

H^p SPACES ON BOUNDED SYMMETRIC DOMAINS⁽¹⁾

BY

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1. Introduction. Let D be a bounded symmetric domain in the complex vector space C^N and $0 \in D$. Any bounded symmetric domain D , furnished with the Bergman metric M , is a hermitian symmetric space (D, M) of noncompact type and is necessarily simply connected [3, p. 311]. Let Γ be the group of holomorphic automorphisms of D ; Γ is transitive on D and extends continuously to the topological boundary ∂D of D [7, p. 269]. The isotropy group $\Gamma_0 = \{\gamma \in \Gamma : \gamma(0) = 0\}$ of Γ is a compact subgroup of Γ and contains no normal subgroup of Γ . Thus D can be identified with the coset space Γ/Γ_0 . This realization of bounded symmetric domains enables us to study the structure of D , using the algebraic machinery of Lie groups. Any bounded symmetric domain may be represented as the topological product of irreducible bounded symmetric domains; the class of irreducible bounded symmetric domains consists of four types of classical Cartan domains and two exceptional ones.

A bounded symmetric domain D is circular and star-shaped with respect to the origin, that is, $tz \in D$ when $z \in D$ and $t \in C$ with $|t| \leq 1$ [7]. It has Bergman-Silov boundary b which is circular and invariant under Γ [7]. The group Γ_0 is transitive on b [13, p. 922] and b has a unique normalized Γ_0 -invariant measure μ , which is given by $d\mu_t = V^{-1} ds_t$, V the euclidean volume of b and ds_t the euclidean volume element at $t \in b$.

A complex-valued function $h: D \rightarrow C$ is harmonic on D if $\Delta h = 0$ for each Γ -invariant differential operator Δ of the hermitian space (D, M) [6, p. 340].

For $p > 0$ the Hardy space H^p is defined on D by

$$H^p \equiv H^p(D) = \left\{ f : f \text{ holomorphic on } D \text{ and } \sup_{0 \leq r < 1} \left(\frac{1}{V} \int_b |f(rt)|^p ds_t \right)^{1/p} = M < \infty \right\}$$

and the space \tilde{H}^p by

$$\tilde{H}^p \equiv \tilde{H}^p(D) = \{ f : f \text{ holomorphic on } D \text{ and } |f(z)|^p \leq h(z) \text{ on } D, h \in \mathfrak{H}(D) \},$$

where

$$\mathfrak{H} \equiv \mathfrak{H}(D) = \left\{ h : h(z) = \int_b P(z, t) \phi(t) ds_t \equiv (P_z, \phi)_b, z \in D, \phi \in L(b) \right\},$$

$P(z, t)$ the Poisson kernel of the domain D .

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In §2 we give several properties of the spaces H^p and \tilde{H}^p and prove that these spaces are equivalent for bounded symmetric domains. Rudin pointed out this equivalence for C^1 in [8]. In §3 we show that H^p is a Banach space for $p \geq 1$ and a complete linear Hausdorff space for $0 < p < 1$, thus generalizing the results for the unit disc [11]. §§4 and 5 consider properties of linear functionals on H^p . For other treatments of H^p spaces ($p \geq 1$) on bounded symmetric domains see [6].

2. Properties of the spaces H^p and \tilde{H}^p .

1. Let

$$D_r = \{rz : z \in D\}, \quad b_r = \{rz : z \in b\}.$$

Since D is star-shaped with respect to 0, $\bar{D}_r \subset D$ if $0 < r < 1$ and $\lim_{r \rightarrow 1} D_r = D$. Also any compact subset K of $D \subset D_r$ for some $r < 1$.

THEOREM 1. Let $u(z)$ be defined on D_r and $v(\zeta) = u(r\zeta R^{-1})$. Then $u \in \mathfrak{S}(D_r)$ if and only if $v \in \mathfrak{S}(D_R)$.

Proof. This follows since under the transformation $z = r^{-1}\zeta R$, $u(z) = (P_{r,z}, \phi)_{b_r}$, $\phi \in L(b_r)$, goes into $v(\zeta) = (P_{R,z}, \psi)_{b_R}$, where $\psi(v) = \phi(rvR^{-1}) \in L(b_R)$ and conversely ($P_{R,z}$ and $P_{r,z}$ the respective Poisson kernels of D_R and D_r).

THEOREM 2. A function $u \in \mathfrak{S}(D)$ is harmonic on D .

