \boldsymbol{Z}^N\to G$ carrying $(\alpha_1,\dots,\alpha_N)\to\alpha_1 g_1+\cdots+\alpha_N g_N$. Λ_0 is a maximal l dimensional lattice in $\boldsymbol{Z}^N$ for some $0\leqq l<N$; furthermore, the orthocomplement P of Λ_0 in $\boldsymbol{R}^N$ has dimension $N-l=k$ (see discussion after Lemma 7.5). Write π for the orthogonal projection of $\boldsymbol{R}^N$ to $P\cong\boldsymbol{R}^k$ and set $\Gamma=\pi(\boldsymbol{Z}^N)$, $D=\pi(S)$. For $\lambda=(\lambda_i)\in pS$, $p=1,2,\dots$ we define the affine combination

$$E(\lambda,p) = \lambda_1E_1+\cdots+\lambda_NE_N+\left(p-\sum_{i=1}^N \lambda_i\right)E_{N+1},$$

which is again an open convex set in V (see [2, Chapter 5]). The sets $E(\lambda,p)$ are uniformly bounded in V as λ ranges through pS since they all lie within pB where $B=\operatorname{conv}(E_1\cup\cdots\cup E_{N+1})$. For any $x\in P$ we define convex set (nonvoid only if $x\in pD=p\cdot\pi(S)=\pi(pS)$)

$$E(x,p) = \bigcup\{E(\lambda,p) : \lambda\in\pi^{-1}(x)\cap pS\}\subset V.$$

For brevity we set $E(x)=E(x,1)$. Next define the functions

$$\psi(x,p) = |E(x,p)|, \qquad \psi(x) = \psi(x,1).$$

We show in the Appendix that these functions (indexed on $p=1,2,\dots$) are continuous (see A.7).

We may now identify the constant of our theorem as

$$A = \frac{1}{|\Gamma|}\int_D \psi(\lambda)\,d\lambda$$

where $|\Gamma|$ is the measure of a fundamental region Σ for the lattice Γ (note: A is unaltered by a change of Haar measure in P). Since the $\{E_i\}$ are convex: $E_i^p = pE_i$ and we have

$$\begin{aligned} C^p &= \bigcup_{g\in F^p}\Big(g+\bigcup\Big\{p_1E_1+\cdots+p_NE_N+\Big(p-\sum_i p_i\Big)E_{N+1} : p_1g_1+\cdots+p_Ng_N = g\Big\}\Big) \\ &= \bigcup_{x\in\Gamma\cap pD}\Big(g+\bigcup\Big\{p_1E_1+\cdots+\Big(p-\sum_i p_i\Big)E_{N+1} : (p_i)\in\pi^{-1}(x)\cap pS\cap \mathbf{Z}^N\Big\}\Big) \end{aligned}$$

where g is the element $p_1g_1+\cdots+p_Ng_N$, which is independent of the choice $(p_i)\in\pi^{-1}(x)\cap pS\cap\mathbf{Z}^N$. Notice that the correspondence $\Gamma\to G$ is injective, by construction, and hence all the sets $g+\bigcup\{\cdots\}$ in the latter union lie in disjoint cosets of V in H; therefore

$$\begin{aligned} |C^p| &= \sum_{x\in\Gamma\cap pD}\Big|\bigcup\Big\{p_1E_1+\cdots+\Big(p-\sum_i p_i\Big)E_{N+1} : (p_i)\in\pi^{-1}(x)\cap pS\cap\mathbf{Z}^N\Big\}\Big| \\ &= \sum_{x\in\Gamma\cap pD}\big|\bigcup\{E(\lambda,p) : \lambda\in\pi^{-1}(x)\cap pS\cap\mathbf{Z}^N\}\big|. \end{aligned} \tag{10}$$

We immediately obtain an upper bound for $|C^p|$ by allowing λ to range over *all* points in $\pi^{-1}(x)\cap pS$, which is the same as allowing λ/p to range over all points of $\pi^{-1}(x/p)\cap S$; this is just $p\cdot E(x/p,1)=pE(x/p)$ by definition, so that

$$|C^p| \leqq p^m\sum\{\psi(x/p) : x\in\Gamma\cap pD\}.$$

The difficult part of the estimate is to find good lower bounds for the individual terms in the sum (10). Let us denote:

$$A(x,p) = \bigcup\{E(\lambda,p) : \lambda\in\pi^{-1}(x)\cap pS\cap\mathbf{Z}^N\}\subset V.$$

Let d_0 be the "mesh" of Λ_0 in the subspace $M\subset\mathbf{R}^N$ spanned by Λ_0—so no point in M can be further than a distance d_0 from a point in Λ_0, and consequently, if $x\in\Gamma$, every point in the coset $\pi^{-1}(x)$ of M lies within distance d_0 of some point in $\mathbf{Z}^N\cap\pi^{-1}(x)$, which is a coset of $\mathbf{Z}^N/\Lambda_0$. Define $r=2Nd_0\operatorname{diam}(B)$ where

$$B = \operatorname{conv}(E_1\cup\cdots\cup E_{N+1}\cup\{0\}).$$

We consider $x\in\Gamma\cap pD$ and define:

$$B(x,p) = \bigcup\{E(\lambda,p) : \lambda\in(\pi^{-1}(x)\cap pS)_{d_0}\}\subset pB\subset V.$$

If $a>0$ and $\lambda=(\lambda_i)\in pS\subset\mathbf{R}^N$, we define the modified sum of sets

$$E_a(\lambda,p) = \lambda_1(E_1)_a+\cdots+\lambda_N(E_N)_a+\Big(p-\sum_{i=1}^N\lambda_i\Big)(E_{N+1})_a\subset V.$$

Now if $\lambda=(\lambda_1,\ldots,\lambda_N)\in(\pi^{-1}(x)\cap pS)_{d_0}$, and $U'(d_0)$ the ball in M of radius d_0, then $\lambda+U'(d_0)\subset\pi^{-1}(x)\cap pS$, and our definition of "mesh" d_0 insures that there is a $\lambda'\in\pi^{-1}(x)\cap pS\cap \boldsymbol{Z}^N$ with $\|\lambda-\lambda'\|\leqq d_0$, which $\Rightarrow$

$$|\lambda_i-\lambda_i'| \leqq d_0 \qquad (1 \leqq i \leqq N),$$

and also

$$\left|\left(p-\sum_{i=1}^{N}\lambda_i\right)-\left(p-\sum_{i=1}^{N}\lambda_i'\right)\right| \leqq Nd_0.$$

