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COHOMOLOGY FOR ERGODIC GROUPOIDS[1]

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Introduction. We construct a cohomology theory for the category of Mackey's ergodic groupoids and homomorphisms (but excluding certain homomorphisms considered by Mackey). Then various results of Mackey on ergodic groupoids and ergodic actions of groups are interpreted in the framework of this cohomology theory.

Preliminary definitions regarding Borel groupoids are given in §1. In §2 we define cohomology groups and induced homomorphisms for Borel groupoids. This is seen to be a generalization of the cohomology theory for Borel groups as done by C. C. Moore in [4] (assuming the action on the coefficient group is trivial). Next, in §3 we define the cohomology for ergodic groupoids, essentially by identifying Borel n-cochains which agree on some "inessential contraction." An exact sequence relates the ergodic cohomology and the Borel cohomology for the underlying Borel groupoid. For an ergodic groupoid, $\mathscr{F}$, C, and abelian ergodic coefficient group, A, we find $H^0(\mathscr{F}; A) \cong A$, and $H^1(\mathscr{F}; A) = \{\text{homomorphisms}: \mathscr{F} \to A\}$ mod similarity (or equivalently, mod principal homomorphisms). If ϕ and ψ are similar homomorphisms (as defined in this paper) $\mathscr{F} \to \mathscr{G}$ (ergodic groupoids) then we find that the induced maps ϕ^* and ψ^* $H^n(\mathscr{G}; A) \to H^n(\mathscr{F}; A)$ are the same. Accordingly, the cohomology theory may be of use in providing new invariants for classifying virtual groups, in connection with the first part of the program at the end of [3]. Finally, in §4 we consider the relative cohomology for a monomorphism $h: \mathscr{F} \to \mathscr{G}$ (ergodic groupoids). An exact sequence relates the relative cohomology, $H^n(\mathscr{G}, \mathscr{F}; A)$, to that of $\mathscr{G}$ and $\mathscr{F}$. The relative cohomology may be useful in classifying "imbeddings" of virtual groups into groups (which we convert into monomorphisms of ergodic groupoids), in connection with the second part of the program described at the end of [3]. In particular the constructions of §8 in [3] are related to $H^1(\mathscr{G}, \mathscr{F}; A)$ for the appropriate monomorphism, h.

§1

1.0 DEFINITION. A *Borel groupoid*, $\mathscr{F}$, consists of the following:

1.01 A set, $\mathscr{F}$, with a (not-necessarily transitive) groupoid structure, as defined for example in [1, pp. 132–133, omitting condition 6].

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1.02. A Borel structure on $\mathcal{F}$ such that (a) The domain of composition, $\mathcal{F}^2$, is a Borel subset of $\mathcal{F}\times\mathcal{F}$ (product Borel structure) and (b) the law of composition, $c: \mathcal{F}^2 \to \mathcal{F}$, and the inverse map, $I: \mathcal{F} \to \mathcal{F}$, are Borel maps.

1.1 REMARKS. We use upper case Greek letters for elements of $\mathcal{F}$ (except for the units of $\mathcal{F}$, which may be denoted by lower case letters), and we usually write $c(\Phi, \Psi)$ as $\Phi\cdot\Psi$ and $I(\Phi)$ as Φ^{-1}. Given an element $\Phi_i \in \mathcal{F}$ we denote the left unit of Φ_i by q_i and the right unit by p_i. Then $q_i=\Phi_i\cdot\Phi_i^{-1}$ and $p_i=\Phi_i^{-1}\cdot\Phi_i$, and $(\Phi_i, \Phi_j)\in\mathcal{F}^2$ iff $p_i=q_j$. Definition 1.0 agrees with that in [2], except that $\mathcal{F}$ is not required to be an analytic Borel space here.

Given a Borel groupoid, $\mathcal{F}$, we construct a sequence of Borel spaces, $\mathcal{F}^n$, for $n\geqq 0$.