Proof. By definition the Poisson kernel is

$$(1) \quad P(z, t) = |S(z, \bar{t})|^2 / S(z, \bar{z}),$$

where S is the Szegő (or Cauchy) kernel of D . By [5, p. 88]

$$(2) \quad S(z, \bar{t}) = \sum_{k=0}^{\infty} \sum_{v=1}^{m_k} \phi_v^{(k)}(z) \overline{\phi_v^{(k)}(t)}, \quad ((z, t) \in D \times b),$$

where $\{\phi_v^{(k)}\}$ is a complete orthonormal system of homogeneous polynomials on D , orthonormalized with respect to b , and $\phi_0 = \phi_0^{(0)} = V^{-1}$. Since the convergence of series (2) is uniform on compact subsets of $D \times \bar{D}$ [5, p. 89], $S(z, \bar{t})$ is holomorphic in (z, \bar{t}) on $D \times D$ and continuous on $D \times \bar{D}$. In particular, $S(z, \bar{z})$ is holomorphic in (z, \bar{z}) for $z \in D$ and from (2)

$$(3) \quad S(z, \bar{z}) \geq |\phi_0(z)|^2 = V^{-1}.$$

By the Weierstrass theorem [4, p. 6] $D_j S(z, \bar{t})$ is holomorphic on $D \times D$ and continuous on $D \times \bar{D}$ and $D_{j\bar{k}}^2 S(z, \bar{z})$ is holomorphic in (z, \bar{z}) for $z \in D$. Thus the derivatives $D_{j\bar{k}}^2 P(z, t)$ are bounded on $K \times b$, K a compact subset of D , and by the Lebesgue dominated convergence theorem $\Delta_z u = (\Delta_z P_z, \phi) = 0$ ($\Delta = \Delta_z$ at z), since $P(z, t)$ is harmonic in z for $z \in D$, $t \in b$ [6, Theorem 3.5].

2. Equivalence of H^p and \tilde{H}^p .

LEMMA 1. Let u be a plurisubharmonic (psh) function on D and set $u_r(z) = u(rz)$ for $0 < r < 1$ and $z \in D$. Then

$$(4) \quad u_r(z) \leq (P_z, u_r).$$

Proof. Since u is psh on D , u_r is psh on \bar{D} for $r < 1$. Thus

$$(5) \quad u_r(0) \leq \frac{1}{2\pi} \int_0^{2\pi} u_r(te^{i\theta}) d\theta$$

for $t \in b$ [10, p. 73]. Let $\gamma \in \Gamma$ and $\gamma(z) = 0$. Set $U_r(w) = u_r(\gamma^{-1}(w))$. Since psh functions are invariant under biholomorphic mappings [10, p. 81], U_r is psh on \bar{D} so that $U_r(0)$ satisfies (5). Integrate over b and use Fubini's theorem

$$(6) \quad U_r(0)V \leq \frac{1}{2\pi} \int_b ds_t \int_0^{2\pi} U_r(te^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_b U_r(te^{i\theta}) ds_t = \int_b U_r(t') ds_{t'},$$

since b is circular and $ds_{t'} = ds_t$ under $t' = te^{i\theta}$. Thus

$$u_r(z) \leq \frac{1}{V} \int_b u_r(\gamma^{-1}(t')) ds_{t'}.$$

Set $t = \gamma^{-1}(t')$. Then $b = \gamma^{-1}(b)$ and by the invariance of the measure $P(z, t) ds_t$ under γ

$$(7) \quad (1/V) ds_{t'} = P(0, t') ds_{t'} = P(z, t) ds_t$$

[6, p. 339]. Thus (4) follows.

LEMMA 2. *The function*

$$(8) \quad M(r) = \frac{1}{V} \int_b |f(rt)|^p ds_t$$

is a monotone nondecreasing function of r on $[0, 1)$.

Proof. Since f is holomorphic on D , f_r is holomorphic on \bar{D} and $|f_r|^p$ is psh on \bar{D} for $p > 0$ [10, p. 74]. As in (6)

$$(9) \quad I = \frac{1}{2\pi} \int_b ds_t \int_0^{2\pi} |f(rte^{i\theta})|^p d\theta = \int_b |f(rt)|^p ds_t = VM(r).$$

By the definition of psh, $|f(\lambda t)|^p$ is subharmonic with respect to λ in every component of the open set $O_t = \{\lambda : \lambda t \in D\}$. From [10, p. 62] the function

$$(10) \quad m(rt) = \frac{1}{2\pi} \int_0^{2\pi} |f(rte^{i\theta})|^p d\theta,$$

which is the mean value of a subharmonic function, is a nondecreasing function of r and convex with respect to $\log r$ for all t . By (9) and (10) for $r < r'$

$$M(r) = \frac{1}{V} \int_b m(rt) ds_t \leq \frac{1}{V} \int_b m(r't) ds_t = M(r').$$

Also $M(r)$ is convex with respect to $\log r$ in $(0, 1)$.