For convenience we shall occasionally write $\lambda_{N+1}=p-\sum_{i=1}^{N}\lambda_i$ if $(\lambda_i)\in pS\subset\boldsymbol{R}^N$. We assert that, in this context,

$$E_{r/p}(\lambda,p) = \lambda_1(E_1)_{r/p}+\cdots+\left(p-\sum_i\lambda_i\right)(E_{N+1})_{r/p} \subset E(\lambda',p): \tag{11}$$

in fact, if $\xi\in E_{r/p}(\lambda,p)$

$$\begin{aligned}\xi &= \lambda_1e_1+\cdots+\lambda_{N+1}e_{N+1} \qquad (e_i\in(E_i)_{r/p})\\ &= (\lambda_1'e_1+\cdots+\lambda_{N+1}'e_{N+1})+((\lambda_1-\lambda_1')e_1+\cdots+(\lambda_{N+1}-\lambda_{N+1}')e_{N+1})\end{aligned}$$

and the right-hand term has norm at most $\sum_i|\lambda_i-\lambda_i'|\,\|e_i\|\leqq N(d_0\operatorname{diam}(B))+Nd_0\operatorname{diam}(B)=r$. Let us write $U(\xi,\delta)$ for the open ball of radius $\delta>0$ about a point $\xi\in V$, abbreviating $U=U(0,1)$; for $X\subset V$ write

$$U^+(X,\delta) = \{x\in V : \operatorname{dist}(x,X) < \delta\}$$

and $X_\delta=U^-(X,\delta)=\{x\in V: x\in X, \operatorname{dist}(x,\operatorname{bdry}X)>\delta\}$. We have just shown that $E_{r/p}(\lambda,p)\subset E_{r/p}(\lambda',p)+rU$; some elementary manipulation of the constructs U^+, U^- gives (11)—see A.4 in the Appendix. From (11) it follows that: if $x\in\Gamma\cap pD$, and if we write

$$B'(x,p) = \bigcup\{E_{r/p}(\lambda,p) : \lambda\in(\pi^{-1}(x)\cap pS)_{d_0}\}$$

we have

$$B'(x,p)\subset A(x,p), \quad \text{all } x\in(\Gamma\cap pD), \text{ all } p.$$

Next we assert that there is an $r_1>0$ and an integer p_0 such that

$$B(x,p)\subset B'(x,p)+r_1U\subset A(x,p)+r_1U, \tag{12}$$

for all $p\geqq p_0$. In fact the elementary Lemma A.1(ii) in the Appendix gives

$$E_i\subset U^+((E_i)_{r/p},(r/p)\mu(E_i)) = (E_i)_{r/p}+(r/p)\mu(E_i)U$$

(recall $\mu(E_i)=\operatorname{diam}(E_i)/r_{\text{in}}(E_i))$ for all sufficiently large p (say $p\geqq p(E_i)$). Hence if $p_0\geqq p(E_i)$ all $1\leqq i\leqq N+1$, this inclusion is true for $p\geqq p_0$ and $1\leqq i\leqq N+1$. Hence if $p\geqq p_0$ and $\lambda\in pS$ is any point, we have

$$\begin{aligned}E(\lambda,p) &= \sum_{i=1}^{N+1}\lambda_iE_i \subset \sum_{i=1}^{N+1}\lambda_iU^+((E_i)_{r/p},(r/p)\mu(E_i))\\ &= \sum_{i=1}^{N+1}\lambda_i(E_i)_{r/p}+\left(\sum_{i=1}^{N+1}\lambda_i(r/p)\mu(E_i)\right)\cdot U\\ &\subset \sum_{i=1}^{N+1}\lambda_i(E_i)_{r/p}+r_1U = U^+(E_{r/p}(\lambda,p),r_1)\end{aligned}$$

where $r_1 = \max\{r\mu(E_i)\}$—notice that $\sum_{i=1}^{N+1} (\lambda_i/p) = 1$ if $(\lambda_1, \ldots, \lambda_N) \in pS$, as required to prove (12).

Finally, we want to compare $B(x, p)$ with:

$$E(x, p) = \bigcup \{E(\lambda, p) : \lambda \in \pi^{-1}(x) \cap pS\} \qquad (x \in pD).$$

It is this comparison which gives our lower bound (and error estimates) for $|C^p|$. By Lemma A.1(ii) (see Appendix) we have, in $\boldsymbol{R}^N$,

$$(13) \qquad (\pi^{-1}(x) \cap pS) \subset (\pi^{-1}(x) \cap pS)_{d_0} + d_0\mu(\pi^{-1}(x) \cap pS) \cdot U$$

for all $x \in pD$ such that the inradius $r_{\text{in}}(\pi^{-1}(x) \cap pS) > d_0$.

By Lemma 7.5, since $r_{\text{in}}(\pi^{-1}(x) \cap pS) \geqq \operatorname{diam}^*(x)$ for $x \in \operatorname{int}(pD)$, we have

$$(14) \qquad r_{\text{in}}(\pi^{-1}(x) \cap pS) > \frac{1}{\mu(S)}\delta \quad \text{for } x \in (pD)_\delta$$

for all δ with $0 < \delta < r_{\text{in}}(pS) = pr_{\text{in}}(S)$. Note that $\mu(pS) = \mu(S)$.

But clearly $\operatorname{diam}(\pi^{-1}(x) \cap pS) \leqq \operatorname{diam}(pS) = p \operatorname{diam}(S)$, all $x \in pD$, which $\Rightarrow$

$$(15) \quad \mu(\pi^{-1}(x) \cap pS) \leqq \frac{p \operatorname{diam}(S)\mu(S)}{\delta} = (p/\delta)(\operatorname{diam}(S)\mu(S)), \quad \text{all } x \in (pD)_\delta.$$

Hence (13) becomes

$$(13') \qquad \pi^{-1}(x) \cap pS \subset (\pi^{-1}(x) \cap pS)_{d_0} + (p/\delta)(\operatorname{diam}(S)\mu(S))U,$$

for all $x \in (pD)_\delta$ with $d_0 < r_{\text{in}}(\pi^{-1}(x) \cap pS)$, which is guaranteed if we take $\delta > \mu(S)d_0$ (by 14).