1.2 DEFINITION. $\mathcal{F}^0$ = the set of units of $\mathcal{F}$ with the quotient Borel structure from $\sigma: \mathcal{F}\to\mathcal{F}^0$; $\sigma(\Phi_0)=q_0$. $\mathcal{F}^2$ is the domain of composition in $\mathcal{F}\times\mathcal{F}$ with the relative Borel structure (cf. 1.02a). $\mathcal{F}^1$ is defined $=\mathcal{F}$, and for $n\geqq 3$, $\mathcal{F}^n$ is defined as the Borel subspace of

$$\underbrace{\mathcal{F}\times\mathcal{F}\cdots\times\mathcal{F}}_{n \text{ copies}}$$

given inductively by $\mathcal{F}^n=\mathcal{F}^1\times\mathcal{F}^{n-1}\cap\mathcal{F}^{n-1}\times\mathcal{F}^1$.

1.21 REMARKS. We note that $\mathcal{F}^n$ for $n>0$ consists of the sequences of n elements from $\mathcal{F}$ such that composition is defined for adjacent elements. The Borel structure on $\mathcal{F}^0$ is easily seen to be the relative Borel structure for $\mathcal{F}^0$ as a subset of $\mathcal{F}$.

1.3 DEFINITION. The map

$$\sigma^n: \mathcal{F}^n \to \underbrace{\mathcal{F}^0\times\mathcal{F}^0\cdots\times\mathcal{F}^0}_{n+1 \text{ copies}}, \quad \text{for } n>0,$$

is defined by $\sigma^n(\Phi_0,\ldots,\Phi_{n-1})=(q_0,\ldots,q_{n-1},p_{n-1})$.

1.31 REMARKS. σ^n is a Borel map. σ^1 is injective iff $\mathcal{F}$ is a *principal* groupoid (cf. [3]). σ^1 is a surjective map iff $\mathcal{F}$ is a *transitive* groupoid. $\sigma^1(\mathcal{F}^1)$ is an equivalence relation in $\mathcal{F}^0\times\mathcal{F}^0$.

§2

2.0 DEFINITION. Let $\mathcal{F}$ be a Borel groupoid and A be an abelian Borel group (additive notation) such that $\{\Phi\}$ is a Borel set for every $\Phi\in A$. The *Borel cochain complex* for $\mathcal{F}$ with coefficients in A is

2.01. $C^n_B(\mathcal{F}; A)$ = {Borel maps $f: \mathcal{F}^n\to A$ such that if $n>0$ and $\Phi_i\in\mathcal{F}^0$ for some i, $0\leqq i\leqq n-1$, then $f(\Phi_0,\Phi_1,\ldots,\Phi_{n-1})=0$}, with an abelian group structure defined by pointwise addition of maps, and

2.02. The coboundary, $\delta^n: C^n_B(\mathcal{F}; A)\to C^{n+1}_B(\mathcal{F}; A)$, for $n>0$, is given by

$$\delta^n f(\Phi_0,\ldots,\Phi_n) = f(\Phi_1,\ldots,\Phi_n)+\sum_{i=1}^{n}(-1)^i f(\Phi_0,\ldots,\Phi_{i-1}\cdot\Phi_i,\ldots,\Phi_n)$$
$$+(-1)^{n+1}f(\Phi_0,\ldots,\Phi_{n-1})$$
$$\cdot\,\delta^0 f(\Phi_0)=f(p_0)-f(q_0).$$

2.03 REMARKS. $\delta^n f$ is a Borel map for $n \geqq 0$ by the Borel conditions on composition, inverse, and the choice of Borel structure for $\mathscr{F}^0$. The "normalization conditions" are easily verified for $\delta^n f$. The result $\delta^n \cdot \delta^{n-1} = 0$ is obtained by the usual computation, and then the cohomology groups are defined as usual—$Z^n_B(\mathscr{F}; A) = \{$"Borel n-cocycles"$\}$ = kernel of δ^n, $B^n_B(\mathscr{F}; A) = \{$"Borel n-boundaries"$\}$ = image of δ^{n-1} if $n \geqq 1$ and $= \{0\}$ if $n = 0$, and $H^n_B(\mathscr{F}; A) = Z^n_B(\mathscr{F}; A)/B^n_B(\mathscr{F}; A)$.

2.1 EXAMPLES. (a) $\mathscr{F}^2 = \mathscr{F} \times \mathscr{F}$ iff $\mathscr{F}^0 = \{e\}$ iff $\mathscr{F}$ is a group. If $\mathscr{F}$ is a group then the Borel cochain complex defined here is essentially the same as in [4], with the restriction that the action of $\mathscr{F}$ on A is trivial. I have in mind a cochain complex based on an "action" of a Borel groupoid on an "abelian Borel groupoid" which would extend the group cohomology theory of [4], and which may have interesting applications, but we avoid such complications here.