THEOREM 3. *For $p > 0$, $f \in H^p$ if and only if $f \in \tilde{H}^p$.*

Proof. Since $|f|^p$ is psh on D for $p > 0$, (4) holds for $|f_r|^p$, $0 < r < 1$.

Let $f \in H^p$. Since ds_t is a finite Borel measure and b is circularly invariant, by a result of Bochner [1, Theorems 2, 3] there exists a function ψ , measurable on b , such that

$$\lim_{r \rightarrow 1} \int_b |f(rt) - \psi(t)|^p ds_t = 0.$$

Since $P_z(t)$ is uniformly continuous and nonnegative on the compact set b ,

$$(11) \quad \lim_{r \rightarrow 1} \int_b |f(rt) - \psi(t)|^p P(z, t) ds_t = 0.$$

From (11) follows by Minkowski's inequality for $p \geq 1$ and the inequality

$$(12) \quad (a + b)^p \leq a^p + b^p,$$

$a, b \geq 0$ and $0 < p < 1$ [11] that

$$(13) \quad \lim_{r \rightarrow 1} (P_z, |f_r|^p) = (P_z, |\psi|^p).$$

In particular since $P(0, t) = V^{-1}$ for $t \in b$, $\psi \in L^p(b)$. Let $r \rightarrow 1$. From the continuity of f on D and (4) and (13) follows

$$(14) \quad 0 \leq |f(z)|^p = \lim_{r \rightarrow 1} |f_r(z)|^p \leq \lim_{r \rightarrow 1} (P_z, |f_r|^p) = (P_z, |\psi|^p) = u^*(z).$$

Since $u^* \in \mathfrak{H}(D)$, $f \in \tilde{H}^p$.

REMARK. (14) is proved in [12] with reference to [1] for a method of proof of inequality (4) for $|f_r|^p$. However no details are given in [1].

Conversely let $f \in \tilde{H}^p$. Then $|f(z)|^p \leq h(z) = (P_z, \phi)$, $\phi \in L(b)$ and hence $|f_r(t)|^p \leq h_r(t)$ on \bar{D} for $r < 1$. Integrate over b and use Fubini's theorem. Then

$$\begin{aligned} \frac{1}{V} \int_b |f_r(t)|^p ds_t &\leq \frac{1}{V} \int_b h_r(t) ds_t = \frac{1}{V} \int_b \int_b P(rt, v) \phi(v) ds_v ds_t \\ &= \frac{1}{V} \int_b \phi(v) ds_v = h(0) < \infty, \end{aligned}$$

since $P(rt, v) = P(rv, t)$ for $v, t \in b$ [5, Theorem 4.5.2] and $\int_b P(rv, t) ds_t = 1$. Thus $f \in H^p$.

Another necessary and sufficient condition that $f \in H^p$ is given by

THEOREM 4. Let $z_0 \in D$ and f be holomorphic on D . Let r , $0 < r < 1$, be such that $z_0 \in D_r$. Then $f \in H^p(D)$ if and only if there exists a constant $B(z_0)$, independent of r , such that

$$(15) \quad (P_{z_0}, |f_r|^p) \leq B(z_0).$$

Proof. The necessity of (15) follows from the uniform continuity of $P_{z_0}(t)$ on b and Lemma 2, namely

$$(16) \quad (P_{z_0}, |f_r|^p) \leq \max_{t \in b} P_{z_0}(t) VM(r) \equiv b_1(z_0) VM(r) \leq b_1(z_0) MV \equiv B(z_0).$$

Conversely, let $b_2(z_0) = \min_{t \in b} P_{z_0}(t)$; $b_2(z_0) > 0$ since $P_{z_0}(t) > 0$ on b . Proof. For any $t \in b$ there exists a holomorphic automorphism γ_t of D which takes $z_0 \rightarrow 0$ and $b \rightarrow b$. Let $t \rightarrow t'$. By (7) $P_{z_0}(t) > 0$. Since $t \in b$ was arbitrary $P_{z_0}(t) > 0$ on b . Then by (14)

$$\int_b |f_r|^p ds_t \leq b_2^{-1}(z_0)(P_{z_0}, |f_r|^p) \leq b_2^{-1}(z_0)B(z_0)$$

so that $f \in H^p$.

3. The topology of H^p spaces. For $p > 0$ set

$$\|f\|_p = \sup_{0 \leq r < 1} \left(\frac{1}{V} \int_b |f(rt)|^p ds_t \right)^{1/p}.$$

REMARK From (2.14) with $z=0$ and Lemma 2 follows $\|f\|_p = u^*(0)^{1/p}$, where $u^* \in \mathfrak{F}(D)$.