Now consider the decomposition

$$(16) \qquad \Gamma \cap pD = \bigcup_{i=1}^{N(p)} ((\Gamma \cap pD)_{i-1} \sim (\Gamma \cap pD)_i) \cup (\Gamma \cap pD)_{N(p)}$$

where

$$N(p) = [(p/2)r_{\text{in}}(D)].$$

Choose any $i > \mu(S)d_0$, and fix any $x \in (\Gamma \cap pD)_i$. Then if $\lambda \in \pi^{-1}(x) \cap pS$, by (13′) there exists a $\lambda' \in (\pi^{-1}(x) \cap pS)_{d_0}$ satisfying $\|\lambda - \lambda'\| \leqq (p/i)(d_0 \operatorname{diam}(S)\mu(S))$. Hence by the same method as used in proving (11), there is a constant r^* (independent of p, i, x, λ, etc.). Such that $E(\lambda, p) \subset E(\lambda', p) + (p/i)r^*U$: thus (12) gives

$$(17) \qquad E(x, p) \subset B(x, p) + (p/i)r^*U \subset A(x, p) + (r_1 + (p/i)r^*)U,$$

for all $p \geqq p_0$, $x \in (\Gamma \cap pD)_i$, $i > \mu(S)d_0$.

Therefore, by A.1(iii), $U^-(E(x, p), r_1 + (p/i)r^*) \subset A(x, p)$; thus by A.2 and A.3:

$$(18) \qquad \begin{aligned} |A(x, p)| &\geqq |U^-(E(x, p), r_1 + (p/i)r^*)| \\ &\geqq |E(x, p)| - \frac{(r_1 + (p/i)r^*)N}{r_{\text{in}}(E(x, p))} |E(x, p)| \\ &\geqq |E(x, p)| - \alpha\left(\frac{r_1}{p} + \frac{r^*}{i}\right)|E(x, p)|. \end{aligned}$$

(Here $r_{\text{in}}(E(x,p))=p\cdot r_{\text{in}}(E(x/p))$ and $r_{\text{in}}(E(x/p))\geqq\min\{r_{\text{in}}(E_i)\}=1/\alpha$.) With (18) in hand our lower estimate for $|C^p|$ follows directly.

In fact, for $p\geqq p_0$ (p_0 determined as in (12)),

$$
\begin{aligned}
|C^p| &= \sum\{|A(x,p)| : x\in\Gamma\cap pD\}\\
&= \sum_{i=1}^{N(p)}\sum\{|A(x,p)| : x\in(\Gamma\cap pD)_{i-1}\sim(\Gamma\cap pD)_i\}\\
&\quad +\sum\{|A(x,p)| : x\in(\Gamma\cap pD)_{N(p)}\}\\
&\geqq \sum_{\mu(S)d_0<i\leqq N(p)}\sum\{|E(x,p)| : x\in(\Gamma\cap pD)_{i-1}\sim(\Gamma\cap pD)_i\}\\
&\quad +\sum\{|E(x,p)| : x\in(\Gamma\cap pD)_{N(p)}\}\\
&\quad -\alpha\sum_{\mu(S)d_0<i\leqq N(p)}\left(\frac{r_1}{p}+\frac{r^*}{i}\right)\{|E(x,p)| : x\in(\Gamma\cap pD)_{i-1}\sim(\Gamma\cap pD)_i\}\\
&\quad -\alpha\left(\frac{r_1}{p}+\frac{r^*}{N(p)}\right)\sum\{|E(x,p)| : x\in(\Gamma\cap pD)_{N(p)}\}\\
&= \sum\{|E(x,p)| : x\in(\Gamma\cap pD)\}\\
&\quad -\sum\{|E(x,p)| : x\in(\Gamma\cap pD)\sim(\Gamma\cap pD)_{\mu(S)d_0}\}\\
&\quad +O\left(\frac{1}{p}\sum\{|E(x,p)| : x\in(\Gamma\cap pD)\}\right)\\
&\quad +O\left(\sum_{i=1}^{N(p)}\frac{1}{i}\{|E(x,p)| : x\in(\Gamma\cap pD)_{i-1}\sim(\Gamma\cap pD)_i\}\right).
\end{aligned}\tag{19}
$$

But $|E(x,p)|=p^m|E(x/p)|=p^m\psi(x/p)$ by definition, and for

$$1\leqq i\leqq N(p)=[(p/2)r_{\text{in}}(D)]$$

we have $r_{\text{in}}((pD)_i)>(p/2)r_{\text{in}}(D)$. Therefore

$$|(\Gamma\cap pD)_{i-1}\sim(\Gamma\cap pD)_i| = O(p^{k-1}),\qquad 1\leqq i\leqq N(p),\tag{20}$$

since, if d^* is the diameter of Γ,

$$
\begin{aligned}
|(\Gamma\cap pD)_{i-1}\sim(\Gamma\cap pD)_i| &\leqq |(pD)_{i-d^*-1}\sim(pD)_{i+d^*}|\\
&\leqq \frac{N|pD|(2d^*+1)}{(p/2)r_{\text{in}}(D)} = O(p^{k-1}).
\end{aligned}
$$

But each term $|E(x,p)|=O(p^m)$ (uniformly in x), so that

$$
\begin{aligned}
\sum_{i=1}^{N(p)}\frac{1}{i}\{|E(x,p)| : x\in(\Gamma\cap pD)_{i-1}\sim(\Gamma\cap pD)_i\} &= O\left(p^{m+k-2}\sum_{i=1}^{N(p)}\frac{1}{i}\right)\\
&= O(p^{m+k-1}\log p).
\end{aligned}
$$

Similarly

$$\sum \{|E(x,p)| : x \in (\Gamma \cap pD) \sim (\Gamma \cap pD)_{\mu(S)d_0}\} = O(p^{m+k-1}).$$

Finally,

$$(21) \qquad |C^p| = (p^m + O(p^{m-1})) \sum \{\psi(x/p) : x \in \Gamma \cap pD\} + O(p^{m+k-1} \log p).$$

Finally, by A.5 of the appendix,

$$\sum \{\psi(x/p) : x \in \Gamma \cap pD\} = \frac{p^k}{|\Gamma|} \int_D \psi(x)\, dx + O(p^{k-1} \log p);$$

upon substituting this in (21), (9) follows and Theorem 7.6 has been proved when C satisfies the auxilliary hypothesis that its V-components $E_1, \ldots, E_{n+1}$ are all *convex*.