(b) $C^0_B(\mathscr{F}; A) = \{$Borel maps $f: \mathscr{F}^0 \to A\}$, and $\delta f \equiv 0$ iff f = constant on equivalence classes of $\sigma^1(\mathscr{F}^1)$, i.e. $f(\Phi \cdot \Phi^{-1}) = f(\Phi^{-1} \cdot \Phi)$ for all $\Phi \in \mathscr{F}$.

(c) $Z^1_B(\mathscr{F}; A) = \{F: \mathscr{F} \to A$ such that F is a Borel homomorphism$\}$, and $B^1_B(\mathscr{F}; A) = \{$Borel homomorphisms of the form $\Phi_0 \to f(p_0) - f(q_0)$ for a Borel map $f: \mathscr{F}^0 \to A\}$ = $\{$"principal Borel homomorphisms"$\}$. Borel homomorphisms F and F' are Borel similar by Definition 2.2 iff $F - F' \in B^1_B(\mathscr{F}; A)$.

Suppose $\mathscr{F}$ and $\mathscr{G}$ are Borel groupoids and ϕ and ψ are Borel homomorphisms of $\mathscr{F}$ into $\mathscr{G}$.

2.2 DEFINITION. ϕ and ψ are *Borel similar* iff there exists a Borel map $\theta: \mathscr{F}^0 \to \mathscr{G}$ such that $\theta(q_0) \cdot \phi(\Phi_0) = \psi(\Phi_0) \cdot \theta(p_0)$ for all $\Phi_0 \in \mathscr{F}$.

2.21 REMARKS. The Borel homomorphisms ϕ and ψ induce maps ϕ^n and $\psi^n: \mathscr{F}^n \to \mathscr{G}^n$, and accordingly induce maps $(\phi^n)^*$ and $(\psi^n)^*: C^n_B(\mathscr{G}; A) \to C^n_B(\mathscr{F}; A)$, which are compatible with the coboundary homomorphisms, and hence induce ϕ^{n*} and $\psi^{n*}: H^n_B(\mathscr{G}; A) \to H^n_B(\mathscr{F}; A)$.

2.3 THEOREM. *If ϕ and ψ are Borel similar then $\phi^{n*} = \psi^{n*}$ for $n \geqq 1$.*

Proof. We obtain a cochain homotopy, $k^n: C^n_B(\mathscr{G}; A) \to C^{n-1}_B(\mathscr{F}; A)$, for $n \geqq 1$, as follows: $k^1 f(q) = f(\theta(q)^{-1})$ and for $n > 1$,

$$\begin{aligned} k^n f(\Phi_0, \ldots, \Phi_{n-2}) &= f(\theta(q_0)^{-1}, \psi(\Phi_0), \ldots, \psi(\Phi_{n-2})) \\ &\quad + \sum_{h=1}^{n-2} (-1)^h f(\phi(\Phi_0), \ldots, \phi(\Phi_{h-1}), \theta(q_h)^{-1}, \psi(\Phi_h), \ldots, \psi(\Phi_{n-2})) \\ &\quad + (-1)^{n-1} f(\phi(\Phi_0), \ldots, \phi(\Phi_{n-2}), \theta(p_{n-2})^{-1}). \end{aligned}$$

Extensive computations show that

$$\delta^{n-1} k^n f + k^{n+1} \delta^n f = ((\psi^n)^* - (\phi^n)^*) f.$$

2.4 EXAMPLE. Suppose $\mathscr{G} = X \times G \times X$, where G is a Borel group, X is a Borel space, $\mathscr{G}$ has the product Borel structure, and $(x, \Phi, y) \circ (y', \Psi, z)$ is defined for $y = y'$ to be $(x, \Phi \cdot \Psi, z)$, and $(x, \Phi, y)^{-1} = (y, \Phi^{-1}, x)$. Then $\mathscr{G}$ is a transitive Borel groupoid.

2.5 THEOREM. *Given* $\mathcal{G}$ *as above in Example* 2.4, $H^n_B(\mathcal{G}; A) \cong H^n_B(G; A)$ *for* $n \geqq 1$.