For $p \geq 1$ the triangle inequality follows for $f, g \in H^p$ by Minkowski's inequality and properties of sup. For $0 < p < 1$ Minkowski's inequality does not hold but (2.12) applied to f, g gives

$$(1) \quad \|f+g\|_p^p \leq \|f\|_p^p + \|g\|_p^p \quad (0 < p < 1).$$

LEMMA 3. Let $f \in H^p$. For any $z \in D$ there exists a constant $C(z)$, depending on z, p , and D but not on f , such that

$$(2) \quad |f(z)| \leq C(z)\|f\|_p.$$

For any compact set K of D there exists a constant $C = C(D, K, p)$, depending on D, K and p but not on f , such that for $z \in K$

$$(3) \quad |f(z)| \leq C\|f\|_p.$$

Hence if $\|f - f_n\|_p \rightarrow 0$ as $n \rightarrow \infty$, then $f_n \rightarrow f$ uniformly on compact subsets of D .

Proof. From (2.8) and (2.14) follows $|f_r(z)|^p \leq b_1(z)M(r) \leq b_1(z)V\|f\|_p^p$ for $z \in D$. Letting $r \rightarrow 1$ gives (2) with $C(z) = V^{1/p}b_1(z)^{1/p}$. If K is a compact subset of D , then

$$|f_r(z)|^p \leq \max_{z \in K} P(z, t)V\|f\|_p^p \equiv C^p(K, D, p)\|f\|_p^p$$

and (3) follows when $r \rightarrow 1$.

From (2), $\|f\|_p = 0$, implies $f(z) = 0$ on D and conversely. Thus H^p is a metric space for $p \geq 1$ and satisfies all the axioms except the triangle inequality for $0 < p < 1$. As usual a subset O of H^p is said to be open if for every $f_0 \in O$ there exists $\rho > 0$ such that $\{f : \|f - f_0\|_p < \rho\} \subset O$; for $p \geq 1$ this gives the usual topology induced by the metric. It is easy to prove that the Hausdorff separation axiom holds. Thus H^p is a linear Hausdorff space. From the last statement of Lemma 3 the completeness of H^p follows by well-known procedures from the completeness of C^1 , the triangle inequality for $p \geq 1$, and inequality (1) for $0 < p < 1$.

Hence:

THEOREM 5. For $p \geq 1$ H^p is a Banach space and for $0 < p < 1$ a complete linear Hausdorff space.

THEOREM 6. H^p is equivalent to a closed subspace of $L^p(b)$ (that is, there exists an algebraic isomorphism σ of H^p onto a closed subspace of $L^p(b)$, the isomorphism being norm-preserving).

Proof. From the monotonicity of $M(r)$ and (2.14) with $z=0$ follows

$$\|f\|_p = \lim_{r \rightarrow 1} M(r)^{1/p} = \left(\frac{1}{V} \int_b |\psi(t)|^p ds_t \right)^{1/p} \equiv \|\psi\|_p.$$

Define a mapping σ by $\sigma(f) = \psi$. From $\|f\|_p = \|\psi\|_p$ follows σ is 1-1 from H^p onto a subspace of $L^p(b)$. Also $\sigma(H^p)$ is closed in $L^p(b)$ [11, p. 802].

A final property is that H^p spaces are perfectly separable as follows by the same proof as in [11].

4. Linear functionals. Let γ be a functional in H^p . Then $\gamma \in (H^p)^*$ if and only if γ is bounded on the unit sphere in H^p . Topologize $(H^p)^*$ by setting $\|\gamma\| = \sup_{\|f\|=1} |\gamma(f)|$. Then $(H^p)^*$ is a Banach space [11]. The class $[L^p(0, 2\pi)]^*$ of linear functionals on $L^p(0, 2\pi)$, $0 < p < 1$, contains only the zero functional but, as in the case of the unit disc, $(H^p)^*$ contains other elements [11]. Set

$$\gamma_{n,z}^{(v)}(f) = (1/n!) D_z^n f(z), \quad v = 1, \dots, m_n, \quad D_z^n = \partial^n / \partial z_1^{n_1} \cdots \partial z_N^{n_N},$$

where m_n is the number of derivatives of order n . Similar to Theorem 6 in [11], we obtain precise bounds for the norms $\|\gamma_{n,z}^{(v)}\|$. Such bounds will be used in the proof of Theorem 9 and later papers.