We now wish to extend our result to general relatively compact open sets C which do not necessarily have convex components. We first introduce the following notation: if $C = \bigcup (E_i + g_i)$ is the decomposition of C by cosets of B, we define the convex hull of C, conv (C) by the formula

$$\operatorname{conv} C = \bigcup (\operatorname{conv} (E_i) + g_i).$$

We then have the following analogue of Theorem 5.2:

THEOREM 7.7. *Let C be a relatively compact open subset of $H = V \times G$ (notation as in Theorem 7.6). Then there is a point $t = t(C) \in H$ and an integer $p_0 \geqq 1$ such that*:

$$(22) \qquad t + (\operatorname{conv}(C))^{p-p_0} \subset C^p \subset (\operatorname{conv}(C))^p, \quad \textit{all } p \geqq p_0.$$

Proof. Proceeding as in the first paragraphs of 7.6, we obtain a formula similar to (10):

$$(10') \quad C^p = \bigcup_{x \in \Gamma \cap pD} (g + \bigcup \{E_1^{p_1} + \cdots + E_N^{p_N} + E_{N+1}^{(p-\sum_i p_i)} : (p_i) \in \pi^{-1}(x) \cap pS \cap Z^N\}.$$

Now by Theorem 5.2 there are integers $q_i \geqq 1$ and vectors $t_i \in V$ such that:

$$t_i + (q - q_i) \operatorname{conv} (E_i) \subset E_i^q \subset q \operatorname{conv} (E_i)$$

for all $1 \leqq i \leqq N+1$ and all $q \geqq q_i$. Thus one sees:

$$\begin{aligned} &(t_1 + \cdots + t_{N+1}) + (p_1 - q_1) \operatorname{conv} (E_1) + \cdots + (p_n - q_n) \operatorname{conv} (E_N) \\ (23) \qquad &+ \left(p - \sum_{i=1}^N p_i - q_{N+1}\right) \operatorname{conv} (E_{N+1}) \\ &\subset E_1^{p_1} + \cdots + E_N^{p_N} + E_{N+1}^{(p-\sum_i p_i)} \subset p_1 \operatorname{conv} (E_1) + \cdots + \left(p - \sum_i p_i\right) \operatorname{conv} (E_{N+1}), \end{aligned}$$

where, if any $p_i < q_i$, we interpret the first term to be empty. By (10′) we clearly have $C^p \subset (\text{conv}(C))^p$ for all $p \geqq 1$. Set $t = (t_1 + \cdots + t_{N+1}) - (q_1 g_1 + \cdots + q_N g_N)$, and $p_0 = q_1 + \cdots + q_{N+1}$. Then for $p \geqq p_0$,

$$t + (\text{conv } C)^{p-p_0}$$
$$= \bigcup_{x \in \Gamma \cap (p-p_0)D} \left(g + t + \bigcup \left\{ p_1' \text{ conv } E_1 + \cdots + \left(\left(p - \sum_i p_i' \right) - p_0 \right) \text{conv } E_{N+1} : (p_i') \in \pi^{-1}(x) \cap (p-p_0)S \cap \mathbf{Z}^N \right\} \right).$$

Now let $p_i = p_i' + q_i$ $(1 \leqq i \leqq N+1$; recall $g = \sum_{i=1}^N p_i' g_i)$; then

$$t + (\text{conv } C)^{p-p_0}$$
$$(24) \qquad = \bigcup_{x \in \Gamma \cap pD} \left(g + \bigcup \left\{ (t_1 + \cdots + t_{N+1}) + (p_1 - q_1) \text{ conv } (E_1) + \cdots + \left(p - \sum_i p_i \right) \text{conv } (E_{N+1}) : (p_i) \in \pi^{-1}(x) \cap pS \cap \mathbf{Z}^N \right\} \right);$$

(22) follows from (23) and (24). Q.E.D.

Now to return to Theorem 7.6, we see that (22) implies

$$|(\text{conv}(C))^{p-p_0}| \leqq |C^p| \leqq |(\text{conv } C)^p|, \quad \text{all } p \geqq p_0.$$

Hence by (9), which is known to be valid for the open set conv (C), we immediately obtain the same estimate for $|C^p|$ since $(p-p_0)^{m+k} = p^{m+k} + O(p^{m+k-1})$, etc.; the constant A corresponding to C is the same constant as that corresponding to conv (C). Q.E.D.

We conclude this section with the basic results of this paper, which are immediately obtained from the preceding:

THEOREM 7.8. *Any abelian group H is strongly amenable; in fact, if C is any relatively compact open subset of H, we obtain the asymptotic estimate*

$$(25) \qquad |C^p| = Ap^n + O(p^{n-1} \log p)$$

for suitable $A > 0$ and integer $n \geqq 0$. Furthermore, if K is any compact group and H is abelian, $G = K \times H$ is strongly amenable. In fact, the estimate (25) *prevails for relatively compact open $C \subset G$.*

REMARK. The integer n need *not* be bounded (i.e. it may depend on which set C we look at). Of course, the O-constant depends on (the choice of) G.

Proof. If H' is the compactly generated open subgroup of H generated by C, we may as well consider the behavior of C^p with respect to H', since Haar measure restricts properly to H'. But by the structure theorem for compactly generated Abelian groups, we have topological isomorphism

$$H' \cong K \times V \times \mathbf{Z}^N \qquad (\text{some integer } N)$$

where K is compact abelian and V is a vector group. Our assertion now follows from Theorems 7.6 and 6.8. Our second statement follows immediately, upon appealing once more to Theorem 6.8.

It is possible to generalize Theorem 7.6 to deal with very general relatively compact sets C (not necessarily measurable) and prove:

THEOREM 7.9. *If G is abelian and C is any relatively compact subset such that some power of C has nonvoid interior, then*

$$|C^p|^+/|C^{p-1}|^- \to 1 \quad \text{as } p\to\infty.$$

Furthermore, there is a constant $A>0$ and an integer $k\geqq 0$ such that

$$|C^p|^+ \sim |C^p|^- \sim Ap^k \quad \text{as } p\to\infty.$$

We omit the proof, which is based on the results in Theorem 7.6 but requires no reference to the mechanics of that proof.