Proof. Choose a point $e \in X$, and define $\phi: G \to \mathcal{G}$ by $\phi(\Phi) = (e, \Phi, e)$. Define $\psi: \mathcal{G} \to G$ by $\psi(x, \Phi, y) = \Phi$. Then $\psi \cdot \phi$ is the identity map: $G \to G$ and $\phi \cdot \psi$ is Borel similar to the identity map: $\mathcal{G} \to \mathcal{G}$.

§3

From here on we assume that $\mathcal{F}$ is provided with a measure class, C, and that $\mathcal{F}, C$ is an ergodic groupoid as defined by Mackey in [3]. If S^0 is a Borel set in $\mathcal{F}^0$ such that $\sigma^{-1}(\mathcal{F}^0 - S^0)$ is a C-null set in $\mathcal{F}$, we obtain the inessential contraction (i.c.), $\mathcal{S} = \{\Phi_0 \in \mathcal{F} : q_0 \text{ and } p_0 \in S^0\} = \mathcal{F} \restriction S^0$ in the notation of [3]. Then $\mathcal{S}$ with the relative structure from $\mathcal{F}$ is also an ergodic groupoid with $\mathcal{S}^0 = S^0$ and $\mathcal{S}^n \subseteq \mathcal{F}^n$.

3.0 DEFINITIONS. The *i.c. cochain complex* for $\mathcal{F}, C$ with coefficients in A is given by 3.01 $C^n_{ic}(\mathcal{F}; A) = \{f \in C^n_B(\mathcal{F}; A) \text{ such that } f \equiv 0 \text{ on } \mathcal{S}^n \text{ for some i.c. } \mathcal{S} \text{ of } \mathcal{F}\}$. The coboundaries for 3.01 and 3.02 are the homomorphisms induced by δ^n. The *ergodic cochain complex* for $\mathcal{F}, C$ with coefficients in A is given by

$$C^n(\mathcal{F}; A) = C^n_B(\mathcal{F}; A)/C^n_{ic}(\mathcal{F}; A). \tag{3.02}$$

3.1 REMARKS. The short exact sequences,

$$0 \to C^n_{ic}(\mathcal{F}; A) \to C^n_B(\mathcal{F}; A) \to C^n(\mathcal{F}; A) \to 0$$

for $n \geqq 0$, yield the long exact sequence

$$\cdots \to H^n_{ic}(\mathcal{F}; A) \to H^n_B(\mathcal{F}; A) \to H^n(\mathcal{F}; A) \to H^{n+1}_{ic}(\mathcal{F}; A) \to \cdots \tag{3.11}$$

of the corresponding cohomology groups.

The cohomology groups, $H^n(\mathcal{F}; A)$ are of interest in the study of virtual groups, as defined in [3].

3.2 EXAMPLES. (a) Let $\mathcal{G} = X \times G \times X$ be the Borel groupoid of Example 2.4, assuming X and G are analytic. Let $\tilde{C}$ be a measure class on X, and C' be a measure class on G invariant under left composition and the inverse map. Then $\mathcal{G}$ with the product measure class, $\tilde{C} \times C' \times \tilde{C}$, is a transitive ergodic groupoid.

(b) Suppose we have an ergodic action of G on X (with respect to $\tilde{C}$). Let $\mathcal{F} = \{(x, \Phi, y) \in X \times G \times X \text{ such that } \Phi(y) = x\}$. Then $\mathcal{F}$ is the graph of the action of G on X, and hence is a Borel subset of $X \times G \times X$. Also, $\mathcal{F}$ is closed under composition and inverse and hence inherits a groupoid structure from $X \times G \times X$. The map $\mathcal{F} \to X \times G$; $(x, \Phi, y) \to (x, \Phi)$ is a Borel isomorphism and with the measure class C carried back from the product measure class $\tilde{C} \times C'$, $\mathcal{F}$ is an ergodic groupoid. This is the example obtained in [3] and [2] for an ergodic action.

Suppose $\mathcal{F}, C$ and $\mathcal{G}, D$ are ergodic groupoids.

3.3 DEFINITION. A map $\phi: \mathcal{F} \to \mathcal{G}$ is a *strict homomorphism* iff ϕ is a Borel homomorphism of the underlying Borel groupoids and $\phi^{-1}(\mathcal{S})$ contains an i.c. of $\mathcal{F}$ whenever $\mathcal{S}$ is an i.c. of $\mathcal{G}$.