THEOREM 7. $\gamma_{n,z}^{(v)} \in (H^p)^*$ ($n=0, 1, 2, \dots; v=1, \dots, m_n$) for $z \in D$ and

$$(1) \quad \|\gamma_{0,z}\| \leq 1/(1-r_z)^{2N/p},$$

$$(2) \quad \|\gamma_{n,z}^{(v)}\| \leq \frac{r_{n,z}^{N/2} V^{1/2}}{n! (r_{n,z} - r_z)^{N/2} (1 - r_{n,z})^{2N/p}} [D_{z,z}^{2n} S(Z_{n,r}, \bar{Z}_{n,r})]^{1/2} \quad (n > 0),$$

($Z_{n,r} = (r_{n,z} r_z)^{-1/2} z$), where r_z , $0 < r_z < 1$, depends on z only and $r_{n,z}$ is the value of r on $(r_z, 1)$, which minimizes the right side of (8). (If $z=0$, replace r_z by $\frac{1}{2}r_{n,z}$.)

Proof. (1) and (2) imply that $\gamma_{n,z}^{(v)} \in (H^p)^*$.

Proof of (1). Fix $r_1 \in (0, 1)$. By (2.4) applied to $|f_{r_1}|^p$ and (2.16)

$$(3) \quad |f_{r_1}(z)|^p \leq \max_{t \in b} P(z, t) V \|f\|_p^p.$$

From (2.1) and (2.2) and the fact that $\phi_v^{(k)}$ is homogeneous of degree k

$$P(z, t) = S(z, \bar{z})^{-1} \lim_n \left| \sum_{k=0, v}^n \phi_v^{(k)}(z r^{-1}) \overline{\phi_v^{(k)}(t) r^k} \right|^2,$$

where r is chosen so that $zr^{-1} \in \bar{D}$. By the maximum principle and Schwarz inequality the right side is

$$\begin{aligned} &\leq S(z, \bar{z})^{-1} \lim_n \max_{z \in b} \sum_{k=0, v}^n |\phi_v^{(k)}(z)|^2 r^k \sum_{k=0, v}^n |\phi_v^{(k)}(t)|^2 r^k \\ &\leq 1/V(1-r)^{2N} \end{aligned}$$

by (2.3) and [5, Theorem 4.5.1]. Since D is circular and star-shaped, it is clear that there is a unique $r_z \in [0, 1)$ such that $z \in \bar{D}_{r_z}$ and $z \notin D_r$ for $r < r_z$.

Thus

$$(4) \quad P(z, t) \leq 1/V(1-r_z)^{2N};$$

and from (3) and (4)

$$|f_{r_1}(z)| \leq (1-r_z)^{-2N/p} \|f\|_p.$$

(1) follows by letting $r_1 \rightarrow 1$.

Proof of (2). Since f is holomorphic on \bar{D}_r for $r < 1$, the Cauchy integral formula gives $f(z) = (S_{r,z}, f)_{b_r}$, ($z \in D_r$), $S_{r,z}$ the Szegő kernel of D_r . By [4, p. 7, Corollary 2] D_z^n and \int_{b_r} can be interchanged, giving

$$(5) \quad |D_z^n f(z)| \leq \int_{b_r} |D_z^n S_r(z, \bar{t})| |f(t)| ds_t \leq \max_{t \in b_r} |D_z^n S_r(z, \bar{t})| \int_{b_r} |f(t)| ds_t.$$

But

$$S_r(z, \bar{t}) = \sum_{k, v} \phi_v^{(k)}(r^{-1}z) \overline{\phi_v^{(k)}(r^{-1}t)} = \sum_{k, v} \phi_v^{(k)}(z) \overline{\phi_v^{(k)}(r^{-1}t)} r^{-k},$$

where the convergence is uniform for $z \in$ compact subsets of D_r and $t \in b_r$. By Weierstrass's theorem [4, p. 6]

$$D_z^n S_r(z, \bar{t}) = \sum_{k \geq n, v} D_z^n \phi_v^{(k)}(z) \overline{\phi_v^{(k)}(r^{-1}t)} r^{-k} = \sum_{k \geq n, v} D_z^n \phi_v^{(k)}(r_z^{-1}z) \overline{\phi_v^{(k)}(r^{-1}t)} \left(\frac{r_z}{r}\right)^k,$$

if $z, \neq 0, \in \bar{D}_{r_z} \subset D_r$. By the Schwarz inequality

$$\begin{aligned} (6) \quad |D_z^n S_r(z, \bar{t})|^2 &\leq \sum_{k \geq n, v} |D_z^n \phi_v^{(k)}(r_z^{-1}z)|^2 \left(\frac{r_z}{r}\right)^k \sum_{k, v} |\phi_v^{(k)}(r^{-1}t)|^2 \left(\frac{r_z}{r}\right)^k \\ &\leq \frac{r^N \mathcal{S}_n((rr_z)^{-1/2}z)}{V(r-r_z)^N}, \end{aligned}$$