APPENDIX. This discussion of families of convex sets in a Euclidean space extends ideas discussed in Eggleston [2]; this material does not seem to be discussed in the literature. We consider $\boldsymbol{R}^N$, with its Euclidean norm, and let $\Omega(\boldsymbol{R}^N)$ denote the family of all compact convex sets in $\boldsymbol{R}^N$—evidently Ω is closed under addition of sets and affine transformations (see [2, §5.1]). We define $U(x, r)$ to be the open ball of radius $r>0$ about $x\in\boldsymbol{R}^N$ and write $U=U(0, 1)$. If a set $X\subset\boldsymbol{R}^N$ and $\delta>0$ is given there are two constructs of interest: the δ-retract of X

$$X_\delta = U^-(X, \delta) = \{x\in\boldsymbol{R}^N : x\in X, \text{dist}(x, \text{bdry } X) > \delta\}$$

and the δ-neighborhood (also referred to as a "parallel body"—see [10])

$$U^+(X, \delta) = \{x\in\boldsymbol{R}^N : \text{dist}(x, X) < \delta\} = X+\delta U.$$

Notice that if X is convex, then so are $U^+(X, \delta)$ and $U^-(X, \delta)$, although the latter may be empty (see [2, Theorem 2]). The *inradius* $r_{\text{in}}(X)$ of a convex set $X\in\Omega$ is defined in the obvious way [2, §4.5]. An important parameter in all our considerations is

$$\mu(X) = \text{diam}(X)/r_{\text{in}}(X), \quad \text{all } X\in\Omega(\boldsymbol{R}^N),$$

which is invariant under scaling automorphisms and translations.

PROPOSITION A.1. *If $X\in\Omega(\boldsymbol{R}^N)$ and $\delta>0$, then*

(i) *$U^-(U^+(X, \delta), \delta)=X$ all $\delta>0$.*

(ii) *$U^+(U^-(X, \delta), \delta\mu(X))=X_\delta+\delta\mu(X)\cdot U\supset X$ all $0<\delta<r_{\text{in}}(X)$.*

(iii) *If $X\subset U^+(X', \delta)$, then $X_\delta\subset X'$, all $\delta>0$.*

Proof. Notice the lack of commutativity of constructions U^+, U^-. In (i) it is trivial (and in fact true for any $X\subset\boldsymbol{R}^N$) that $X\subset(X+\delta U)_\delta$; for the converse, suppose there exists an $x\in((X+\delta U)_\delta\sim X)$ and let $y\in X$ be chosen so that $\text{dist}(x, X)=\text{dist}(x, y)>0$. There is an $N-1$ dimensional hyperplane H tangent to X at y (y must be in bdry X) such that $x-y$ is orthogonal to H. Furthermore,

$x \in (X+\delta U)_\delta \Rightarrow x+\delta U \subset X+\delta U \Rightarrow x' = x+(\delta/\|x-y\|)(x-y) \in X+\delta U$. But we have

$$\operatorname{dist}(x', X) \geqq \operatorname{dist}(x', H) = \operatorname{dist}(x', y) = \delta + \operatorname{dist}(x, y) > \delta,$$

which $\Rightarrow x' \notin X+\delta U$, a contradiction.

For (ii), let $x_0 \in X$ be taken so $x_0 + r_{\text{in}}(X)U \subset X$. Fix any $\delta > 0$ with $0 < \delta < r_{\text{in}}(X)$, let $x \in X$, and write $r = r_{\text{in}}(X)$. Then the cone with vertex x and "base" $x_0 + rU$:

$$\{(1-\alpha)x + \alpha(x_0 + rU) : 0 \leqq \alpha \leqq 1\}$$

lies within X. In particular, since $0 < \delta/r < 1$, the sphere

$$(1-(\delta/r))x + (\delta/r)(x_0 + rU) = (1-(\delta/r))x + (\delta/r)x_0 + \delta U$$

lies within X, which $\Rightarrow x' = (1-(\delta/r))x + (\delta/r)x_0 \in X_\delta$. But $\|x - x'\| = (\delta/r)\|x - x_0\| \leqq (\delta/r) \operatorname{diam}(X) = \delta\mu(X)$, which $\Rightarrow x \in x' + \delta\mu(x)U \subset (X)_\delta + \delta\mu(X)U$, as required.

For (iii) note that $X \subset X' + \delta U \Rightarrow X_\delta \subset (X' + \delta U)_\delta = X'$ by (i). Q.E.D.

Proposition A.2. *Let us write $|X|$ for the measure of a set $X \in \Omega(\mathbf{R}^N)$. Then*

(i) *If $X \in \Omega(\mathbf{R}^N)$ then*

$$0 \leqq |X| - |X_\delta| \leqq (N|X|/r_{\text{in}}(X))\delta, \quad \textit{all } \delta > 0.$$

(ii) *Let $\delta_0 > 0$ and $X \in \Omega(\mathbf{R}^N)$ be given. Then there is a constant K such that*

$$|\{x \in \mathbf{R}^N : \operatorname{dist}(x, \operatorname{bdry}(pX)) \leqq \delta\}| \leqq K\delta p^{N-1}$$

all $0 < \delta \leqq \delta_0$, all p.

Note. The set in (ii) is a fringe of radius δ about the $N-1$ dimensional set $\operatorname{bdry}(pX)$. For large p it is intuitively clear why such an estimate must hold.