3.31 REMARKS. (a) If $\phi: \mathscr{F} \to \mathscr{G}$ is a strict homomorphism and $f: \mathscr{G}^n \to A$ is a Borel n-cochain $=0$ on $\mathscr{S}^n$ for an i.c., $\mathscr{S}$, of $\mathscr{G}$, then $(\phi^n)^*f=0$ on $\mathscr{R}^n$ for some i.c., $\mathscr{R} \subseteq \phi^{-1}(\mathscr{S})$, of $\mathscr{F}$. Accordingly, ϕ induces maps $\phi^{n*}: H^n(\mathscr{G}; A) \to H^n(\mathscr{F}; A)$ for $n \geqq 0$.

(b) The Definition 3.3 for a strict homomorphism requires more than that of [3] in some cases. For example, if $\{e\}$ is a $\tilde{C}$-null set in X in Example 3.2(a), then $G \to X \times G \times X$; $\Phi \to (e, \Phi, e)$ is not a strict homomorphism by 3.3 but is a strict homomorphism in [3].

The following definitions are as in [3], except for the reservation discussed in 3.31(b).

3.4 DEFINITION. A map $\phi: \mathscr{F} \to \mathscr{G}$ (ergodic groupoids) is a *homomorphism* iff ϕ is a Borel map and $\phi|_{\mathscr{S}}$ is a strict homomorphism for some i.c. $\mathscr{S}$ of $\mathscr{F}$.

3.41 DEFINITION. Strict homomorphisms ϕ and $\psi: \mathscr{F} \to \mathscr{G}$ (ergodic groupoids) are *strictly similar* iff the corresponding Borel homomorphisms are similar as in Definition 2.2.

3.42 DEFINITION. Homomorphisms ϕ and ψ are *similar* iff there exists an i.c., $\mathscr{S}$, of $\mathscr{F}$ such that $\phi|_{\mathscr{S}}$ and $\psi|_{\mathscr{S}}$ are strictly similar strict homomorphisms.

3.43 REMARKS. (a) If $\mathscr{F}$ and $\mathscr{G}$ are groups then ϕ and ψ are similar iff they differ by an inner automorphism of $\mathscr{G}$.

(b) Similarity is an equivalence relation.

(c) Referring to Example 2.1(c), $H^1(\mathscr{F}, A) = \{\text{homomorphisms}: \mathscr{F} \to A$ (satisfying 3.4)$\}$ mod similarity (as defined in 3.42). Here we regard A as an ergodic group. So A requires an appropriate measure class—but the measure class is irrelevant since any i.c. of A will $=A$.

3.5 THEOREM. *If $\mathscr{S}$ is an inessential contraction* (i.c.) *of $\mathscr{F}$ then the inclusion map $\phi: \mathscr{S} \to \mathscr{F}$ is a homomorphism and it induces isomorphisms $\phi^{*n}: H^n(\mathscr{F}; A) \to H^n(\mathscr{S}; A)$ for $n \geqq 0$.*

Proof. Note that ϕ induces bijective maps $(\phi^*)^n: C^n(\mathscr{F}; A) \to C^n(\mathscr{S}; A)$ for $n \geqq 0$.

3.51 THEOREM. $H^0(\mathscr{F}; A) \cong A$.

Proof. $f \in Z^0(\mathscr{F}; A) = H^0(\mathscr{F}; A)$ iff $f(q_0) = f(p_0)$ for almost all $\Phi_0 \in \mathscr{F}$. The ergodic condition (vi) on p. 205 of [3] implies $f=$constant a.e. The map $f \to$ the value of f assumed a.e.: $H^0(\mathscr{F}; A) \to A$ is bijective.

3.52 THEOREM. *If $\phi: \mathscr{F} \to \mathscr{G}$ (ergodic groupoids) is a homomorphism, then*

$$\begin{array}{ccc} H^0(\mathscr{G}; A) & \cong & A \\ \phi^{0*}\downarrow & & \vdots \\ H^0(\mathscr{F}; A) & \cong & A \end{array}$$

commutes where $\cong$ is defined as in the proof of 3.51.

3.53 Theorem. *If* $\mathcal{G}=X\times G\times X$ *is as in Example* 3.2(a), *then* $H^n_{\text{ic}}(\mathcal{G}; A)=0$ *for* $n\geqq 1$.