where $\mathcal{S}_n(Z) = \sum_{k \geq n, v} |D_z^n \phi_v^{(k)}(Z)|^2$. If $z=0$, r_z can be replaced by $\frac{1}{2}r$, say. Now if $t \in b_r$, then $t \in \bar{D}_r$ and $\notin D_\rho$ for $\rho < r$. Thus in (4) we can take $r_t = r$ for all t . Hence by (3) with $r_1 = 1$ and (4)

$$(7) \quad |f(t)| \leq \|f\|_p / (1-r)^{2N/p}.$$

Using (6) and (7) in (5) gives

$$(8) \quad |D_z^n f(z)| \leq \frac{V^{1/2} r^{N/2} \mathcal{S}_n^{1/2}(Z)}{(r-r_z)^{N/2} (1-r)^{2N/p}} \|f\|_p \quad (Z = (rr_z)^{-1/2}z).$$

On $(r_z, 1)$ the function

$$Y(r) = r^{N/2}(r-r_z)^{-N/2}(1-r)^{-2N/p}\mathcal{S}_n^{1/2}(Z)$$

is positive and continuous and $\rightarrow \infty$ as $r \rightarrow 1^-$ and $\rightarrow r_z^+$, where $0 < \mathcal{S}_n^{1/2}(zr_z^{-1}) \leq \infty$. Hence $Y(r)$ assumes its minimum value on $(r_z, 1)$ on the compact set $I_\sigma = \{r_z + \sigma, 1 - \sigma\}$ if $\sigma > 0$ is sufficiently small, and this minimum value is positive. Suppose Y assumes its minimum value at $r = r_{n,z}$. Since $Z_{n,r} = (r_{n,z}r_z)^{-1/2}z \in D$, $\mathcal{S}_n(Z_{n,r}) = D_{z,z}^{2n}S(Z_{n,r}, \bar{Z}_{n,r})$ and (2) follows.

COROLLARY 1 [11]. *There exists a countable collection of linear functionals $\{\eta_n\}$ on H^p such that if $f \in H^p$ and $f \neq 0$, then there is an n with $\eta_n(f) \neq 0$.*

Proof. Let $z_0 \in D$ and set $\eta_n = \gamma_{n,z_0}$. Since $f \neq 0$ on D , by the identity theorem $f(z) \neq 0$ in any polydisc neighborhood N of $z_0 \in D$. Hence by the power series expansion of f in N some derivative $D_{z_0}^n f(z_0) \neq 0$. Thus $\gamma_{n,z_0}(f) \neq 0$.

THEOREM 8. *Let $F = \{f \in H^p : \gamma(f) \text{ is bounded on } F \text{ for fixed } \gamma \in (H^p)^*\}$. Then there exists $B > 0$, independent of f , such that*

$$\begin{aligned} (a) \quad & |\gamma(f)| \leq B\|\gamma\|, \\ (9) \quad (b) \quad & |f(z)| \leq B/(1-r_z)^{2N/p}, \\ (c) \quad & |\gamma_{n,z}^{(v)}(f)| \leq \frac{BV^{1/2}r_{n,z}^{N/2}}{n!(r_{n,z}-r_z)^{N/2}(1-r_{n,z})^{2N/p}} \{D_{z,z}^{2n}S(Z_{n,r}, \bar{Z}_{n,r})\}^{1/2} \quad (n > 0), \end{aligned}$$

$(Z_{n,r} = (r_{n,z}r_z)^{-1/2}z)$ for all $f \in F$. (If $z = 0$, set $r_0 = \frac{1}{2}r_{n,0}$.)

Proof. The proof of (a) uses only functional analysis and is the same as the proof of Theorem 7 in [11]. Inequalities (b) and (c) follow by setting $\gamma = \gamma_{n,z}^{(v)}$ and using (1) and (2).

5. Weak convergence. A sequence $\{f_n\} \subset H^p$ converges weakly to $f \in H^p$, $f_n \rightarrow {}^w f$, if $\lim_n \gamma(f_n) = \gamma(f)$ for every $\gamma \in (H^p)^*$. By Corollary 1 the limit is unique.

The following lemma is more general than necessary but has some independent interest. Let Δ_R be a polydisc of radius R and center 0.