Proof. We assume $0 < \delta < r_{\text{in}}(X)$, for the inequality in (i) is trivial if $\delta \geqq r_{\text{in}}(X)$. Let $x_0 \in X$ be the center of some ball $U(x_0, r_{\text{in}})$ lying within X. We may assume $x_0 = 0$ since translation does not affect our measure considerations. Write $r = r_{\text{in}}(X)$. As in the proof of A.1(ii), $x \in X \Rightarrow (1-(\delta/r))x \in X_\delta$ (since $x_0 = 0$ here), which $\Rightarrow (1-(\delta/r))X \subset X_\delta$, which $\Rightarrow |X_\delta| \geqq |(1-(\delta/r))X|$; thus

$$0 \leqq |X| - |X_\delta| \leqq (1-(1-(\delta/r))^N)|X| \leqq N|X|(\delta/r),$$

as required. For (ii) let us write $Y(p, \delta) = \{x \in \mathbf{R}^N : \operatorname{dist}(x, \operatorname{bdry}(pX)) \leqq \delta\} = (pX + \delta U) \sim (pX)_\delta$. Clearly

$$\begin{aligned} |Y(p, \delta)| &= |(pX+\delta U) \sim pX| + |pX \sim (pX)_\delta| \\ &= (|pX+\delta U| - |pX|) + (|pX| - |(pX)_\delta|). \end{aligned}$$

By (i), if $0 < \delta < r_{\text{in}}(pX) = p \cdot r_{\text{in}}(X) = pr$, then we have

$$0 \leqq |pX| - |(pX)_\delta| \leqq N|X|(\delta/r)p^{N-1}.$$

Moreover, for any $\delta > 0$ we have implications: $rU = U(x_0, r_{\text{in}}(X)) \subset X \Rightarrow \delta U \subset (\delta/r)X \Rightarrow pX + \delta U \subset pX + (\delta/r)X = (p+(\delta/r))X$, which $\Rightarrow |pX+\delta U| - |pX| \leqq \{(p+(\delta/r))^N - p^N\}|X|$ for all $p = 1, 2, \ldots$. If we restrain $0 < \delta \leqq \delta_0$ we can take a constant K_0 such that the last number here is dominated by $K_0\delta p^{N-1}$ all $0 < \delta \leqq \delta_0$, all p. If we set $p_0=\delta/r$, $K_1=K_0+N|X|r^{-1}$, we get $|Y(p,\delta)|\leqq K_1\delta p^{N-1}$ all $0<\delta\leqq\delta_0$, $p\geqq p_0$. If we take $K=K_1p_0^N$, say, we get $|Y(p,\delta)|\leqq K\delta p^{N-1}$ all $0<\delta\leqq\delta_0$, all p, and (ii) is proved.

REMARK. In this context we recall the following classic formula due to Steiner: $|X+\delta U|=|X|+A(X)\delta+M(X)\delta^2+(4\pi/3)\delta^3$, where $A(X)$ is the surface area of convex set X, $M(X)$ is the integral of mean curvature of X (see [10] for discussion and proof); however, in our present context we do not need the precision of this result.

Now we must discuss linear families of convex sets, which are discussed to some extent in [2, Chapter 5]. Here we shall consider a specified finite collection of compact convex sets with interior: $\{E_1,\ldots,E_{N+1}\}$ in a vector space $V\cong\boldsymbol{R}^l$. We define a mapping $E:\boldsymbol{R}^{N+1}\to\Omega(V)$ which assigns to a vector $\lambda=(\lambda_i)\in\boldsymbol{R}^{N+1}$ the affine combination $E(\lambda)=\lambda_1E_1+\cdots+\lambda_{N+1}E_{N+1}$. Notice that $E(\lambda)$ is convex closed and has interior (unless $\lambda=0$). For any subset $X\subset\boldsymbol{R}^{N+1}$ we define

$$E(X)=\bigcup\{E(\lambda):\lambda\in X\}\subset V,$$

and observe that $E(X)$ is convex if X is convex and compact if X is compact, and $E(\alpha X)=\alpha E(X)$ for all $\alpha\in\boldsymbol{R}$. Our first result concerns the invariant

$$\mu(X)=\operatorname{diam}X/r_{\text{in}}(X)\quad\text{for } X\in\Omega(V).$$

PROPOSITION A.3. *If* $\lambda\neq0$ *in* $\boldsymbol{R}^{N+1}$ *then* $\mu(E(\lambda))\leqq\max\{\mu(E_i):1\leqq i\leqq N+1\}$. *There is a constant* $\mu_0>0$ *such that* $\mu(E(X))\leqq\mu_0$ *for all* $X\in\Omega(\boldsymbol{R}^{N+1})$ *with* $X\neq\varnothing$.

Proof. The set $E(\lambda)=\lambda_1E_1+\cdots+\lambda_{N+1}E_{N+1}$ contains a closed ball of radius $|\lambda_1|r_{\text{in}}(E_1)+\cdots+|\lambda_{N+1}|r_{\text{in}}(E_{N+1})$, which $\Rightarrow r_{\text{in}}(E(\lambda))\geqq\sum_i|\lambda_i|r_{\text{in}}(E_i)$. Also

$$\operatorname{diam}(E(\lambda))\leqq\sum_i|\lambda_i|\operatorname{diam}(E_i);$$

therefore

$$\mu(E(\lambda))=\frac{\operatorname{diam}(E(\lambda))}{r_{\text{in}}(E(\lambda))}\leqq\frac{\sum_i|\lambda_i|\operatorname{diam}(E_i)}{\sum_i|\lambda_i|r_{\text{in}}(E_i)}$$
$$\leqq\max\{\mu(E_i):1\leqq i\leqq N+1\}$$

since, in general, if $a_i,b_i>0$ we have $(\sum_i a_i)/(\sum_i b_i)\leqq\max\{a_i/b_i:i=1,2,\ldots,N+1\}$. For (ii) we write $\|\lambda\|_1=\sum_{i=1}^{N+1}|\lambda_i|$ for $\lambda\in\boldsymbol{R}^{N+1}$, and $\|X\|_1=\max\{\|\lambda\|_1:\lambda\in X\}$ for a set $X\in\Omega(\boldsymbol{R}^{N+1})$. Then $r_{\text{in}}(E(X))\geqq\|X\|_1\min\{r_{\text{in}}(E_1):1\leqq i\leqq N+1\}$ by reasoning similar to that in (i) above. Also, $X\subset\|X\|_1U$ (U the unit ball about 0 in the Euclidean norm), which

$$\Rightarrow\operatorname{diam}(E(X))\leqq\operatorname{diam}(E(\|X\|_1U))=\operatorname{diam}(\|X\|_1\cdot E(U))=\|X\|_1\operatorname{diam}(E(U)),$$

which $\Rightarrow$

$$\mu(E(X))=\frac{\operatorname{diam}E(X)}{r_{\text{in}}(E(X))}\leqq\frac{\|X\|_1\operatorname{diam}(E(U))}{\|X\|_1\min\{r_{\text{in}}(E_i)\}}=\frac{\operatorname{diam}E(U)}{\min\{r_{\text{in}}(E_i)\}}=\mu_0.\quad\text{Q.E.D.}$$

Using the notation $E_\delta(\lambda)=\lambda_1(E_1)_\delta+\cdots+\lambda_{N+1}(E_{N+1})_\delta$ for $\lambda\in\boldsymbol{R}^{N+1}$ and $\delta>0$, we note the following interpolation result.