Proof. For $f\in Z^n_{\text{ic}}(\mathcal{G}; A)$ choose $e\in\mathcal{S}^0$, where $\mathcal{S}$ is an i.c. of $\mathcal{G}$ such that $f|_{\mathcal{S}^n}\equiv 0$. Let $\phi\colon\mathcal{G}\to\mathcal{G}$ be defined by $\phi(x,\Phi,y)=(e,\Phi,e)$. Then $f\cdot\phi^n\equiv 0$ on $\mathcal{G}^n$. Since ϕ is Borel similar to the identity map, by the proof of Theorem 2.3 we have $\delta^{n-1}(k^n f)=f$ and $k^n f\in C^{n-1}_{\text{ic}}(\mathcal{G}; A)$, hence $f\in B^n_{\text{ic}}(\mathcal{G}; A)$ for $n\geqq 1$.

3.54 Theorem. *If* $\mathcal{G}=X\times G\times X$ *as in Example* 3.2(a), *then* $H^n(\mathcal{G}; A)\cong H^n(G; A)$ *for* $n\geqq 0$.

Proof. For $n=0$, we apply Theorem 3.52. For $n\geqq 1$, the exactness of (3.11) and the fact that $H^n_{\text{ic}}(\mathcal{G}, A)=0$ imply that $H^n(\mathcal{G}; A)\cong H^n_{\text{B}}(\mathcal{G}; A)$. Then the results follow from Theorem 2.5 and the fact that $H^n_{\text{B}}(G; A)=H^n(G; A)$.

3.55 Theorem. *If* ϕ *and* $\psi\colon\mathcal{F}\to\mathcal{G}$ *(ergodic groupoids) are similar homomorphisms then* ϕ^{n*} *and* $\psi^{n*}\colon H^n(\mathcal{G}, A)\to H^n(\mathcal{F}; A)$ *are the same induced homomorphisms for* $n\geqq 0$.

Proof. By Remark 3.31(a) the induced homomorphisms are well defined. For $n\geqq 1$, the proof is the same as that of Theorem 2.3, after taking a suitable i.c. (which induces an isomorphism by Theorem 3.5). For $n=0$, refer to Theorem 3.52.

§4

Suppose $h\colon\mathcal{F}\to\mathcal{G}$ is an injective homomorphism (monomorphism) of ergodic groupoids and $h(\mathcal{F})$ is a Borel set in $\mathcal{G}$.

4.0 Definition. The *relative cochain complex* for $h=C^n(h; A)=C^n(\mathcal{G},\mathcal{F}, A)$ (to exhibit the ergodic groupoids) $=\{f\in C^n(\mathcal{G}; A)$ such that $f\cdot h\equiv 0\}$. The coboundary is the restriction of δ^n for $C^n(\mathcal{G}; A)$.

4.1 Remarks. The short exact sequences

$$0\longrightarrow C^n(h; A)\xrightarrow{\text{inclusion}} C^n(\mathcal{G}; A)\xrightarrow{h^*} C^n(\mathcal{F}; A)\longrightarrow 0$$

yield the long exact sequence

$$\begin{aligned}(4.11)\qquad &0\to H^1(h; A)\to H^1(\mathcal{G}; A)\to H^1(\mathcal{F}; A)\to H^2(h; A)\to\cdots\\ &\to H^n(\mathcal{G}; A)\to H^n(\mathcal{F}; A)\to H^{n+1}(h; A)\to H^{n+1}(\mathcal{G}; A)\to\cdots,\end{aligned}$$

noting that $0\to H^0(\mathcal{G}; A)\cong H^0(\mathcal{F}; A)\to 0$.

4.2 Example. Let $\mathcal{G}$ be as in Example 3.2(a), $\mathcal{F}$ be as in Example 3.2(b), and $h\colon\mathcal{F}\to\mathcal{G}$ be the inclusion map. Then $H^n(\mathcal{G}; A)$ in the exact sequence (4.11) may be replaced by the Borel group cohomology groups $H^n(G; A)$ by Theorem 3.54. If G is abelian we find that $H^1(\mathcal{G},\mathcal{F}; G_2)=A_{G_2}$ as defined in [3, §8]. Hence (as discussed in [3, §8]) if X admits a finite G-invariant measure, μ, then $H^1(\mathcal{G},\mathcal{F}; T)$ $\cong$ point spectrum of the action of G induced on $L_2(X,\mu)$ (where T is the circle group).

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