LEMMA 4 (VITALI'S CONVERGENCE THEOREM FOR C^N). *Let $\{f_n\}$ be a sequence of holomorphic functions on the closed polydisc $\bar{\Delta}_R$, which are bounded independently of z and n on $\bar{\Delta}_R$. Also $f_n \rightarrow a$ limit as $n \rightarrow \infty$ on a set $\{z'\}$ with limit point 0 and such that for each j , $1 \leq j \leq N$, $\{z_j'\}$ is an infinite set. Then $\{f_n\}$ tends uniformly to a limit on compact subsets of Δ_R .*

Proof. It is sufficient to consider the case $N=2$. Then f_n , holomorphic on $\bar{\Delta}_R$, has a power series representation

$$f_n(z) = \sum_{j,k=0}^{\infty} a_{jk}^{(n)} z_1^j z_2^k \quad (z \in \bar{\Delta}_R),$$

where the convergence is absolute and uniform on compact subsets of $\bar{\Delta}_R$. Following the proof in [9] for $N=1$, we show that $\lim_n a_{jk}^{(n)}$ exists for each j, k and equals a_{jk} , say. By Schwarz's Lemma [2] and the uniform boundedness it follows as in [9] that $\{a_{00}^{(n)}\}$ is a Cauchy sequence and hence convergent. The function

$$g_{10}^{(n)}(z_1) = \sum_{j=1}^{\infty} a_{j0}^{(n)} z_1^{j-1} = \frac{f_n(z_1, 0) - a_{00}^{(n)}}{z_1}$$

($z_1 \neq 0$) satisfies the same hypotheses as $f_n(z)$.

Proof. The functions $f_n(z_1, z'_2)$ are holomorphic and uniformly bounded independently of n and z_1 on the closed disc $|z_1| \leq R$. Also 0 is a limit point of the set $\{z'_2\}$ and $\lim_n f_n(z'_1, z'_2)$ exists. Thus by Vitali's convergence theorem for $N=1$, $\lim_n f_n(z_1, z'_2)$ exists uniformly on compact subsets of $|z_1| < R$. Since also $\lim_{z'_2 \rightarrow 0} f_n(z_1, z'_2) = f_n(z_1, 0)$, the hypotheses of the Moore-Osgood theorem are satisfied so that $\lim_n \lim_{z'_2 \rightarrow 0} f_n(z'_1, z'_2) = \lim_n f_n(z'_1, 0)$ exists. Thus $\lim g_{10}^{(n)}(z')$ exists. Also $g_{10}^{(n)}$ is holomorphic on $\bar{\Delta}_R$ with a removable singularity at $z_1=0$. Now $|g_{10}^{(n)}(z_1)| \leq 2MR^{-1}$ on $|z_1| = R$ so that by the maximum principle $|g_{10}^{(n)}(z_1)| \leq 2MR^{-1}$ on $|z_1| \leq R$. Hence similarly as for $\{a_{00}^{(n)}\}$, $\lim_n a_{10}^{(n)}$ exists. By an analogous argument $\lim_n a_{j0}^{(n)}$ and $\lim_n a_{0j}^{(n)}$ exist for all $j \geq 1$. Next set

$$(1) \quad g_{11}^{(n)}(z) = \sum_{j,k=1}^{\infty} a_{jk}^{(n)} z_1^{j-1} z_2^{k-1} = \frac{f_n(z_1, z_2) - f_n(z_1, 0) - f_n(0, z_2) + f_n(0, 0)}{z_1 z_2},$$

$g_{11}^{(n)}(z)$ satisfies the same hypotheses as $f_n(z)$. $\lim_n g_{11}^{(n)}(z')$ exists since the four limits on the right side of (1) exist. Also $g_{11}^{(n)}$ is holomorphic on $\bar{\Delta}_R$ if $z_1 z_2 \neq 0$ and is locally bounded in the neighborhood of points $(z_1, 0)$, $(0, z_2)$, $(0, 0)$ since

$$\lim_{z_2 \rightarrow 0} g_{11}^{(n)}(z) = \sum_{j=1}^{\infty} a_{j1}^{(n)} z_1^{j-1} = z_1^{-1} [\partial f_n(z_1, 0) / \partial z_2 - \partial f_n(0, 0) / \partial z_2]$$

for $z_1 \neq 0$ and similarly for the other two limits. Thus by Riemann's theorem on removable singularities [2] $g_{11}^{(n)}$ is holomorphic on $\bar{\Delta}_R$. Also on $c = \{z : |z_j| = R, j=1, 2\}$, $|g_{11}^{(n)}(z)| \leq 4MR^{-2}$ and by the maximum principle for polydiscs $|g_{11}^{(n)}(z)| \leq 4MR^{-2}$ on $\bar{\Delta}_R$. Thus as above $\lim_n a_{11}^{(n)}$ exists. Similarly $\lim_n a_{jk}^{(n)}$ exists.