LEMMA A.4. *If $\lambda\neq 0$ in $\boldsymbol{R}^{N+1}$ and $\|\lambda\|_1=|\lambda_1|+\cdots+|\lambda_{N+1}|$, then*

$$E_{r/\|\lambda\|_1}(\lambda)+rU\subset E(\lambda)$$

all $r>0$.

Proof. Let $x_i\in(E_i)_{r/\|x\|_1}$ for $1\leqq i\leqq N+1$, and let $x\in rU$. Then

$$x_i' = x_i+\operatorname{sgn}(\lambda_i)(x/\|x\|_1)\in E_i, \qquad 1\leqq i\leqq N+1,$$

which $\Rightarrow$ $\lambda_1x_1+\cdots+\lambda_{N+1}x_{N+1}+x=\lambda_1x_1'+\cdots+\lambda_{N+1}x_{N+1}'\in E(\lambda)$. Q.E.D.

We define a metric $\Delta(X, Y)$ on the collection of all compact subsets of a vector space V:

$$\Delta(X, Y) = \inf\{\delta > 0 : U^+(X,\delta)\supset Y,\ U^+(Y,\delta)\supset X\}.$$

This is a metric and the subfamily $\Omega(V)$ becomes a complete metric space, as indicated in [2, pp. 59,H]. Furthermore,

PROPOSITION A.5. *Let convex compact sets with interior $\{E_1,\ldots,E_{N+1}\}$ be given in vector space $V\cong\boldsymbol{R}^l$ and define E: $\Omega(\boldsymbol{R}^{N+1})\to\Omega(V)$ as above. Then E is continuous and in fact satisfies a Lipschitz condition with respect to the natural metrics. Let us define $\psi(X)=|E(X)|$ for $X\in\Omega(\boldsymbol{R}^{N+1})$. If we are given a bounded set $W\subset\boldsymbol{R}^{N+1}$ then there is a constant $\alpha=\alpha(W,\delta)$ such that $|\psi(X)-\psi(X')|<\alpha\Delta(X,X')$ all $X, X'\in\Omega(\boldsymbol{R}^{N+1})$ with $X\subset W$, $X'\subset W$; i.e. ψ satisfies a Lipschitz condition.*

Proof. For the first assertion, let $X, X'\in\Omega(\boldsymbol{R}^{N+1})$, write $\Delta(X,X')=\varepsilon\geqq 0$, and set $Y=E(X)$, $Y'=E(X')$. Then for any $x\in X$, there is some $x'\in X'$ with $\|x-x'\|\leqq\varepsilon$, so that $|x_i-x_i'|\leqq\varepsilon$ for $1\leqq i\leqq N+1$. Now let $y\in Y$, say $y=x_1e_1+\cdots+x_{N+1}e_{N+1}$ where $x=(x_1,\ldots,x_N)\in X$ and $e_i\in E_i$; then $y'=x_1'e_1+\cdots+x_{N+1}'e_{N+1}\in Y'$ and we have

$$\|y-y'\| \leqq \sum_i |x_i-x_i'|\,\|e_i\| \leqq \beta\varepsilon \quad \text{where } \beta = (N+1)\max\Big\{\|e\| : e\in\bigcup_i E_i\Big\}.$$

Thus $Y\subset Y'+\varepsilon\beta U$, and similarly if Y, Y' are reversed, so $\Delta(Y,Y')\leqq\beta\varepsilon=\beta\Delta(X,X')$ as required.

In the second part it suffices to consider W a ball about the origin: $W=mU$ for some $m>0$. Clearly there is a constant m' such that, $Y=E(X)\subset m'U$ (in vector space V) for any $X\subset mU$ (take $X'=\{0\}$ above). Now let $X, X'\in\Omega(\boldsymbol{R}^{N+1})$ with $X, X'\subset mU$ and set $Y=E(X)$, $Y'=E(X')$. Then, as above, there is a constant β such that $\Delta(Y,Y')<\beta\Delta(X,X')$, hence if we write $\varepsilon=\Delta(X,X')$, we have $Y\subset Y'+\beta\varepsilon U$ and $Y'\subset Y+\beta\varepsilon U$, so that $(Y)_{\beta\varepsilon}\subset Y'$ and $(Y')_{\beta\varepsilon}\subset Y$ by A.1(iii). Thus $|(Y)_{\beta\varepsilon}|\leqq|Y'|$ and $|(Y')_{\beta\varepsilon}|\leqq|Y|$. But by A.2(i) and A.3 we have (μ_0 as in A.3):

$$|(Y)_{\beta\varepsilon}| \geqq |Y|-\frac{N|Y|}{r_{\text{in}}(Y)}\beta\varepsilon \geqq |Y|-N\beta\mu_0\frac{|Y|}{\operatorname{diam} Y}\varepsilon,$$

and similarly for Y'. But since $Y \subset m'U$, we have $|Y|/\text{diam } Y \leqq m''$ for a suitably chosen m'', which can be taken independent of the $X \subset mU$ used to define $Y = E(X)$ (since $\text{diam }(\sum_i x_i E_i) \geqq \max\{|x_i|\} \min\{\text{diam } E_i\} \geqq \|x\| \min\{\text{diam } E_i\}$ for $x \in \boldsymbol{R}^{N+1}$; if we exclude the trivial case $X = \{0\}$ and set $\rho = \max\{\|x\| : x \in X\}$, we have $\text{diam }(Y) \geqq \rho \cdot \min\{\text{diam } E_i\}$, while $|E(X)| \leqq |E(\rho U)| = \rho^l |E(U)|$). Thus we see

$$||Y| - |Y'|| \leqq c\varepsilon = c\Delta(X, X')$$

with $c = (N\beta m''\mu_0)$. Q.E.D.

Now we prove the critical integral approximation needed to conclude the proof of Theorem 7.6. The proof of A.6 runs parallel to the discussion in the main section of Theorem 7.6.