Finally we show that

$$(2) \quad \lim_n \sum_{j,k=0}^{\infty} a_{jk}^{(n)} z_1^j z_2^k = \sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k.$$

By the Cauchy inequality for derivatives [2] and the uniform boundedness follows $|a_{jk}^{(n)}| \leq MR^{-j-k}$ and hence $|a_{jk}| \leq MR^{-j-k}$. Thus the series on the right of (2) converges absolutely and uniformly on compact subsets of Δ_R . Now given $\varepsilon > 0$ there exists $K = K(\sigma, \varepsilon)$ such that

$$\sum_{j=K+1}^{\infty} \left(\frac{R-\sigma}{R} \right)^j < \frac{1}{2} \varepsilon A_{\sigma}, \quad A_{\sigma} = \frac{\sigma}{4RM} \quad (\sigma > 0),$$

and $N = N(K, \varepsilon)$ such that for $n > N$

$$|a_{jk}^{(n)} - a_{jk}| < \frac{1}{2}\varepsilon B_K, \quad B_K = \left(\frac{R-1}{R^{K+1}-1}\right)^2 \quad (j, k = 1, \dots, K).$$

Then

$$\begin{aligned} & \left| \sum_{j,k=0}^{\infty} a_{jk}^{(n)} z_1^j z_2^k - \sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k \right| \\ & \leq \left| \sum_{j,k=0}^K (a_{jk}^{(n)} - a_{jk}) z_1^j z_2^k \right| + \left| \left(\sum_{j=K+1}^{\infty} \sum_{k=0}^{\infty} + \sum_{j=0}^K \sum_{k=K+1}^{\infty} \right) (a_{jk}^{(n)} - a_{jk}) z_1^j z_2^k \right| \\ & \leq \sum_{j,k=0}^K |a_{jk}^{(n)} - a_{jk}| R^{j+k} + 2M \left(\sum_{j=K+1}^{\infty} \sum_{k=0}^{\infty} + \sum_{j=0}^K \sum_{k=K+1}^{\infty} \right) \left(\frac{R-\sigma}{R}\right)^j \left(\frac{R-\sigma}{R}\right)^k < \varepsilon \end{aligned}$$

for $n > N$. Thus (2) holds and Lemma 4 is proved.

We have

THEOREM 9. *If $f_n \rightarrow {}^w f$ in H^p , then $\lim_n f_n(z) = f(z)$ uniformly on compact subsets of D .*

Proof. Since $\lim_n \gamma(f_n) = \gamma(f)$ for $\gamma \in (H^p)^*$, $\{\gamma(f_n)\}$ is bounded independently of n . From inequality (4.9b) follows $|f_n(z)| \leq B(1-r)^{-2N/p}$ for $z \in \bar{D}_r$, which bound is independent of n and z . In particular $\gamma_{0,z}(f_n) \rightarrow \gamma_{0,z}(f)$, that is, $f_n(z) \rightarrow f(z)$ for $z \in \bar{D}_r$. Hence by Lemma 4, $f_n(z) \rightarrow f(z)$ uniformly on compact subsets of D_r ($r < 1$). Hence $f_n(z) \rightarrow f(z)$ uniformly on compact subsets of D . (Lemma 4 was proved for a polydisc but the compact set \bar{D}_r can be covered by a finite number of closed polydiscs and the conclusion of Lemma 4 will hold for \bar{D}_r also.)

REFERENCES

1. S. Bochner, *Classes of holomorphic functions of several complex variables in circular domains*, Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 721-723.
2. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.
3. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.
4. M. Hervé, *Several complex variables, local theory*, Oxford Univ. Press, Oxford, 1963.
5. L. K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*, Transl. Math. Monographs, Vol. 6, Amer. Math. Soc., Providence, R. I., Oxford, 1963.
6. A. Korányi, *The Poisson integral for generalized half-planes and bounded symmetric domains*, Ann. of Math. **82** (1965), 332-350.
7. A. Korányi and J. A. Wolf, *Realization of hermitian symmetric spaces as generalized half-planes*, Ann. of Math. **81** (1965), 265-288.
8. W. Rudin, *Analytic functions of class H_p* , Trans. Amer. Math. Soc. **78** (1955), 46-66.
9. E. C. Titchmarsh, *The theory of functions*, 2nd. ed., Oxford Univ. Press, Oxford, 1939.
10. V. S. Vladimirov, *Methods of the theory of functions of several complex variables*, MIT Press, Cambridge, Mass., 1966.

11. S. S. Walters, *The space H^p with $0 < p < 1$* , Proc. Amer. Math. Soc. **6** (1950), 800–805.
12. N. J. Weiss, *An isometry of H^p spaces*, Proc. Amer. Math. Soc. **19** (1968), 1083–1086.
13. J. A. Wolf and A. Korányi, *Generalized Cayley transformations of bounded symmetric domains*, Amer. J. Math. **87** (1965), 899–939.

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