THEOREM A.6. *In the notation of Theorem* 7.6 *we have*

$$\sum \left\{ \psi\left(\frac{1}{p}\lambda\right) : \lambda \in \Gamma \cap pD \right\} = \frac{p^k}{|\Gamma|} \int_D \psi(x)\, dx + O(p^{k-1} \log p).$$

REMARK. In our proof we will see that ψ is continuous on D, so ψ is integrable.

Proof. Recall our definition: $\psi(x) = |E(\pi^{-1}(x) \cap S)|$, all $x \in D$. By a minor alteration of A.5(ii) (compose E there with $(x_1, \ldots, x_N) \to (x_1, \ldots, x_N, 1 - \sum_i x_i)$ to get E as used in Theorem 7.6) we see that the map

(26) $$\pi^{-1}(x) \cap S \to \psi(x) \qquad (\text{note } \pi^{-1}(x) \cap S \in \Omega(\boldsymbol{R}^N))$$

satisfies a Lipschitz condition.

Now consider the mapping

(27) $$D \to \Omega(\boldsymbol{R}^N), \qquad x \to \pi^{-1}(x) \cap S,$$

where S is *any fixed compact convex set with interior* in $\boldsymbol{R}^N$ and $D = \pi(S)$. Let $\lambda_1 \neq \lambda_2$ both lie in $(pD)_\delta \subset \boldsymbol{R}^k$ for some $\delta > 0$ and consider the line L through λ_1, λ_2. Let λ_1', λ_2' be the points where L meets bdry (pD)—labelling things so the points appear in order $\lambda_1', \lambda_1, \lambda_2, \lambda_2'$ on L. Choose any points $x_1' \in \pi^{-1}(\lambda_1') \cap pS$ and $x_2 \in \pi^{-1}(\lambda_2) \cap pS$; we assert that there exists a point $x_1 \in \pi^{-1}(\lambda_1) \cap pS$ such that

$$\|x_1 - x_2\| < (p/\delta) \text{ diam } S \|\lambda_1 - \lambda_2\|.$$

In fact, we have $\alpha x_1' + (1-\alpha)x_2 \in pS$ for all $0 \leqq \alpha \leqq 1$; consider

$$x_1 = \frac{\|\lambda_1' - \lambda_1\|}{\|\lambda_1' - \lambda_2\|} x_2 + \frac{\|\lambda_1 - \lambda_2\|}{\|\lambda_1' - \lambda_2\|} x_1'.$$

This is in the fiber $\pi^{-1}(\lambda_1) \cap pS$ since

$$\frac{\|x_2 - x_1\|}{\|x_1' - x_2\|} = \frac{\|\lambda_1 - \lambda_2\|}{\|\lambda_1' - \lambda_2\|};$$

and clearly we have $\|x_1' - x_2\| < p \text{ diam } S$. Since $\lambda_2 \in (pD)_\delta$ we have $\|\lambda_1' - \lambda_2\| > \delta$ and our assertion follows. Therefore

$$(\pi^{-1}(\lambda_2) \cap pS) \subset (\pi^{-1}(\lambda_1) \cap pS) + (p/\delta) \text{ diam } S \|\lambda_1 - \lambda_2\| U$$

and similarly if we reverse λ_1, λ_2. Thus:

$$\Delta(\pi^{-1}((1/p)\lambda_1) \cap S, \pi^{-1}((1/p)\lambda_2) \cap S) \leqq (1/\delta) \operatorname{diam} S \|\lambda_1 - \lambda_2\|,$$

all $\lambda_1, \lambda_2 \in (pD)_\delta$. Hence (see (26)) there is a constant c such that

$$\left|\psi\left(\frac{1}{p}\lambda_1\right) - \psi\left(\frac{1}{p}\lambda_2\right)\right| \leqq (c/\delta)\|\lambda_1 - \lambda_2\| \quad \text{all } \lambda_1, \lambda_2 \in (pD)_\delta.$$

Now let Σ be a fundamental domain for Γ and choose constant c^* so

(i) $(\lambda + c^* U) \cap \Gamma \neq \varnothing$ all $\lambda \in \boldsymbol{R}^k$,

(ii) $\operatorname{diam}(\Sigma) < c^*$.

Then for each $\lambda \in (\Gamma \cap pD)_{\delta + c^*}$ we have $\lambda + \Sigma \subset (pD)_\delta$, which $\Rightarrow$ $(1/p)\lambda + (1/p)\Sigma \subset (D)_{\delta/p}$. But we see that

$$\begin{aligned}\int_{(1/p)\lambda + (1/p)\Sigma} \psi(x)\, dx &= \int_{(1/p)\lambda + (1/p)\Sigma} \psi((1/p)\lambda)\, dx + \int_{(1/p)\lambda + (1/p)\Sigma} \{\psi(x) - \psi((1/p)\lambda)\}\, dx \\ &= |\Gamma| p^{-k} \psi((1/p)\lambda) \\ &\quad + O(p^{-k} \max\{|\psi(x) - \psi((1/p)\lambda)| : x \in (1/p)\lambda + (1/p)\Sigma\}) \\ &= |\Gamma| p^{-k}\psi((1/p)\lambda) + O(1/\delta p^k),\end{aligned}$$

if $\lambda \in (\Gamma \cap pD)_{\delta + c^*}$. Upon using essentially the same decomposition of $\Gamma \cap pD$ employed in the discussion of Theorem 7.6, we obtain

$$\begin{aligned}\sum \left\{\psi\left(\frac{1}{p}\lambda\right) : \lambda \in \Gamma \cap pD\right\} &= \frac{p^k}{|\Gamma|}\int_D \psi(x)\, dx + O\left(p^{k-1} \sum_{i=1}^{N(p)} \frac{1}{i}\right) \\ &= \frac{p^k}{|\Gamma|}\int_D \psi(x)\, dx + O(p^{k-1}\log p)\end{aligned}$$

as asserted. Q.E.D.

REMARK. In case S is a *polyhedron*, which is the case in the context of Theorem 7.6, one may actually prove that the mapping (27) satisfies a Lipschitz condition (this generally fails to be true if S is not a polyhedron), and consequently one may improve the error theorem in A.6 to $O(p^{k-1})$. However, this will not improve the error term in Theorem 7.6, since the error there is dominated by the error term in formula (21). The existence of Lipschitz conditions on cross-sections of convex bodies will be discussed elsewhere.

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